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ON FRAMINGS OF 3-MANIFOLDS

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§1. INTRODUCTION

RECENTLY Witten [7] has generalized to arbitrary 3-manifolds the Jones invariants for knots in the 3-sphere [4]. In Witten's theory a crucial role is played by framings of the 3-manifold. These framings are intimately related to the central extensions of the mapping class groups which occur in the Hamiltonian version of the theory. The purpose of this note is to present an elementary treatment of these questions in the framework of standard algebraic topology.

I shall show (Proposition 1) that every oriented 3-manifold has a *canonical* 2-framing, and so Witten's theory leads to a 3-manifold invariant. This is presumably related to the recent work of Reshetikin and Turaev [6] in which framings of 3-manifolds do not enter explicitly (but signatures of bounding 4-manifolds enter instead).

The essential ideas involved are all due to Witten. In particular he recognized the role of what I have called 2-framings and the central extension $\hat{\Gamma}$ of the mapping class group Γ introduced in §3. The only additional novelty here is that I relate these ideas to the signature cocycle of Meyer [5] and its elaboration as developed in [1]. This leads to a novel computation of the extension class of $\hat{\Gamma}$ (Proposition 2). I am also indebted to G. B. Segal for explaining to me his somewhat different approach to the subject.

§2. CANONICAL FRAMINGS

Let Y be a compact connected oriented 3-dimensional and let T_Y denote its tangent bundle. Then

$$2T_Y = T_Y \oplus T_Y$$

has a natural spin structure arising from the lift to Spin (6) of the diagonal embedding

$$\mathrm{SO}(3) \to \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathrm{SO}(6).$$

Since T_Y is trivial so is $2T_Y$. A homotopy class of trivializations of $2T_Y$ (as a Spin (6) bundle) will be called a *2-framing* of Y. Two such 2-framings differ by a homotopy class of maps

$$Y \to \mathrm{Spin}(6).$$

Such maps deform down to $\mathrm{Spin}(3) \subset \mathrm{Spin}(6)$ and so are determined by an integer, the corresponding degree. This integer can be computed as follows. Lift $2T_Y$ to $Y \times I$ and trivialize it by α, β at the two ends $Y \times 0$ and $Y \times 1$. This enables us to define the relative Pontrjagin class (or number)

$$p_1 \in H^4(Y \times I,\ Y \times \partial I) \cong H^3(Y) \cong Z.$$

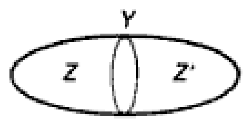
For spin bundles p_1 is always even (because of the relation $p_1 \equiv w_2^2 \bmod 2$) and the integer difference between α and β is $\frac{1}{2}p_1$.

Now let Z be an oriented 4-manifold with boundary Y (this always exists). Given any 2-framing α or Y we can define

$$\sigma(\alpha) = \operatorname{Sign} Z - \tfrac{1}{6}p_1(2T_Z, \alpha) \tag{2.1}$$

where the relative Pontrjagin number p_1 is defined as above by using the trivialization α on $Y = \partial Z$. If Z' is another choice, then we can form the closed 4-manifold W

$$W = Z \cup_Y (-Z')$$



Since both the signature and relative p_1 are additive the Hirzebruch signature formula for W

$$\operatorname{Sign} W = \tfrac{1}{3}p_1(T_W) = \tfrac{1}{6}p_1(2T_W)$$

shows that $\sigma(\alpha)$ is independent of the choice of Z and hence depends only on α (as the notation indicates).

By altering α we can change $p_1(2T_Z, \alpha)$ by the corresponding integer. It follows that there is a unique choice of α making $\sigma(\alpha)=0$. Thus we have established:

Proposition 1. *There is a canonical 2-framing α of any compact connected oriented 3-manifold Y, characterized by the property that the Hirzebruch signature formula holds for any 4-manifold Z with boundary Y:*

$$\operatorname{Sign} Z = \tfrac{1}{6}p_1(2T_Z, \alpha).$$

Remarks

(1) The Hirzebruch signature formula continues to hold even when $Y = \partial Z$ is not connected, provided each component of Y is given its canonical 2-framing.

