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EQUIVARIANT K-THEORY AND COMPLETION

M. F. ATIYAH & G. B. SEGAL

1. Introduction

It was shown in [3] that, for any *finite* group G, the completed character ring $R(G)^\wedge$ was isomorphic to $K^*(B_G)$ where B_G denotes a classifying space for G. The corresponding result for compact *connected* Lie groups was established in [2], and a combination of the methods of [2] and [3] (together with certain basic properties of $R(G)$ given in [18]) can be used to derive the theorem for general compact Lie groups. Such a proof however would be extremely lengthy, the worst part being in fact the treatment for finite groups where one climbs up via cyclic and Sylow subgroups.

The purpose of this paper is to give a new and much simpler proof of the theorem about $K^*(B_G)$ which applies directly to all compact Lie groups G. The main feature of our new proof is that we generalize the whole problem in a rather natural way by working with the equivariant K-theory developed in [17]. We shall formulate and prove a general theorem about the completion $K_G^*(X)^\wedge$ for any compact G-space X. The theorem about $R(G)$ then follows by taking X to be a point.

The proof consists of four steps. First we deal with the case when $G = T$ is the circle group. Because of the simple model for B_T given by the (infinite) complex projective space this case is easily dealt with directly. The second step is to pass from the circle to a general torus, and this is done in an obvious way by induction on the dimension of the torus. The third and key step shows how to reduce the case of the unitary group $U(n)$ to its maximal torus; this depends on the analytical methods, using elliptic operators, developed in [6]. The fourth and final step reduces the case of a general group G to the case of a unitary group by means of an embedding $G \subset U$; we replace the G-space X by the U-space $Y = U \times_G X$. Thus, *even if we are only interested in the case when G is finite and X is a point*, we are forced at this stage to consider the Lie group U and the U-space U/G.

Using the spectral sequence of [17] it would in principle be possible to pass from the case of a point to general X. However, as we have just explained, there is nothing to be gained by this procedure because the proof we give applies naturally to the general case.

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The real K-theory of B_G has been determined by D. W. Anderson [1] starting from the result in the complex case; the results are expressed in terms of the real representations of G. Anderson's method was to use various exact sequences relating real and complex K-theory. Our new approach however makes such an indirect approach unnecessary. Provided we work with the KR-theory of [4] the proofs apply directly in the real case also.

There are a number of familiar technical difficulties associated with the fact that the classifying space B_G is not a compact space and is also not strictly unique. Since B_G is a limit of compact subspaces B_G^n we can of course consider the inverse system of rings $K^*(B_G^n)$, but because we need to vary the models for B_G and B_G^n we find it convenient to go one stage further and introduce the pro-ring associated to these inverse systems. This is formally similar to the procedure of passing from a filtration to the associated topology and it has the same advantages.

In § 2 we state our main Theorem (2.1) and deduce a number of corollaries. The proof of the theorem is then given in § 3. In § 4 we consider the "genuine" K^* of B_G—not just the K-groups of compact subsets—and we show how (2.1) can be reformulated in these terms; the resulting Proposition (4.2) is perhaps the most attractive version of the theorem. In § 5 we give another and somewhat weaker version concerned with G-maps which are homotopy equivalences.

The remainder of the paper is devoted to the real case. Thus in § 7 we review the equivariant form of KR-theory while in § 8 we state and prove theorem (7.1)—the real analogue of (2.1). We also examine in some detail the special case of (7.1) where X is a point, and show how to recover the results of Anderson [1].

2. Statement of the theorems

We consider a compact G-space X, where G is compact Lie group. Let B_G be a classifying-space for G, and E_G the corresponding universal G-space. To X is associated a space $X_G = (X \times E_G)/G$, determined up to homotopy, which is fibred over B_G with fibre X.

Because the space X_G is not compact there is some choice as to the definition of $K(X_G)$; this will be discussed in § 4. Until then all the propositions we shall prove will involve only compact spaces, and statements concerning K of non-compact spaces should be interpreted for the moment as suggestive rather than precise.

Let F be a G-vector-bundle [17] on X; then $(F \times E_G)/G$ is a vector-bundle on X_G. The assignment $F \mapsto (F \times E_G)/G$ is additive, so it induces a homomorphism $\alpha: K_G(X) \to K(X_G)$. We propose to prove that the groups $K_G(X)$ and $K(X_G)$ can be given topologies so that α is continuous, $K(X_G)$ is complete, and in suitable circumstances α induces a topological isomorphism of the completion $K_G(X)^\wedge$ with $K(X_G)$. Thus the theorem tells one how K_G of a

G-space can be approximated by K of an auxiliary space. When X is a point, $K_G(X) = R(G)$, the representation-ring, and $X_G = B_G$, and the theorem reduces to that of [2] and [3].

Because G acts freely on $X \times E_G$ one can identify $K(X_G)$ with $K_G(X \times E_G)$ [17]. Then α becomes the homomorphism of rings $K_G(X) \to K_G(X \times E_G)$ induced by the projection $X \times E_G \to X$. In this guise it can be manipulated more conveniently.

Two extreme cases may be worth pointing out. If G acts freely on X then X_G is fibred over X/G with fibre E_G, which is contractible; so X_G is homotopy-equivalent to X/G, and the theorem reduces to the elementary fact [17] that $K_G(X) \cong K(X/G)$ when G acts freely on X. This is of course true without completion.

On the other hand, if G acts trivially on X, then X_G is just $X \times B_G$, and $K_G(X) \cong K(X) \otimes R(G)$ [17], so the theorem is the composite of the Künneth formula $K(X \times B_G) \cong K(X) \otimes K(B_G)$, and the isomorphism $K(B_G) \cong R(G)^\wedge$.

We shall prove the theorem in a somewhat different form from the one we have been discussing. This is partly for convenience—so that we can stay in the category of compact spaces—but also because the statement we shall obtain is a little more precise.

We shall use Milnor's model [15] for the universal space of G. Thus E_G is the direct limit[1] of the sequence of subspaces $E_G^n = G * \cdots * G$, the join of n copies of G; and $B_G^n = E_G^n/G$ is the union of the n contractible subsets U_i, U_i being the set where the i-th join-coordinate does not vanish. This means that the product of any n elements of the reduced group $\tilde{K}^*(B_G^n)$ is zero. Now $\tilde{K}^*(B_G^n)$ is the kernel of the augmentation $\varepsilon\colon K^*(B_G^n) \to Z$. The composite homomorphism $R(G) = K_G^*(\text{point}) \xrightarrow{\alpha_n} K_G^*(E_G^n) = K^*(B_G^n) \xrightarrow{\varepsilon} Z$, where α_n is induced by $E_G^n \to$ (point), is the usual augmentation of $R(G)$, whose kernel is the *augmentation-ideal* I_G. So it appears that the natural map α_n: $R(G) \to K_G^*(E_G^n)$ factorizes through $R(G)/I_G^n$.

