EQUIVARIANT K-THEORY AND COMPLETION

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1. Introduction

It was shown in [3] that, for any finite group $G$, the completed character ring $R(G)$ was isomorphic to $K^*(B_G)$ where $B_G$ denotes a classifying space for $G$. The corresponding result for compact connected Lie groups was established in [2], and a combination of the methods of [2] and [3] (together with certain basic properties of $R(G)$ given in [18]) can be used to derive the theorem for general compact Lie groups. Such a proof however would be extremely lengthy, the worst part being in fact the treatment for finite groups where one climbs up via cyclic and Sylow subgroups.

The purpose of this paper is to give a new and much simpler proof of the theorem about $K^*(B_G)$ which applies directly to all compact Lie groups $G$. The main feature of our new proof is that we generalize the whole problem in a rather natural way by working with the equivariant $K$-theory developed in [17]. We shall formulate and prove a general theorem about the completion $K(X)^G$ for any compact $G$-space $X$. The theorem about $R(G)$ then follows by taking $X$ to be a point.

The proof consists of four steps. First we deal with the case when $G = T$ is the circle group. Because of the simple model for $B_T$ given by the (infinite) complex projective space case this is easily dealt with directly. The second step is to pass from the circle to a general torus, and this is done in an obvious way by induction on the dimension of the torus. The third and final step shows how to reduce the case of the unitary group $U(n)$ to its maximal torus; this depends on the analytical methods, using elliptic operators, developed in [6].

The fourth and final step reduces the case of a general group $G$ to the case of a unitary group by means of an embedding $G \subseteq U$; we replace the $G$-space $X$ by the $U$-space $Y = U \simeq G \times X$. Thus, even if we are only interested in the case when $G$ is finite and $X$ is a point, we are forced at this stage to consider the Lie group $U$ and the $U$-space $U/G$.

Using the spectral sequence of [17] it would in principle be possible to pass from the case of a point to general $X$. However, as we have just explained, there is nothing to be gained by this procedure because the proof we give applies naturally to the general case.

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The real $K$-theory of $B_0$ has been determined by D. W. Anderson [1] starting from the result in the complex case; the results are expressed in terms of the real representations of $G$. Anderson's method was to use various exact sequences relating real and complex $K$-theory. Our new approach however makes such an indirect approach unnecessary. Provided we work with the $KR$-theory of (4) the proofs apply directly in the real case also.

There are a number of familiar technical difficulties associated with the fact that the classifying space $B_0$ is not a compact space and is also not strictly unique. Since $B_0$ is a limit of compact subspaces $B_k$, we can of course consider the inverse system of rings $K^n(B_k)$, but because we need to vary the models for $B_0$ and $B_2$ we find it convenient to go one stage further and introduce the pre-ring associated to these inverse systems. This is formally similar to the procedure of passing from a filtration to the associated topology and it has the same advantages.

In § 2 we state our main Theorem (2.1) and deduce a number of corollaries. The proof of the theorem is then given in § 3. In § 4 we consider the "genuine" $K^n$ of $B_0$, not just the $K$-groups of compact subsets—and we show how (2.1) can be reformulated in these terms; the resulting Proposition (4.2) is perhaps the most attractive version of the theorem. In § 5 we give another and somewhat weaker version concerned with $G$-maps which are homotopy equivalences.

The remainder of the paper is devoted to the real case. Thus in § 7 we review the equivariant form of $KR$-theory while in § 8 we state and prove theorem (7.1)—the real analogue of (2.1). We also examine in some detail the special case of (7.1) where $X$ is a point, and show how to recover the results of Anderson [1].

2. Statement of the theorems

We consider a compact $G$-space $X$, where $G$ is compact Lie group. Let $B_0$ be a classifying-space for $G$, and $E_0$ the corresponding universal $G$-space. To $X$ is associated a space $X_0 = (X \times E_0)/G$, determined up to homotopy, which is fibred over $B_0$ with fibre $X$.

Because the space $X_0$ is not compact there is some choice as to the definition of $K(X_0)$; this will be discussed in § 4. Until then all the propositions we shall prove will involve only compact spaces, and statements concerning $K$ of non-compact spaces should be interpreted for the moment as suggestive rather than precise.

Let $F$ be a $G$-vector-bundle [17] on $X$; then $(F \times E_0)/G$ is a vector-bundle on $X_0$. The assignment $F \mapsto (F \times E_0)/G$ is additive, so it induces a homomorphism $\alpha: K_0(X) \rightarrow K(X_0)$. We propose to prove that the groups $K_0(X)$ and $K(X_0)$ can be given topologies so that $\alpha$ is continuous, $K(X_0)$ is complete, and in suitable circumstances $\alpha$ induces a topological isomorphism of the completion $K_0(X)^\hat{}$ with $K(X_0)$. Thus the theorem tells one how $K_0$ of a
G-space can be approximated by K of an auxiliary space. When X is a point, \( K_\alpha (X) = R(G) \), the representation-ring, and \( X_\alpha = B_\alpha \), and the theorem reduces to that of \([2]\) and \([3]\).

Because G acts freely on \( X \times E_\alpha \), one can identify \( K(X) \) with \( K_\alpha (X \times E_\alpha) \) \([17]\). Then \( \alpha \) becomes the homomorphism of rings \( K_\alpha (X) \rightarrow K_\alpha (X \times E_\alpha) \) induced by the projection \( X \times E_\alpha \rightarrow X \). In this guise it can be manipulated more conveniently.

Two extreme cases may be worth pointing out. If G acts freely on X then \( X_\alpha \) is fibré over \( X \times G \) with fiber \( E_\alpha \), which is contractible; so \( X_\alpha \) is homotopy-equivalent to \( X \times G \), and the theorem reduces to the elementary fact \([17]\) that \( K_\alpha (X) \cong K(X \times G) \) when G acts freely on X. This is of course true without completion.

On the other hand, if G acts trivially on X, then \( X_\alpha \) is just \( X \times B_\alpha \), and \( K_\alpha (X) \cong K(X) \otimes R(G) \) \([17]\), so the theorem is the composite of the Künneth formula \( K(X \times B_\alpha) \cong K(X) \otimes K(B_\alpha) \), and the isomorphism \( K(B_\alpha) \cong R(G) \).

We shall prove the theorem in a somewhat different form from the one we have been discussing. This is partly for convenience—so that we can stay in the category of compact spaces—but also because the statement we shall obtain is a little more precise.

We shall use Milnor's model \([15]\) for the universal space of G. Thus \( E_G \) is the direct limit of the sequence of subspaces \( E_2 = G \times \cdots \times G \), the join of \( n \) copies of \( G \); and \( E_2 = E_2 \cap (e) \) is the union of the \( n \) contractible subsets \( U_i \), \( i \in I \), being the set where the \( i \)-th joint-coordinate does not vanish. This means that the product of any \( n \) elements of the reduced group \( R(G \text{-} R(B_\alpha)) \) is zero. Now \( R(G \text{-} R(B_\alpha)) \) is the kernel of the augmentation \( a: K^*(R(B_\alpha)) \rightarrow Z \), where \( a_\alpha \) is induced by \( E_\alpha \rightarrow \{ \text{point} \} \). The usual augmentation of \( R(G) \), whose kernel is the augmentation-ideal \( I_{E_G} \). So it appears that the natural map \( a_\alpha \): \( R(G) \rightarrow R(\text{K}^*(E_\alpha)) \), factorizes through \( R(G) \rightarrow I_{E_G} \).

