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An Introduction to Topological Quantum Field Theories

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1. Introduction

Over the past 20 years there has been a remarkable interaction between new ideas in theoretical physics, centred around gauge theories, and various branches of geometry. In particular the notion of a topological quantum field theory (QFT) has emerged. On the one hand this is the most primitive part of QFT, concerned with vacuum structures rather than real particles and dynamics. On the other hand it has interesting topological content which has opened up new areas in classical geometry.

The most exciting developments have been in 3 dimensions (Jones invariants of knots) and in 4 dimensions (Donaldson theory and Seiberg-Witten theory). There are also significant applications in 2 dimensions to Riemann surface theory.

In this lecture, addressed to mathematicians, I would like to concentrate on the most elementary case, that of a topological QFT in 2 dimensions. I will first give an axiomatic approach, valid for all dimensions, following [2]. I will then specialize this down to 2 dimensions and show that in this case the theory is entirely equivalent to a certain algebra. This should provide some degree of confidence and familiarity.

2. The axioms

Here I shall essentially follow [2]. Before beginning it is helpful to recall that homology theory has been given an axiomatic formulation and that this has proved extremely useful. The motivation is geometric and there are many geometric constructions for homology (simplicial, singular, Čech, de Rham) which are important for applications. However the purely formal properties are best studied independently of any geometric realization. The same will apply to topological QFT. In fact, producing a rigorous geometric or analytic construction may not yet be possible in all cases, so that the axioms provide a framework to aim for.

Pursuing our analogy with homology let us recall that the starting point of homology theory is that, to every topological space X (of a suitable type), we associate functorially a vector space (or abelian group) $H(X)$. Some initial key properties are

(1) homotopy invariance, expressed using the product $X \times I$ (with I the unit interval)
(2) additivity on disjoint sums and $H(\phi) = 0$

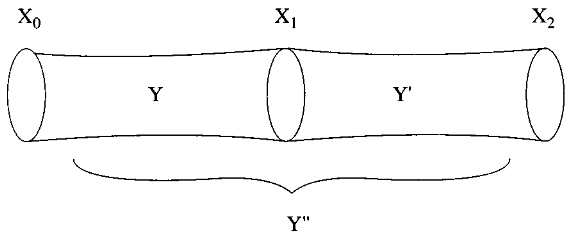
A topological QFT also assigns functorially, to each X, a vector space $H(X)$. Key differences from homology however are the following

(a) X is a manifold of a **fixed dimension,**
(b) the homotopy axiom is strengthened using cobordisms more general than the cylinder $X \times I$,
(c) $H(X)$ is multiplicative on disjoint sums and $H(\phi) = \mathbb{C}$.

So, with this preparation, let us proceed. We fixed an integer d and consider smooth compact oriented manifolds Y of dimension $d+1$ (with boundary) and also smooth closed oriented manifolds X of dimension d. To each X we associate functorially a **finite-dimensional complex vector space** $H(X)$, and to each Y whose boundary is X we associate a **vector** $F(Y) \in H(X)$. These are subject to the following properties (axioms).

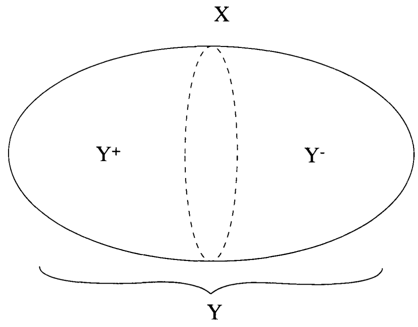
Orientation	$H(X^*) \cong H(X)^*$, where X^* is X with the opposite orientation and H^* is the dual space.
Multiplicativity	$H(X_1 \cup X_2) \cong H(X_1) \otimes H(X_2)$
Transitivity	Let $\partial Y = X_0^* \cup X_1$ (with X_i not necessarily connected), and consider $F(Y) \in H(X_0^*) \otimes H(X_1) \cong \mathrm{Hom}(H(X_0), H(X_1))$

Then, in the following picture,



we have $F(Y'') = F(Y')F(Y)$.
[**Note:** Y and Y' are not necessarily cylinders as the picture might suggest.] These axioms in particular imply **homotopy invariance,** in other words, for a cylinder $X \times I$, $F(Y) : H(X) \to H(X)$ is independent of the length of I. Moreover the transitivity ensures that $F(Y)$ is an idempotent. There is little loss of generality in assuming further that $F(Y) = 1$.