For any G-space X, $K_G^*(X)$ is a module over $R(G) = K_G^*(\text{point})$, and by naturality the homomorphism $\alpha_n\colon K_G^*(X) \to K_G^*(X \times E_G^n)$ induced by $X \times E_G^n \to X$ factorizes through

$$(*) \qquad \alpha_n\colon K_G^*(X)/I_G^n \cdot K_G^*(X) \to K_G^*(X \times E_G^n)\ .$$

Let us recall that if R is a commutative ring with an ideal I, and K is an R-module, then K can be given the I-adic topology, for which the submodules $I^n \cdot K$ form a basis of the neighbourhoods of 0. The Hausdorff completion $\hat{K}$ of K for this topology can be identified with $\varprojlim K/I^n \cdot K$ [7, Ch. X].

We propose to prove that the system of homomorphisms $(*)$ induces an

[1] Milnor gives E_G a different topology. The limit topology is more convenient for our purposes, and (G being compact) it is still true that the G-action is continuous and that E_G is contractible.

isomorphism of the inverse-limits $K_G^*(X)^\wedge \to \varprojlim K_G^*(X \times E_G^n)$. But somewhat more than that is true, and to formulate it it is convenient to work with what are called *pro-objects* [13].

If C is any category, one can form a new category *Pro* (C) whose objects are inverse-systems $\{A_\alpha\}_{\alpha \in S}$ of objects of C indexed by directed sets S. To define a morphism from $\{A_\alpha\}_{\alpha \in S}$ to $\{B_\beta\}_{\beta \in T}$ one prescribes a map $\theta\colon T \to S$ (not necessarily order-preserving) and morphisms $f_\beta\colon A_{\theta\beta} \to B_\beta$ of C for each $\beta \in T$, subject to the condition that if $\beta \leq \beta'$ in T then for some $\alpha \in S$ such that $\alpha \geq \theta\beta$, $\alpha \geq \theta\beta'$, the diagram

$$\begin{array}{ccccc} & & A_{\theta\beta} & \xrightarrow{f_\beta} & B_\beta \\ & \nearrow^{a_{\alpha,\theta\beta}} & & & \uparrow{\scriptstyle b_{\beta',\beta}} \\ A_\alpha & \xrightarrow[a_{\alpha,\theta\beta'}]{} & A_{\theta\beta'} & \xrightarrow{f_{\beta'}} & B_{\beta'} \end{array}$$

commutes ($a_{\alpha\alpha'}\colon A_\alpha \to A_{\alpha'}$ and $b_{\beta\beta'}\colon B_\beta \to B_{\beta'}$ being the structural maps of the inverse-systems). But one identifies the morphisms $(\theta; f_\beta)$ and $(\theta'; f'_\beta)$ if for each β there is an $\alpha \in S$ such that $\alpha \geq \theta\beta$, $\alpha \geq \theta'\beta$, and $f_\beta a_{\alpha,\theta\beta} = f'_\beta a_{\alpha\theta\beta'}$.

A pro-group should be thought of as much the same kind of thing as a topological group. In fact if A is a topological group one can associate naturally to it the pro-group $\{A/I_\alpha\}$, where $\{I_\alpha\}$ is the family of all open subgroups of A. And if $\{A_\alpha\}$ is a pro-group one can associate to it the group $\varprojlim A_\alpha$ topologized as a subgroup of the product ΠA_α, where each A_α is given the discrete topology. In this way we obtain two functors, P: (topological groups) $\to$ (pro-groups), and Q: (pro-groups) $\to$ (topological groups); moreover $Q \circ P\{A\} \cong A$ if and only if A is Hausdorff and complete and has a neighbourhood-basis at its neutral element consisting of subgroups; while $P \circ Q\{A_\alpha\} \cong \{A_\alpha\}$ if and only if $\{A_\alpha\}$ satisfies the Mittag-Leffler condition (see below). All the pro-objects which occur in this paper do satisfy the Mittag-Leffler condition, but that emerges from our proof, and is not evident a priori.

The system of homomorphisms (*) can be regarded as a morphism $(id; \alpha_n)$ in the category of pro-rings. Our main theorem is:

Theorem 2.1. *Let X be a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Then the homomorphisms*

$$\alpha_n\colon K_G^*(X)/I_G^n \cdot K_G^*(X) \to K_G^*(X \times E_G^n)$$

induce an isomorphism of pro-rings.

That is to say, we shall prove that for each n one can find k and a homomorphism $\beta_n\colon K_G^*(X \times E_G^{n+k}) \to K_G^*(X)/I_G^n \cdot K_G^*(X)$ such that the diagram

$$\begin{array}{ccc} K_G^*(X)/I_G^{n+k}\cdot K_G^*(X) & \xrightarrow{\alpha_{n+k}} & K_G^*(X \times E_G^{n+k}) \\ \downarrow & \swarrow_{\beta_n} & \downarrow \\ K_G^*(X)/I_G^n\cdot K_G^*(X) & \xrightarrow[\alpha_n]{} & K_G^*(X \times E_G^n) \end{array}$$

commutes.

Remark. It can be shown [17, Prop. (5.4)] that $K_G^*(X)$ is finite over $R(G)$ when X is a compact differentiable manifold on which G acts smoothly, and, more generally, when X is locally G-contractible and X/G has finite covering dimension.

Theorem (2.1) has several immediate corollaries as follows:

Corollary 2.2. *The homomorphisms* $\{\alpha_n\}$ *induce an isomorphism* $K_G^*(X)^\wedge \to \varprojlim K_G^*(X \times E_G^n)$, *the completion being in the* I_G*-adic topology.*

Corollary 2.3. *Let* K_n *be the kernel of* $\alpha_n\colon K_G^*(X) \to K_G^*(X \times E_G^n)$. *Then the sequence of ideals* $\{K_n\}$ *defines the* I_G*-adic topology on* $K_G^*(X)$.

Proof. We know that $I_G^n\cdot K_G^*(X) \subset K_n$. Theorem (2.1) implies that for each n there is a k such that $K_{n+k} \subset I_G^n\cdot K_G^*(X)$.

Let us recall [12, Chap. 0, § 13] and [3, § 3] that an inverse-system $\{A_\alpha\}_{\alpha\in S}$ is said to satisfy the *Mittag-Leffler condition* if for each α the image of the map $A_\beta \to A_\alpha$ is constant for large β, i.e. for all β greater than some β_0. It is easy to show that $\{A_\alpha\}_{\alpha\in S}$ satisfies this condition if and only if it is isomorphic as pro-object to a system $\{B_\beta\}_{\beta\in T}$ for which $B_{\beta'} \to B_\beta$ is surjective for all $\beta \leq \beta'$. So we have

Corollary 2.4. *The inverse-system* $\{K_G^*(X \times E_G^n)\}$ *satisfies the Mittag-Leffler condition.*

This corollary will be important in § 4.

