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THEORY OF BRAIDS

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A theory of braids leading to a classification was given in my paper "Theorie der Zöpfe" in vol. 4 of the Hamburger Abhandlungen (quoted as Z). Most of the proofs are entirely intuitive. That of the main theorem in §7 is not even convincing. It is possible to correct the proofs. The difficulties that one encounters if one tries to do so come from the fact that projection of the braid, which is an excellent tool for intuitive investigations, is a very clumsy one for rigorous proofs. This has lead me to abandon projections altogether. We shall use the more powerful tool of braid coordinates and obtain thereby farther reaching results of greater generality.

A few words about the initial definitions. The fact that we assume of a braid string that it ends in a straight line is of course unimportant. It could be replaced by limit assumptions or introduction of infinite points. The present definition was selected because it makes some of the discussions easier and may be replaced any time by another one. I also wish to stress the fact that the definition of s-isotopy is of a provisional character only and is replaced later (Definition 3) by a general notion of isotopy.

More than half of the paper is of a geometric nature. In this part we develop some results that may escape an intuitive investigation (Theorem 7 to 10).

We do not prove (as has been done in Z) that the relations (18) (19) are defining relations for the braid group. We refer the reader to a paper by F. Bohnenblust[1] where a proof of this fact and of many of our results is given by purely group theoretical methods.

Later the proofs become more algebraic. With the developed tools we are able to give a unique normal form for every braid[2] (Theorem 17, fig. 4 and remark following Theorem 18). In Theorem 19 we determine the center of the braid group and finally we give a characterisation of braids of braids.

I would like to mention in this introduction a few of the more important of the unsolved problems:

1) Assume that two braids can be deformed into each other by a deformation of the most general nature including self intersection of each string but avoiding intersection of two different strings. Are they isotopic? One would be inclined to doubt it. Theorem 8 solves, however, a special case of this problem.

2) In Definition 3, we introduce a notion of isotopy that is already very general. What conditions must be put on a many to many mapping so that the result of Theorem 9 still holds?

[1] F. Bohnenblust, The algebraical braid group, Ann. Math., vol. 48, (1947), pp. 127–136.

[2] The freedom of the group of k-pure braids has been proved with other methods in: W. FRÖHLICH, *Über ein spezielles Transformationsproblem bei einer besonderen Klasse von Zöpfen*, Monatshefte für Math. und Physik, vol. 44 (1936), p. 225.

3) Determine all automorphisms of the braid group.

4) With what braids is a given braid commutative?

5) Decide for any two given braids whether they can be transformed into each other by an inner automorphism of the group. Concerning applications of braid theory this is by far the most important problem.

The last three of our questions seem to require an extensive study of the automorphisms of free groups.

We shall have to consider numerous functions of several variables. All of them are meant to be continuous in all the variables involved so that the statement of continuity is always omitted. Only more stringent conditions shall be mentioned.

Let x, y, z be the Cartesian coordinates of a 3-space. By a braid string we mean a curve that has precisely one point of intersection with each plane $z = a$ so that z may be used as parameter. Denoting by X the two dimensional vector (x, y) we may therefore describe the string by a vector function $X = X(z)$. In addition to that we assume the existence of two constants a, b such that $X(z)$ assumes a constant value X^- for all $z \leqq a$ and a constant value X^+ for all $z \geqq b$. X^- and X^+ are called the ends of the string.

By an n-braid we mean a set of n strings $X_i(z)$ $(i = 1, 2, \cdots n)$ without intersections (hence $X_i(z) \neq X_k(z)$ for $i \neq k$) where the numbering of the strings is considered unessential.

Two n-braids $X_i(z)$ and $Y_i(z)$ are called strongly isotopic (s-isotopic) if n vector functions $X_i(z, t)$ can be found with the following properties: They are defined for all z and for all t of a certain interval $c \leqq t \leqq d$, give an n-braid for each special t and are $X_i(z)$ for $t = c$, $Y_i(z)$ for $t = d$. They are constant in z and t if z is large enough and also constant if $-z$ is large enough. We remark at once that the ends remain fixed.

