K-THEORY AND REALITY

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Introduction

THE K-theory of complex vector bundles (2, 5) has many variants and refinements. Thus there are:

(1) K-theory of real vector bundles, denoted by KO,
(2) K-theory of self-conjugate bundles, denoted by KC (1) or KSC (7),
(3) K-theory of G-vector bundles over G-spaces (6), denoted by K_G.

In this paper we introduce a new K-theory denoted by KR which is, in a sense, a mixture of these three. Our definition is motivated partly by analogy with real algebraic geometry and partly by the theory of real elliptic operators. In fact, for a thorough treatment of the index problem for real elliptic operators, our KR-theory is essential. On the other hand, from the purely topological point of view, KR-theory has a number of advantages and there is a strong case for regarding it as the primary theory and obtaining all the others from it. One of the main purposes of this paper is in fact to show how KR-theory leads to an elegant proof of the periodicity theorem for KO-theory, starting essentially from the periodicity theorem for K-theory as proved in (3). On the way we also encounter, in a natural manner, the self-conjugate theory and various exact sequences between the different theories. There is here a considerable overlap with the thesis of Anderson (1) but, from our new vantage point, the relationship between the various theories is much easier to see.

Recently Karoubi (8) has developed an abstract K-theory for suitable categories with involution. Our theory is included in this abstraction but its particular properties are not developed in (8), nor is it exploited to simplify the KO-periodicity.

The definition and elementary properties of KR are given in § 1. The periodicity theorem and general cohomology properties for KR are discussed in § 2. Then in § 3 we introduce various derived theories—KR with coefficients in certain spaces—ending up with the periodicity theorem for KO. In § 4 we discuss briefly the relation of KR with Clifford algebras on the lines of (4), and in particular we establish a lemma which is used in § 3. The significance of KR-theory for the topological study of real elliptic operators is then briefly discussed in § 5.

This paper is essentially a by-product of the author's joint work with I. M. Singer on the index theorem. Since the results are of independent topological interest it seemed better to publish them on their own.

1. The real category

By a *space with involution* we mean a topological space X together with a homeomorphism $\tau: X \to X$ of period 2 (i.e. $\tau^2 =$ Identity). The involution τ is regarded as part of the structure of X and is frequently omitted if there is no possibility of confusion. A space with involution is just a Z_2-space in the sense of (6), where Z_2 is the group of order 2. An alternative terminology which is more suggestive is to call a space with involution a *real* space. This is in analogy with algebraic geometry. In fact if X is the set of complex points of a real algebraic variety it has a natural structure of real space in our sense, the involution being given by complex conjugation. Note that the fixed points are just the real points of the variety X. In conformity with this example we shall frequently write the involution τ as complex conjugation:

$$\tau(x) = \bar{x}.$$

By a *real vector bundle* over the *real space* X we mean a complex vector bundle E over X which is also a real space and such that

(i) the projection $E \to X$ is real (i.e. commutes with the involutions on E, X);

(ii) the map $E_x \to E_{\bar{x}}$ is anti-linear, i.e. the diagram

$$\begin{array}{ccc} \mathbf{C} \times E_x & \to & E_x \\ \downarrow & & \downarrow \\ \mathbf{C} \times E_{\bar{x}} & \to & E_{\bar{x}} \end{array}$$

commutes, where the vertical arrows denote the involution and $\mathbf{C}$ is given its standard real structure ($\tau(z) = \bar{z}$).

It is important to notice the difference between a vector bundle in the category of real spaces (as defined above) and a complex vector bundle in the category of Z_2-spaces. In the definition of the latter the map

$$E_x \to E_{\tau(x)}$$

is assumed to be complex-linear. On the other hand note that if E is a real vector bundle in the category of Z_2-spaces its complexification can be given two different structures, depending on whether

$$E_x \to E_{\tau(x)}$$

is extended linearly or anti-linearly. In the first it would be a bundle in the real category, while in the second it would be a complex bundle in the Z_2-category.

At a fixed point of the involution on X (also called a real point of X) the involution on E gives an anti-linear map

$$\tau_x : E_x \to E_x$$

with $\tau_x^2 = 1$. This means that E_x is in a natural way the complexification of a real vector space, namely the $+1$-eigenspace of τ_x (the real points of E_x). In particular if the involution on X is trivial, so that all points of X are real, there is a natural equivalence between the category $\mathscr{E}(X)$ of real vector bundles over X (as space) and the category $\mathscr{F}(X)$ of real vector bundles over X (as real space):† define $\mathscr{E}(X) \to \mathscr{F}(X)$ by $E \mapsto E \otimes_{\mathbf{R}} \mathbf{C}$ ($\mathbf{C}$ being given its standard real structure) and $\mathscr{F}(X) \to \mathscr{E}(X)$ by $F \mapsto F_R$ (F_R being the set of real points of F). This justifies our use of 'real vector bundle' in the category of real spaces: it may be regarded as a natural extension of the notion of real vector bundle in the category of spaces.

If E is a real vector bundle over the real space X then the space $\Gamma(E)$ of cross-sections is a complex vector space with an anti-linear involution: if $s \in \Gamma(E)$, $\bar{s}$ is defined by

$$\bar{s}(x) = \overline{s(\bar{x})}.$$

Thus $\Gamma(E)$ has a real structure, i.e. $\Gamma(E)$ is the complexification of the real vector space $\Gamma(E)_R$.

If E, F are real vector bundles over the real space X a morphism $\phi: E \to F$ will be a homomorphism of complex vector bundles commuting with the involutions, i.e.

$$\phi(\bar{e}) = \overline{\phi(e)} \quad (e \in E).$$

$E \otimes_{\mathbf{C}} F$ and $\mathrm{Hom}_{\mathbf{C}}(E, F)$ have natural structures of real vector bundles. For example if $\phi_x \in \mathrm{Hom}_{\mathbf{C}}(E_x, F_x)$ we define $\overline{\phi_x} \in \mathrm{Hom}_{\mathbf{C}}(E_{\bar{x}}, F_{\bar{x}})$ by

$$\overline{\phi_x}(u) = \overline{(\phi_x \bar{u})} \quad (u \in E_{\bar{x}}).$$

It is then clear that a morphism $\phi: E \to F$ is just a real section of $\mathrm{Hom}_{\mathbf{C}}(E, F)$, i.e. an element of $(\Gamma\,\mathrm{Hom}_{\mathbf{C}}(E, F))_R$.