For any G-space \( X \), \( K_\alpha (X) \) is a module over \( R(G) = \oplus E_\alpha \). (point), and by naturality the homomorphism \( a_\alpha \): \( K_\alpha (X) \rightarrow K_\alpha (X \times E_\alpha) \) induced by \( X \times E_\alpha \rightarrow X \) factorizes through \( K_\alpha (X) \rightarrow K_\alpha (X \times E_\alpha) \).

(*)

Let us recall that if \( R \) is a commutative ring with an ideal \( I \), and \( K \) is an \( R \)-module, then \( K \) can be given the \( I \)-adic topology, for which the submodules \( P^n \cdot K \) form a basis of the neighborhoods of \( 0 \). The Hausdorff completion \( \hat{K} \) of \( K \) for this topology can be identified with \( \lim_n K/P^n \cdot K \) \([7, Ch. X] \).

We propose to prove that the system of homomorphisms \( \alpha_\alpha \) induces an

\(^{1}\)Milnor gives \( E_\alpha \) a different topology. The limit topology is more convenient for our purposes, and \( G \) being compact it is still true that the \( G \)-action is continuous and that \( E_\alpha \) is contractible.
isomorphism of the inverse-limits $K\xi(X)^{\sim} \to \lim K\xi(X \times E_\xi)$. But somewhat more than that is true, and to formulate it it is convenient to work with what are called pro-objects [13].

If $C$ is any category, one can form a new category $\text{Pro}(C)$ whose objects are inverse-systems $\{A_n\}_{n \in I}$ of objects of $C$ indexed by directed sets $I$. To define a morphism from $\{A_n\}_{n \in I}$ to $\{B_n\}_{n \in J}$ one prescribes a map $\theta: I \to J$ (not necessarily order-preserving) and morphisms $\iota_\theta: A_{\theta(n)} \to B_n$ for each $n \in I$, subject to the condition that if $n \leq \theta(n)$ then for some $\mu \in I$ such that $\theta(n) \leq \mu \leq \theta(n)$, the diagram

\[
\begin{array}{ccc}
A_{\theta(n)} & \xrightarrow{\iota_\theta} & B_n \\
\downarrow a_{\theta(n)} & & \downarrow b_{\theta(n)} \\
A_n & \xrightarrow{\iota_\theta} & B_{\theta(n)}
\end{array}
\]

commutes ($a_{\theta(n)}: A_{\theta(n)} \to A_n$ and $b_{\theta(n)}: B_n \to B_{\theta(n)}$ being the structural maps of the inverse-systems). But one identifies the morphisms $(\theta; f_\theta)$ and $(\theta'; f_{\theta'})$ if for each $\beta$ there is an $\alpha \leq \beta$ such that $\alpha \geq \theta(\alpha)$, $\alpha \geq \theta'(\alpha)$, and $f_{\alpha, \beta} = f_{\alpha, \beta}'$. A pro-group should be thought of as much the same kind of thing as a topological group. In fact if $A$ is a topological group one can associate naturally to the pro-group $\{A/I_n\}$, where $I_n$ is the family of all open subgroups of $A$. And if $\{A_n\}$ is a pro-group one can associate to it the group $\lim A_n$, topologized as a subgroup of the product $\Pi A_n$, where each $A_n$ is given the discrete topology.

In this way we obtain two functors, $P$: (topological groups $\to$ (pro-groups), and $Q$: (pro-groups) $\to$ (topological groups); moreover $Q \circ P[A] \equiv A$ if and only if $A$ is Hausdorff and complete and has a neighbourhood-basis at its neutral element consisting of subgroups; while $P \circ Q[A_n] = \{A_n\}$ if and only if $\{A_n\}$ satisfies the Mittag-Leffler condition (see below). All the pro-objects which occur in this paper do satisfy the Mittag-Leffler condition, but that emerges from our proof, and is not evident a priori.

The system of homomorphisms $(\star)$ can be regarded as a morphism $(id; \varepsilon_\xi)$ in the category of pro-rings. Our main theorem is:

**Theorem 2.1.** Let $X$ be a compact $G$-space such that $K\xi^2(X)$ is fortiss over $R(G)$. Then the homomorphisms

\[ a_{\xi} : K\xi^2(X)/I_2 \to K\xi^2(X \times E_\xi) \]

induce an isomorphism of pro-rings.

That is to say, we shall prove that for each $n$ one can find $k$ and a homomorphism $\beta_n: K\xi^2(X \times E_\xi) \to K\xi^2(X)/I_2$, $K\xi^2(X)$ such that the diagram
\[ \mathbb{K}_\mathcal{X}(X)/\mathbb{I}_G \cong \mathbb{K}_G(X \times \mathbb{E}_G) \]
\[ \mathbb{K}_G(X)/\mathbb{I}_G \rightarrow \mathbb{E}_G \]
\[ \mathbb{K}_G(X)/\mathbb{I}_G \rightarrow \mathbb{E}_G \]

commutes.

**Remark.** It can be shown [17, Prop. (5.4)] that \( \mathbb{K}_\mathcal{X}(X) \) is finite over \( R(G) \)
when \( X \) is a compact differentiable manifold on which \( G \) acts smoothly, and, more generally, when \( X \) is locally \( G \)-contractible and \( X/G \) has finite covering dimension.

**Theorem (2.1) has several immediate corollaries as follows:**

**Corollary 2.2.** The homomorphisms \( \mathbb{a}_n \) induce an isomorphism \( \mathbb{K}_\mathcal{X}(X)^n \rightarrow \mathbb{K}_\mathcal{X}(X \times \mathbb{E}_G) \), the completion being in the \( I_\mathcal{X} \)-adic topology.

**Corollary 2.3.** Let \( K_n \) be the kernel of \( \mathbb{a}_n \); \( \mathbb{K}_\mathcal{X}(X) \rightarrow \mathbb{K}_\mathcal{X}(X \times \mathbb{E}_G) \). Then the sequence of ideals \( (K_n) \) defines the \( I_\mathcal{X} \)-adic topology on \( \mathbb{K}_\mathcal{X}(X) \).

**Proof.** We know that \( \mathbb{E}_G \mathbb{K}_\mathcal{X}(X) \subset K_n \). Theorem (2.1) implies that for each \( n \) there is a \( k \) such that \( K_{n+k} \subset \mathbb{K}_\mathcal{X}(X) \).

Let us recall [12, Chap. 0, §13] and [3, §3] that an inverse-system \( \{A_{\alpha} \}_{\alpha \leq \beta} \) is said to satisfy the Mittag-Leffler condition if for each \( \alpha \) the image of the map \( A_\beta \rightarrow A_\alpha \) is constant for large \( \beta \), i.e., for all \( \beta \) greater than some \( \beta_0 \). It is easy to show that \( \{A_{\alpha} \}_{\alpha \leq \beta} \) satisfies this condition if and only if it is isomorphic as \( \mathbb{E}_G \)-object to a system \( \{B_\beta \}_{\beta \leq \gamma} \) for which \( B_\beta \rightarrow B_\gamma \) is surjective for all \( \beta \leq \gamma \).

