Algebraic Topology and Operators in Hilbert Space

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Introduction.

In recent years considerable progress has been made in the global theory of elliptic equations. This has been essentially of a topological character and it has brought to light some very interesting connections between the topology and the analysis. I want to talk here about a simple but significant aspect of this connection, namely the relation between the index of Fredholm operators and the theory of vector bundles.

§1. Fredholm operators.

Let H be a separable complex Hilbert space. A bounded linear operator

$$F : H \longrightarrow H$$

is called a Fredholm operator if its kernel and cokernel are both finite-dimensional. In other words the homogeneous equation $Fu = 0$ has a finite number of linearly independent solutions, and the inhomogeneous equation $Fu = v$ can be solved provided v satisfies a finite number of linear conditions. Such operators occur frequently in various branches of analysis particularly in connection with elliptic problems. A useful criterion characterizing Fredholm operators is the following

PROPOSITION 1.1. F is a Fredholm operator if and only if it is invertible modulo compact operators, i.e. $\exists G$ with $FG-1$ and $GF-1$ both compact.

This proposition is essentially a reformulation of the classical result of Fredholm that, if K is compact, $1+K$ is a Fredholm operator. In the theory of elliptic differential equations the approximate inverse G

*Supported in part by Air Force Office of Scientific Research grant AF-AFOSR-359-66.

is called a "parametrix."

To reformulate (1.1) in geometrical terms let $\mathcal{A}$ denote the Banach algebra of all bounded operators on H. The compact operators form a closed 2-sided ideal $\mathcal{K}$ of $\mathcal{A}$ so that $\mathcal{B} = \mathcal{A}/\mathcal{K}$ is again a Banach algebra. Let $\mathcal{B}^*$ denote the group of invertible elements in $\mathcal{B}$. Then (1.1) asserts that the space $\mathcal{F}$ of Fredholm operators is just the inverse image $\pi^{-1}(\mathcal{B}^*)$ under the natural map $\pi : \mathcal{A} \longrightarrow \mathcal{B}$. In particular this shows that $\mathcal{F}$ is an open set in $\mathcal{A}$, closed under composition, taking adjoints and adding compact operators.

The <u>index</u> of a Fredholm operator F is defined by

$$\text{index } F = \dim \operatorname{Ker} F - \dim \operatorname{Coker} F \ .$$

<u>Remark</u>. A simple application of the closed-graph theorem (see Lemma (2.1)) shows that a Fredholm operator has a closed range. Thus

$$\operatorname{Coker} F = H/F(H) \cong \operatorname{Ker} F^*$$

where F^* is the adjoint of F. Hence

$$\text{index } F = \dim \operatorname{Ker} F - \dim \operatorname{Ker} F^* \ .$$

As we have observed the composition $F' \circ F$ of two Fredholm operators is again Fredholm. A simple algebraic argument shows at once that

$$\text{index } F' \circ F = \text{index } F' + \text{index } F \ .$$

Suppose now that $\{e_0, e_1, \dots\}$ is an orthonormal base for H and let H_n denote the closed subspace spanned by the e_i with $i \geq n$. The orthogonal projection P_n onto H_n is then Fredholm and, since it is self-adjoint, it has index zero. Thus $F_n = P_n \circ F$ is Fredholm and has the same index as F. Now choose n so that $e_0, e_1, \dots, e_{n-1}$ and F(H) span H -- which is possible because $\dim H/F(H)$ is finite. Then $F_n(H) = H_n$ and so $\operatorname{Ker} F_n^* = H_n^\perp$. Thus

(1.2) $$\text{index } F = \text{index } F_n = \dim \operatorname{Ker} F_n - n \ .$$

This is a convenient way of calculating the index because we have fixed the dimension of the cokernel to be n, and so we have only one unknown dimension to compute, namely $\dim \operatorname{Ker} F_n$.

§2. Fredholm families.

We want now to investigate families of Fredholm operators depending continuously on some parameter. Formally this means that we have a topological space X (the space of parameters) and a continuous map

$$F : X \longrightarrow \mathcal{F}$$

where $\mathcal{F}$ is topologized as a subspace of the bounded operators $\mathcal{Q}$ (with the norm topology). Thus for each $x \in X$ we have a Fredholm operator $F(x)$ and

$$\| F(x) - F(x_0) \| < \varepsilon$$

for x sufficiently close to x_0.