THEOREM 1. *s-isotopy is reflexive, symmetric and transitive.*

This allows us to unite s-isotopic braids into one class. We also obtain without difficulty:

THEOREM 2. *Let $g(z, t)$ be a numerical function defined for all z and all t of a certain t-interval. Assume that it tends with z to ∞ uniformly in t (the sign is unimportant). If $X_i(z)$ is a braid then the new braids $X_i(g(z, t))$ for different values of t are all s-isotopic. (They need not be s-isotopic to $X_i(z)$ itself.)*

COROLLARY 1. *If the numerical function $g(z)$ satisfies $\lim_{z=+\infty} g(z) = +\infty$ and $\lim_{z=-\infty} g(z) = -\infty$, then the braids $X_i(z)$ and $X_i(g(z))$ are s-isotopic.*

PROOF: Put $g(z, t) = (1 - t)\, z + t\, g(z)$ for $0 \leqq t \leqq 1$.

COROLLARY 2. *$X_i(z)$ is s-isotopic to any z-translation $X_i(z + t)$.*

COROLLARY 3. *The braids $X_i(\,|\, z \,|\, + t)$ are s-isotopic among themselves for different values of t. For large positive values of t all braid strings are constant $= X_i^+$. For large negative values of t the part of the braid above the xy-plane looks like a z-translation of $X_i(z)$, the part below this plane like its reflection on the xy-plane.*

We mention a few theorems of which little use shall be made in this paper, but that are useful if braids are to be studied by means of their projection on the yz-plane from the positive x-direction. By lexicographical arrangement of vectors we mean, as usual, their arrangement according to the size of their y-coordinate and, in case the y-coordinates are equal, according to their x-coordinate.

Two braids are said to have the same z-pattern of projection if the lexicographical arrangement of the vectors $X_i(z)$ is for each value of z the same as that of the vectors $Y_i(z)$.

LEMMA. *Two braids with the same z-pattern of projection are s-isotopic.*

PROOF: Put $X_i(z, t) = (1 - t)X_i(z) + t\,Y_i(z)$ for $0 \leqq t \leqq 1$. Assume we could find values z, t and $i \neq k$ such that $X_i(z, t) = X_k(z, t)$. This means that $(1 - t)(X_i(z) - X_k(z)) + t(Y_i(z) - Y_k(z)) = 0$. Let $X_i(z) - X_k(z) = (a, b)$ and $Y_i(z) - Y_k(z) = (a', b')$. The y-coordinate of our equation shows that b and b' can not have the same sign hence $b = b' = 0$. Now the x-coordinate of the equation leads to $a = a' = 0$. But then $X_i(z)$ would not be a braid.

Two braids $X_i(z)$ and $Y_i(z)$ are said to have the same pattern of projection if a monotonically increasing function $g(z)$ with infinite limits exists such that $X_i(z)$ and $Y_i(g(z))$ have the same z-pattern of projection.

THEOREM 3. *Two braids with the same pattern of projection are s-isotopic.*

PROOF: Use the lemma and Corollary 1 of Theorem 2.

THEOREM 4. *Let d be less than half the minimal distance between two of the strings of $X_i(z)$. Let $Y_i(z)$ be a braid with the same ends as $X_i(z)$ and assume that equally numbered strings of the two braids have at each z level a distance less than d. Then the braids are s-isotopic.*

The proof is done with the same device as in the lemma and is trivial.

DEFINITION: Two braids $X_i(z)$ and $Y_i(z)$ are called composable if:

1) They have the same number of strings.

2) After a suitable change in the numbering of the strings we have $Y_i^+ = X_i^-$. So the upper ends of $Y_i(z)$ must fit the lower ends of $X_i(z)$.

If (after a suitable translation) we join these ends we obtain a new braid which is said to be composed of $X_i(z)$ and $Y_i(z)$ (in this order). The formal definition would be:

Select an a and a b such that $X_i(z + a)$ is constant for negative z and $Y_i(z + b)$ is constant for positive z. Put $Z_i(z) = X_i(z + a)$ for positive z and $= Y_i(z + b)$ for negative z. This braid and any translation of it is called the composed braid. We still have a great freedom in the selection of a, b and the translation. All these braids are however s-isotopic.

If we replace both braids by others that are s-isotopic to them we obtain braids that are s-isotopic to $Z_i(z)$. The proof follows from the fact that after suitable selection of a and b the necessary deformation can be carried out independently in both sections of the composed braid. This leads to:

DEFINITION 1. Two classes of braids A and B are called *composable* if the braids in these classes are composable. The resulting braids form a class denoted by AB. It is to be remarked that if A is composable with B then B may not be composable with A. And even if the classes are composable from both sides then the commutative law may not hold. But the associative law does. If A and B as well as AB and C are composable then B and C as well as A and BC are composable and we have $A(BC) = (AB)C$.