So we have

**Corollary 2.4.** The inverse-system \( \{K_n(X \times \mathbb{E}_G)\} \) satisfies the Mittag-Leffler condition.

This corollary will be important in §4.

Before proving the result of (2.1) we should make a remark about \( E_\alpha \). If \( E_\alpha \) is another universal G-space, G-homotopy-equivalent to \( E_\alpha \), which is the direct limit of a sequence of compact subspaces \( \{E_\beta\} \), then the G-homotopy-equivalences \( E_\beta \rightarrow E_\alpha \) take each \( E_\beta \) into some \( E_{\beta'} \). Each \( E_\alpha \) is defined by \( E_\alpha \). This means that the inverse-systems \( \{\mathbb{K}_\mathcal{X}(X \times \mathbb{E}_G)\} \) and \( \{\mathbb{K}_\mathcal{X}(X \times \mathbb{E}_G)\} \) define isomorphic \( \mathbb{E}_G \)-objects. Furthermore, one need not restrict oneself to sequences of subspaces: One can usually replace the family of all compact subspaces of \( E_\alpha \) by any cofinal subfamily.

We shall return to the version of the theorem involving \( E_\alpha \) or \( X_\alpha \) in §4.

3. **Proof of the theorem**

The proof consists of four steps.

**Step 1.** Proof when \( G \) is the circle-group \( \mathcal{T} \).
For the sake of step 2 we shall prove the slightly more general statement.

**Lemma 3.1.** Let $G$ be a compact Lie group, and $X$ a compact $G$-space such that $K_2(X)$ is finite over $R(G)$. Let $\theta: G \to T$ be a homomorphism by which $G$ acts on $E_p$. Then the homomorphisms
\[ \alpha_n: K_n(X)/\Theta_n \to K_n(X \times E_p) / \Theta_n, \]
induced by the projections $X \times E_p \to X$, define an isomorphism of pro-rings, (here $K_2(X)$ is regarded as an $R(T)$-module by means of $\theta$).

**Proof.** We can identify $E_p = \mathbb{T} \cdot \cdots \cdot \mathbb{T}$ with $S^{n-1}$, the unit sphere in $\mathbb{C}^n$, on which $T$ acts as a subgroup of the multiplicative group of $C^1$.

Consider the exact sequence for the pair $(X \times D^{n_1}, X \times S^{n-1})$. Since $D^{n_1}$ is contractible we have $K_n(X \times D^{n_1}) \simeq K_n(X)$, and by the Thom isomorphism [17] we have $K_n(X \times D^{n_1}, X \times S^{n-1}) \simeq K_n(X)$. Making these identifications the restriction $K_n(X \times D^{n_1}, X \times S^{n-1}) \to K_n(X \times D^{n_1})$ becomes multiplication by $\pi_1(C^n) = (1 - \rho)^n = \rho^n$, where $\rho$ is the standard one-dimensional representation of $T$. Thus one has an exact sequence
\[ 0 \to \mathbb{K}/\mathbb{K}^+ \to K_n(X \times S^{n-1}) \to \mathbb{K} \to 0, \]
where $\mathbb{K} = E_p$, and $\mathbb{K}$ is $\{ x \in \mathbb{K}; x = 0 \}$.

Because $\mathbb{K}$ generates the augmentation-ideal $I_\mathbb{K}$ to prove the lemma we must find for each $n$ a homomorphism $\beta_n: K_n(X \times S^{n-1}) \to \mathbb{K}/\mathbb{K}^+$ (for some $k$) making the diagram
\[ 0 \to \mathbb{K}/\mathbb{K}^+ \xrightarrow{\beta_n} K_n(X \times S^{n-1}) \xrightarrow{\rho_n} \mathbb{K} \to 0 \]
commute. But $K$ is a finitely generated module over the noetherian ring $R(G)$ [18, (3.3)], so one can find $k$ such that $\rho_n K = \rho_k K = \cdots$. Then $\rho^n$ annihilates $\rho_n K$ for any $n$, and the existence of $\beta_n$ is clear.

**Step 2.** Proof when $G$ is a torus $T^n$.

As in step 1 we shall prove the more general statement that when $G$ acts on $E_p$ by a homomorphism $\theta: T \to \mathbb{R}$ the homomorphisms $\alpha_n: K_n(X/E_p) \to K_n(X \times E_p)$ define an isomorphism of pro-rings. We proceed by induction on $m$, and write $T^n = T \times \mathbb{R}$.

In virtue of the remark at the end of § 2 we can replace $E_p$ by $E_p \times E_p$, and can use the cofinal system $\{ E_p \times E_p \}$ of compact subspaces. Furthermore we have $I_{E_p} \simeq a + b$, where $a$ and $b$ are the ideals of $R(T^n)$ generated by $I_{E_p}$ and $I_{E_p}$ respectively; and for any $R(T^n)$-module $K$ (here is an isomorphism of pro-rings $K/I_{E_p} K \simeq K(a + b)^{\infty}$). Because $a^n + b^n \subset (a + b)^n$ and
(a + b)^{p+1} \subset a^p + b^p. Thus we have to prove that the homomorphisms $\sigma_{pq}: K_{r}(a^p + b^p) \rightarrow K_{q}(X \times E_{r} \times E_{q})$, where $K_{r} = K_{r}(X)$, define an isomorphism of pro-rings.

Now $K_{r}(a^p + b^p) \cong K \otimes_{R} (R/a^p) \otimes_{R} (R/b^p)$, where $R = R(T^p)$; and $\sigma_{pq}$ can be factorized as:

$$K \otimes (R/a^p) \otimes (R/b^p) \xrightarrow{\text{iso}} K_{r}^Y(X \times E_{r} \otimes (R/b^p)) \xrightarrow{\text{iso}} K_{q}(X \times E_{r} \times E_{q}).$$

For each fixed $q$ the homomorphisms $\sigma_{pq}$ define, by step 1, an isomorphism of the "partial" pro-objects indexed by $p \leq q$. Similarly, if one holds $p$ constant, $\phi_{pq}$ induces an isomorphism of the partial pro-objects indexed by $q$ in virtue of the inductive hypothesis. (It follows from the exact sequence used in step 1 that $K_{r}^Y(X \times E_{r})$ is finite over $R(G)$ if $K_{r}(X)$ is.) So the result we want follows from two applications of

**Lemma 3.2.** If one has a homomorphism of inverse-systems $f: [M_{pq}]_{p \leq q < \infty} \rightarrow [N_{pq}]_{p \leq q < \infty}$ which for each $p$ induces an isomorphism of pro-objects $f_{p}: [M_{pq}]_{p \leq q < \infty} \rightarrow [N_{pq}]_{p \leq q < \infty}$ then $f$ is itself an isomorphism of pro-objects.

The proof of the lemma is trivial: in fact the pro-object $M_{q} = [M_{pq}]_{p \leq q < \infty}$ is the inverse-limit in the category of pro-objects of the pro-objects $M_{pq} = [M_{pq}]_{p \leq q < \infty}$.