If now X is compact then exactly as in (3) [§ 1] we deduce the homotopy property of real vector bundles. The only point to note is that a real section s over a real subspace Y of X can always be extended to a real section over X; in fact if t is any section extending s then $\frac{1}{2}(t+\bar{t})$ is a real extension.

† The morphisms in $\mathscr{F}(X)$ will be defined below.

Suppose now that X is a real algebraic space (i.e. the complex points of a real algebraic variety) then, as we have already remarked, it defines in a natural way a real topological space $X_{\text{alg}} \mapsto X_{\text{top}}$. A *real algebraic* vector bundle can, for our purposes, be taken as a complex algebraic vector bundle $\pi: E \to X$ where X, E, π, and the scalar multiplication $\mathbf{C} \times E \to E$ are all defined over $\mathbf{R}$ (i.e. they are given by equations with real coefficients). Passing to the underlying topological structure it is then clear that E_{top} is a real vector bundle over the real space X_{top}.

Consider as a particular example $X = P(\mathbf{C}^n)$, $(n-1)$-dimensional complex projective space. The standard line-bundle H over $P(\mathbf{C}^n)$ is a real algebraic bundle. In fact H is defined by the exact sequence of vector bundles

$$0 \to E \to X \times \mathbf{C}^n \to H \to 0,$$

where $E \subset X \times \mathbf{C}^n$ consists of all pairs $((z), u) \in X \times \mathbf{C}^n$ satisfying

$$\sum u_i z_i = 0.$$

Since this equation has real coefficients E is a real bundle and this then implies that H is also real. Hence H defines a real bundle over the real space $P(\mathbf{C}^n)$.

As another example consider the affine quadric

$$\sum_{i=1}^{n} z_i^2 + 1 = 0.$$

Since this is affine a real vector bundle may be defined by projective modules over the affine ring $A_+ = \mathbf{R}[z_1, \dots, z_n]/(\sum z_i^2 + 1)$. Now the intersection of the quadric with the imaginary plane is the sphere

$$\sum_{1}^{n} y_i^2 = 1,$$

the involution being just the anti-podal map $y \mapsto -y$. Thus projective modules over the ring A_+ define real vector bundles over S^{n-1} with the anti-podal involution. If instead we had considered the quadric

$$\sum z_i^2 - 1 = 0$$

then its intersection with the real plane would have been the sphere with trivial involution, so that projective modules over

$$A_- = \frac{\mathbf{R}[z_1, \dots, z_n]}{(\sum z_i^2 - 1)}$$

define real vector bundles over S^{n-1} with the trivial involution (and so these are real vector bundles in the usual sense). The significance of S^{n-1} in this example is that it is a deformation retract of the quadric in our category (i.e. the retraction preserving the involution).

The Grothendieck group of the category of real vector bundles over a real space X is denoted by $KR(X)$. Restricting to the real points of X we obtain a homomorphism

$$KR(X) \to KR(X_R) \cong KO(X_R).$$

In particular if $X = X_R$ we have

$$KR(X) \cong KO(X).$$

For example taking $X = P(\mathbf{C}^n)$ we have $X_R = P(\mathbf{R}^n)$ and hence a restriction homomorphism

$$KR(P(\mathbf{C}^n)) \to KR(P(\mathbf{R}^n)) = KO(P(\mathbf{R}^n)).$$

Note that the image of $[H]$ in this homomorphism is just the standard real Hopf bundle over $P(\mathbf{R}^n)$.

The tensor product turns $KR(X)$ into a ring in the usual way.

If we ignore the involution on X we obtain a natural homomorphism

$$c\colon KR(X) \to K(X).$$

If $X = X_R$ then this is just complexification. On the other hand if E is a complex vector bundle over X, $E \oplus \tau^* \bar{E}$ has a natural real structure and so we obtain a homomorphism

$$r\colon K(X) \to KR(X).$$

If $X = X_R$ then this is just 'realization', i.e. taking the underlying real space.

2. The periodicity theorem

We come now to the periodicity theorem. Here we shall follow carefully the proof in (**3**) [§ 2] and point out the modifications needed for our present theory.

If E is a real vector bundle over the real space X then $P(E)$, the projective bundle of E, is also a real space. Moreover the standard line-bundle H over $P(E)$ is a real line-bundle. Then the periodicity theorem for KR asserts:

THEOREM 2.1. *Let L be a real line-bundle over the real compact space X, H the standard real line-bundle over the real space $P(L \oplus 1)$. Then, as a $KR(X)$-algebra, $KR(P(L \oplus 1))$ is generated by H, subject to the single relation*

$$([H]-[1])([L][H]-[1]) = 0.$$

First of all we choose a metric in L invariant under the involution. The unit circle bundle S is then a real space. The section z of $\pi^*(L)$ defined by the inclusion $S \to L$ is a *real* section. Hence so are its powers z^k. The isomorphism
$$H^k \cong (1, z^{-k}, L^{-k}) \quad [(3)\ 2.5]$$
is an isomorphism of real bundles. Finally we assert that, if f is a real section of $\operatorname{Hom}(\pi^* E^0, \pi^* E^\infty)$ then its Fourier coefficients a_k are real sections of $\operatorname{Hom}(L^k \otimes E^0, E^\infty)$. In fact we have
$$\begin{aligned}\bar{a}_k(x) = \overline{a_k(\bar{x})} &= -\frac{1}{2\pi i}\overline{\int_{S_{\bar{x}}} f_{\bar{x}} z_{\bar{x}}^{-k-1}\, dz_{\bar{x}}} \\ &= \frac{1}{2\pi i}\int_{S_x} \overline{f_{\bar{x}}}(\overline{z_{\bar{x}}})^{-k-1}\, d\overline{z_{\bar{x}}} \quad \text{(since the involution reverses the orientation of } S) \\ &= \frac{1}{2\pi i}\int_{S_x} f_x z_x^{-k-1}\, dz_x \quad \text{(since } f \text{ and } z \text{ are real)} \\ &= a_k(x).\end{aligned}$$
It may be helpful to consider what happens at a real point of X. The condition that f_x is real then becomes
$$f_x(e^{-i\theta}) = \overline{f_x(e^{i\theta})}$$
which implies at once that the Fourier coefficients are real.