THEOREM 5. *The classes of n-braids form a groupoid under our composition.*

PROOF: The postulates that we must verify are:

1) The kind of associative law we just have mentioned.

2) The existence of two classes U and U' for each given class A (dependent on A) such that $AU = U'A = A$. U is obviously the class containing a constant braid with the same lower ends as A and U' the similar class connected with the upper ends of A.

3) The existence of a class A^{-1} such that $A^{-1}A = U$ and $AA^{-1} = U'$. Corollary 3 shows that the reflection of a braid on the x, y plane gives such a class.

4) If A and B are given classes there exists a class C such that AC as well as CB can be formed. This just means to construct an example of a braid with given ends.

If U is one of the unit classes call G_U the set of all A that have U as left as well as right unit. They form a group. If V is another unit and C a class such that $UC = CV = C$ then $G_V = C^{-1}G_UC$ and the transformation thus indicated is an isomorphism. The knowledge of one of these groups reveals the structure of all and as a matter of fact the structure of the whole groupoid. The braids in such a group are those whose upper ends are only a permutation of the lower ends.

Next we prove that s-isotopy can be extended to the whole space:

THEOREM 6. *Let $X_i(z, t)$ be the n functions describing an s-isotopy. Then we can find a function $F(X, z, t)$ defined for all X and z and all necessary t, whose value is a vector, and that has the following properties:*

1) *For any fixed z it is a deformation of the plane. That means that it is a one to one correspondence of the plane if t also is fixed and it is identity for $t = a$ if that is the beginning of the t-interval.*

2) *Should for any special value of z the original functions $X_i(z, t)$ be independent of t, then $F(X, z, t) = X$ for that z and all X and t.*

3) *If a point (X, z) of the 3-space has a sufficiently large distance from the origin then $F(X, z, t) = X$ for all t.*

4) *$F(X_i(z, a), z, t) = X_i(z, t)$. So the deformation of the space moves the braid-strings precisely as the s-isotopy does.*

PROOF: 1) Select an $r > 0$ such that $| X_i(z, t) - X_k(z, t) | < 3r$ for all $i \neq k$ and all z and t. We first construct an auxiliary function $G(X, P_\nu, Q_\nu)$ of X and $2n$ points P_ν, Q_ν ($\nu = 1, 2, \cdots n$) of the plane. The points P_ν are restricted by the condition that their mutual distance shall always be greater than $3r$, the points Q_ν by the condition that Q_i lies in the interior of a circle C_i of radius r around P_i. The value of G shall be X if X is outside of all the circles or on the periphery of one. For $X = P_i$ the value shall be Q_i. If X is in the interior of C_i but different from P_i draw a radius through X and call R its intersection with C_i. Define the functionvalue as that point on the straight line segment

$R\,Q_i$ that bisects it in the same ratio as X bisects $R\,P_i$. Our function is continuous in all the variables, is a one to one correspondence of the plane for fixed P_ν and Q_ν and reduces to identity if $P_\nu = Q_\nu$ for all ν.

2) Divide the t interval into a finite number of parts $t_i \leqq t \leqq t_{i+1}$ such that the variation of every $X_i(z, t)$ in that interval is less than r for fixed z.

In $a = t_0 \leqq t \leqq t_1$ we define $F(X, z, t) = G(X, X_\nu(z, a), X_\nu(z, t))$. It has all the necessary properties. Assume that we succeeded to define $F(X, z, t)$ for all t of $t_0 \leqq t \leqq t_m$ and to check on the required properties. For $t_m \leqq t \leqq t_{m+1}$ we define:

$$F(X, z, t) = G(F(X, z, t_m), X_\nu(z, t_m), X_\nu(z, t)).$$

For $t = t_m$, we get the old value so it is a continuous continuation. The properties 1, 2, 3 follow immediately. For $X = X_i(z, a)$ we get $F(X, z, t_m) = X_i(z, t_m)$ hence $F(X, z, t) = X_i(z, t)$.

The extension of an s-isotopy to the whole space is not the only use of Theorem 6. We also use it to introduce new coordinates for the points of the space called braid coordinates. They are much more flexible in dealing with braids and the principal tool in the proofs of most of the following theorems.