Before beginning the proof of (2.1) we should make a remark about E_G. If $\tilde{E}_G$ is another universal G-space, G-homotopy-equivalent to E_G, which is the direct limit of a sequence of compact subspaces $\{\tilde{E}_G^n\}$, then the G-homotopy-equivalences $\tilde{E}_G \to E_G$, $E_G \to \tilde{E}_G$ take each $\tilde{E}_G^n$ into some E_G^{n+k}, and each E_G^n into some $\tilde{E}_G^{n+m}$. This means that the inverse-systems $\{K_G^*(X \times E_G^n)\}$ and $\{K_G^*(X \times \tilde{E}_G^n)\}$ define isomorphic pro-objects. Furthermore, one need not restrict oneself to *sequences* of subspaces: One can equally well use the family of all compact subspaces of E_G, or any cofinal subfamily; this will be convenient in the proof. In fact, the family of compact G-subspaces of E_G can even be replaced by the category of all compact free G-spaces if one is prepared to admit pro-objects indexed by directed categories instead of directed sets.

We shall return to the version of the theorem involving E_G or X_G in § 4.

3. Proof of the theorem

The proof consists of four steps.

Step 1. *Proof when G is the circle-group* ***T***

For the sake of step 2 we shall prove the following slightly more general statement.

Lemma 3.1. *Let G be a compact Lie group, and X a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Let $\theta\colon G \to T$ be a homomorphism by which G acts on E_T. Then the homomorphisms*

$$\alpha_n\colon K_G^*(X)/I_T^n \cdot K_G^*(X) \to K_G^*(X \times E_G^n)\ ,$$

induced by the projections $X \times E_T^n \to X$, define an isomorphism of pro-rings. (Here $K_G^*(X)$ is regarded as an $R(T)$-module by means of θ.)

Proof. We can identify $E_T^n = T * \cdots * T$ with S^{2n-1}, the unit sphere in C^n, on which T acts as a subgroup of the multiplicative group of C.

Consider the exact sequence for the pair $(X \times D^{2n}, X \times S^{2n-1})$. Since D^{2n} is contractible we have $K_G^*(X \times D^{2n}) \cong K_G^*(X)$, and by the Thom isomorphism [17] we have $K_G^*(X \times D^{2n}, X \times S^{2n-1}) \cong K_G^*(X)$. Making these identifications the restriction $K_G^*(X \times D^{2n}, X \times S^{2n-1}) \to K_G^*(X \times D^{2n})$ becomes multiplication by $\lambda_{-1}(C^n) = (1-\rho)^n = \xi^n$, where ρ is the standard one-dimensional representation of T. Thus one has an exact sequence

$$(**) \qquad 0 \to K/\xi^n \cdot K \xrightarrow{\alpha_n} K_G^*(X \times S^{2n-1}) \to {}_{\xi^n}K \to 0\ ,$$

where $K = K_G^*(X)$, and ${}_{\xi^n}K = \{x \in K\colon \xi^n x = 0\}$.

Because ξ generates the augmentation-ideal I_T, to prove the lemma we must find for each n a homomorphism $\beta_n\colon K_G^*(X \times S^{2n+2k-1}) \to K/\xi^n K$ (for some k) making the diagram

$$\begin{array}{ccccccccc}
0 \to & K/\xi^{n+k}\cdot K & \xrightarrow{\alpha_{n+k}} & K_G^*(X \times S^{2n+2k-1}) & \to & {}_{\xi^{n+k}}K & \to 0 \\
 & \downarrow & \swarrow^{\beta_n} & \downarrow & & \downarrow{\times \xi^k} & \\
0 \to & K/\xi^n\cdot K & \xrightarrow[\alpha_n]{} & K_G^*(X \times S^{2n-1}) & \to & {}_{\xi^n}K & \to 0
\end{array}$$

commute. But K is a finitely generated module over the noetherian ring $R(G)$ [18, (3.3)], so one can find k such that ${}_{\xi^k}K = {}_{\xi^{k+1}}K = \cdots$. Then ξ^k annihilates ${}_{\xi^{n+k}}K$ for any n, and the existence of β_n is clear.

Step 2. Proof when G is a torus T^m

As in step 1 we shall prove the more general statement that when G acts on E_{T^m} by a homomorphism $\theta\colon G \to T^m$ the homomorphisms $\alpha_n\colon K_G^*(X)/I_{T^m}^n \cdot K_G^*(X) \to K_G^*(X \times E_{T^m}^n)$ define an isomorphism of pro-rings. We proceed by induction on m, and write $T^m = T \times H$.

In virtue of the remark at the end of § 2 we can replace E_{T^m} by $E_T \times E_H$, and can use the confinal system $\{E_T^p \times E_H^q\}$ of compact subspaces. Furthermore we have $I_{T^m} = a + b$, where a and b are the ideals of $R(T^m)$ generated by I_T and I_H respectively; and for any $R(T^m)$-module K there is an isomorphism of pro-rings $\{K/I_{T^m}^n \cdot K\} \cong \{K/(a^p + b^q)\cdot K\}$ because $a^n + b^n \subset (a+b)^n$ and

$(a + b)^{p+q-1} \subset a^p + b^q$. Thus we have to prove that the homomorphisms $\alpha_{pq}\colon K/(a^p + b^q)\cdot K \to K_G^*(X \times E_T^p \times E_H^q)$, where $K = K_G^*(X)$, define an isomorphism of pro-rings.

Now $K/(a^p + b^q)\cdot K \cong K \otimes_R (R/a^p) \otimes_R (R/b^q)$, where $R = R(T^m)$; and α_{pq} can be factorized

$$K \otimes (R/a^p) \otimes (R/b^q) \xrightarrow{\varphi_{pq}} K_G^*(X \times E_T^p) \otimes (R/b^q)$$
$$\xrightarrow{\psi_{pq}} K_G^*(X \times E_T^p \times E_H^q) .$$

For each fixed q the homomorphisms φ_{pq} define, by step 1, an isomorphism of the "partial" pro-objects indexed by $p \in N$. Similarly, if one holds p constant, ψ_{pq} induces an isomorphism of the partial pro-objects indexed by q in virtue of the inductive hypothesis. (It follows from the exact sequence used in step 1 that $K_G^*(X \times E_T^p)$ is finite over $R(G)$ if $K_G^*(X)$ is.) So the result we want follows from two applications of

Lemma 3.2. *If one has a homomorphism of inverse-systems* $f\colon \{M_{pq}\}_{(p,q)\in N\times N} \to \{N_{pq}\}_{(p,q)\in N\times N}$ *which for each q induces an isomorphism of pro-objects* $f_q\colon \{M_{pq}\}_{p\in N} \to \{N_{pq}\}_{p\in N}$, *then f is itself an isomorphism of pro-objects.*

The proof of the lemma is trivial: in fact the pro-object $M_{..} = \{M_{pq}\}_{(p,q)\in N\times N}$ is the inverse-limit in the category[2] of pro-objects of the pro-objects $M_{.q} = \{M_{pq}\}_{p\in N}$.

Step 3. *Proof when G is a unitary group U(m)*

This step is the most important and it depends on the use of elliptic operators in [6]. More precisely we shall need Proposition (4.9) of [6] whose statement we now recall

Proposition. *Let $j\colon T \to U$ be the inclusion of the maximal torus in the unitary group $U = U(m)$. For any compact U-space let $j^*\colon K_U(x) \to K_T(x)$ be the map induced by j. Then there is a functorial homomorphism of $K_U(X)$-modules*

$$j_*\colon K_T(X) \to K_U(X)$$

which is a left inverse of j^*.