Since the linearization procedure of (3) [§ 3] involves only the a_k and and the z^k it follows that the isomorphisms obtained there are all real isomorphisms.

The projection operators Q^0 and Q^∞ of (3) [§ 4] are also real, provided p is real. In fact
$$\begin{aligned}\bar{Q}_x^0 = \overline{Q_{\bar{x}}^0} &= -\frac{1}{2\pi i}\overline{\int_{S_{\bar{x}}} p_{\bar{x}}^{-1}\, dp_{\bar{x}}} \\ &= \frac{1}{2\pi i}\int_{S_x} (\overline{p_{\bar{x}}})^{-1}\, d\overline{p_{\bar{x}}} \\ &= \frac{1}{2\pi i}\int_{S_x} p_x^{-1}\, dp_x, \qquad \text{since } p \text{ is real.}\end{aligned}$$
Similarly for Q^∞. The bundle $V_n(E^0, p, E^\infty)$ is therefore real and (4.6) is an equation in $KR(P)$. The proof in § 5 now applies quite formally.

We are now in a position to develop the usual cohomology-type theory, using relative groups and suspensions. There is, however, one new feature here which is important. Besides the usual suspension, based on $\mathbf{R}$ with trivial involution, we can also consider $\mathbf{R}$ with the involution $x \mapsto -x$. It is often convenient to regard the first case as the real axis $\mathbf{R} \subset \mathbf{C}$ and the second as the imaginary axis $i\mathbf{R} \subset \mathbf{C}$, the complex numbers $\mathbf{C}$ always having the standard real structure given by complex conjugation. We use the following notation:

$$R^{p,q} = \mathbf{R}^q \oplus i\mathbf{R}^p,$$
$$B^{p,q} = \text{unit ball in } R^{p,q},$$
$$S^{p,q} = \text{unit sphere in } R^{p,q}.$$

Note that $R^{p,p} \cong \mathbf{C}^p$. Note also that, with this notation, $S^{p,q}$ has dimension $p+q-1$.

The relative group $KR(X,Y)$ is defined in the usual way as $\widetilde{KR}(X/Y)$ where $\widetilde{KR}$ is the kernel of the restriction to base point. We then define the (p,q) suspension groups

$$KR^{p,q}(X,Y) = KR(X \times B^{p,q}, X \times S^{p,q} \cup Y \times B^{p,q}).$$

Thus the usual suspension groups KR^{-q} are given by

$$KR^{-q} = KR^{0,q}.$$

As in (2) one then obtains the exact sequence for a real pair (X,Y)

$$\ldots \to KR^{-1}(X) \to KR^{-1}(Y) \to KR(X,Y) \to KR(X) \to KR(Y). \quad (2.2)$$

Similarly one has the exact sequence of a real triple (X,Y,Z). Taking the triple $(X \times B^{p,0},\ X \times S^{p,0} \cup Y \times B^{p,0},\ X \times S^{p,0})$ one then obtains an exact sequence

$$\ldots \to KR^{p,1}(X) \to KR^{p,1}(Y) \to KR^{p,0}(X,Y) \to KR^{p,0}(X) \to KR^{p,0}(Y)$$

for each integer $p \geqslant 0$.

The ring structure of $KR(X)$ extends in a natural way to give external products

$$KR^{p,q}(X,Y) \otimes KR^{p',q'}(X',Y') \to KR^{p+p',q+q'}(X'',Y''),$$

where $X'' = X \times X'$, $Y'' = X \times Y' \cup X' \times Y$. By restriction to the diagonal these define internal products.

We can reformulate Theorem 2.1 in the usual way. Thus let

$$b = [H] - 1 \in KR^{1,1}(\text{point}) = KR(B^{1,1}, S^{1,1}) = KR(P(\mathbf{C}^2))$$

and denote by β the homomorphism

$$KR^{p,q}(X,Y) \to KR^{p+1,q+1}(X,Y)$$

given by $x \mapsto b.x$. Then we have

THEOREM 2.3. *β: $KR^{p,q}(X, Y) \to KR^{p+1,q+1}(X, Y)$ is an isomorphism.*

Note also that the exact sequence of a real pair is compatible with the periodicity isomorphism. Hence if we define

$$KR^p(X,Y) = KR^{p,0}(X,Y) \quad \text{for } p \geqslant 0$$

it follows that the exact sequence (2.2) for (X, Y) can be extended to infinity in both directions. Moreover we have natural isomorphisms $KR^{p,q} \cong KR^{p-q}$.

We consider now the general Thom isomorphism theorem as proved for K-theory in (2) [§ 2.7]. We recall that the main steps in the proof proceed as follows:

(i) for a line-bundle we use (2.1),

(ii) for a decomposable vector bundle we proceed by induction using (2.1),

(iii) for a general vector bundle we use the splitting principle.

An examination of the proof in (2) [§ 2.7] shows that the only point requiring essential modification is the assertion that a vector bundle is locally trivial and hence locally decomposable. Now a real vector bundle has been defined as a vector bundle with a real structure. Thus it has been assumed locally trivial as a vector bundle in the category of spaces. What we have to show is that it is also *locally trivial in the category of real spaces*. To do this we have to consider two cases.

(i) $x \in X$ a real point. Then $E_x \cong \mathbf{C}^n$ in our category. Hence by the extension lemma there exists a real neighbourhood U of x such that $E|U \cong U \times \mathbf{C}^n$ in the category.

(ii) $x \neq \bar{x}$. Take a complex isomorphism $E_x \cong \mathbf{C}^n$. This induces an isomorphism $E_{\bar{x}} \cong \mathbf{C}^n$. Hence we have a real isomorphism
$$E|Y \cong Y \times \mathbf{C}^n,$$
where $Y = \{x, \bar{x}\}$. By the extension lemma there exists a real neighbourhood U of Y so that $E|U \cong U \times \mathbf{C}^n$.