**Step 3.** Proof when $G$ is a unitary group $U(m)$.

This step is the most important and it depends on the use of elliptic operators in [6]. More precisely we shall need Proposition (4.9) of [6] whose statement we now recall

**Proposition.** Let $\iota: T \rightarrow U$ be the inclusion of the maximal torus in the unitary group $U = U(m)$. For any compact $U$-space let $i^{jk}: K_{j}(X) \rightarrow K_{k}(X)$ be the map induced by $j$. Then there is a functorial homomorphism of $K_{j}(X)$-modules

$$i^{jk}: K_{j}(X) \rightarrow K_{k}(X)$$

which is a left inverse of $j^{k}$.

**Remarks.**

1) If $X$ is a point, $i^{jk}: R(T) \rightarrow R(U)$ is essentially "holomorphic induction": it assigns to a positive weight $j$ the irreducible representation of $U$ with maximal weight $j$. The construction of $i^{jk}$ for general $X$ given in [6] amounts to "holomorphic induction over a parameter space $X$".

2) Replacing $X$ by $X \times S$ we see that $K$ in the Proposition can be replaced by $K^\times$.

\footnote{Note that the embedding of a category in a corresponding pro-category does not commute with inverse-limits.}
3) A routine application of this Proposition (or alternatively of the closely related Thom isomorphism (4.8) of [6]) gives the full structure of \( K_\mathbb{Z}(X) \) over \( K_\mathbb{Z}(X) \): it is a free module of rank \( m' \). The arguments are identical with those in [5], (21)-one first deals with projective space bundles as in [6, (4.9, Remark 3)] or [17, (3.9)] and then decomposes the flag manifold into projective spaces as in the passage from (4.8) to (4.9) in [6]. All we shall need in fact is that \( K_\mathbb{Z}(X) \) is a finite module over \( K_\mathbb{Z}(X) \).

Because of \( \iota_\mathbb{Z} \) we know that for any compact \( U \)-space, \( K_\mathbb{Z}(X) \) is a natural canonical direct summand of \( K_\mathbb{Z}(X) \). Then from the diagram

\[
\begin{array}{ccc}
K_\mathbb{Z}(X) / \iota_\mathbb{Z} K_\mathbb{Z}(X) & \xrightarrow{\alpha_\mathbb{Z}} & K_\mathbb{Z}(X \times \mathbb{E}_2) \\
& \downarrow {_{P^2 }} & \downarrow {_{P^2 }} \\
K^* (X) / \iota_\mathbb{Z} K^* (X) & \xrightarrow{\gamma_\mathbb{Z}} & K^* (X \times \mathbb{E}_2)
\end{array}
\]

we find that \( \alpha_\mathbb{Z} \) defines an isomorphism of pro-rings if and only if \( \gamma_\mathbb{Z} \) does.

But in the diagram

\[
\begin{array}{ccc}
K^* (X) / \iota_\mathbb{Z} K^* (X) & \xrightarrow{\alpha_\mathbb{Z}} & K^* (X \times \mathbb{E}_2) \\
& \downarrow {_{P^2 }} & \downarrow {_{P^2 }} \\
K^* (X) / \iota_\mathbb{Z} K^* (X) & \xrightarrow{\gamma_\mathbb{Z}} & K^* (X \times \mathbb{E}_2)
\end{array}
\]

\( \alpha_\mathbb{Z} \) defines an isomorphism by step 2, \( \iota_\mathbb{Z} \) defines an isomorphism because the \( l_\mathbb{Z} \)-adic and \( l_\mathbb{Z} \)-adic topologies coincide on any \( R(T) \)-module (i.e., \( I_\mathbb{Z} \subseteq I_\mathbb{Z} \) and \( I_\mathbb{Z} = I_\mathbb{Z} \cdot R(T) \) for some \( k \), [18, (3.9)]), and \( \gamma_\mathbb{Z} \) defines an isomorphism by the remark at the end of §2, for \( E_\mathbb{Z} \) is a universal space for \( T \), and \( E_\mathbb{E}_2 \) is a cofinal system of compact \( T \)-subspaces. So \( \gamma_\mathbb{Z} \) defines an isomorphism as desired.

**Step 4. The general case**

This is now very simple. Embed \( G \) in a unitary group \( U \). If \( X \) is a compact \( G \)-space, then \( X = U \times \mathbb{X} = (U \times \mathbb{X}) / G \) is a compact \( U \)-space, and \( K^* (X) \) is naturally isomorphic to \( K^* (X) / \iota_\mathbb{Z} K^* (X) \) as \( (U \mathbb{Z}) \)-module [17, § 2 Example (iii)].

This completes the proof of Theorem (2.1).

To conclude this section we shall give an example to show that the finiteness-condition in the theorem is necessary, and cannot be replaced, for example,
by the assumption that $X$ is finite-dimensional. Observe first that the exact sequence (**) holds equally well when $G$ is replaced by a finite cyclic group. Taking inverse limits and remarking that $(K, K)$ has $R^i K = 0$ we get the exact sequence

$$0 \to K / \pi K \to \lim \pi K / \pi K(\mathbb{Z} \times \\mathbb{Z}) \to \lim \pi K \to 0.$$ 

Now take $G = \mathbb{Z}/2$, so that $R(G) = \mathbb{Z}[1/\phi(2)]$. If $G$ acts trivially on $X$, then $K(X) \cong K^*(X) \otimes K(G)$, and $\lim \pi K(\mathbb{Z} \times \mathbb{Z}) \cong \lim \pi K^*(X)$. We shall produce a two-dimensional compact space $X$ for which the last group does not vanish; in fact, $K^*(X) \cong \mathbb{Z} \oplus \mathbb{Z}[1/2]$. So $K^*(X)$ is a trivial $G$-space the homomorphism $K^*(X) \to \lim K_*(X \times \mathbb{Z})$ is accordingly not surjective. To construct $X$, first form the dyadic solenoid $S^1$, the inductive limit of the system $S^1 \to S^1 \to S^1 \cdots$, where $f$ has degree 2. Because $K^*$ is continuous, $K^*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}[1/2]$ The desired space $X$ is the mapping-torus of the projection $Y \to S^1$.

4. The space $X_0$

In this section we shall return to the original formulation of our theorem. We must begin by defining $K^*(X)$ for spaces $X$ which are not compact. We shall use the representable definition: i.e. $K^*(X) = \{X; F^*\}$, the group of homotopy-classes of maps $X \to F^*$, where $F^*$ is a suitable $H$-space. This leads to a satisfactory $\mathbb{Z}/2$-graded cohomology-theory on the category of paracompact spaces, but that does not concern us here. The only property we shall need is the following:

**Proposition 4.1 (Milnor).** If the space $X$ is the limit of an expanding sequence of compact subspaces $X_0 \subset X_1 \subset \ldots \ldots$, then there is a natural exact sequence

$$0 \to K^*(X_0) \to K^*(X) \to \lim K^*(X_n) \to 0.$$ 

($\lim$ is the derived functor of $\lim$).