If Y is closed, so that $\partial Y = \phi$, then $F(Y) \in \mathbb{C}$ is just a **numerical invariant** of Y. Transitivity means that we can calculate Y by cutting along any codimension-one submanifold X as indicated



$$F(Y^{+}) \in H(X^{+}) \qquad F(Y^{-}) \in H(X^{-})$$
$$F(Y) = (F(Y^{+}), F(Y^{-}))$$

where (,) is the pairing between the dual spaces $H(X^{+})$ and $H(X^{-})$.

The functoriality of H implies as usual that any group G acting on X induces a representation on $H(X)$. The character of this representation is actually included in the data provided by the vectors F. To see this, let α be a diffeomorphism of X and let Z be the closed $(d+1)$- manifold obtained from $X \times I$ by using α to identify the two ends. The transitivity axiom, applied to two copies of $X \times I$, shows that

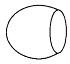
$$F(Z) = \text{Trace } \alpha_H$$

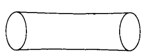
where α_H is the action of α on $H(X)$.

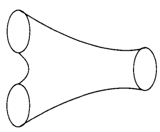
There are a number of generalizations or modifications of the axiom that are useful. For example we can replace vector spaces by mod 2 graded vector spaces, the tensor product being modified accordingly. We can also, as in cobordism theory put additional conditions or data (e.g.spin) on the manifolds being considered.

3. Dimension 2

We shall now restrict attention to the case $d = 1$, $d + 1 = 2$, so that we consider surfaces with boundary. Here we can make a complete analysis of the situation because all surfaces can be built up from the following three basic building blocks

The disc 

The cylinder 

The pair of pants 

More formally these are just the 2-sphere with n holes for n=1,2,3.

Since a closed 1-manifold is just a disjoint union of circles the vector space $H(X)$ is just the tensor product of a number of copies of the basic vector space H associated to one circle. Since we can reflect a circle to reverse its orientation we have a natural symmetric isomorphism $H \cong H^*$, or equivalently a nondegenerate quadratic form on H.

Since every surface is built up from our three basic surfaces the vectors $F(Y)$ are entirely determined by their values for these three cases. For the disc Y, the vector $F(Y) \in H$ will be denoted by 1, for reasons that will become apparent.

For the cylinder, we already know quite generally that $F(Y) \in H^* \otimes H$ corresponds to the identity homomorphism $H \to H$.

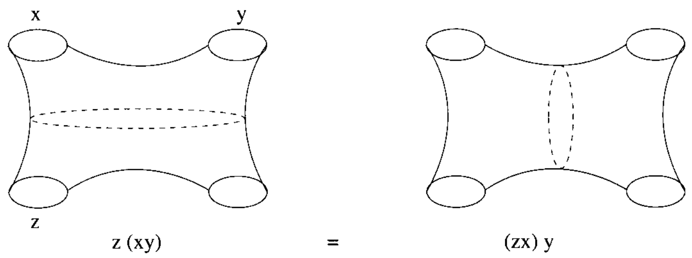
For the pair of pants, with an appropriate orientation of the ends, $F(Y)$ can be viewed as a vector

$$F(Y) \in H^* \otimes H^* \otimes H \cong \mathrm{Hom}(H \otimes H, H).$$

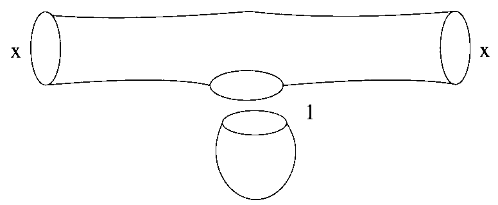
We consider this as **multiplication**. The axioms then imply that this has the following properties.