Thus we have

THEOREM 2.4 (Thom Isomorphism Theorem). *Let E be a real vector bundle over the real compact space X. Then*
$$\phi\colon KR(X) \to \widetilde{KR}(X^E)$$
is an isomorphism where $\phi(x) = \lambda_E . x$ and λ_E is the element of $\widetilde{KR}(X^E)$ defined by the exterior algebra of E.

Among other results of (2) [§ 2.7] we note the following:
$$\begin{aligned} KR(X \times P(\mathbf{C}^n)) &\cong KR(X)[t]/t^n - 1 \\ &\cong KR(X) \otimes_{\mathbf{Z}} K(P(\mathbf{C}^n)). \end{aligned}$$

We leave the computation of KR for Grassmannians and Flag manifolds as exercises for the reader. The determination of KR for quadrics is a more interesting problem, since the answer will depend on the signature of the quadratic form.

We conclude with the following observation. Consider the inclusion

$$R^{0,1} = \mathbf{R} \xrightarrow{i} \mathbf{C} = R^{1,1}.$$

This induces a homomorphism

$$\begin{array}{ccc} K^{1,1}(\text{point}) & \xrightarrow{i^*} & K^{0,1}(\text{point}) \\ \wr\!\!\mid\!\mid & & \wr\!\!\mid\!\mid \\ \widetilde{KR}(P(\mathbf{C}^2)) & \rightarrow & \widetilde{KR}(P(\mathbf{R}^2)). \end{array}$$

Since $i^*[H]$ is the real Hopf bundle over $P(\mathbf{R}^2)$ it follows that $\eta = i^*(b) = i^*([H]-1)$ is the reduced Hopf bundle over $P(\mathbf{R}^2)$.

3. Coefficient theories

If Y is a fixed real space then the functor $X \mapsto KR(X \times Y)$ gives a new cohomology theory on the category of real spaces which may be called *KR-theory with coefficients in Y*. We shall take for Y the spheres $S^{p,0}$ (where the involution is the anti-podal map). A theory F will be said to have period q if we have a natural isomorphism $F \cong F^{-q}$. Then we have

Proposition 3.1. *KR-theory with coefficients in $S^{p,0}$ has period*

$$\begin{array}{l} 2 \textit{ if } p = 1, \\ 4 \textit{ if } p = 2, \\ 8 \textit{ if } p = 4. \end{array}$$

Proof. Consider R^p as one of the three fields $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ ($p = 1, 2,$ or 4). Then for any real space X the map

$$\mu_p \colon X \times S^{p,0} \times R^{0,p} \rightarrow X \times S^{p,0} \times R^{p,0}$$

given by $\mu_p(x, s, u) = (x, s, su)$, where su is the product in the field, is a real isomorphism. Hence it induces an isomorphism

$$\mu_p^* \colon KR^{p,0}(X \times S^{p,0}) \rightarrow KR^{0,p}(X \times S^{p,0}).$$

Replacing X by a suspension gives an isomorphism

$$\mu_p^* \colon KR^{p,q}(X \times S^{p,0}) \rightarrow KR^{0,p+q}(X \times S^{p,0}).$$

Taking $q = p$ and using the isomorphism

$$\beta^p \colon KR \rightarrow KR^{p,p}$$

given by Theorem 2.1, we obtain finally an isomorphism

$$\begin{array}{rcl} \mu_p^* \beta^p \colon KR(X \times S^{p,0}) & \rightarrow & KR^{0,2p}(X \times S^{p,0}) \\ & & \mid\mid \\ & & KR^{-2p}(X \times S^{p,0}). \end{array}$$

Remark. μ^* is clearly a $KR(X)$-module homomorphism. Since the same is true of β this implies that the periodicity isomorphism

$$\gamma_p = \mu_p^* \beta^p : KR(X \times S^{p,0}) \to KR^{-2p}(X \times S^{p,0})$$

is multiplication by the image c_p of 1 in the isomorphism

$$KR(S^{p,0}) \to KR^{-2p}(S^{p,0}).$$

This element c_p is given by

$$c_p = \gamma_p(1) = \mu^*(b^p \,.\, 1), \quad 1 \in KR(S^{p,0}).$$

For any Y the projection $X \times Y \to X$ will give rise to an exact coefficient sequence involving KR and KR with coefficients in Y. When Y is a sphere we get a type of Gysin sequence:

PROPOSITION 3.2. *The projection* $\pi: S^{p,0} \to$ point *induces the following exact sequence*

$$\ldots \to KR^{p-q}(X) \xrightarrow{\chi} KR^{-q}(X) \xrightarrow{\pi^*} KR^{-q}(X \times S^{p,0}) \xrightarrow{\delta} \ldots$$

where χ *is the product with* $(-\eta)^p$, *and* $\eta \in KR^{-1}(\text{point}) \cong \widetilde{KR}(P(\mathbf{R}^2))$ *is the reduced real Hopf bundle.*

Proof. We replace π by the equivalent inclusion $S^{p,0} \to B^{p,0}$. The relative group is then $KR^{p,q}(X)$. To compute χ we use the commutative diagram

$$\begin{tikzcd} KR^{p,q}(X) \arrow[r, "\chi"] \arrow[d, "\beta^p"] \arrow[dr, dashed, "\phi"] & KR^{0,q}(X) \arrow[d, "\beta^p"] \\ KR^{2p,p+q}(X) \arrow[r, "\chi"] & KR^{p,p+q}(X) \end{tikzcd}$$

Let θ be the automorphism of $K^{2p,p+q}(X)$ obtained by interchanging the two factors $R^{p,0}$ which occur. Then the composition $\chi\theta\beta^p$ is just multiplication by the image of b^p in

$$KR^{p,p}(\text{point}) \to KR^{0,p}(\text{point}).$$

But this is just η^p. It remains then to calculate θ. But the usual proof given in (2) [§ 2.4] shows that $\theta = (-1)^{p^2} = (-1)^p$.