Remarks. 1) If X is a point, $j_*\colon R(T) \to R(U)$ is essentially "holomorphic induction": it assigns to a positive weight λ the irreducible representation of U with maximal weight λ. The construction of j_* for general X given in [6] amounts to "holomorphic induction over a parameter space X".

2) Replacing X by $X \times S^1$ we see that K in the Proposition can be replaced by K^*.

[2] Notice that the embedding of a category in the corresponding pro-category does not commute with inverse-limits.

3) A routine application of this Proposition (or alternatively of the closely related Thom isomorphism (4.8) of [6]) gives the full structure of $K_T^*(X)$ over $K_U^*(X)$: it is a free module of rank $m!$. The arguments are identical with those in [5, (2)]-one first deals with projective space bundles as in [6, (4.9, Remark 3)] or [17, (3.9)] and then decomposes the flag manifold into projective spaces as in the passage from (4.8) to (4.9) in [6]. All we shall need in fact is that *$K_U^*(X)$ is a finite module over $K_T^*(X)$.*

Because of j_* we know that for any compact U-space X, $K_U^*(X)$ is a natural canonical direct summand of $K_T^*(X)$. Then from the diagram

$$\begin{array}{ccc} K_U^*(X)/I_U^n\cdot K_U^*(X) & \xrightarrow{\alpha_n} & K_U^*(X\times E_U^n) \\ {\scriptstyle j^*}\downarrow\uparrow{\scriptstyle j_*} & & {\scriptstyle j^*}\downarrow\uparrow{\scriptstyle j_*} \\ K_T^*(X)/I_U^n\cdot K_T^*(X) & \xrightarrow{\eta_n} & K_T^*(X\times E_U^n) \end{array}$$

we find that $\{\alpha_n\}$ defines an isomorphism of pro-rings if and only if $\{\eta_n\}$ does. But in the diagram

$$\begin{array}{ccc} K_T^*(X)/I_U^n\cdot K_T^*(X) & \xrightarrow{\eta_n} & K_T^*(X\times E_U^n) \\ \downarrow{\scriptstyle\lambda_n} & & \downarrow{\scriptstyle\rho_n} \\ K_T^*(X)/I_T^n\cdot K_T^*(X) & \xrightarrow{\alpha_n} & K_T^*(X\times E_T^n) \end{array}$$

$\{\alpha_n\}$ defines an isomorphism by step 2, $\{\lambda_n\}$ defines an isomorphism because the I_U-adic and I_T-adic topologies coincide on any $R(T)$-module (i.e. $I_U \subset I_T$ and $I_T^k \subset I_U\cdot R(T)$ for some k [18, (3.9)]), and $\{\rho_n\}$ defines an isomorphism by the remark at the end of § 2, for E_U is a universal space for T, and $\{E_U^n\}$ is a cofinal system of compact T-subspaces. So $\{\eta_n\}$ defines an isomorphism as desired.

Step 4. The general case

This is now very simple. Embed G in a unitary group U. If X is a compact G-space, then $\bar{X} = U\times_G X = (U\times X)/G$ is a compact U-space, and $K_U^*(\bar{X})$ is naturally isomorphic to $K_G^*(X)$ as $R(U)$-module [17, § 2 Example (iii)]. $K_U^*(\bar{X})$ is finite over $R(U)$ if and only if $K_G^*(X)$ is finite over $R(G)$, for $R(G)$ is finite over $R(U)$ [18, (3.2)]. Also $\bar{X}\times E_U^n = (U\times_G X)\cong U\times_G (X\times E_U^n)$ so that $K_U^*(\bar{X}\times E_U^n)\cong K_G^*(X\times E_U^n)$. Thus Theorem (2.1) for the U-space $\bar{X}$ tells one that the homomorphisms $K_G^*(X)/I_U^n\cdot K_G^*(X)\to K_G^*(X\times E_U^n)$ define an isomorphism of pro-rings; and the proof is completed by observing that E_U is a universal space for G, and that the I_U-adic and I_G-adic topologies coincide on any $R(G)$-module [18, (3.9)].

This completes the proof of Theorem (2.1).

To conclude this section we shall give an example to show that the finiteness-condition in the theorem is necessary, and cannot be replaced, for example, by the assumption that X is finite-dimensional. Observe first that the exact sequence (**) holds equally well when G is replaced by a finite cyclic group. Taking inverse limits and remarking that $\{K/\xi^n K\}$ has $R^1 \varprojlim = 0$ we get the exact sequence

$$0 \to \varprojlim K/\xi^n K \to \varprojlim K_G^*(X \times E_G^n) \to \varprojlim {}_{\xi^n}K \to 0 .$$

Now take $G = Z/2$, so that $R(G) = Z[\xi]/(\xi^2 - 2\xi)$. If G acts trivially on X, then $K_G^*(X) \cong K^*(X) \otimes R(G)$, and $\varprojlim {}_{\xi^n}K_G^*(X) \cong \varprojlim {}_{2^n}K^*(X)$. We shall produce a two-dimensional compact space X for which the last group does not vanish; in fact, $K^*(X) \cong Z \oplus Z[1/2]/Z$, where $Z[1/2]$ means the dyadic rationals, so $\varprojlim {}_{2^n}K^*(X) \cong \varprojlim Z/2^n$. When X is regarded as a trivial G-space the homomorphism $K_G(X)^\wedge \to \varprojlim K_G(X \times E_G^n)$ is accordingly not surjective. To construct X, first form the dyadic solenoid Y, the inverse-limit of the system $S^1 \xleftarrow{f} S^1 \xrightarrow{f} S^1 \xleftarrow{f} \cdots$, where f has degree 2. Because K^* is continuous, $K^*(Y) \cong Z \oplus Z[1/2]$. The desired space X is the mapping-cone of the projection $Y \to S^1$.

4. The space X_G

In this section we shall return to the original formulation of our theorem. We must begin by defining $K^*(X)$ for spaces X which are not compact. We shall use the representable definition: i.e. $K^q(X) = [X; F^q]$, the group of homotopy-classes of maps $X \to F^q$, where F^q is a suitable H-space. This leads to a satisfactory $Z/2$-graded cohomology-theory on the category of paracompact spaces, but that does not concern us here. The only property we shall need is the following:

Proposition 4.1 (*Milnor*). *If the space X is the limit of an expanding sequence of compact subspaces X_n, then there is a natural exact sequence*

$$0 \leftarrow R^1 \varprojlim K^{q-1}(X_n) \to K^q(X) \to \varprojlim K^q(X_n) \to 0.$$

($R^1 \varprojlim$ is the derived functor of $\varprojlim$ [16].)

Proof. Milnor has proved [16] that there is an exact sequence

$$0 \to R^1 \varprojlim K^{q-1}(X_n) \to K^q(T) \to \varprojlim K^q(X_n) \to 0 ,$$

where T is the telescope formed from the sequence $X_0 \subset X_1 \subset \cdots$. So one has only to show that the projection $T \to X$ induces an isomorphism $[X; F^q] \to [T; F^q]$. It is easy to see that this will be true providing all compact pairs have the homotopy extension property with respect to F^q. But that is immediate, as one can suppose that the F^q are open subsets of a Banach space, or alternatively, that they are ANR's, or have the homotopy-type of CW-complexes.