Commutativity. This follows from the diffeomorphism that interchanges two of the ends of Y.

Associativity. This is more interesting and it follows by looking at the sphere with 4 holes and cutting it in half in two different ways as indicated in the diagram



Unit. The element of H, coming from the disc, which we have denoted by 1 acts as a unit for multiplication. This follows from the diagram



Inner Product. As we have noted H has a symmetric (complex linear) inner product. This is related to the multiplication by

$$(xy, z) = (x, yz)$$

In fact this expression in x, y, z is totally symmetric and its dual in H^* is the linear form on $H \otimes H \otimes H$ given by the pair of pants. Physicists call it the "3-point function".

An algebra with all the above properties is called a Frobenius algebra. It is not hard to show that any Frobenius algebra defines a 2-dimensional topological QFT. All that has to be shown is that the axiomatic properties of F, H for a general surface Y follow consistently from those for the basic surfaces.

Thus the notions of 2-dim topological QFT and Frobenius algebra are equivalent. In particular this demonstrates that there are many examples satisfying the axioms since Frobenius algebras are quite numerous. In the next section we shall review some standard examples.

4. Examples of Frobenius algebras

Our first example relates to the work of Frobenius and is constructed from a finite group G. We take H to be the vector space of class functions on G, with multiplication given by convolution. The unit 1 is given by the character of the regular representation. H is usually given an anti-linear inner product, but since we have a natural complex conjugation on H, we can convert this into a complex linear inner product. Essentially this algebra is defined over the real numbers.

Explicitly, the characters χ_α of the irreducible representations α form a basis of H and multiplication is essentially given by coordinate-wise multiplication. Thus if

$$x = \sum x_\alpha \chi_\alpha, \; y = \sum y_\alpha \chi_\alpha$$

then

$$xy = \sum \frac{x_\alpha y_\alpha}{\dim \alpha} \chi_\alpha$$

Note that

$$1 = \sum (\dim \alpha) \chi_\alpha.$$

It is now an amusing exercise to compute the QFT invariants of closed surfaces. For the 2-sphere S^2, by cutting it into two discs, we see that

$$F(S^2) = (1,1) = \sum (\dim \alpha)^2 = |G|$$

For a torus $S^1 \times S^1$ we have

$F(S^1 \times S^1) = \dim H =$ class number of G.

For a surface Y_g of genus $g \geq 2$ we cut it into 2g-2 pairs of pants and 2 cylinders. We then find

$$F(Y_g) = \sum \frac{1}{(\dim \alpha)^{2g-2}}$$

Note that this also holds for the two special cases $g = 0, 1$.

As is well known the representation theory of finite groups extends to that of compact Lie groups, except that there are now infinitely many irreducible characters. Thus if we allow H to be infinite-dimensional we again get a Frobenius algebra. Note that

$$F(Y_g) = \sum \frac{1}{(\dim\ \alpha)^{2g-2}}$$

is now an infinite series which converges for $g \geq 2$. However (if G is not finite) it diverges for $g = 0, 1$ which is consistent with $\dim H = \infty$ and the unit element being the delta function which is now not in L^2.

For $G = SU(2)$ we get

$$F(Y_g) = \sum \frac{1}{n^{2g-2}} = \zeta(2g-2)$$

with ζ the Riemann zeta-function

If we allow mod 2 graded algebras then we get Frobenius algebras from the cohomology rings of compact oriented manifolds. The non-degeneracy of the quadratic form, given by evaluating products on the fundamental homology class, is ensured by Poincaré duality.

The simplest example (which actually has only even degrees and so is strictly commutative) is given by the cohomology of the complex projective space $P_{n-1}(\mathbb{C})$. The ring is the truncated polynomial ring $\mathbb{C}[z]/z^n$ and the inner product is got by picking out the coefficient of z^{n-1} in the product.