We proceed to consider in more detail each of the theories in (3.1). For $p = 1$, $S^{p,0}$ is just a pair of conjugate points $\{+1, -1\}$. A real vector bundle E over $X \times \{+1, -1\}$ is entirely determined by the complex vector bundle E_+ which is its restriction to $X \times \{+1\}$. Thus we have

PROPOSITION 3.3. *There is a natural isomorphism*

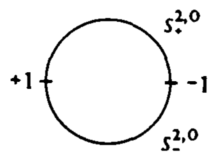
$$KR(X \times S^{1,0}) \cong K(X).$$

Note in particular that this does not depend on the real structure of X but just on the underlying space. The period 2 given by (3.1) confirms what we know about $K(X)$. The exact sequence of (3.2) becomes now

$$\ldots \to KR^{1-q}(X) \xrightarrow{\chi} KR^{-q}(X) \xrightarrow{\pi^*} K^{-q}(X) \xrightarrow{\delta} KR^{2-q}(X) \to \ldots \qquad (3.4)$$

where χ is multiplication by $-\eta$ and $\pi^* = c$ is complexification. We leave the identification of δ as an exercise for the reader. This exact sequence is well-known (when the involution on X is trivial) but it is always deduced from the periodicity theorem for the orthogonal group. Our procedure has been different and we could in fact use (3.4) to prove the orthogonal periodicity. Instead we shall deduce this more easily later from the case $p = 4$ of (3.1).

Next we consider $p = 2$ in (3.1). Then $KR^{-q}(X \times S^{2,0})$ has period 4. We propose to identify this with a self-conjugate theory. If X is a real space with involution τ a *self-conjugate* bundle over X will mean a complex vector bundle E together with an isomorphism $\alpha\colon E \to \overline{\tau^* E}$. Consider now the space $X \times S^{2,0}$ and decompose $S^{2,0}$ into two halves $S^{2,0}_+$ and $S^{2,0}_-$ with intersection $\{\pm 1\}$.



It is clear that to give a real vector bundle F over $X \times S^{2,0}$ is equivalent to giving a complex vector bundle F_+ over $X \times S^{2,0}_+$ (the restriction of F) together with an isomorphism

$$\phi\colon F|X \times \{+1\} \to \tau^*(\overline{F}|X \times \{-1\}).$$

But $X \times \{+1\}$ is a deformation retract of $X \times S^{2,0}_+$ and so [cf. (3) 2.3] we have an isomorphism

$$\theta\colon F_+|X \times \{-1\} \to F_+|X \times \{+1\}$$

unique up to homotopy. Thus to give ϕ is equivalent, up to homotopy, to giving an isomorphism

$$\alpha\colon E \to \overline{\tau^* E},$$

where E is the bundle over X induced from F_+ by $x \mapsto (x, 1)$ and

$$\alpha_x = \theta_{(x,-1)} \phi_{(x,1)}.$$

In other words *isomorphism classes of real bundles over $X \times S^{2,0}$ correspond bijectively to homotopy classes of self-conjugate bundles over X.* Moreover this correspondence is clearly compatible with tensor products.

Now let $KSC(X)$ denote the Grothendieck group of homotopy classes of self-conjugate bundles over X. If τ is trivial this agrees with the definitions of (1) and (7). Then we have established

PROPOSITION 3.5. *There is a natural isomorphism of rings*

$$KSC(X) \to KR(X \times S^{2,0}).$$

The exact sequence of (3.2), with $p = 2$, then gives an exact sequence

$$\dots \to KR^{2-q}(X) \xrightarrow{\chi} KR^{-q}(X) \xrightarrow{\pi^*} KSC^{-q}(X) \xrightarrow{\delta} KR^{3-q}(X) \to \dots \qquad (3.6)$$

where χ is multiplication by η^2 and π^* is the map which assigns to any real bundle the associated self-conjugate bundle (take $\alpha = \tau$). The periodicity in KSC is given by multiplication by a generator of $KSC^{-4}(\text{point})$.

Finally we come to the case $p = 4$. For this we need

LEMMA 3.7. *Let* $\eta \in KR^{-1}(\text{point})$ *be the element defined in* § 2. *Then* $\eta^3 = 0$.

Proof. This can be proved by linear algebra. In fact we recall [(4) § 11] the existence of a homomorphism $\alpha: A_k \to KR^{-k}(\text{point})$ where the A_k are the groups defined by use of Clifford algebras. Then η is the image of the generator of $A_1 \cong Z_2$ and $A_3 = 0$. Since the homomorphisms α_k are multiplicative [(4) § 11.4] this implies that $\eta^3 = 0$.

COROLLARY 3.8. *For any* $p \geqslant 3$ *we have short exact sequences*

$$0 \to KR^{-q}(X) \xrightarrow{\pi^*} KR^{-q}(X \times S^{p,0}) \xrightarrow{\delta} KR^{p+1-q}(X) \to 0.$$

Proof. This follows from (3.7) and (3.2).

According to the remark following (3.1) the periodicity for $KR(X \times S^{4,0})$ is given by multiplication with the element

$$c_4 = \mu_4^*(b^4 . 1) \in KR^{-8}(S^{4,0}).$$

Now recall [(4) Table 2] that $A_8 \cong \mathbf{Z}$, generated by an element λ (representing one of the irreducible graded modules for the Clifford algebra C_8). Applying the homomorphism

$$\alpha: A_8 \to KR^{-8}(\text{point})$$

we obtain an element $\alpha(\lambda) \in KR^{-8}(\text{point})$. The connexion between c_4 and $\alpha(\lambda)$ is then given by the following lemma:

LEMMA 3.9. *Let* 1 *denote the identity of* $KR(S^{4,0})$. *Then*

$$c_4 = \alpha(\lambda) . 1 \in KR^{-8}(S^{4,0}).$$

The proof of (3.9) involves a careful consideration of Clifford algebras and is therefore postponed until § 4 where we shall be discussing Clifford algebras in more detail.