We now ask how the vector spaces $\operatorname{Ker} F(x)$ and $\operatorname{Ker} F(x)^*$ vary with x. It is easy to see that their dimensions are not continuous (i.e. locally constant) functions of x: they are only semi-continuous, that is

$$\dim \operatorname{Ker} F(x_0) \geq \dim \operatorname{Ker} F(x)$$

for all x sufficiently close to x_0. On the other hand, as we shall see below in (2.1), index $F(x)$ is locally constant. Thus, although $\dim \operatorname{Ker} F(x)$ and $\dim \operatorname{Ker} F(x)^*$ can jump as $x \longrightarrow x_0$, they always jump by the same amount so that their difference remains constant. This invariance of the index under perturbation is its most significant property: it brings it into the realm of algebraic topology.

To go beyond questions of dimension it will be convenient to introduce the operators

$$F_n(x) = P_n \circ F(x)$$

defined as in §1. For large n this has the effect of fixing the dimension of the spaces $\operatorname{Ker} F_n(x)$. This is a necessary preliminary if we want these spaces to vary reasonably with x. More precisely we have the following result on the local continuity properties of these kernels:

LEMMA (2.1). *Let* $F : X \longrightarrow \mathcal{F}$ *be a continuous family of Fredholm operators and let* $x_0 \in X$. *Choose* n *so that* $F_n(x_0)(H) = H_n$. *Then there exists a neighborhood* U *of* x_0 *so that, for all* $x \in U$,

$$F_n(x)(H) = H_n \quad .$$

Moreover $\dim \operatorname{Ker} F_n(x)$ *is then a constant* d (*for* $x \in U$) *and we can find* d *continuous functions*

$$s_i : X \longrightarrow H$$

such that, for $x \in U$, $s_1(x), \ldots, s_d(x)$ *is a basis of* $\operatorname{Ker} F_n(x)$.

The proof of the lemma is shorter than its enunciation. Let $S = \operatorname{Ker} F_n(x_0)$ and define an operator

$$G(x) : H \longrightarrow H_n \oplus S$$

by $G(x)u = (F_n(x)u, P_S u)$ where P_S is the projection on S. Clearly $G(x_0)$ is an isomorphism.* Since $G(x)$ is continuous in x it follows that† $G(x)$ is an isomorphism for all x in some neighborhood U of x_0. This proves the lemma. In fact we clearly have $F_n(x)(H) = H_n$ for $x \in U$ and, if $e_1, \ldots, e_d$ is a basis of S,

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*Algebraically and hence, by the closed-graph theorem, topologically.

†Modulo an identification of $H_n \oplus S$ with H this is just the assertion that the invertible elements in the Banach algebra $\mathcal{A}$ form an open set.

$$s_i(x) = G(x)^{-1} e_i \qquad i \leq i \leq d$$

give a basis for Ker $F_n(x)$.

In view of (1.2) Lemma (2.1) shows at once that index $F(x) = d-n$ is a locally constant function of x. However it gives more information which we proceed to exploit. Suppose now that our parameter space X is compact. The first part of (2.1), combined with the compactness of X, implies that we can find an integer n so that

$$F_n(x)H = H_n \text{ for all } x \in X \ .$$

Choose such an integer and consider the family of vector spaces $S(x) = \text{Ker } F_n(x)$ for $x \in X$. If we topologize $S = \bigcup_{x \in X} S(x)$ in the natural way as a subspace of $X \times H$ the last part of (2.1) implies that S is a locally trivial family of d-dimensional vector spaces, i.e. in a neighborhood of any point x_0, we can find d continuous maps $s_i : X \longrightarrow S$ such that $s_1(x), \ldots, s_d(x)$ lie in $S(x)$ and form a basis.

A locally trivial family of vector spaces parametrized by X is called a vector bundle over X. Thus a vector bundle consists of a topological space S mapped continuously by a map π onto X so that each 'fibre' $S(x) = \pi^{-1}(x)$ has a vector space structure and locally we can find continuous bases $s_1(x), \ldots, s_d(x)$ as above.