\textbf{Proof.} Milnor has proved [16] that there is an exact sequence

$$0 \to K^*(X_0) \to K^* (T) \to \lim K^*(X_n) \to 0,$$

where $T$ is the telescope formed from the sequence $X_0 \subset X_1 \subset \ldots \ldots$. So one has only to show that the projection $T \to X$ induces an isomorphism $[K; F^*] \to [T; F^*]$. It is easy to see that this will be true providing all compact pairs have the homotopy extension property with respect to $F^*$. But that is immediate,
as one can suppose that the \( \mathcal{Y} \) are open subsets of a Banach space, or alternatively, that they are ANR's, or have the homotopy-type of CW-complexes.

**Remarks.** (a) The terms of the exact sequence \( \alpha \) do not depend on the sequence of compact subspaces \( X_\alpha \), for any such sequence is cofinal in any other, in the obvious sense.

(b) The proposition is true, with the same proof, in equivariant K-theory.

For \( K^*_G(X) \) representable: if \( X \) is compact then \( K^*_G(X) \cong \{ \alpha \mid \alpha \in \text{Map}(X; F_0) \} \), the group of G-homotopy-classes of G-maps \( X \to F_0 \), where \( F_0 \) is the G-space of Fredholm operators in a fixed separable G-Hilbert-space in which each simple G-module occurs with infinite multiplicity.

Now we come to our theorem.

**Proposition 4.2.** Let \( X \) be a compact G-space such that \( K^*_G(X) \) is finite over \( R(G) \). Then the homomorphism

\[
\alpha: K^*_G(X) \to K^*(X_0)
\]

induces an isomorphism of the \( I_0 \)-adic completion of \( K^*_G(X) \) with \( K^*(X_0) \).

**Proof.** The space \( X_0 \) is the limit of the sequence of compact subspaces \( X_\alpha = X \times \alpha E_0 \). From (2.1) we know that the sequence of groups \( K^*_G(X_\alpha) \)

\[
= K^*_G(X \times \alpha E_0)
\]

satisfies the Mittag-Leffler condition. Hence \( K^*_G(X) \) is isomorphic to its completion and hence complete and Hausdorff. But one can make a stronger statement.

**Proposition 4.3.** For any compact G-space \( X \) the following properties are equivalent:

1. \( K^*_G(X) \) is discrete in the \( I_0 \)-adic topology.
2. \( K^*_G(X) \) is complete and Hausdorff.
3. \( K^*_G(X) \) is complete\(^2\).
4. \( G \) acts freely on \( X \).

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) trivially.

(3) \( \Rightarrow \) (4). Let \( K^*_G(X) \) be complete and suppose that \( G \) does not act freely. Then there is a subgroup \( H \) of prime order \( p \) and a point \( x \in X \) with \( Hx = x \). The restriction homomorphism

\[
K^*_G(X) \to K^*_G(X^H) = R(H)
\]

makes \( R(H) \) into a \( K^*_G(X) \)-module, and a topological module when both rings are given the \( I_0 \)-adic topology. It is a finitely-generated module because \( R(H) \)

\(^2\)But not necessarily Hausdorff, i.e. Cauchy sequences have limits but these may not be unique.
is finite even over R(G) [18, (3.2)]. This implies [7, (10.13)] that R(H) is complete in the \( I_v \)-adic topology and hence in the \( I_v \)-adic topology, since these topologies coincide [18, (5.9)]. But, for a \( p \)-group, the \( I_v \)-adic topology on \( I_v \subset R(G) \) is just the \( p \)-adic topology on \( A_1 \subset K_1 \) and \( I_v \) is certainly not \( p \)-adically complete. This gives the required contradiction, so that \( G \) must act freely.

\((4) \Rightarrow (1)\). Because \( R(G) \) is noetherian the ideal \( I_v \) has a finite number of generators whose images in \( K_2(X) \) generate \( I_v, K_2(X) \) as an ideal in \( K_2(X) \). On the other hand \( K_2(X) \subset K(X \setminus G) \) because \( G \) acts freely; also the images in \( K(X \setminus G) \) of the generators of \( I_v \) have augmentation zero, and so are nilpotent by [5, (3.1.6)]. Thus the ideal \( I_v, K_2(X) \) is nilpotent, and \( K_2(X) \) is discrete.

5. Another version of the theorem

Proposition (4.2) implies the following result.

Proposition 5.1. Let \( X \) and \( Y \) be compact G-spaces such that \( K_2(X) \) and \( K_2(Y) \) are finite over \( R(G) \). Let \( f : Y \to X \) be a G-map which is \( \text{a homotopy-equivalence} \), but not necessarily a \( G \)-homotopy-equivalence. Then \( f^* : K_2(Y) \to K_2(X) \) induces \( \text{an isomorphism of the} \ I_v \text{-adic completions.} \)

Proof. By (4.2) it suffices to show that the induced map \( f_\theta : Y_\theta \to X_\theta \) is a homotopy-equivalence. But \( I_v \) is a map of fibre-spaces over \( B_\theta \), and is a homotopy-equivalence on each fibre. It follows from theorems of Dold [11, Theorems (6.3) and (6.4)] that \( I_v \) is a homotopy-equivalence.

Remark. Proposition (4.2) states that the projection \( X \times E_\theta \to X \), which is an example of a homotopy-equivalence but not of a \( G \)-homotopy-equivalence, induces an isomorphism \( K_2(X \times E_\theta) \to K_2(X \times E_\theta) \); but because \( E_\theta \) is not compact we cannot quite deduce (4.2) from (5.1). This version of the theorem should be complemented by two examples. First of all, if \( X \) and \( Y \) are as in (5.1) and \( f_\theta \); \( Y \to X \) are two G-maps which are homotopic but not \( G \)-homotopic, then it is not true that \( f^\theta \) and \( f^\theta \) necessarily induce the same isomorphism \( K_2(Y^\theta) \to K_2(X^\theta) \). For example, let \( Y \) be a point and \( X = M^\theta \), the one-point compactification of a complex G-module \( M \), and let \( f_\theta \); \( Y \to M^\theta \) take \( Y \) to \( 0 \) and \( \in \) respectively. There is a canonical element \( f_\theta \) in \( K_0(M) \) = \( K_0(M^\theta, \infty) \), defined by the exterior algebra [17, §3]. Let \( E_\theta \) be its image in \( K_0(M^\theta) \). Then \( f^\theta(E_\theta) = 0 \), while \( f^\theta(E_\theta) = \Sigma(-1)^\theta \wedge(M \otimes \in) \). and the latter element is in general not in the kernel of \( R(G) \to R(G^\theta) \).