Using (3.9) we are now ready to establish

THEOREM 3.10. *Let* $\lambda \in A_8$, $\alpha(\lambda) \in KR^{-8}(\text{point})$ *be as above. Then multiplication by* $\alpha(\lambda)$ *induces an isomorphism*

$$KR(X) \to KR^{-8}(X)$$

Proof. Multiplying the exact sequence of (3.8) by $\alpha(\lambda)$ we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & KR^{-q}(X) & \to & KR^{-q}(X \times S^{4,0}) & \to & KR^{5-q}(X) & \to 0 \\ & \downarrow\phi_q & & \downarrow\psi_q & & \downarrow\phi_{5-q} & \\ 0 \to & KR^{-q-8}(X) & \to & KR^{-q-8}(X \times S^{4,0}) & \to & KR^{-3-q}(X) & \to 0. \end{array}$$

By (3.9) we know that ψ_q coincides with the periodicity *isomorphism* γ_4. Hence ϕ_q is a monomorphism for all q. Hence ϕ_{5-q} in the above diagram is a monomorphism, and this, together with the fact that ψ_q is an isomorphism, implies that ϕ_q is an epimorphism. Thus ϕ_q is an isomorphism as required.

Remark. If the involution on X is trivial, so that $KR(X) = KO(X)$, this is the usual 'real periodicity theorem'.

By considering the various inclusions $S^{q,0} \to S^{p,0}$ we obtain interesting exact sequences. For the identification of the relative group we need

LEMMA 3.11. *The real space (with base point)* $S^{p,0}/S^{q,0}$ *is isomorphic to*

$$S^{p-q,0} \times B^{q,0}/S^{p-q,0} \times S^{q,0}.$$

Proof. $S^{p,0} - S^{q,0}$ is isomorphic to $S^{p-q,0} \times R^{q,0}$. Now compactify.

COROLLARY 3.12. *We have natural isomorphisms*:

$$KR(X \times S^{p,0}, X \times S^{q,0}) \cong KR^{0,q}(X \times S^{p-q,0}).$$

In view of (3.8) the only interesting cases are for low values of p, q. Of particular interest is the case $p = 2$, $q = 1$. This gives the exact sequence [cf. (1)]

$$\ldots \to K^{-1}(X) \to KSC(X) \to K(X) \to K(X) \to \ldots .$$

The exact sequence of (3.8) does in fact split canonically, so that (for $p \geqslant 3$)

$$KR^{-q}(X \times S^{p,0}) \cong KR^{-q}(X) \oplus KR^{p+1-q}(X). \qquad (3.13)$$

To prove this it is sufficient to consider the case $p = 3$, because the general case then follows from the commutative diagram ($p \geqslant 4$)

$$\begin{array}{ccccc} 0 \to & KR(X) & \to & KR(X \times S^{p,0}) \\ & \downarrow & & \downarrow \\ 0 \to & KR(X) & \to & KR(X \times S^{3,0}) \end{array}$$

obtained by restriction. Now $S^{3,0}$ is the 2-sphere with the anti-podal involution and this may be regarded as the conic $\sum_0^2 z_i^2 = 0$ in $P(\mathbf{C}^3)$. In § 5 we shall give, without proof, a general proposition which will imply that, when Y is a quadric,

$$KR(X) \to KR(X \times Y)$$

has a canonical left inverse. This will establish (3.13).

4. Relation with Clifford algebras

Let $\mathrm{Cliff}(R^{p,q})$ denote the Clifford algebra (over $\mathbf{R}$) of the quadratic form

$$-\Big(\sum_1^p y_i^2 + \sum_1^q x_j^2\Big)$$

on $R^{p,q}$. The involution $(y,x) \mapsto (-y,x)$ of $R^{p,q}$ induces an involutory automorphism of $\mathrm{Cliff}(R^{p,q})$ denoted by† $a \mapsto \bar{a}$.

Let $M = M^0 \oplus M^1$ be a complex Z_2-graded $\mathrm{Cliff}(R^{p,q})$-module. We shall say that M is a *real* Z_2-graded $\mathrm{Cliff}(R^{p,q})$-module if M has a real structure (i.e. an anti-linear involution $m \mapsto \bar{m}$) such that

(i) the Z_2-grading is compatible with the real structure, i.e.

$$\bar{M}^i = M^i \quad (i = 0, 1),$$

(ii) $\overline{am} = \bar{a}\bar{m}$ for $a \in \mathrm{Cliff}(R^{p,q})$ and $m \in M$.

Note that if $p = 0$, so that the involution on $\mathrm{Cliff}(R^{p,q})$ is trivial, then

$$M_R = M_R^0 \oplus M_R^1 = \{m \in M \,|\, \bar{m} = m\}$$

is a real Z_2-graded module for the Clifford algebra in the usual sense [a C_q-module in the notation of (4)].

The basic construction of (4) carries over to this new situation. Thus a real graded $\mathrm{Cliff}(R^{p,q})$-module $M = M^0 \oplus M^1$ defines a triple (M^0, M^1, σ) where $\sigma\colon S^{p,q} \times M^0 \to S^{p,q} \times M^1$ is a real isomorphism given by

$$\sigma(s, m) = (s, sm).$$

In this way we obtain a homomorphism

$$h\colon M(p,q) \to KR^{p,q}(\text{point})$$

where $M(p,q)$ is the Grothendieck group of real graded $\mathrm{Cliff}(R^{p,q})$-modules. If M is the restriction of a $\mathrm{Cliff}(R^{p,q+1})$-module then σ extends over $S^{p,q+1}$. Since the projection

$$S_+^{p,q+1} \to B^{p,q}$$

† This notation diverges from that of (4) [§ 1] where (for $q = 0$) this involution is called α and 'bar' is reserved for an anti-automorphism.