Remarks. (a) The terms of the exact sequence do not depend on the sequence of compact subspaces X_n, for any such sequence is cofinal in any other, in the obvious sense.

(b) The proposition is true, with the same proof, in equivariant K-theory. For K_G^* is representable: if X is compact then $K_G(X) \cong [X; F_G]_G$, the group of G-homotopy-classes of G-maps $X \to F_G$, where F_G is the G-space of Fredholm operators in a fixed separable G-Hilbert-space in which each simple G-module occurs with infinite multiplicity.

Now we come to our theorem.

Proposition 4.2. *Let X be a compact G-space such that $K_G^*(X)$ is finite over $R(G)$. Then the homomorphism*

$$\alpha: K_G^*(X) \to K^*(X_G)$$

induces an isomorphism of the I_G-adic completion of $K_G^*(X)$ with $K^*(X_G)$.

Proof. The space X_G is the limit of the sequence of compact subspaces $X_n = X \times_G E_G^n$. From (2.1) we know that the sequence of groups $K^*(X_n) = K_G^*(X \times E_G^n)$ satisfies the Mittag-Leffler condition. Hence $R^1 \varprojlim K^*(X_n) = 0$ [12, (0.13)] and so by (4.1) we have $K^*(X_G) \cong \lim K_G^*(X \times E_G^n)$. The proposition now follows from (2.1).

If G acts freely on X then, as we have mentioned before, $K_G^*(X)$ and $K^*(X_G)$ are both isomorphic to $K^*(X/G)$. Combined with (4.2) this means that $K_G^*(X)$ is isomorphic to its completion and hence complete and Hausdorff. But one can make a stronger statement.

Proposition 4.3. *For any compact G-space X the following properties are equivalent:*

(1) *$K_G^*(X)$ is discrete (in the I_G-adic topology),*

(2) *$K_G^*(X)$ is complete and Hausdorff,*

(3) *$K_G^*(X)$ is complete*[3],

(4) *G acts freely on X.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) trivially.

(3) $\Rightarrow$ (4). Let $K_G^*(X)$ be complete and suppose that G does not act freely. Then there is a subgroup H of prime order p and a point $x \in X$ with $Hx = x$. The restriction homomorphism

$$K_G^*(X) \to K_G^*(x) = R(H)$$

makes $R(H)$ into a $K_G^*(X)$-module, and a topological module when both rings are given the I_G-adic topology. It is a finitely-generated module because $R(H)$

[3] But not necessarily Hausdorff, i.e. Cauchy sequences have limits but these may not be unique.

is finite even over $R(G)$ [18, (3.2)]. This implies [7, (10.13)] that $R(H)$ is complete in the I_G-adic topology and hence in the I_H-adic topology, since these topologies coincide [18, (3.9)]. But, for a p-group, the I_H-adic topology on $I_H \subset R(H)$ is just the p-adic topology [8, III (1.1)] and I_H is certainly not p-adically complete. This gives the required contradiction, so that G must act freely.

(4) $\Rightarrow$ (1). Because $R(G)$ is noetherian the ideal I_G has a finite number of generators whose images in $K_G^*(X)$ generate $I_G \cdot K_G^*(X)$ as an ideal in $K_G^*(X)$. On the other hand $K_G^*(X) \cong K^*(X/G)$ because G acts freely; also the images in $K^*(X/G)$ of the generators of I_G have augmentation zero, and so are nilpotent by [5, (3.1.6)]. Thus the ideal $I_G \cdot K_G^*(X)$ is nilpotent, and $K_G^*(X)$ is discrete.

5. Another version of the theorem

Proposition (4.2) implies the following result.

Proposition 5.1. *Let X and Y be compact G-spaces such that $K_G^*(X)$ and $K_G^*(Y)$ are finite over $R(G)$. Let $f\colon Y \to X$ be a G-map which is a homotopy-equivalence, but not necessarily a G-homotopy-equivalence. Then $f^*\colon K_G^*(X) \to K_G^*(Y)$ induces an isomorphism of the I_G-adic completions.*

Proof. By (4.2) it suffices to show that the induced map $f_G\colon Y_G \to X_G$ is a homotopy-equivalence. But f_G is a map of fibre-spaces over B_G, and is a homotopy-equivalence on each fibre. It follows from theorems of Dold [11, Theorems (6.3) and (6.4)] that f_G is a homotopy-equivalence.

Remark. Proposition (4.2) states that the projection $X \times E_G \to X$, which is an example of a homotopy-equivalence but not of a G-homotopy-equivalence, induces an isomorphism $K_G(X)^\wedge \to K_G(X \times E_G)^\wedge$; but because E_G is not compact we cannot quite deduce (4.2) from (5.1).

This version of the theorem should be complemented by two examples. First of all, if X and Y are as in (5.1) and $f_0, f_1\colon Y \to X$ are two G-maps which are homotopic but not G-homotopic, then it is not true that f_0^* and f_1^* necessarily induce the same homomorphism $K_G^*(X)^\wedge \to K_G^*(Y)^\wedge$. For an example, let Y be a point and $X = M^+$, the one-point compactification of a complex G-module M, and let $f_0, f_\infty\colon Y \to M^+$ take Y to 0 and ∞ respectively. There is a canonical element λ_M in $K_G(M) = K_G(M^+, \infty)$, defined by the exterior algebra [17, § 3]. Let λ'_M be its image in $K_G(M^+)$. Then $f_\infty^*(\lambda'_M) = 0$, while $f_0^*(\lambda'_M) = \lambda_{-1}M = \Sigma(-1)^k \wedge^k M \in R(G)$; and the latter element is in general not in the kernel of $R(G) \to R(G)^\wedge$.

The second example is more subtle. We shall produce a G-map $f\colon Y \to X$ satisfying the conditions of (5.1) and such that $f^*\colon K_G(X) \to K_G(Y)$ is not an isomorphism, though it becomes one on completion. Let $G = \{1, g\}$ be cyclic of order two. For any odd prime number p one can find a 5-sphere Y with a differentiable involution whose fixed-point-set F is a lens-space $S^3/(Z/p)$ [10]

or [14]. Let x be a point of F, T the tangent-space of Y at x, and $X = T^+$ its one-point-compactification. The group G acts linearly on T, and one can choose a G-isomorphism between T and a neighbourhood of x in Y. This isomorphism induces a G-map $Y \to T^+ = X$ which is of degree one, and therefore a homotopy-equivalence. We have[4] $K_G(X) \cong R(G) \oplus K_G(T) \cong R(G) \oplus K_G^{-3}(\boldsymbol{R}^2)$, where g acts as -1 on $\boldsymbol{R}^2$. So $K_G(X) \cong R(G) \oplus K_G^{-3}(\text{point}) \cong R(G)$. We can show that $K_G(Y)$ is not isomorphic to $R(G)$ as follows. In $R(G)$ let $\mathscr{P}$ be the prime ideal of characters χ such that $\chi(g)$ is divisible by p. The "support" of $\mathscr{P}$ (in the sense of [17] or [18]) is G itself, so by the localization-theorem of [17, § 4] the restriction $K_G(Y)_{\mathscr{P}} \to K_G(F)_{\mathscr{P}}$ is an isomorphism. Now $K_G(F)_{\mathscr{P}} \cong K(F) \otimes R(G)_{\mathscr{P}}$, and $K(F) = K_{\boldsymbol{Z}/p}(S^3)$, which can be calculated from the exact sequence for the pair (D^4, S^3) as in § 3, and is found to be

$$\begin{aligned} R(\boldsymbol{Z}/p)/(\lambda_{-1}C^2) &\cong \boldsymbol{Z}[\theta]/(\theta^p - 1, (\theta - 1)^2) \\ &= \boldsymbol{Z}[\theta]/(p(\theta - 1), (\theta - 1)^2) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/p \,. \end{aligned}$$