The second example is more subtle. We shall produce a G-map \( Y \to X \) satisfying the conditions of (5.1) and such that \( f^\theta : K_2(Y) \to K_2(X) \) is not an isomorphism, though it becomes one on completion. Let \( G = \{ 1, g \} \) be cyclic of order two. For any odd prime number \( p \) one can find a 5-sphere \( Y \) with a differentiable involution whose fixed-point set \( F \) is a lens-space \( S^4(\mathbb{Z}/p) \) [10]
or [14]. Let \( x \) be a point of \( F \), \( T \) the tangent-space of \( Y \) at \( x \), and \( \mathcal{X} = T^* \) its one-point-compactionification. The group \( G \) acts linearly on \( T \), and one can choose a \( G \)-isomorphism between \( T \) and a neighbourhood of \( x \) in \( Y \). This isomorphism induces a \( G \)-map \( Y \to \mathcal{X} \) which is \( \mathcal{C} \), degree one, and therefore a homotopy-equivalence. We have \( K_0(X) \cong R(G) \otimes K_0(T) \cong R(G) \otimes K_0(\mathbb{R}^n) \), where \( g \) acts as \(-1 \) on \( \mathbb{R}^n \). So \( K_0(X) \cong R(G) \otimes K_0(\mathbb{R}^n) \cong R(G) \). We can show that \( K_0(Y) \) is not isomorphic to \( R(G) \) as follows. In \( R(G) \), let \( \mathcal{P} \) be the prime ideal of characters \( \chi \) such that \( \chi(a) \) is divisible by \( p \). The "support" of \( \mathcal{P} \) (in the sense of [17] or [18]) is \( G \) itself, so by the localization-theorem of [17, §4] the restriction \( K_0(T)_{\mathcal{P}} \to K_0(F)_{\mathcal{P}} \) is an isomorphism. Now \( K_0(F)_{\mathcal{P}} \cong K_0(F) \otimes R(G)_{\mathcal{P}} \) and \( K_0(F) \cong K_0(S^1) \), which can be calculated from the exact sequence for the pair \((D^2, S^1)\) as in [3], and is found to be

\[
R(Z; p) / (1, C) \cong \mathbb{Z}[a] / (a^p - 1, \theta - 1) \cong \mathbb{Z} / (Z^p)
\]

Now \( R(G) \) is \( \mathcal{P} \)-local, and the ideal \( \mathcal{P} \) is \( (p, \theta + 1) \); so \( R(G)_{\mathcal{P}} \cong \mathbb{Z}_p \) is \( \mathcal{P} \)-localized at \( p \), by the map taking \( a \) to \( -1 \). Thus \( K_0(T)_{\mathcal{P}} \cong \mathbb{Z}_p \otimes \mathbb{Z}/p \), which is different from \( R(G)_{\mathcal{P}} \).

6. Real equivariant \( K \)-theory

In this section and the next we shall prove the theorem in real \( K \)-theory which corresponds to Theorem (2.1). We use the term "real \( K \)-theory" in the sense of [4], and henceforth shall write \( \text{Real} \) with a capital to avoid confusion with the ordinary use of the word. Thus, for a compact space \( X \) with an involution \( x \mapsto x \), one defines \( \text{KR}(X) \) by considering complex vector-bundles \( E \) on \( X \) with a given involution \( E \mapsto E \) which takes fibres \( E_\xi \) antilinearly on to \( E_\xi \), for each \( \xi \in X \). But we need here an \( \text{equivariant} \) version of the theory.

In conformity with the spirit of [4], a \( \text{Real Lie group} \) will mean here a Lie group \( G \) with an involution \( g \mapsto g \). As usual we shall consider only compact Lie groups \( G \). A \( \text{Real G-space} \) is a \( G \)-space \( X \) with an involution such that \( g \cdot x = x \). A \( \text{Real G-vector-bundle} \) on \( X \) is a complex \( G \)-vector-bundle which is also a \( \text{Real space} \), and such that the projection is a \( \text{Real map} \). \( \text{KRG}(X) \) is the abelian group associated to the \( \text{setgroup of isomorphism-classes of Real G-vector-bundles on X} \). As usual, it becomes a ring under tensor-product. Beginning with \( \text{KRG}(X) \) one can define an \( \text{equivariant cohomology-theory} \) in the usual way: one uses both \( G \) and its involution act trivially on the suspension-coordinates. It turns out that \( \text{KRG}(X) \) is periodic in \( \xi \) with period \( b \), so the groups can be defined for all \( q \) [4], [6].

\(^{a}\) As in [57] we adopt the convention that \( K_0 \) means \( *K_0 \) with compact supports.

\(^{b}\) Thus \( K_0(T^*) \) means \( K_0(T^*)^{\text{eq}} \).

\(^{c}\) But as usual \( \text{KR}_G(X) \) will mean \( \bigoplus_{\xi} \text{KR}_G(\xi) \).
We should like to emphasize that we use the generalized Real $K$-theory and consider groups with involution not just for the sake of additional generality but because our proof can only be carried through in the wider setting.

Before proceeding to the theorem we should say something about the base-
ring $KR_{\sigma}$ (point) of the theory $KR_{\tau}$. This is the Real representation-ring $R_{\sigma}(G)$ of $G$, the free abelian group generated by the classes of simple Real $G$-modules, i.e., of complex $G$-modules with an antilinear involution compatible with that of $G$. If the involution of $G$ is trivial, $R_{\sigma}(G)$ is the usual real representation-
ring. In any case there is a "forgetful" map $\rho: R_{\sigma}(G) \to R(G)$, a homomorphism of rings, which one ordinarily calls "complexification", and also an additive homomorphism $\mu: R(G) \to R_{\sigma}(G)$ which takes the complex $G$-module $M$ to the "Real" $G$-module $M \oplus M$, on which $G$ acts by $g \cdot (x, x') = (g(x), g(x'))$, and whose involution is $(x, x') \mapsto (\overline{x}, \overline{x})$. The homomorphism $\mu$ is injective, because $\mu$ is multiplication by $2$; so one can identify $R_{\sigma}(G)$ with a subring of $R(G)$, and we shall do this.

Let us summarize a few facts about $R_{\sigma}(G)$ which correspond to properties of $R(G)$ established in [18].

**Proposition 6.1.**
(i) $R_{\sigma}(T) = R(T) \cong Z[X, X^{-1}]$, where $T$ is the group of complex numbers if modules $1$ with the usual complex conjugation.
(ii) $R_{\sigma}(U(n)) = R(U(n)) \cong Z[\lambda_1, \ldots, \lambda_n]$, where $U(n)$ is the unitary group with the usual complex conjugation.
(iii) Any Real Lie group $G$ has a Real embedding in $U(n)$ for some $n$.
(iv) If $H$ is a Real subgroup of $G$, then the restriction $R_{\sigma}(G) \to R_{\sigma}(H)$ makes $R_{\sigma}(H)$ a finite $R_{\sigma}(G)$-module, and the $IR_{\sigma}$-adic topology on $R_{\sigma}(H)$ coincides with its $IR_{\sigma}$-adic topology. ($IR_{\sigma} = R_{\sigma}(G) \cap IR_{\sigma}$)
(v) $R(G)$ is a finite module over $R_{\sigma}(G)$.
(vi) $R_{\sigma}(G)$ is a noetherian ring.

**Proof.** (i) and (ii) are true because the standard one-dimensional repre-
tsentation $X$ of $T$ and the exterior powers $\wedge^i$ of the standard representations of $U(n)$ are Real representations in our sense.
(iii) Let $M$ be a faithful complex $G$-module. Then $\rho(M) = M \oplus M$ is a faithful Real $G$-module, and $G$ is embedded in the unitary group of $M \oplus M$.
(iv) Because $G$ can be embedded in $U(n)$ it is sufficient to consider the case $G = U(n)$. Then $R_{\sigma}(G) = R(G)$, and is noetherian. The ring $R_{\sigma}(H)$ is a subring of $R(H)$, which is finite over $R(G)$ [18, 5.2.3], so $R_{\sigma}(H)$ is finite over $R(G)$. Furthermore $R(H)$ is finite over $R_{\sigma}(H)$, so by the Artin-Rees lemma [1, (10.11)] $R_{\sigma}(H)$ is a topological submodule of $R(H)$ when both are given the $IR_{\sigma}$-adic topology. But the $IR_{\sigma}$-adic topology on $R(H)$ lies between the $I_0$-adic and $I_1$-adic topologies; and these last two coincide by [18, 5.9].
So the $IR_{\sigma}$-adic and $IR_{\sigma}$-adic topologies coincide on $R(H)$, and hence on $R_{\sigma}(G)$. 