Let $X_i(z)$ be a given braid, constant for $z \leqq a$ and for $z \geqq b$. Consider the braids $X_i(z + t)$ for $0 \leqq t \leqq b - a$. They are all isotopic and let $F(X, z, t)$ be the extension of this isotopy to the whole space. Then we have:

$$F(X_i(z), z, t) = X_i(z + t), \qquad 0 \leqq t \leqq b - a. \tag{1}$$

With each point (x, z) of the space we associate now a 2 dimensional vector $Y = Y(X, z)$ in the following way:

For $z \leqq a$ let $Y = X$.

For $a \leqq z \leqq b$ let Y be the unique solution of $F(Y, a, z - a) = X$. For $z = a$ we have $F(X, a, 0) = X$, therefore $X = Y$ again.

For $z \geqq b$ put $Y(X, z) = Y(X, b)$.

If z is fixed then the mapping $Y = Y(X, z)$ is a one to one correspondence of the plane that is certainly identity, outside a large circle whose radius does not depend on z. It is identity for all X if $z \leqq a$. For $z \geqq b$ it is in general not identity but at least is the same mapping for all $z \geqq b$.

$F(X_i(a), a, z - a) = X_i(a + (z - a)) = X_i(z)$ if $a \leqq z \leqq b$ because of (1). This shows that the Y for the point $(X_i(z), z)$ of the i^{th} string is $X_i(a) = X_i^-$. The same is true for $z \leqq a$ and for $z \geqq b$ for trivial reasons.

We associate now with the point (X, z) the corresponding combination $\{Y, z\}$ and call it the braid coordinates of that point. They equal the ordinary coordinates for all $z \leqq a$ and also for all large $|X|$. All points on the i^{th} string have the simple braid coordinates $\{X_i^-, z\}$.

Another way to look upon the braid coordinates is this: Interpret them as ordinary coordinates of a point of a 3-space. Then our 3-space is mapped by a one to one correspondence onto this new one and the braid strings are mapped onto vertical lines with the same lower ends.

Let now u be any real number. The mapping $\{Y, z\} \rightarrow \{Y, z + u\}$ in our old space is a one to one correspondence that has all the essential features of a translation and shall therefore be called translation by u along the braid. Each single string remains fixed as a whole. For large $| X |$ and also if $| z |$ is large in comparison to $| u |$ it is an ordinary translation. For the other points the z-coordinate does at least behave in the ordinary way.

We can of course also find the inverse function $X = H(Y, z)$ that describes the passage from Y back to X. Let now $X(z)$ be any other braidstring (it may intersect the strings of our braid). Apply to it a translation by u along our braid. The braid coordinates of $(X(z), z)$ are $\{Y, z\}$ where $Y = Y(X(z), z)$. The translation moves it into $\{Y, z + u\}$. The ordinary coordinates are $(X', z + u)$ where $X' = H(Y(X(z), z), z + u)$. Looking at the new string as a whole, we may replace z by $z - u$ and obtain the new braidstring

$$X(z, u) = H(Y(X(z - u), z - u), z)$$

such that $(X(z, u), z)$ are the points of the new braidstring. Letting now the u change, we see that we have before us an s-isotopic change where the points move only in horizontal planes. Should the original string not intersect the braidstrings, then its translation does not either and the braid formed out of the old braid by adding our new string undergoes an s-isotopy under translation.

We use this in the following way: Let $X_i(z)$ be an n-braid and replace its n^{th} string by any other string $X'_n(z)$ with the same ends. The new string may intersect $X_n(z)$ but shall not intersect the other strings of our braid. This gives a new n-braid and we apply now to our old braid a translation by a large u along our new braid. What happens? The $n - 1$ first strings remain fixed. If u is sufficiently large then a very low portion of the n^{th} string will now be in the main part of the braid. In that very low portion the string $X_n(z) = X'_n(z)$. The string $X'_n(z)$ does not change under our translation. This shows that in the main portion of our braid $X_n(z)$ has moved into the position $X'_n(z)$. This is of course compensated by the fact that the n^{th} string is now entangled in the other strings above the main portion of the braid. But above the main section of the braid, the first $n - 1$ strings are very simple, namely parallel lines. Remember finally that we have shown in the preceding paragraph that the translation can also be considered as a horizontal motion, as an s-isotopy.