There is† a natural notion of isomorphism for vector bundles over X. Let us denote by Vect(X) the set of isomorphism classes of all vector bundles over X. If X is a point a vector bundle over X is just a single vector space and Vect(X) is the set of non-negative integers $\mathbf{Z}^+$. For general spaces X however one can give simple examples of non-trivial vector bundles so that the isomorphism class of a vector bundle is not determined by its dimension.

Given two vector bundles S, T over the same space X one can form their direct sum $S \oplus T$. This is again a vector bundle over X and

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† For definitions and elementary properties of vector bundles see [1, Chapter I].

the fibre of $S \oplus T$ at x is just $S(x) \oplus T(x)$. This induces an abelian semi-group structure on Vect(X), generalizing the semi-group structure on $\mathbb{Z}^+$.

Let us return now to the consequences of (2.1). We have seen that, given the Fredholm family

$$F : X \longrightarrow \mathcal{F}$$

(with X compact), we can choose an integer n so that $\bigcup_{x \in X} \operatorname{Ker} F_n(x)$ is a vector bundle over X. We denote this vector bundle by $\operatorname{Ker} F_n$. Moreover $\operatorname{Ker} F_n^*(x) = H_n^\perp$ for all x so that $\operatorname{Ker} F_n^*$ is the trivial bundle $X \times H_n^\perp$. In view of formula (1.2) for the index of a single Fredholm operator it is now rather natural to try to define a more general notion of index for a Fredholm family F by putting

$$\begin{aligned} \text{index } F &= [\operatorname{Ker} F_n] - [\operatorname{Ker} F_n^*] \\ &= [\operatorname{Ker} F_n] - [X \times H_n^\perp] \end{aligned} \tag{2.2}$$

where [ ] denotes the isomorphism class in Vect(X). Unfortunately this does not quite make sense because Vect(X), like $\mathbb{Z}^+$, is only a semi-group and not a group so that subtraction is not admissible. However the way out is fairly clear: we must generalize the construction of passing from the semi-group $\mathbb{Z}^+$ to the group $\mathbb{Z}$ and associate to the semi-group Vect(X) an abelian group K(X). There is in fact a routine construction which starts from an abelian semi-group A and produces an abelian group B. One way is to define B to consist of all pairs (a_1, a_2) with $a_i \in A$ modulo the equivalence relation generated by

$$(a_1, a_2) \sim (a_1 + a, a_2 + a) \qquad a \in A$$

(we think of (a_1, a_2) as the difference $a_1 - a_2$). The only point to watch is that the natural map $A \longrightarrow B$ need not be injective.

Having introduced our group K(X) formula (2.2) now makes sense and defines the index of a Fredholm family as an element of K(X). In this definition we had to choose a sufficiently large integer n. However if we replace n by n+1 it is easy to see that

$$\text{Ker } F_{n+1} \cong \text{Ker } F_n \oplus E_n$$
$$\text{Ker } F^*_{n+1} \cong \text{Ker } F^*_n \oplus E_n$$

where E_n is the trivial 1-dimensional vector bundle generated by the extra basis vector e_n. Thus

$$\text{index } F \in K(X)$$

is well-defined independent of the choice of n.

Remarks. 1. Our definition of index F was dependent on a fixed orthonormal basis. It is a simple matter, which we leave to the reader, to show that the choice of basis is irrelevant.

2. Our introduction of K(X) here was motivated by the requirement of finding a natural "value group" for the index of Fredholm families. Historically the motivation for K(X) was somewhat different and arose from the work of Grothendieck in algebraic geometry. For this reason K(X) is referred to as the Grothendieck group of vector bundles over X.

Although our definition of the index as an element of K(X) may seem plausible, it is not clear at first how trivial or non-trivial it is. To show that it really is significant I will mention the following result.