Now $R(G)$ is $\boldsymbol{Z}[\sigma]/(\sigma^2 - 1)$, and the ideal $\mathscr{P}$ is $(p, \sigma + 1)$; so $R(G)_{\mathscr{P}} \cong \boldsymbol{Z}_{(p)} = \{\boldsymbol{Z} \text{ localized at } p\}$, by the map taking σ to -1. Thus $K_G(Y)_{\mathscr{P}} \cong \boldsymbol{Z}_{(p)} \oplus \boldsymbol{Z}/p$, which is different from $R(G)_{\mathscr{P}}$.

6. Real equivariant *K*-theory

In this section and the next we shall prove the theorem in real K-theory which corresponds to Theorem (2.1). We use the term "real K-theory" in the sense of [4], and henceforth shall write Real with a capital to avoid confusion with the ordinary use of the word. Thus, for a compact space X with an involution $x \mapsto \bar{x}$, one defines $KR(X)$ by considering complex vector-bundles E on X with a given involution $E \to \bar{E}$ which takes the fibre E_x antilinearly on to $E_{\bar{x}}$, for each $x \in X$. But we need here an equivariant version of the theory.

In conformity with the spirit of [4], a *Real Lie group* will mean here a Lie group G with an involution $g \mapsto \bar{g}$. As usual we shall consider only *compact* Lie groups G. A *Real G-space* is a G-space X with an involution such that $\overline{g \cdot x} = \bar{g} \cdot \bar{x}$. A *Real G-vector-bundle* on X is a complex G-vector-bundle which is also a Real space, and such that the projection is a Real map. $KR_G(X)$ is the abelian group associated to the semigroup of isomorphism-classes of Real G-vector-bundles on X. As usual, it becomes a ring under tensor-product. Beginning with $KR_G(X)$ one can define an equivariant cohomology-theory in the usual way: one lets both G and the involution act trivially on the suspension-coordinates. It turns out that KR_G^{-q} is periodic in q with period 8, so the groups can be defined[5] for all q [4], [6].

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[4] As in [17] we adopt the convention that K_G means "K_G with compact supports". Thus $K_G(T)$ means $K_G(T^+, \infty)$.

[5] But as usual $KR_G^*(X)$ will mean $\bigoplus_{q=0}^{7} KR_G^{-q}(x)$.

We should like to emphasize that we use the generalized Real K-theory and consider groups with involution not just for the sake of additional generality but because our proof can only be carried through in the wider setting.

Before proceeding to the theorem we should say something about the base-ring KR_G(point) of the theory KR_G. This is the Real representation-ring $R_{\mathbf{R}}(G)$ of G, the free abelian group generated by the classes of simple Real G-modules, i.e. of complex G-modules with an antilinear involution compatible with that of G. If the involution of G is trivial, $R_{\mathbf{R}}(G)$ is the usual real representation-ring. In any case there is a "forgetful" map $i: R_{\mathbf{R}}(G) \to R(G)$, a homomorphism of rings, which one ordinarily calls "complexification", and also an additive homomorphism $\rho: R(G) \to R_{\mathbf{R}}(G)$ which takes the complex G-module M to the "Real" G-module $M \oplus \bar{M}$, on which G acts by $g\cdot(\xi_1, \xi_2) = (g\xi_1, \bar{g}\xi_2)$, and whose involution is $(\xi_1, \xi_2) \mapsto (\xi_2, \xi_1)$. The homomorphism i is injective, because ρi is multiplication by 2; so one can identify $R_{\mathbf{R}}(G)$ with a subring of $R(G)$, and we shall do this.

Let us summarize a few facts about $R_{\mathbf{R}}(G)$ which correspond to properties of $R(G)$ established in [18].

Proposition 6.1.

(i) $R_{\mathbf{R}}(T) = R(T) \cong \mathbf{Z}[X, X^{-1}]$, *where* T *is the group of complex numbers of modulus* 1 *with the usual complex conjugation.*

(ii) $R_{\mathbf{R}}(U(n)) = R(U(n)) \cong \mathbf{Z}[\lambda_1, \cdots, \lambda_n, \lambda_n^{-1}]$, *where* $U(n)$ *is the unitary group with the usual complex conjugation.*

(iii) *Any Real Lie group* G *has a Real embedding in* $U(n)$ *for some* n.

(iv) *If* H *is a Real subgroup of* G, *then the restriction* $R_{\mathbf{R}}(G) \to R_{\mathbf{R}}(H)$ *makes* $R_{\mathbf{R}}(H)$ *a finite* $R_{\mathbf{R}}(G)$*-module, and the* IR_G*-adic topology on* $R_{\mathbf{R}}(H)$ *coincides with its* IR_H*-adic topology.* ($IR_G = R_{\mathbf{R}}(G) \cap I_G$ *is the augmentation-ideal of* $R_{\mathbf{R}}(G)$.)

(v) $R(G)$ *is a finite module over* $R_{\mathbf{R}}(G)$.

(vi) $R_{\mathbf{R}}(G)$ *is a noetherian ring.*

Proof. (i) and (ii) are true because the standard one-dimensional representation X of T and the exterior powers λ^i of the standard representations of $U(n)$ are Real representations in our sense.

(iii) Let M be a faithful complex G-module. Then $\rho(M) = M \oplus \bar{M}$ is a faithful Real G-module, and G is embedded in the unitary group of $M \oplus \bar{M}$.

(iv) Because G can be embedded in $U(n)$ it is sufficient to consider the case $G = U(n)$. Then $R_{\mathbf{R}}(G) = R(G)$, and is noetherian. The ring $R_{\mathbf{R}}(H)$ is a subring of $R(H)$, which is finite over $R(G)$ [18, (3.2)], so $R_{\mathbf{R}}(H)$ is finite over $R(G)$. Furthermore $R(H)$ is finite over $R_{\mathbf{R}}(H)$, so by the Artin-Rees lemma [7, (10.11)] $R_{\mathbf{R}}(H)$ is a topological submodule of $R(H)$ when both are given the IR_H-adic topology. But the IR_H-adic topology on $R(H)$ lies between the I_H-adic and I_G-adic topologies; and these last two coincide by [18, (3.9)]. So the IR_H-adic and IR_G-adic topologies coincide on $R(H)$, and hence on $R_{\mathbf{R}}(H)$.