(v) has appeared in the course of the proof of (iv).
(vi) follows from (iv) because \( R_u(G) \) is finite over \( R(U(n)) \), which is noetherian.

Note. The groups \( T \) and \( U(n) \) will henceforth always be regarded as Real groups with the involutions described above.

7. The theorem in the real case

Let \( G \) be a compact Real Lie group. The involution of \( G \) induces an involution of the universal space \( E_G \) and of the classifying space \( B_G \), making them Real spaces. Then we have

**Theorem 7.1.** Let \( X \) be a compact Real \( G \)-space such that \( KR^*_G(X) \) is finite over \( R_u(G) \). Then the natural maps

\[
\alpha_n: KR^*_G(X) / I_R^n KR^*_G(X) \to KR^*_G(X \times E_G)
\]

induce an isomorphism of pro-rings.

**Remarks.**
(i) The theorem has a family of corollaries corresponding to those of (2.1), e.g. \( KR^*_G(X)_n \cong KR^*_G(X)_C \), where the completion is \( I_R \)-adic.

We shall not list them here.

(ii) In the next section we shall see that \( KR^*_G(\text{point}) \) is finite over \( R_u(G) \).

So \( KR^*_G(X) \) is finite over \( R_u(G) \) if and only if it is finite over \( KR^*_G(\text{point}) \).

The proof of (7.1) is the same as that of (2.1). Let us recapitulate briefly the steps.

Step 1. The proof when \( G = T \). Then \( E_T \) is \( S^{n-1} \), the unit sphere in \( C^n \), and its involution is complex conjugation. To determine \( KR^*_T(X \times S^{n-1}) \) we use the Thom isomorphism \( KR^{-}_T(X) \cong KR^*_T(X \times C^n) \), which is defined by the exterior algebra in the usual way, and is an isomorphism in accordance with the equivariant version of the real periodicity theorem (§6).

Step 2. The proof for a torus \( T^n \) presents nothing new.

Step 3. The passage from \( T^n \) to \( U(n) \) is made on Proposition (5.2) of [6] asserting the existence of a homomorphism \( T^n \rightarrow KR^*_G(X) \), such that \( \eta^n \) is as described in (6.1) this depends essentially on the fact that we are using KR-theory (and that \( U(n) \) is its natural Real structure). The fact that the corresponding result for \( KO^* \) is false was due to the difficulties encountered in trying to imitate the proof of [2] for KO-theory.

The remarks made in discussing Step 3 in the complex case apply here also.

Step 4. One treats a general Real group \( G \) by embedding it as a Real subgroup of \( U(n) \), using (6.1 (iii)).

8. The case when \( X \) is a point

When \( X \) is a point, Theorem (7.1) tells one that \( KR^*_G(\text{point}) \cong KR^*_G(B_G) \).

We know of course that \( KR^*_G(\text{point}) = R_u(G) \). The object of this section is to
describe explicitly the other groups \(KR_{\ast}(\text{point})\). But before doing so we must discuss one of the few significant departures of \(KR_{\ast}\)-theory from \(E_{\ast}\)-theory.

Recall that if \(G\) acts trivially on the space \(X\) then \(K_0(X) \cong \mathbb{Z} \otimes K(X)\), because any \(G\)-vector-bundle \(E\) on \(X\) can be decomposed canonically as \(\otimes \mathcal{M} \otimes \text{Hom}^G(M; E)\), where \(\mathcal{M}\) is the set of classes of simple \(G\)-modules, \(\otimes\) is a vector-bundle on \(X\) on which \(G\) acts trivially. In the real or complex theory the situation is more complicated. If \(M\) is a simple \(\mathbb{R}\) or \(\mathbb{C}\) module its commuting-field, i.e. the set of complex linear endomorphisms which commute with both \(G\) and the involution, is \(C_r\) or \(H\). (The three cases arise according as the underlying complex \(G\)-module is (i) simple (ii) sum of two non-isomorphic conjugate simple modules (iii) sum of two copies of a self-conjugate simple module.) If \(E\) is a \(G\)-vector-bundle on a \(\mathbb{R}\)-or \(\mathbb{C}\)-manifold \(X\), whose fibre at \(x \in X\) consists of the \(G\)-linear homomorphisms \(M \rightarrow E_x\) which commute with \(G\) but not necessarily with the involution, which takes \(E_x\) into \(E_x\), is a \(C_r\)-Complex, or Quaternionic, according to the type of \(M\). (A Compact (resp. Quaternionic) vector-bundle on a \(\mathbb{R}\)-manifold is a \(\mathbb{R}\)-vector-bundle together with an additional \(C\)-linear action of \(C\) (resp. \(H\)) in each fibre which commutes with both \(G\) and the involution.) If we break up the free sheaf group \(\mathbb{R}_G(G)\) as \(\mathcal{A}_G \oplus \mathcal{B}_G \oplus \mathcal{C}_G\), where the parts correspond to the commuting-fields \(R, C, H\), respectively, then \(KR_{\ast}(X)\) can be decomposed as follows.

**Proposition 8.1.** If \(G\) acts trivially on the \(\mathbb{R}\)-manifold \(X\), then

\[
KR_{\ast}(X) \cong A_G \oplus KR(X) \oplus B_G \oplus KC(X) \oplus C_G \oplus KH(X)
\]

In the statement \(KC(X)\) and \(KH(X)\) are the Grothendieck groups formed from the Complex and Quaternionic vector-bundles on \(X\). The proof is just as in the complex case [17, (2.2)]: For a complex \(G\)-module, \(E\) can be decomposed naturally by the isomorphism

\[
\otimes \mathcal{M} \otimes \text{Hom}^G(M; E_x) \cong \bigoplus \mathcal{A}_G \otimes \bigotimes \mathcal{B}_G \otimes \bigotimes \mathcal{C}_G
\]

where \(\mathcal{M}\) runs through the simple \(G\)-modules, and \(\mathcal{B}_G\) is the complexification of the commuting-field of \(M\).

**Remark.** In fact, \(KC(X)\) is naturally isomorphic to \(K(X)\), and so does not depend on the involution \(\theta\) of \(X\). For a Complex vector-bundle decomposes canonically as \(E_x \otimes \theta E_x\), where \(E_x\) is the sub-bundle of \(E\) on which its two complex-structures coincide.

Now we can describe \(KR_{\ast}(\text{point})\). The result can be stated in a number of ways, of which the following is probably the most systematic. Let \(C\) be the...
Clifford algebra of the vector-space $R^n$ with the standard negative-definite quadratic form. It is a $(Z/2)$-graded algebra. A Real graded $C_r(G)$-module is a $(Z/2)$-graded complex vector-space with:

(i) a $C$-linear action of $C_r$ making it a graded $C_r$-module,

(ii) an antilinear involution $\xi \mapsto \bar{\xi}$ of degree zero commuting with $C_r$,

(iii) a $C$-linear action of $G$ commuting with $C_r$ and such that $g^{-1}\bar{\xi}g = \bar{g}\cdot \bar{\xi}$.