(2) If we had worked with conventional framings (instead of the more exotic 2-framings) this proposition would have failed for two reasons: (a) it would only have held for a sub-class of 3-manifolds, namely those for which the invariant $S(Y)$ introduced in [3; (4.19)] is zero (as observed in [3], $2S(Y)$ is the number (mod 2) of 2-primary summands in $H^2(Y, Z)$; (b) different framings correspond to maps $Y \to SO(3)$ and these are distinguished not only by an integer degree but also by an element of $H^1(Y, Z_2)$ which is related to spin structures on Y. For the Abelian version of the Witten theory, spin structures are necessary, but for the non-abelian version (defined by a compact semi-simple Lie group G) they are an unnecessary encumbrance.

(3) Applying a reflection converts a 2-framing α of Y into a 2-framing $-\alpha$ of $-Y$ (i.e. Y with orientation reversed). Clearly the canonical 2-framing of Proposition 1 is consistent with reflection.

(4) Determining explicitly the canonical 2-framing of a given 3-manifold Y involves finding an explicit 4-manifold Z with $Y = \partial Z$. We can then always modify Z (by taking a connected sum with an appropriate number of complex projective planes with the relevant

orientation) to make sign $Z = 0$. The canonical 2-framing of Y is then the unique one which extends to Z, so that the relative $p_1 = 0$. For the 3-sphere therefore the canonical 2-framing is the one extending to the ball and this is how (from Witten's viewpoint) the Jones invariants of knots are normalized. A general 3-manifold Y can be constructed from a series of surgeries along a set of links in S^3. These involve an explicit set of cobordisms and hence a definite final choice of Z with $\partial Y = Z$. In this way the canonical 2-framing of Y can be made explicit.

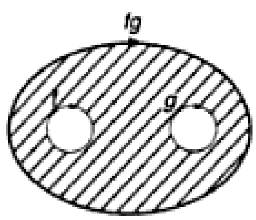
§3. DIFFEOMORPHISMS OF SURFACES

Let X be a compact connected oriented surface and let Γ be the group of components of $\mathrm{Diff}^+(X)$ (the orientation preserving diffeomorphisms of X). For any $f \in \mathrm{Diff}^+(X)$ we can form the oriented 3-manifold $Y = X_f$ fibred over the circle with fibre X and monodromy f. More precisely we form Y_f from $X \times I$ by using f to identify $X \times 0$ and $X \times 1$. The isomorphism class of the fibration $Y \to S^1$ depends only on the class of f in Γ. Note that

$$X_{f^{-1}} = -X_f$$

where $-X_f$ denotes X_f with orientation reversed.

Now let $\hat{\Gamma}$ be the set of isomorphism classes of fibrations $Y \to S^1$ with fibre endowed with a choice of 2-framing α on Y. Thus $\hat{\Gamma}$ is essentially a set of pairs (γ, α) with $\gamma \in \Gamma$ and α a 2-framing on a representative X_f (with $(f) = \gamma$). There is a natural group law on $\hat{\Gamma}$ which may be defined as follows. Given $f, g \in \mathrm{Diff}^+(X)$ we construct the 4-manifold Z fibred (with fibre X) over a plane region B as indicated



B has 3 circles as boundaries and correspondingly

$$\partial Z = X_f + X_g - X_{fg}.$$

Given α, β 2-framings on X_f, X_g respectively there is then a unique 2-framing γ on X_{fg} so that the relative $p_1(2T_Z)$, for the trivialization α, β, γ on ∂Z, vanishes. We define

$$[(f), \alpha][(g), \beta] = [(fg), \gamma].$$

The obvious additivity of relative p_1 ensures the associativity of the product in $\hat{\Gamma}$, while

$$[(f), \alpha]^{-1} = [(f^{-1}), -\alpha].$$

An alternative way (due to Witten) of describing the group $\hat{\Gamma}$ is to represent its elements by pairs (f, ϕ_t) with $f \in \mathrm{Diff}^+(X)$ and ϕ_t, $0 \leqslant t \leqslant 1$, a path in the space of trivializations of $2T_X$ such that $\phi_1 = f^*\phi_0$. The group law is then given by composition of paths (one must first make a homotopy so that the end of one path is the beginning of the next).