(v) has appeared in the course of the proof of (iv).

(vi) follows from (iv) because $R_{\boldsymbol{R}}(G)$ is finite over $R(U(n))$, which is noetherian.

Note. The groups $\boldsymbol{T}$ and $U(n)$ will henceforth always be regarded as Real groups with the involutions described above.

7. The theorem in the real case

Let G be a compact Real Lie group. The involution of G induces an involution of the universal space E_G and of the classifying space B_G, making them Real spaces. Then we have

Theorem 7.1. *Let X be a compact Real G-space such that $KR_G^*(X)$ is finite over $R_{\boldsymbol{R}}(G)$. Then the natural maps*

$$\alpha_n \colon KR_G^*(X)/IR_G^n \cdot KR_G^*(X) \to KR_G^*(X \times E_G^n)$$

induce an isomorphism of pro-rings.

Remarks. (i) The theorem has a family of corollaries corresponding to those of (2.1), e.g. $KR_G^*(X)^\wedge \cong KR^*(X_G)$, where the completion is IR_G-adic. We shall not list them here.

(ii) In the next section we shall see that $KR_G^*(\text{point})$ is finite over $R_{\boldsymbol{R}}(G)$. So $KR_G^*(X)$ is finite over $R_{\boldsymbol{R}}(G)$ if and only if it is finite over $KR_G^*(\text{point})$.

The proof of (7.1) is the same as that of (2.1). Let us recapitulate briefly the steps.

Step 1. *The proof when* $G = \boldsymbol{T}$. Then E_G^n is S^{2n-1}, the unit sphere in $\boldsymbol{C}^n$, and its involution is complex conjugation. To determine $KR_G^*(X \times S^{2n-1})$ we use the Thom isomorphism $KR_G^*(X) \to KR_G^*(X \times \boldsymbol{C}^n)$, which is defined by the exterior algebra in the usual way, and is an isomorphism in accordance with the equivariant version of the real periodicity-theorem [6].

Step 2. The proof for a torus $\boldsymbol{T}^n$ presents nothing new.

Step 3. The passage from $\boldsymbol{T}^n$ to $U(n)$ depends on Proposition (5.2) of [6] asserting the existence of a homomorphism $j_* \colon KR_{\boldsymbol{T}^n}^*(X) \to KR_{U(n)}^*(X)$ such that $j_* j^* = id$. As remarked in [6] this depends essentially on the fact that we are using KR-theory (and that $U(n)$ is given its natural Real structure). The fact that the corresponding result for KO^* is false was one of the difficulties encountered in trying to imitate the proof of [2] for KO-theory.

The remarks made in discussing Step 3 in the complex case all apply here also.

Step 4. One treats a general Real group G by embedding it as a Real subgroup of $U(n)$, using (6.1 (iii)).

8. The case when X is a point

When X is a point, Theorem (7.1) tells one that $KR_G^*(\text{point})^\wedge \cong KR^*(B_G)$. We know of course that $KR_G^0(\text{point}) = R_{\boldsymbol{R}}(G)$. The object of this section is to describe explicitly the other groups KR_G^{-q}(point). But before doing so we must discuss one of the few significant departures of KR_G-theory from K_G-theory.

Recall that if G acts trivially on the space X then $K_G(X) \cong R(G) \otimes K(X)$, because any G-vector-bundle E on X can be decomposed canonically as $\bigoplus_{M \in \hat{G}} M \otimes \mathrm{Hom}^G(M; E)$, where $\hat{G}$ is the set of classes of simple G-modules, and $\mathrm{Hom}^G(M; E)$ is a vector-bundle on X on which G acts trivially. In the real or Real theory the situation is more complicated. If M is a simple Real G-module its commuting-field, i.e. the set of complex linear endomorphisms which commute with both G and the involution, is $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$. (The three cases arise according as the underlying complex G-module is (i) simple (ii) sum of two non-isomorphic conjugate simple modules (iii) sum of two copies of a self-conjugate simple module.) If E is a Real G-vector-bundle on a Real space X on which G acts trivially, the vector-bundle $\mathrm{Hom}^G(M; E)$, whose fibre at $x \in X$ consists of the $\boldsymbol{C}$-linear homomorphisms $M \to E_x$ which commute with G (but not necessarily with the involution, which takes E_x into $E_{\bar{x}}$), is Real, Complex, or Quaternionic, according to the type of M. (A *Complex* (resp. *Quaternionic*)[6] vector-bundle on a Real space is a Real vector-bundle together with an additional $\boldsymbol{C}$-linear action of $\boldsymbol{C}$ (resp. $\boldsymbol{H}$) in each fibre which commutes with both G and the involution.) If we break up the free abelian group $R_{\boldsymbol{R}}(G)$ as $A_G \oplus B_G \oplus C_G$, where the parts correspond to the commuting-fields $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$, respectively, then $KR_G(X)$ can be decomposed as follows.

Proposition 8.1. *If G acts trivially on the Real space X, then*

$$KR_G(X) \cong A_G \otimes KR(X) \oplus B_G \otimes KC(X) \oplus C_G \otimes KH(X) .$$

In this statement $KC(X)$ and $KH(X)$ are the Grothendieck groups formed from the Complex and Quaternionic vector-bundles on X. The proof is just as in the complex case [17, (2.2)]: For a (complex) G-module, E_x can be decomposed naturally by the isomorphism

$$\bigoplus_M M \otimes_{F_M} \mathrm{Hom}^G(M; E_x) \xrightarrow{\cong} E_x ,$$

where M runs through the simple Real G-modules, and F_M is the complexification of the commuting-field of M.

Remark. In fact, $KC(X)$ is naturally isomorphic to $K(X)$, and so does not depend on the involution θ of X. For a Complex vector-bundle decomposes canonically as $E_0 \oplus \theta^* \bar{E}_0$, where E_0 is the sub-bundle of E on which its two complex-structures coincide.

Now we can describe KR_G^{-q}(point). The result can be stated in a number of ways, of which the following is probably the most systematic. Let C_q be the

[6] Such bundles (with a slightly different definition) are considered in a paper by J. L. Dupont to appear in Math. Scand.

Clifford algebra of the vector-space R^q with the standard negative-definite[7] quadratic form. It is a $(Z/2)$-graded algebra. A Real graded $C_q[G]$-module is a $(Z/2)$-graded complex vector-space with

(i) a C-linear action of C_q making it a graded C_q-module,

(ii) an antilinear involution $\xi \mapsto \bar{\xi}$ of degree zero commuting with C_q,

(iii) a C-linear action of G commuting with C_q and such that $\overline{g\cdot\xi} = \bar{g}\cdot\bar{\xi}$.

Let $M_q^R(G)$ be the Grothendieck group formed from such modules. One defines $M_q^C(G)$ and $M_q^H(G)$ similarly, replacing C_q by $C_q \otimes_R C$ (resp. $C_q \otimes_R H$) in the definition.