THEOREM 7. *It is possible to apply to a braid an s-isotopy moving one string only whereby this string may be brought into any other position provided this is compensated by a motion of the string above the main section of the braid where the $n - 1$ other strings are parallel.*

This suggests the definition:

DEFINITION 2. A braid with the same upper and lower ends is called *i-pure*, if all but the i^{th} string are constant. A class is called i-pure if it contains an i-pure representative.

We see: if B and B' are braids or classes of braids having $n - 1$ strings in common then $B = AB'$ where A is i-pure.

Another useful notion connected with braid coordinates is that of projection along the braid. Consider the plane $z = z_0$. The mapping $\{Y, z\} \rightarrow \{Y, z_0\}$ is this projection. It carries the braidstrings into their intersection with the plane and a point not on the braid into a point different from these intersections.

Let now R_B be the complementary set to B in the 3-space. We introduce the usual notion of homotopy of paths in R_B. Two paths $a = \{Y(t), z(t)\}$, $b = \{Y'(t), z'(t)\}$, $0 \leqq t \leqq 1$ are called homotopic in $R_B : a \sim_B b$ if a function of two variables t and s (interval 0, 1) $\{Y(t, s), z(t, s)\}$ can be found, that is constant for $t = 0$ and $t = 1$, gives the first path for $s = 0$ and the second for $s = 1$. All points of the deformation have to belong to R_B which means simply that the function Y avoids the values X_i^-.

The composition of homotopy classes is introduced in the usual way and leads to a groupoid. If we select a point P in R_B and consider the homotopy classes of those paths that have beginning as well as endpoint at P, we obtain the Poincaré group of R_B. Let z_0 be the z coordinate of P and $a = \{Y(t), z(t)\}$ any element of this group. The projection $a' = \{Y(t), z_0\}$ is homotopic to a as the function $\{Y(t), z_0 + s(z(t) - z_0)\}$ shows. If two paths in this plane are B-homotropic, say by the function $\{Y(t, s), z(t, s)\}$, then the function $\{Y(t, s), z_0\}$ shows that the paths are already homotopic in the plane. The Poincaré group is therefore the same as that of a plane punctured in the n points $X_i(z_0)$. So it is a free group with n generators.

We must now carefully describe the generators we want to use. The plane will be either in the region of z where the braidstrings assume the constant values X_i^- and shall then be called a lower plane or in the region of the X_i^+ when we call it an upper plane. Take an upper plane and draw in it a ray that does not meet any of the upper or lower ends. We intend to take the point P on that ray sufficiently far away. Each of the points X_i^+ shall be connected with the beginning point Q of the ray by a broken line without self intersection such that two of the lines and also the ray have only the endpoint Q in common. By l_i we denote the connection thus established between the beginning point X_i^+ and the point P on the ray. By $l_i(\epsilon)$ we mean the same path but starting with the parameter value ϵ. An orientation of the plane is selected. By $c_i(\epsilon)$ we mean a curve with the winding number 1 around X_i^+ starting and ending at the beginning point of $l_i(\epsilon)$ that stays within a small neighborhood of X_i^+. It is well known that the paths (for small ϵ)

$$t_i = l_i(\epsilon)^{-1}c_i(\epsilon)l_i(\epsilon) \tag{2}$$

are free generators of the Poincaré group of the punctured plane. This pattern of paths is then transferred to all other upper planes by projection (in the ordinary sense) including the point P. We use the same names for the paths in all the upper planes.

In a lower plane we first transfer the ray, the orientation and the point P to it by ordinary projection. Since we now have to take care of the lower ends, new paths are selected denoted by l'_i, $l'_i(\epsilon)$, $c'_i(\epsilon)$. The corresponding generators shall be called t'_i.

What we intend to do with this setup is roughly this: If two braids are s-isotopic they must by necessity have the same ends. It therefore suffices to consider braids with given ends. For all these braids we use the same pattern of paths in the lower and upper planes. If B is a given braid then we project by braid projection the generators t'_i into the upper plane. We obtain paths $\bar{t}_i$ that are now generators in the upper plane. Therefore they can be expressed in terms of the t_i. It turns out that these expressions are a complete set of invariants for the isotopy classes.