THEOREM (2.3). *Let* X *be a compact space and let* $[X, \mathcal{F}]$ *denote the set of homotopy classes of continuous maps* $F : X \longrightarrow \mathcal{F}$. *Then* $F \longmapsto \text{index } F$ *induces an isomorphism*

$$\text{index} : [X, \mathcal{F}] \longrightarrow K(X) \quad .$$

This theorem shows that our index is the only deformation invariant of a

Fredholm family. For example if X is a point the theorem asserts that the connected components of $\mathcal{F}$ correspond bijectively to the integers, the correspondence being given by the ordinary (integer) index. The proof of (2.3) is not deep. It can be reduced quite easily (see [1; Appendix (A6)]) to Kuiper's theorem that the group $\mathcal{Q}^*$ of invertible operators on H is contractible. The proof of Kuiper's theorem, though ingenious, involves only elementary properties of Hilbert space.

§3. K-Theory.

Having introduced the group K(X) I shall now review briefly its elementary properties. In the first place we observe that one can form the tensor product $S \otimes T$ of two vector bundles and this induces a ring structure on K(X). Thus K(X) becomes a commutative ring with an identity element 1 -- corresponding to the trivial bundle $X \times \mathbb{C}^1$. When X is a point this is of course the usual ring structure of the integers $\mathbb{Z}$.

Next we investigate the behavior of K under change of parameter space. If $f : Y \longrightarrow X$ is a continuous map and if S is a vector bundle over X we obtain an induced vector bundle f^*S over Y, the fibre $f^*S(y)$ being $S(f(y))$. Passing to K we get a homomorphism of rings

$$f^* : K(X) \longrightarrow K(Y) \quad .$$

Thus K(X) is a contravariant functor of X. In this and other important respects K(X) closely resembles the cohomology ring H(X).

Finally we have the homotopy invariance of K(X) which asserts that $f^* : K(X) \longrightarrow K(Y)$ depends only on the homotopy class of the map $f : Y \longrightarrow X$. This follows from the fact that, in a continuous family of vector bundles over X, the isomorphism class is locally constant. For a simple proof of this see [1, (1.4.3)].

It is convenient to extend the definition of K(X) to locally compact spaces X by putting

$$K(X) = \mathrm{Ker}\{K(X^+) \xrightarrow{i^*} K(+)\}$$

where X^+ is the one-point compactification of X and $i : + \longrightarrow X^+$ is the inclusion of this "one-point." If X is not compact then $K(X)$ is a ring without identity: it is functorial for proper maps.

In addition to the ring structure in $K(X)$ it is convenient also to consider "external products." First of all, when X, Y are compact, we have a homomorphism

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

obtained by assigning to a vector bundle E over X and a vector bundle F over Y the vector bundle $E \boxtimes F$ over $X \times Y$ whose fibre at a point (x, y) is $E_x \otimes F_y$. This extends at once to locally compact X, Y in view of the fact that we have an exact sequence

$$(3.1) \qquad 0 \longrightarrow K(X \times Y) \longrightarrow K(X^+ \times Y^+) \longrightarrow K(X^+) \oplus K(Y^+)$$

which identifies $K(X \times Y)$ with the subgroup of $K(X^+ \times Y^+)$ vanishing on the two "axes" X^+, Y^+. For a proof of (3.1) see [1, (2.48)].

If $x \in K(X)$, $y \in K(Y)$ the image of $x \otimes y$ in $K(X \times Y)$ is called their external product and written simply as xy.

As an important example of a locally compact space X we can take the Euclidean space $\mathbb{R}^n$, so that X^+ is the sphere S^n. Then we have

$$K(S^n) \cong K(\mathbb{R}^n) \oplus \mathbb{Z}$$

the $\mathbb{Z}$ summand being given by the dimension of a vector bundle. Thus $K(\mathbb{R}^n)$ is really the interesting part of $K(S^n)$. More generally for any X, (3.1) implies a decomposition

$$(3.2) \qquad K(S^n \times X) \cong K(\mathbb{R}^n \times X) \oplus K(X) \quad .$$

The fundamental result of K-theory is the Bott periodicity theorem which asserts that the groups $K(\mathbb{R}^n)$ are periodic in n with period 2, so that

$$K(\mathbb{R}^{2m}) \cong K(\text{point}) = \mathbb{Z}$$
$$K(\mathbb{R}^{2m+1}) \cong K(\mathbb{R}^1) = 0$$

(the last equality depends on the fact that all vector bundles over the circle S^1 are trivial). This periodicity follows by induction from the more general result:

THEOREM 3.3. *For any locally compact space* X *we have a natural isomorphism*

$$K(\mathbb{R}^2 \times X) \cong K(X) \quad .$$

In §2 we explained the significant connection between the functor K and the space $\mathcal{F}$ of Fredholm operators. In view of this it is rather natural to raise the following two questions:

a) Can we use the index to prove the periodicity theorem (3.3)?

b) Can we use (3.3) to help study and compute indices?