is an isomorphism of real spaces (S_+ denotes the upper hemisphere with respect to the last coordinate) it follows that M defines the zero element of $KR^{p,q}$(point). Hence, defining $A(p,q)$ as the cokernel of the restriction

$$M(p,q+1) \to M(p,q),$$

we see that h induces a homomorphism

$$\alpha\colon A(p,q) \to KR^{p,q}(\text{point}).$$

Moreover, as in (4), α is multiplicative. Note that for $p = 0$ this α coincides essentially with that defined in (4), since

$$A(0,q) \cong A_q,$$
$$KR^{0,q}(\text{point}) \cong KO^{-q}(\text{point}).$$

The exterior algebra $\Lambda^*(\mathbf{C}^1)$ defines in a natural way a Cliff$(R^{1,1})$-module by

$$z(1) = ze, \quad z(e) = -\bar{z}1$$

where $1 \in \Lambda^0(\mathbf{C}^1)$ and $e \in \Lambda^1(\mathbf{C}^1)$ are the standard generators. Let $\lambda_1 \in A(1,1)$ denote the element defined by this module. In view of the definition of $b \in KR^{1,1}$(point) we see that

$$\alpha(\lambda_1) = -b$$

and hence, since α is multiplicative,

$$\alpha(\lambda_1^4) = b^4.$$

Let M be a graded Cliff$(R^{4,4})$-module representing λ_1^4 (in fact as shown in (4) [§ 11], we can construct M out of the exterior algebra $\Lambda^*(\mathbf{C}^4)$), and let $w = e_1 e_2 e_3 e_4 \in$ Cliff$(R^{4,4})$ where e_1, e_2, e_3, e_4 are the standard basis of $R^{4,0}$. Then we have

$$w^2 = 1, \quad \bar{w} = w,$$
$$wz = \bar{z}w \quad \text{for } z \in \mathbf{C}^4 = R^{4,4}.$$

Hence we may define a new anti-linear involution $m \mapsto \tilde{m}$ on M by

$$\tilde{m} = -w\bar{m}$$

and we have

$$\widetilde{zm} = -w\overline{zm} = -w\bar{z}\bar{m} = -zw\bar{m}$$
$$= z\tilde{m}.$$

Thus M with this new involution (or real structure) is a real graded Cliff$(R^{0,8})$-module, a C_8-module in the notation of (4): as such we denote it by N. From dimensional considerations [cf. (4) Table 2], we see that it must be one of the two irreducible C_8-modules. But on complexification (i.e. ignoring involutions) it gives the same as M and hence N represents the element of A_8 denoted in (4) by λ.

After these preliminaries we can now proceed to the proof of Lemma 3.9. What we have to show is that under the map

$$\mu_4 \colon S^{4,0} \times \mathbf{R}^8 \to S^{4,0} \times \mathbf{C}^4$$

the element of $KR^{4,4}(S^{4,0})$ defined by M lifts to the element of $KR^{-8}(S^{4,0})$ defined by N. To do this it is clearly sufficient to exhibit a commutative diagram of real isomorphisms

$$\begin{array}{ccc} S^{4,0} \times \mathbf{R}^8 \times N & \xrightarrow{\nu} & S^{4,0} \times \mathbf{C}^4 \times M \\ \downarrow & & \downarrow \\ S^{4,0} \times \mathbf{R}^8 \times N & \xrightarrow{\nu} & S^{4,0} \times \mathbf{C}^4 \times M \end{array} \tag{4.1}$$

where ν is compatible with μ_4 (i.e. $\nu(s, x, y, n) = (s, x+isy, m)$ for some m) and the vertical arrows are given by the module structures (i.e. $(s, x, y, n) \mapsto (s, x, y, (x, y)n)$.

Consider now the algebra $\mathrm{Cliff}(R^{4,0}) = C_4$. The even part C_4^0 is isomorphic to $\mathbf{H} \oplus \mathbf{H}$ [(4) Table 1]. Moreover its centre is generated by 1 and $w = e_1 e_2 e_3 e_4$, the two projections being $\frac{1}{2}(1 \pm w)$. To be quite specific let us define the embedding

$$\xi \colon \mathbf{H} \to \mathrm{Cliff}^0(R^{4,0})$$

by

$$\xi(1) = \frac{1+w}{2},$$

$$\xi(i) = \frac{1+w}{2} e_1 e_2,$$

$$\xi(j) = \frac{1+w}{2} e_1 e_3,$$

$$\xi(k) = \frac{1+w}{2} e_1 e_4.$$

Then we can define an embedding

$$\eta \colon S(\mathbf{H}) \to \mathrm{Spin}(4) \subset \Gamma_4$$

by $\eta(s) = \xi(s) + \frac{1}{2}(1-w)$, where Γ_4 is the Clifford group [(4) 3.1] and $S(\mathbf{H})$ denotes the quaternions of norm 1. It can now be verified that the composite homomorphism

$$S(\mathbf{H}) \to \mathrm{Spin}(4) \to SO(4)$$

defines the natural action of $S(\mathbf{H})$ on $\mathbf{R}^4 = \mathbf{H}$ given by left multiplication.† In other words

$$\eta(s) y \eta(s)^{-1} = sy \qquad (s \in S(\mathbf{H}),\ y \in \mathbf{R}^4). \tag{4.2}$$

If we give $S(\mathbf{H})$ the anti-podal involution then η is *not* compatible with involutions, since the involution on the even part C_4^0 is trivial.

† We identify $1, i, j, k$ with the standard base e_1, e_2, e_3, e_4 in that order.

Regarding Cliff$(R^{4,0})$ as embedded in Cliff$(R^{4,4})$ in the natural way we now define the required map ν by

$$\nu(s,x,y,n) = (s, x+isy, \eta(s)n).$$

From the definition of w it follows that

$$\eta(s)w = -\eta(-s)$$

and so $$\eta(-s)\bar{n} = \eta(-s)\{-w\bar{n}\} = \eta(s)\bar{n} = \overline{\eta(s)n},$$

showing that ν is a *real* map. Equation (4.2) implies that

$$\eta(s)(x,y)n = (x+isy)\eta(s)n,$$

showing that ν is compatible with the module structures. Thus we have established the existence of the diagram (4.1) and this completes the proof of Lemma 3.9.