As a first indication that the study of the homotopy classes must give a solution of our problem let us consider the following special case:

Let C be an $(n - 1)$-braid. Form two n-braids by inserting in C an n^{th} string in two ways but both time with the same ends. Select a z_0 such that the two strings are equal for $z \leqq z_0$ and call P_0 the point on the n^{th} strings at that z_0-level. In a similar fashion select z_1 and P_1 for the upper end. Consider now the two pieces of the n^{th} strings between P_0 and P_1. If they are homotopic relative to C we may call the two n^{th} strings C-homotopic. Then the following rather surprising theorem holds:

THEOREM 8. *Let B and B' be obtained from C by insertion of an n^{th} string with given ends. If the two strings are C-homotopic then B and B' are s-isotopic. Every homotopy class can be realized by a braid. The converse is also true but will be proved later.*

PROOF: 1) We use braid coordinates of C and express the two inserted strings in terms of these coordinates: $Y_n(z)$, $Y'_n(z)$. The fact that they do not intersect the strings of C means just that the functions avoid the values X_i^-. By assumption there exists a function $\{Y(t, s), z(t, s)\}$ defined in $z_0 \leqq t \leqq z_1$, $0 \leqq s \leqq 1$ describing the homotopy of the strings. So $Y(t, s)$ will avoid the values X_i^-, have fixed beginning and end points for all s and will be $Y_n(t)$ for $s = 0$ and $Y'_n(t)$ for $s = 1$. The method consists now in forgetting about the function $z(t, s)$ altogether and to define a function $Y_n(z, s)$ as equal to $Y(z, s)$ for $z_0 \leqq z \leqq z_1$ and equal to X_i^- for all other z. The function avoids X_i^- and reduces to the given strings for $s = 0$ and $s = 1$. So it gives the required s-isotopy.

2) If $\{Y(t), z(t)\}$ is any curve defined in $z_0 \leqq z \leqq z_1$ that avoids the strings of C and joins P_0 and P_1, put as before $Y_n(z) = Y(z)$ in that interval and $=$ X_n^- for all other z. The function $\{Y(t), s \cdot z(t) + (1 - s)t\}$ shows that this n^{th} string is homotopic to the given curve.

REMARK. The s-isotopy of Theorem 8 moves the n^{th} string only.

Let us now return to our upper plane. Join one of the points X_i^+ to P by a curve h that avoids the braid with exception of its beginning point. Define $h(\epsilon)$ as before and let $d(\epsilon)$ be a curve analogous to $c_i(\epsilon)$. Consider the element $t = h(\epsilon)^{-1}d(\epsilon)\, h(\epsilon)$. If we join the beginning point of $l_i(\epsilon)$ to that of $h(\epsilon)$ by a path e that stays in a small neighborhood of X_i^+ then $e^{-1}c_i(\epsilon)e$ is homotopic to

$d(\epsilon)$. The path $S' = l_i(\epsilon)^{-1}e\,h(\epsilon)$ is a certain element of the group and may be expressed in terms of the t_ν. The element

$$S'^{-1}t_iS' = h(\epsilon)^{-1}e^{-1}l_i(\epsilon)l_i(\epsilon)^{-1}c_i(\epsilon)l_i(\epsilon)l_i(\epsilon)^{-1}e\,h\,(\epsilon)$$

is homotopic to t. Assume now that a similar expression $t = S^{-1}t_iS$ is known from some other source and may even not be in a reduced form. We first perform the possible cancellations in S only; a further simplification of the expression $S^{-1}t_iS$ is then possible only if S begins with a power of t_i. Then S^{-1} ends with the reciprocal of that power and we see that this term may indeed be dropped. This shows that S is uniquely determined but for a power of t_i and we have therefore $S' = t_i^rS$ or

$$l_i(\epsilon)^{-1}e\,h(\epsilon) = t_i^rS. \tag{3}$$

Let us now reinsert in our plane the one point X_i^+ and consider homotopies in that new plane, punctured in $n - 1$ points only. This homotopy shall be denoted by $\sim_i$. It amounts to put $t_i = 1$ in all previous expressions. The resulting element shall still be denoted by S. We obtain:

$$l_i(\epsilon)^{-1}e\,h(\epsilon) \sim_i S.$$

The path $l_il_i(\epsilon)^{-1}e\,h(\epsilon)h^{-1}$ is i-homotopic to a closed curve starting at X_i^+ and remaining in a small neighborhood of that point. It is therefore i-homotopic to this point X_i^+. This proves the formula