The answer to both questions is affirmative. The first is remarkably simple and I will explain it in detail. The second is considerably deeper and I shall not comment on it further. What I have in mind will be found in [4]. A general discussion of the relationship between questions (a) and (b) is given in [3].

To carry out the program answering question (a) I will show, in the next section, how to define a homomorphism

$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X)$$

using the index of certain simple Fredholm families. Once α has been constructed it is not difficult to show that it is an isomorphism, thus proving Theorem (3.3). This will be done in §5.

§4. The Wiener-Hopf operator.

I will recall here the discrete analogue of the famous Wiener-Hopf equation.†

Let $f(z)$ be a continuous complex-valued function on the unit circle $|z| = 1$. Let H be the Hilbert space of square-integrable functions on the circle $|z| = 1$ and let H_n be the closed subspace spanned by the functions z^k with $k \geq n$. In particular H_0 is the space of functions $u(z)$ with Fourier series of the form

$$u(z) = \sum_{k \geq 0} u_k z^k \ .$$

Let P denote the projection $H \longrightarrow H_0$ and consider the operator

$$F = Pf : H_0 \longrightarrow H_0$$

where f here stands for multiplication by f. Thus the Fourier coefficients $(Fu)_n$ $(n \geq 0)$ for $u \in H_0$ are given by

(4.1) $$(Fu)_n = \sum_{k \geq 0} f_{n-k} u_k \qquad (n \geq 0)$$

where f_m are the Fourier coefficients of f. The classical Wiener-Hopf equation is the integral counterpart of the discrete convolution equation (4.1), the summation being replaced by $\int_0^\infty \hat{f}(x-y)\hat{u}(y)dy$.

It is clear from the definition of F that

$$\|F\| \leq \sup|f(z)|$$

so that F depends continuously on f (for the norm topology of F and the sup norm topology of f). The basic result about these operators F is

PROPOSITION (4.2). *If $f(z)$ is nowhere zero (on $|z| = 1$) then $F = Pf : H_0 \longrightarrow H_0$ is a Fredholm operator.*

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†For an exhaustive treatment of this topic see [5].

Proof. Let $\mathcal{C}$ denote the Banach algebra of all complex-valued functions f on the circle. Our construction $f \longmapsto F$ defines a continuous linear map

$$\mathcal{C} \longrightarrow \mathcal{A}$$

where $\mathcal{A}$ is the Banach algebra of bounded operators on H_0. Passing to the quotient $\mathcal{B} = \mathcal{A}/\mathcal{K}$ by the ideal $\mathcal{K}$ of compact operators we obtain a continuous linear map $T : \mathcal{C} \longrightarrow \mathcal{B}$. Suppose now that f, $g \in \mathcal{C}$ and have <u>finite</u> Fourier series

$$f(z) = \sum_{-n}^{n} f_k z^k \qquad g(z) = \sum_{-m}^{m} g_k z^k$$

Let $\ell = fg$ and let F, G, L denote the corresponding elements of $\mathcal{A}$. It is then clear that

$$FG(z^k) = L(z^k) \quad \text{for} \quad k \geq m+n$$

i.e. the operators FG and L coincide on the subspace H_{m+n} of H_0. Thus FG - L has finite rank and so is compact. Passing to the quotient algebra $\mathcal{B}$ this means that

$$T(fg) = T(f)T(g) \quad .$$

Thus $T : \mathcal{C} \longrightarrow \mathcal{B}$ is a <u>homomorphism</u> on the subalgebra consisting of finite Fourier series. Since this subalgebra is dense in $\mathcal{C}$ and since T is continuous it follows that T itself is a homomorphism. Since $T(1) = 1$ this implies that T takes invertible elements into invertible elements. Thus, if f(z) is nowhere zero, T(f) is invertible in $\mathcal{B}$ and so, by (1.1), F is a Fredholm operator.