More general Frobenius algebras of this type arise from isolated singularities in algebraic geometry. For example let $f(z_1, \dots z_n)$ be a weighted homogenous polynomial (e.g. $\sum z_i^{d_i} = 0$). Then the variety $f(z) = 0$ has $z = 0$ as an isolated singular point and the ring

$$A_f = \mathbb{C}[z_1, \dots, z_n]/(\tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_n})$$

is finite-dimensional (the dimension is the multiplicity of the singular point).

We can define an evaluation (lienar form) on A_f by

$$\varphi(z) \to \left(\frac{1}{2\pi i}\right)^n \int_{T^n} \frac{\varphi(z) dz_1 \dots dz_n}{\frac{\partial f}{\partial z_1} \cdots \frac{\partial f}{\partial z_n}}$$

where T^n is the n-torus $\left|\frac{\partial f}{\partial zi}\right| = \varepsilon$. It can be shown that evaluation of a product defines a non-degenerate linear form, so that A_f becomes a Frobenius algebra.

If we deform f by adding lower order terms then the isolated singularity may break up into a number of simpler singularities. The evaluation formula still holds and gives a deformation of the original Frobenius algebra.

For example, if $n = 1$ and $f(z) = z^k$, we can deform it by taking

$$f_\lambda = z^k - \lambda z$$

$$\frac{\partial f_\lambda}{\partial z} = kz^{k-1} - \lambda$$

We now have $(k-1)$ simple zeros. The algebra A_λ for $\lambda \neq 0$ is then semi-simple (a sum of copies of $\mathbb{C}$) while A_0 is nilpotent. Note also that the integer grading of A_0 (which corresponds to the cohomology of $P_{k-2}(\mathbb{C})$) gets lost in the general A_λ. These examples show that Frobenius algebras are numerous, that they have an interesting deformation theory and that semi-simple algebras can degenerate into nilpotent ones. Note that the example given by the representation theory of a finite group is semi-simple.

5. Quantum Cohomology

One of the most interesting examples of a topological QFT arises from considering projective algebraic varieties (or more generally compact Kähler manifolds). For physicists such a variety M is the target space of the 2-dimensional fields and the Frobenius algebra that arises involves numerical information about rational curves on M.

Such Frobenius algebras go under the name of **quantum cohomology rings.** The vector space is just the ordinary cohomology ring (regarded as mod 2 graded) and the ring multiplication is a deformation of the usual ring structure. If we think of the small deformation parameter λ as playing the role of Planck's constant then $\lambda \to 0$ is the classical limit. The multiplication adds quantum corrections to the usual multiplication.

Geometrically this can be explained roughly as follows. The classical 3-point function (x, y, z) measures the intersection number of cycles dual to x, y, z in $H(M)$. This is zero unless the dimensions are correct. The quantum corrections also measure intersections with rational curves. Higher degree curves give higher order corrections.

For the complex projective space the quantum cohomology ring turns out to be just the Frobenius algebra introduced in the previous section as a deformation of the usual cohomology.

The quantum cohomology ring of M is defined when the first Chern class is non-negative ($c_1 \geq 0$). For example, for a hypersurface of degree d in P_{n-1} this means $d \geq n$. The cases $c_1 > 0$ and $c_1 = 0$ are somewhat different. In particular there is great interest in the case $d = n = 5$. The number of rational curves of any degree on M is then finite and the quantum cohomology ring encodes these numbers. There is a whole subject "Mirror Symmetry" that deals with this story [1].

In conclusion this brief introduction was meant to introduce topological QFT to mathematicians and to show that, in dimension 2, they are quite familiar and interesting. In dimensions 3 and 4 they are of course much deeper and cannot be reduced to simple algebra, though in 3 dimensions "quantum groups" are introduced as one way of handling the theories.

References

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[2] M. F. Atiyah. Topological quantum field theories. Publ. Math. I.H.E. S., 68 (1989), 175-186.

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