The definitions of $M(p,q)$ and $A(p,q)$ given were the natural ones from our present point of view. However, it may be worth pointing out what they correspond to in more concrete or classical terms. To see this we observe that if M is a real $C(R^{p,q})$-module we can define a new action [ ] of R^{p+q} on M by

$$[x,y]m = xm+iym.$$

Then $$[x,y]^2m = \{-\|x\|^2+\|y\|^2\}m.$$

Moreover for the involutions we have

$$\begin{aligned}\overline{[x,y]m} &= \overline{xm}+\overline{iym}\\ &= x\bar{m}+iy\bar{m} \quad (\text{since } \bar{y} = -y)\\ &= [x,y]\bar{m}.\end{aligned}$$

Thus M_R is now a real module in the usual sense for the Clifford algebra $C_{p,q}$ of the quadratic form

$$Q(p,q) \equiv \sum_1^p y_i^2 - \sum_1^q x_j^2.$$

It is easy to see that we can reverse the process. *Thus $M(p,q)$ can equally well be defined as the Grothendieck group of real graded $C_{p,q}$-modules.* From this it is not difficult to compute the groups $A_{(p,q)}$ on the lines of (4) [§ 4, 5] and to see that they depend only on $p-q$ (mod 8) [cf. also (8)]. Using the result of (4) [11.4] one can then deduce that

$$\alpha\colon A(p,q) \to KR^{p,q}(\text{point})$$

is always an isomorphism. The details are left to the reader. We should perhaps point out at this stage that our double index notation was suggested by the work of Karoubi (8).

The map α can be defined more generally for principal spin bundles as in (4) and we obtain a Thom isomorphism theorem for spin bundles on the lines of (4) [12.3]. We leave the formulation to the reader.

5. Relation with the index

If $\hat{\phi}$ denotes the Fourier transform of a function ϕ then we have

$$\hat{\bar{\phi}}(x) = \overline{\hat{\phi}(-x)}.$$

Since the symbol $\sigma(P)$ of an elliptic differential operator P is defined by Fourier transforms (9) it follows that

$$\sigma(\overline{P})(x, \xi) = \overline{\sigma(P)(x, -\xi)}$$

where $\overline{P}$ is the operator defined by

$$\overline{P}\phi = \overline{P\bar{\phi}}.$$

Here we have assumed that P acts on functions so that $\overline{P\bar{\phi}}$ is defined. More generally if X is a *real differentiable manifold*, i.e. a differentiable manifold with a differentiable involution $x \mapsto \bar{x}$, and if E, F are real differentiable vector bundles over X, then the spaces $\Gamma(E)$, $\Gamma(F)$ of smooth sections have a real structure and for any linear operator

$$P \colon \Gamma(E) \to \Gamma(F)$$

we can define $\overline{P} \colon \Gamma(E) \to \Gamma(F)$ by

$$\overline{P}(\phi) = \overline{P\bar{\phi}}.$$

If P is an elliptic differential operator then

$$\sigma(\overline{P})(x, \xi) = \overline{\sigma(P)(\bar{x}, -\tau^*(\xi))}. \tag{5.1}$$

It is natural to define P to be a *real operator* if $P = \overline{P}$. If the involution on X is trivial this means that P is a differential operator with real coefficients with respect to real local bases of E, F. In any case it follows from (5.1) that the symbol $\sigma(P)$ of a real elliptic operator gives an isomorphism of real vector bundles

$$\pi^* E \to \pi^* F,$$

where $\pi \colon S(X) \to X$ is the projection of the cotangent sphere bundle and we define the involution on $S(X)$ by

$$(x, \xi) \to (\bar{x}, -\tau^*(\xi)).$$

Note that if τ is the identity involution on X the involution on $S(X)$ is not the identity but is *the anti-podal map on each fibre*. This is the basic reason why our KR-theory is needed here. In fact the triple

$$(\pi^* E,\ \pi^* F,\ \sigma(P))$$

defines in the usual way an element

$$[\sigma(P)] \in KR(B(X), S(X))$$

where $B(X)$, the unit ball bundle of $S(X)$, has the associated real structure.†

The kernel and cokernel of a real elliptic operator have natural real structures. Thus the index is naturally an element of $KR(\text{point})$. Of course since

$$KR(\text{point}) \to K(\text{point})$$

is an isomorphism there is no immediate advantage in defining this apparently refined real index. However, the situation alters if we consider instead a *family* of real elliptic operators with parameter or base space Y. In this case a real index can be defined as an element of $KR(Y)$ and

$$KR(Y) \to K(Y)$$

is not in general injective.

All these matters admit a natural extension to real elliptic complexes (9). Of particular interest is the Dolbeault complex on a real algebraic manifold. This is a real elliptic complex because the holomorphic map $\tau: X \to \bar{X}$ maps the Dolbeault complex of $\bar{X}$ into the Dolbeault complex of X. If X is such that the sheaf cohomology groups $H^q(X, \mathcal{O}) = 0$ for $q \geqslant 1$, $H^0(X, \mathcal{O}) \cong \mathbf{C}$, the index, or Euler characteristic, of the Dolbeault complex is 1. Based on this fact one can prove the following result:

PROPOSITION. *Let $f: X \to Y$ be a fibering by real algebraic manifolds, where the fibre F is such that*

$$H^q(F, \mathcal{O}) = 0 \qquad (q \geqslant 1,\ H^0(F, \mathcal{O}) \cong \mathbf{C}),$$

then there is a homomorphism

$$f_*: KR(X) \to KR(Y)$$

which is a left inverse of

$$f^*: KR(Y) \to KR(X).$$

The proof cannot be given here but we observe that a special case is given by taking $X = Y \times F$ where F is a (compact) homogeneous space of a real algebraic linear group. For example we can take F to be a complex quadric, as required to prove (3.13). We can also take $F = SO(2n)/U(n)$, or $SO(2n)/T^n$, the flag manifold of $SO(2n)$. These spaces can be used to establish the splitting principle for orthogonal bundles. It is then significant to observe that the real space

$$\{SO(2n)/U(n)\} \times R^{0,2n}$$

† All this extends of course to integral (or pseudo-differential) operators.

has the structure of a real vector bundle. A point of $SO(2n)/U(n)$ defines a complex structure of R^{2n} and conjugate points give conjugate structures. For $n = 2$ this is essentially† what we used in § 3 to deduce the orthogonal periodicity from Theorem 2.1.

† In (3.1) we used the 3-sphere $S^{4,0}$. We could just as well have used the 2-sphere $S^{3,0}$. This coincides with $SO(4)/U(2)$.

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