<u>Remark</u>. For the Fredholm operator F = Pf of this proposition an integer n for which $F_n = P_n f : H_0 \longrightarrow H_n$ is surjective can be found explicitly. It is enough to take a finite Fourier series

$$g(z) = \sum_{-n}^{n} g_k z^k$$

which approximates $f(z)^{-1}$ sufficiently so that

$$\sup |f(z)g(z) - 1| < 1 .$$

I shall omit the simple verification.

As a simple example consider the function $f(z) = z^m$. It is clear that for $F = Pf$ we then have

$$\text{index } F = -m .$$

Since the index is a locally constant function it follows that, for any continuous map

$$f : S^1 \longrightarrow \mathbb{C}^*$$

(where $\mathbb{C}^*$ denotes the non-zero complex numbers), we have

$$(4.3) \qquad \text{index } F = -w(f)$$

where $w(f)$ is the "winding number" of f -- intuitively the number of times the path f goes round the origin in the complex plane. This simple observation is of fundamental importance for us. In fact the Bott periodicity theorem is, in a sense, a natural generalization of (4.3). We proceed now to introduce, by steps, the generalizations of the Wiener-Hopf operator which are required.

First we make an obvious extension, replacing the scalar-valued functions by vector-valued functions. The function f, which plays a multiplicative role, must be replaced by a matrix and the non-zero condition in (4.2) is replaced by the non-singularity of the matrix. Thus we start from a continuous map

$$f : S^1 \longrightarrow GL(N, \mathbb{C})$$

of the circle into the general linear group of $\mathbb{C}^N$. We take the Hilbert space of L^2-functions on S^1 with values in $\mathbb{C}^N$: if H is our original Hilbert space of scalar functions our new Hilbert space is $H \otimes \mathbb{C}^N$. As before we denote by P the projection

$$H \otimes \mathbb{C}^N \longrightarrow H_0 \otimes \mathbb{C}^N ,$$

and we define the operator

$$F = Pf : H_0 \otimes \mathbb{C}^N \longrightarrow H_0 \otimes \mathbb{C}^N$$

to be the composition of P and multiplication by the matrix function $f(z)$. The proof of (4.2) extends at once and shows that F is a Fredholm operator. The index of F is then a homotopy invariant of f. It determines in fact the element of the fundamental group of $GL(N, \mathbb{C})$ represented by f.

Next we generalize the situation by introducing a compact parameter space X. Thus we now consider a continuous map

$$f : S^1 \times X \longrightarrow GL(N, \mathbb{C}) \tag{4.4}$$

so that $f(z, x)$ is a non-singular matrix depending continuously on two variables z, x. For each $x \in X$ we therefore get a Fredholm operator $F(x)$, acting on the Hilbert space $H_0 \otimes \mathbb{C}^N$. Moreover $F(x)$ is a continuous function of x so that we have a Fredholm family

$$F : X \longrightarrow \mathcal{F} .$$

By the construction of §2 this family has an index in $K(X)$. Thus

$$f \longmapsto F \longmapsto \text{index } F$$

assigns to each continuous map f, as in (4.4), an element of $K(X)$. Moreover this element depends only on the homotopy class of f.

Our final generalization is to allow the vector space $\mathbb{C}^N$ to vary continuously with the parameter x. More precisely we fix an N-dimensional complex vector bundle V over X and we suppose given a function

$$(4.5) \qquad f(z, x) \in \operatorname{Aut} V(x)$$

(where $V(x)$ denotes the fibre of V over x) which is continuous in z, x. Since V is locally trivial our function f is locally of the type we had previously (with $V(x) = \mathbb{C}^N$ for all x), so that there is no problem in defining continuity. Our Hilbert space will now be $H_0 \otimes V(x)$ and so varies from point to point. The operator

$$F(x) : H_0 \otimes V(x) \longrightarrow H_0 \otimes V(x)$$

is defined as before and is a Fredholm operator depending continuously on x. Since our Hilbert space varies this is a somewhat more general kind of family than the Fredholm families of §2, but we can still define the index in $K(X)$ by essentially the same method. Since V is locally trivial we can still apply the purely local Lemma (2.1), replacing H_n by $H_n \otimes V(x)$, and obtaining locally an operator $F_n(x)$ with the properties described in (2.1). The compactness of X then leads to a fixed n for which $\operatorname{Ker} F_n$ is a vector bundle and $\operatorname{Ker} F_n^*$ is a trivial vector bundle. We now define

$$\operatorname{index} F = [\operatorname{Ker} F_n] - [\operatorname{Ker} F_n^*] \in K(X)$$

and prove as before that this is independent of n.