The map $[(f), \alpha] \to (f)$, forgetting about the 2-framing, clearly defines a surjective homomorphism $\hat{\Gamma} \to \Gamma$. Moreover different choices of α differ by integers (relative p_1), so that we have an exact sequence of groups

$$1 \to Z \to \hat{\Gamma} \to \Gamma \to 1. \tag{3.1}$$

An integer $m \in Z$ acts on a pair $[(f), \alpha]$ simply altering α by the integer m. In particular Z is in the centre of $\hat{\Gamma}$ so that (3.1) *is a central extension.* In the next section we shall identify this extension by its cohomology class.

§4. COCYCLES

Applying the canonical 2-framing of §2 to the 3-manifolds Y_f of §3 we see that we get a canonical section s of the central extension

$$Z \to \hat{\Gamma} \underset{s}{\rightleftarrows} \Gamma.$$

This gives us a canonical 2-cocycle for this extension, namely

$$c(f, g) = s(f)s(g)[s(fg)]^{-1}$$

is the integer difference between the two 2-framings $s(f)s(g)$ and $s(fg)$ of X_{fg}. Now consider the 4-manifold Z introduced in §3 whose 3-boundary components are X_f, X_g and $-X_{fg}$. By definition of the product in $\hat{\Gamma}$, the relative p_1 for Z (i.e. $\frac{1}{2}p_1(2TZ)$) with 2-framings $s(f)$, $s(g)$ and $s(f)s(g)$ is zero. By definition of the canonical 2-framings of §2, the relative p_1 for Z with 2-framings $s(f)$, $s(g)$ and $s(fg)$ is 3 Sign Z (see Remark (3) following Proposition 1). Hence

$$c(f, g) = 3\,\mathrm{Sign}\,Z. \tag{4.1}$$

Now the cohomology of the 4-manifold Z (constructed from f, g) depends only on the induced elements f^*, g^* in the symplectic group $Sp(2n, R)$, where n = genus of X, induced by the action of f, g on $H^1(X, R)$. Moreover Sign Z viewed as a function of f^*, g^* is a 2-cocycle for the symplectic group. This follows easily from the additivity of the signature and was observed by Meyer [5], who identified the cohomology class of this signature cocycle as 4 times the standard generator.

Another (elementary) treatment of the signature cocycle can be found in [1; §2] in a context close to the present one. From the results in [1] we have

$$[\mathrm{Sign}] = 4c_1(V) \in H^2(\Gamma, Z) \tag{4.2}$$

where [Sign] is the cohomology class of the signature cocycle and V is the equivariant vector-bundle on Teichmüller space which associates to each complex structure τ on X the space of holomorphic differentials on X_τ.

From (4.1) and (4.2) we deduce immediately

PROPOSITION 2. *The cohomology class of the extension $Z \to \hat{\Gamma} \to \Gamma$ defined in* §3 *is* 12 *times the first Chern class of the bundle V of holomorphic differentials.*

Remarks

1. The factor 12 is well-known in conformal field theory and there are many derivations of it. Our computation was based on the Hirzebruch signature theorem and used only standard topological constructions.

2. Proposition 2 merely identifies the cohomology class of the extension. In fact formula (4.1) gives more precise information identifying the section s in terms of the signature cocycle (and its associated section s'). This can also be expressed in terms of a commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & \Gamma' & \underset{s'}{\overset{\longrightarrow}{\dashleftarrow}} & \Gamma \\ {\scriptstyle 3}\downarrow & & \downarrow & & \| \\ Z & \longrightarrow & \hat{\Gamma} & \overset{\longrightarrow}{\dashleftarrow} & \Gamma \end{array}$$

The sections s and s' are both "natural", i.e. they are invariant under conjugation.

§5. FURTHER COMMENTS

The canonical 2-framing of §2 can also be interpreted in terms of the η-invariant of [2]. Recall that this is a spectral invariant of a Riemannian metric on a 3-manifold Y which refines the "gravitational Chern–Simons" invariant. More precisely the Chern-Simons invariant of a connection A on T_Y

$$\mathrm{CS}(A) \in R/Z$$

is only defined modulo integers, and a trivialization of TY picks a particular branch of $CS(A)$ as a real-valued function. For the Levi–Civita connection A_ρ of a metric ρ it was shown in [3; (4.19)] that

$$\mathrm{CS}(A_\rho) = 3\eta_\rho \bmod Z,$$

so that $3\eta_\rho$ defines a canonical branch of $CS(A_\rho)$. This then extends to give a *canonical branch* of CS(A) for all connections A.