Let $M_r(G)$ be the Grothendieck group formed from such modules. One defines $M_r^c(G)$ and $M_r^0(G)$ similarly, replacing $C_r$ by $C_r \otimes_k C$ (resp. $C_r \otimes_k H$) in the definition.

There is a natural homomorphism $\sigma: M_r^c(G) \to KF_0(D^1, S^{n-1})$, where $F = R, C$ or $H$, defined as follows (cf. [9]). If $M = (M^i, \theta^i)$ represents an element of $M_r^0(G)$, then the pair $E = D^1 \times M^i (i = 0, 1)$ of Real vector-bundles on $D^1$, together with the isomorphism $\psi: E^r[S^{n-1}] \to E|S^{n-1}$ given by $\psi(x, z) = (x, z\bar{x})$, where $z \in D^1 \subset R^2$ is regarded as an element of $C_r$, represents an element of $KF_0(D^1, S^{n-1})$.

Our description of $KR_0^c(\text{point})$ is

**Proposition 8.2.** There is an exact sequence

$$M_r^c(G) \xrightarrow{\sigma} M_r^0(G) \xrightarrow{\sigma} KR_0^c(\text{point}) \to 0.$$ 

**Proof.** $KR_0^c(\text{point})$ is $KR_k(D^1, S^{n-1})$, and $\sigma$ has just been defined; $r$ is restriction. The composition $\sigma \circ r$ is zero, for if an element $M$ of $M_r^c(G)$ comes from $M_r^0(G)$, then the isomorphism $\psi$ above extends over $S^n$, and hence over $D^1$, and so $(E^r, E, \psi)$ defines the zero-element of $KR_0(D^1, S^{n-1})$.

Now

$$KR_0(D^1, S^{n-1}) \cong A_0 \otimes KR(D^1, S^{n-1}) \oplus B_0 \otimes KH(D^1, S^{n-1})$$

by (8.1); and in exactly the same way

$$M_r^0(G) \cong A_0 \otimes M_r^0 \oplus B_0 \otimes M_r^0 \oplus C_0 \otimes M_r^0.$$ 

The maps $r$ and $\sigma$ respect the decomposition into isotypical parts, so in fact it suffices to prove that

$$M_r^c \rightarrow M_r^0 \rightarrow KF(D^1, S^{n-1}) \to 0$$

is exact for $F = R, C$, and $H$. This is done in [9] when $F = R$ and $C$. The case $F = H$ can be reduced to the case $F = R$ as follows. We define an isomorphism $\gamma: KH(D^1, S^{n-1}) \rightarrow KR(D^1, S^{n-1})$ such that the diagram

\[ \text{A positive-definite form will do equally well.} \]
commutes. If we show that \( \tau \) is an isomorphism the exactness of the top row will follow from that of the bottom. But it is known that \( \text{KH}(D^+; \mathcal{C}) \cong \text{KH}(D^+; \mathcal{C}^+) \), and if \( \mathbb{Z}, \mathbb{Z}/2 \), or 0. So it suffices to show that \( \tau \) is surjective; and that is clear from the diagram.

The isomorphism \( \beta \) is defined as follows. If \( \mathcal{A} \) is a (\( \mathcal{C} \otimes \text{H}^+ \))-module, then \( \beta(\mathcal{A}) \) is \( \mathcal{A} \otimes \mathcal{H} \), where \( x \) and \( h \) are regarded as elements of \( \mathcal{C} \otimes \mathcal{H} \). The inverse of \( \beta \) associates to a \( \mathcal{C} \otimes \mathcal{H} \)-module \( \mathcal{B} \) the sub-\( \mathcal{C} \)-module \( B_B \), where \( \mathcal{A} \) is the product of the elements of a basis of \( \mathcal{C} \otimes \mathcal{H} \) with \( x \).

We define \( \tau : \text{KH}(X, A) \rightarrow \text{KH}(X \times D^+, (X \times D^+) \cup (X \times D^+)) \) for any pair of real spaces \((X, A)\). If \((E, E', \phi)\) represents an element of \( \text{KH}(X, A) \), where \( \gamma : E \rightarrow E' \), then \( \tau(E, E', \phi) \) is \((E \times D^+, E', \phi \times (E \otimes E'), D^+ \times (E \otimes E'), \phi) \), where \( \phi_{x,x} = \begin{pmatrix} \phi_x & 0 \\ 0 & \phi_x \end{pmatrix} \). The commutativity of the above diagram is trivial.

Proposition (8.2) gives a systematic description of the groups \( \text{KH}^*(\mathcal{A}) \). It is useful also to have the following more explicit descriptions (for more details cf. [9]).

For \( q = 1, 2, \ldots, 8 \) the even components of the algebras \( \mathcal{C}_q \) are respectively:
\[
R, \mathcal{C}, \mathcal{H}, \mathcal{H} \oplus \mathcal{H}, \mathcal{H}(2), \mathcal{A}(4), \mathcal{R}(8), \mathcal{R}(8) \oplus \mathcal{R}(8).
\]
Accordingly, the groups \( \text{KH}^q(G) \) can be identified with
\[
R_{\mathcal{C}}(G), R(G), R_{\mathcal{H}}(G), R_{\mathcal{H}}(G) \oplus R_{\mathcal{H}}(G), R_{\mathcal{H}}(G), R_{\mathcal{R}}(G), R_{\mathcal{R}}(G) \oplus R_{\mathcal{R}}(G).
\]

Then the successive restriction-maps become \( \rho, \iota, \iota \oplus \iota, \iota \oplus \iota, \eta, \iota, \iota \oplus \iota, \iota \oplus \iota \), where \( \mathcal{R}_i(G) \subset \mathcal{R}(G) \) and \( \eta : \mathcal{R}(G) \rightarrow \mathcal{R}(G) \) have already been defined, and \( \iota : \mathcal{R}_i(G) \subset \mathcal{R}(G) \) and \( \gamma : \mathcal{R}(G) \rightarrow \mathcal{R}(G) \) are analogous. (All these maps are homomorphisms of \( \mathcal{R}_i(G) \)-modules.)

This means that \( \text{KH}^q(\text{Point}) \), for \( 0 \leq q \leq 7 \), is \( R_{\mathcal{C}}(G), R(G), R_{\mathcal{H}}(G), 0, R_{\mathcal{R}}(G), R_{\mathcal{R}}(G) \oplus R_{\mathcal{R}}(G), R_{\mathcal{R}}(G), R_{\mathcal{R}}(G) \). Observe that all these are finite \( \mathcal{R}_i(G) \)-modules. It may be worth pointing out also that the \( \mathcal{R}_i(G) \)-adic and \( \mathcal{R}_i(G) \)-adic topologies coincide on \( \mathcal{R}(G) \); this appeared in the course of the proof of (5.1).

On completing the above groups \( \mathcal{R}_i(G) \)-adically, we have a complete description of \( \text{KH}^*(\mathcal{A}) \).
References


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