Finally therefore we have given a construction

$$(V, f) \longmapsto F \longmapsto \operatorname{index} F$$

which assigns to each pair (V, f) — consisting of a vector bundle V over X and an f as in (4.5) — an element of $K(X)$. Again this depends only on the homotopy class of f.

Now a pair (V, f) as above can be used to construct a vector bundle $E(V, f)$ over $S^2 \times X$. This is done as follows. We regard S^2 as the union of two closed hemi-spheres B^+, B^- meeting on the equator S^1. The vector bundle E is then constructed from the two vector bundles

$$\begin{array}{ccc} B^+ \times V & & B^- \times V \\ \downarrow & & \downarrow \\ B^+ \times X & & B^- \times X \end{array}$$

by identifying, over points $(z, x) \in S^1 \times X$,

$$(z, v) \in B^+ \times V(x) \text{ with } (z, f(z, x)v) \in B^- \times V(x) \quad .$$

When X is a point this is a well known construction for defining vector bundles over the sphere S^2. The parameter space X plays a quite harmless role.

It is not hard to show (see [1; p. 47]) that every vector bundle E over $S^2 \times X$ arises in this way from some pair (V, f). Given E we first take V to be the vector bundle over X induced from E by the inclusion map $x \longmapsto (1, x)$ of X into $S^2 \times X$ (where 1 denotes the point $z = 1$ on $S^1 \subset S^2$). We then observe that, since $B^+ \times X$ retracts onto $\{1\} \times X$, the part E^+ of E over $B^+ \times X$ is isomorphic to $B^+ \times V$. Similarly $E^- \cong B^- \times V$. Then, over $S^1 \times X$, the identification of E^+ and E^- defines f. The map f obtained this way is normalized so that $f(1, x)$ is always the identity of $V(x)$, and its homotopy class is then uniquely determined by E. Hence our construction

$$E \longmapsto (V, f) \longrightarrow F \longmapsto \text{index } F$$

defines a map

$$\text{Vect}(S^2 \times X) \longrightarrow K(X) \quad .$$

This is clearly additive and so it extends uniquely to a homomorphism of groups

$$\alpha' : K(S^2 \times X) \longrightarrow K(X) \ .$$

We recall now formula (3.2) which identifies $K(\mathbb{R}^2 \times X)$ with a subgroup of $K(S^2 \times X)$. Restricting α' to this subgroup we therefore obtain a homomorphism

$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X) \ .$$

Essentially therefore α is given by taking the index of a Wiener-Hopf family of Fredholm operators. From its definition it is clear that it has the following multiplicative property. Let Y be another compact space, then we have a commutative diagram

(4.6)
$$\begin{array}{ccc} K(\mathbb{R}^2 \times X) \otimes K(Y) & \longrightarrow & K(\mathbb{R}^2 \times X \times Y) \\ \downarrow \alpha_X \otimes 1 & & \downarrow \alpha_{X \times Y} \\ K(X) \otimes K(Y) & \longrightarrow & K(X \times Y) \end{array}$$

where the horizontal arrows are given by the external product discussed in §3.

We can now easily extend α to locally compact spaces X by passing to the one-point compactification X^+ and using (3.1). The commutative diagram (4.6) continues to hold for X, Y locally compact — again by appealing to (3.1).

In the next section we shall show how to prove that α is an isomorphism, thus establishing the periodicity theorem (3.3).

§5. Proof of periodicity.