On the other hand the canonical 2-framing of Y also defines a canonical branch of CS(A), using the formula

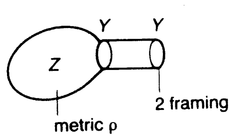
$$\mathrm{CS}(A) = \tfrac{1}{2}\mathrm{CS}(2A),$$

where $2A$ is the induced connection on $2T_Y$.

It is not hard to see that *these two canonical branches* of CS(A) *coincide*. In fact let $\partial Z = Y$, then (by definition)

$$\mathrm{Sign}\, Z = \tfrac{1}{6}p_1(2T_Z, \alpha), \tag{5.1}$$

where α is the canonical 2-framing of Y. Now compute this relative p_1 by using a connection on $2T_Z$ which comes from a Levi–Civita connection on all Z, except for a final cylinder:



This gives

$$\tfrac{1}{2}p_1(2T_Z, \alpha) = \int_Z p_1(A_\rho) - \mathrm{CS}_\alpha(A_\rho),$$

where CS_α is the branch given by α. Substituting in (5.1) and comparing with the main theorem of [2]

$$\operatorname{Sign} Z = \frac{1}{3}\int p_1(A_\rho) - \eta_\rho,$$

shows that

$$\mathrm{CS}_\alpha(A_\rho) = 3\eta_\rho$$

identifying the two canonical choices of the gravitational Chern–Simons function.

In [7] Witten explains that to get a topologically invariant regularization of the Feynman integral with Chern–Simons Lagrangian it is necessary to add a "counter-term" depending on the metric. For this to be unambiguously defined he needs a framing of the 3-manifold Y. In fact a 2-framing is equally good. Our canonical 2-framing would therefore give an invariant for an oriented 3-manifold. As mentioned in §1 this is perhaps relevant in connection with the recent work of Reshetikin and Turaev [6] in which framings do not appear explicitly.

In computing the large k (semi-classical) limit of his partition function Witten gets a sum over representations of $\pi_1(Y)$ multiplied by an overall phase factor

$$\exp\left(\frac{2\pi i d \sigma(\alpha)}{8}\right),$$

where $d = \dim G$ and $\sigma(\alpha)$ is essentially the topological invariant of the 2-framing α on Y defined by (2.1). *Thus if we choose the canonical 2-framing for α this phase factor disappears.*

The canonical 2-framings of 3-manifolds that we have introduced therefore provide a convenient normalization of Witten's 3-manifold invariants. However, from the Hamiltonian point of view, the central extension $\hat{\Gamma}$ of Γ plays an essential role. It is supposed to act on the (finite-dimensional) Hilbert spaces of the theory, while Γ itself only acts projectively. If

$$\hat{\gamma} = [(f), \alpha] \in \hat{\Gamma}$$

then its character for the Hilbert space representation in question is supposed to give the partition function for the 3-manifold X_f with the 2-framing α. Choosing always the canonical 2-framing would mean that we would compute the character of the representation of $\hat{\Gamma}$ on the image $s(\Gamma)$ given by the canonical section $s : \Gamma \to \hat{\Gamma}$. Such formulae would then need to be supplemented by the explicit form of the cocycle c (given by (4.1)) which describes the deviation of s from being a homomorphism.

Finally the case of genus 1 deserves a comment. In that case $\hat{\Gamma} = SL(2, Z)$ and $H^2(\Gamma, Z) \cong Z_{12}$, so that (by Proposition 2) the extension $\hat{\Gamma}$ splits. Moreover since there are no non-zero homomorphisms $\Gamma \to Z$, the splitting $s_1 : \Gamma \to \hat{\Gamma}$ is unique. It is therefore interesting to compare this splitting s_1 (a homomorphism) with the canonical section s of §3 (which is not a homomorphism). The computation of the integer function of $\gamma \in \Gamma$ which gives the difference between $s(\gamma)$ and $s_1(\gamma)$ is the main topic treated in [1] in relation to the Dedekind η-function.

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