There is a natural homomorphism $\alpha\colon M_q^F(G) \to KF_G(D^q, S^{q-1})$, where $F = R, C$ or H, defined as follows (cf. [9]). If $M = (M^0, M^1)$ represents an element of $M_q^F(G)$, then the pair $E^i = D^q \times M^i$ $(i = 0, 1)$ of Real vector-bundles on D^q, together with the isomorphism $\varphi\colon E^0|S^{q-1} \to E^1|S^{q-1}$ given by $\varphi(x, \xi) = (x, x\xi)$, where $x \in D^q \subset R^q$ is regarded as an element of C_q, represents an element of $KF_G(D^q, S^{q-1})$.

Our description of $KR_G^{-q}(\text{point})$ is

Proposition 8.2. *There is an exact sequence*

$$M_{q+1}^R(G) \xrightarrow{\ r\ } M_q^R(G) \xrightarrow{\ \alpha\ } KR_G^{-q}(\text{point}) \to 0 .$$

Proof. $KR_G^{-q}(\text{point})$ is $KR_G(D^q, S^{q-1})$, and α has just been defined; r is restriction. The composition αr is zero, for if an element M of $M_q^R(G)$ comes from $M_{q+1}^R(G)$, then the isomorphism φ above extends over S^q, and hence over D^q, and so (E^0, E^1, φ) defines the zero-element of $KR_G(D^q, S^{q-1})$.

Now

$$KR_G(D^q, S^{q-1}) \cong A_G \otimes KR(D^q, S^{q-1}) \oplus B_G \otimes KC(D^q, S^{q-1}) \oplus C_G \otimes KH(D^q, S^{q-1})$$

by (8.1); and in exactly the same way

$$M_q^R(G) \cong A_G \otimes M_q^R \oplus B_G \otimes M_q^C \oplus C_G \otimes M_q^H .$$

The maps r and α respect the decomposition into isotypical parts, so in fact it suffices to prove that

$$M_{q+1}^F \to M_q^F \to KF(D^q, S^{q-1}) \to 0$$

is exact for $F = R, C$, and H. This is done in [9] when $F = R$ and C. The case $F = H$ can be reduced to the case $F = R$ as follows. We define an isomorphism $\beta\colon M_q^H \to M_{q+4}^R$ and a homomorphism $\gamma\colon KH(D^q, S^{q-1}) \to KR(D^{q+4}, S^{q+3})$ such that the diagram

[7] A positive-definite form will do equally well.

$$\begin{array}{ccccc} M^{H}_{q+1} & \xrightarrow{r} & M^{H}_{q} & \xrightarrow{\alpha} & KH(D^q, S^{q-1}) \to 0 \\ \downarrow \beta & & \downarrow \beta & & \downarrow \gamma \\ M^{R}_{q+5} & \xrightarrow{r} & M^{R}_{q+4} & \xrightarrow{\alpha} & KR(D^{q+4}, S^{q+3}) \to 0 \end{array}$$

commutes. If we show that γ is an isomorphism the exactness of the top line will follow from that of the bottom. But it is known that $KH(D^q, S^{q-1}) \cong KR(D^{q+4}, S^{q+3})$, and is $\boldsymbol{Z}$, $\boldsymbol{Z}/2$, or 0. So it suffices to show that γ is surjective; and that is clear from the diagram.

The isomorphism β is defined as follows. If M is a graded $(C_q \otimes \boldsymbol{H})$-module, then $\beta(M)$ is $M \oplus \tilde{M}$, where $\tilde{M}$ is obtained from M by interchanging the grading. An element (x, h) of $\boldsymbol{R}^q \oplus \boldsymbol{H} \cong \boldsymbol{R}^{q+4}$ acts on $M \oplus \tilde{M}$ as $\begin{pmatrix} x & h \\ -\bar{h} & -x \end{pmatrix}$, where x and h are regarded as elements of $C_q \otimes \boldsymbol{H}$. The inverse of β associates to a C_{q+4}-module N the sub-C_q-module $N_0 = \{\xi \in N : \omega\xi = \xi\}$, where ω is the product of the elements of a basis of $\boldsymbol{R}^4 \subset \boldsymbol{R}^{q+4} \subset C_{q+4}$.

We define $\gamma: KH(X, A) \to KR(X \times D^4, (X \times S^3) \cup (A \times D^4))$ for any pair of Real spaces (X, A). If (E^0, E^1, φ) represents an element of $KH(X, A)$, where $\varphi: E^0|A \xrightarrow{\cong} E^1|A$, then $\gamma(E^0, E^1, \varphi)$ is $(D^4 \times (E^0 \oplus E^1), D^4 \times (E^1 \oplus E^0), \psi)$, where $\psi_{(x,h)} = \begin{pmatrix} \varphi_x & h \\ -\bar{h} & -\varphi_x \end{pmatrix}$. The commutativity of the above diagram is trivial.

Proposition (8.2) gives a systematic description of the groups $KR_G^{-q}(\text{point})$, but it is useful also to have the following more explicit descriptions (For more details cf. [9].)

For $q = 1, 2, \cdots, 8$ the even components of the algebras C_q are respectively: $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$, $\boldsymbol{H} \oplus \boldsymbol{H}$, $\boldsymbol{H}(2)$, $\boldsymbol{C}(4)$, $\boldsymbol{R}(8)$, $\boldsymbol{R}(8) \oplus \boldsymbol{R}(8)$. Accordingly, the groups $M_q^{\boldsymbol{R}}(G)$ can be identified with

$$R_{\boldsymbol{R}}(G), R(G), R_{\boldsymbol{H}}(G), R_{\boldsymbol{H}}(G) \oplus R_{\boldsymbol{H}}(G), R_{\boldsymbol{H}}(G), R(G), R_{\boldsymbol{R}}(G), R_{\boldsymbol{R}}(G) \oplus R_{\boldsymbol{R}}(G) .$$

Then the successive restriction-maps become ρ, j, $id \oplus id$, $id \oplus id$, η, i, $id \oplus id$, where $i: R_{\boldsymbol{R}}(G) \subset R(G)$ and $\rho: R(G) \to R_{\boldsymbol{R}}(G)$ have already been defined, and $j: R_{\boldsymbol{H}}(G) \subset R(G)$ and $\eta: R(G) \to R_{\boldsymbol{H}}(G)$ are analogous. (All these maps are homomorphisms of $R_{\boldsymbol{R}}(G)$-modules.)

This means that $KR_G^{-q}(\text{point})$, for $0 \leq q \leq 7$, is $R_{\boldsymbol{R}}(G)$, $R_{\boldsymbol{R}}(G)/\rho R(G)$, $R(G)/R_{\boldsymbol{H}}(G)$, 0, $R_{\boldsymbol{H}}(G)$, $R_{\boldsymbol{H}}(G)/\eta R(G)$, $R(G)/R_{\boldsymbol{H}}(G)$, 0. Observe that all these are finite $R_{\boldsymbol{R}}(G)$-modules. It may be worth pointing out also that the IR_G-adic and I_G-adic topologies coincide on $R(G)$; this appeared in the course of the proof of (5.1).

On completing the above groups IR_G-adically, we have a complete description of $KR^{-q}(B_G)$.

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