We begin by defining a basic element b in $K(\mathbb{R}^2)$. We take the 1-dimensional vector bundle E_m over S^2 defined, as in §4, by the function $f(z) = z^m$. We put

$$b = [E_{-1}] - [E_0] \in K(S^2) \ .$$

Since E_{-1} and E_0 both have dimension 1 it follows that b lies in the summand $K(\mathbb{R}^2)$ of $K(S^2)$. As we have already observed, if F_m is the Wiener-Hopf operator defined by the function z^m, we have

$$\text{index } F_m = -m \ .$$

Thus

$$\begin{aligned} \alpha(b) &= \text{index } F_{-1} - \text{index } F_0 \\ &= 1 \ . \end{aligned}$$

We now define, for any X, the homomorphism

$$\beta : K(X) \longrightarrow K(\mathbb{R}^2 \times X)$$

to be external multiplication by $b \in K(\mathbb{R}^2)$. Thus, for $x \in K(X)$, $\beta(x)$ is the image of $b \otimes x$ under the homomorphism

$$K(\mathbb{R}^2) \otimes K(X) \longrightarrow K(\mathbb{R}^2 \times X) \ .$$

With these preliminaries we can state a more precise version of the periodicity theorem.

THEOREM (5.1). <u>For any locally compact space</u> X, <u>the homomorphisms</u>

$$\beta : K(X) \longrightarrow K(\mathbb{R}^2 \times X)$$
$$\alpha : K(\mathbb{R}^2 \times X) \longrightarrow K(X)$$

<u>are inverses of each other</u>.

As we shall see the proof of (5.1) is now a simple consequence of the formal properties of α, β. First we apply the diagram (4.6) with X = point, Y = X. This gives, for any $x \in K(X)$,

$$\begin{aligned} \alpha\beta(x) &= \alpha(b)x \\ &= x \qquad \text{since } \alpha(b) = 1 \ . \end{aligned}$$

Thus α is a left inverse of β. To prove that it is also a right inverse we apply (4.6) with $Y = \mathbb{R}^2$. Thus we have the commutative diagram

$$\begin{array}{ccc} K(\mathbb{R}^2 \times X) \otimes K(\mathbb{R}^2) & \longrightarrow & K(\mathbb{R}^2 \times X \times \mathbb{R}^2) \\ \downarrow \alpha_X & & \downarrow \alpha_{X \times \mathbb{R}^2} \\ K(X) \otimes K(\mathbb{R}^2) & \longrightarrow & K(X \times \mathbb{R}^2) \end{array} .$$

Hence for any element $u \in K(\mathbb{R}^2 \times X)$ we have

$$\alpha(ub) = \alpha(u)b \in K(X \times \mathbb{R}^2) \ . \tag{5.2}$$

Consider now the map τ of $\mathbb{R}^2 \times X \times \mathbb{R}^2$ into itself which interchanges the two copies of $\mathbb{R}^2$. On $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ this is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Since this has determinant +1 it lies in the identity component of $GL(4, \mathbb{R})$. Thus the map τ is homotopic to the identity (through homeomorphisms) and so τ^* is the identity of $K(\mathbb{R}^2 \times X \times \mathbb{R}^2)$. On the other hand we have

$$\tau^*(ub) = b\widetilde{u}$$

where $\widetilde{u} \in K(X \times \mathbb{R}^2)$ corresponds to $u \in K(\mathbb{R}^2 \times X)$ under the obvious identification. Thus

$$\alpha(ub) = \alpha(\tau^*(ub)) = \alpha(b\widetilde{u}) = \alpha\beta(\widetilde{u}) = \widetilde{u}$$

since α is a left inverse of β. Combined with (5.2) this gives

$$\widetilde{u} = \alpha(u)b \in K(X \times \mathbb{R}^2) \quad .$$

Switching back to $K(\mathbb{R}^2 \times X)$ this is equivalent to

$$u = b\alpha(u) = \beta\alpha(u) \in K(\mathbb{R}^2 \times X) \quad ,$$

proving that α is also a right inverse of β. This completes the proof of the theorem.

Concluding remarks.

I have given this proof of the periodicity theorem in such detail because I wanted to show how simple it really was. I hope I have demonstrated that the index of Fredholm families, which was used to construct the map α, plays a natural and fundamental role in K-theory.

This proof of periodicity has the advantage that it extends, with little effort, to various generalizations of the theorem. For full details on this I refer to [2].

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