$$\bar{l}_i^{-1}h \sim_i S. \tag{4}$$

Let now B be a braid with the given ends. If we apply braid projection to a generator t_i' of a lower plane onto an upper plane and assume that the point P has been selected sufficiently far out, then the image $\bar{t}_i$ will be an element of the Poincaré group for P. If $\bar{l}_i$ is the projection of l_i' we obtain equations of the form:

$$\bar{t}_i = S_i^{-1}t_iS_i \tag{5}$$

$$l_i(\epsilon)^{-1}e_i\,\overline{l_i(\epsilon)} \sim t_i^rS_i \tag{6}$$

$$\bar{l}_i^{-1}l_i \sim_i S_i. \tag{7}$$

It is to be remarked that the properties of the braid coordinates show that the form of the equation (5) does not depend on the precise location of the upper and the lower plane. It is also clear that it does not depend on ϵ provided it is only small enough. The position of P plays also no role in it provided that it is far enough out. The equations (6) and (7) change of course their meaning and the exponent r may depend on ϵ and e_i.

We may look upon this process in yet another way. Call g the straight line segment that connects P with its projection in the lower plane and put $\tau_i = gt_i'g^{-1}$. They are elements of the Poincaré group for P and as a matter of fact a set of generators. If we subject them to braid projection they will go over again into the $\bar{t}_i$. But being elements with the same beginning and end point τ_i is, as we have seen before, homotopic to $\bar{t}_i$. So equation (5) becomes

$$\tau_i \sim S_1^{-1} t_i S_i . \tag{8}$$

We are now ready for a generalized notion of isotopy:

DEFINITION 3. A braid is called *isotopic to another braid* if the space can be mapped into itself in such a way that points on the first braid but no other points of the space are mapped onto points of the second braid. In addition to this we assume that the mapping is identity outside of a certain sphere. Inside that sphere the mapping must of course be continuous but need not be one to one.

Consider now two isotopic braids B and C. Locate the lower and the upper plane outside the sphere and select P so that g is also outside this sphere. The two elements τ_i and its B-projection $\bar{t}_i$ are B-homotopic. The surface connecting the two paths does not meet B. Its image under our mapping will therefore avoid C. This shows that the images of our paths are C-homotopic. But our paths remain fixed. So τ_i and $\bar{t}_i$ are C-homotopic. τ_i on the other hand is C-homotopic to its C-projection. This C-projection is therefore C-homotopic to $\bar{t}_i$ and this proves:

THEOREM 9. *If B is isotopic to C then the exexpressions in formula* (5) *are the same for B and C.*

The i-homotopy of formula (7) may be interpreted as homotopy with respect to the braid resulting from a cancellation of the i^{th} string. Denote by Σ_i the piece of the i^{th} string that starts at the upper plane and ends at the lower plane. The path $l_1^{-1}\, \Sigma_i l_i' g^{-1}$ is a closed path starting at P and as such i-homotopic to its projection onto the upper plane. But this projection is obviously the left side of (7). Computing Σ_i out of the resulting homotopy we get:

$$\Sigma_i \sim_i l_i S_i g l_i'^{-1} \tag{9}$$

and this shows that S_i determines the homotopy class of the i^{th} string.

(9) Interprets the i^{th} string but it would not completely explain S_i since it is only an i-homotopy. Let Σ_i' be any path connecting the beginning points of $l_i(\epsilon)$ and $l_i'(\epsilon)$ that stays in the immediate vicinity of the n^{th} string without intersecting it. Its projection is then a curve that may be used as e_i. Now the projection of $l_i(\epsilon)^{-1}\Sigma_i' l_i'(\epsilon) g$ is the left side of (6):

$$t_i^r S_i \sim l_i(\epsilon)^{-1} \Sigma_i' l_i'(\epsilon) g \tag{10}$$

which provides the full geometric meaning of S_i. The converse of Theorem 8 is:

THEOREM 10. *Let B and C be two braids with the same ends and with the same first $n - 1$ strings. Assume either that B and C are s-isotopic or that they are isotopic or that the expressions* (5) *are the same for both braids. Then the n^{th} strings are n-homotopic and there exists an s-isotopy moving the n^{th} string only.*

PROOF: If they are s-isotopic then they are isotopic because of Theorem 6. If they are isotopic then the expressions (5) are the same because of Theorem 9.

If (5) is the same even for $i = n$ only then the n^{th} strings are n-homotopic because of formula (9). The remark to Theorem 8 shows the rest of the contention.

THEOREM 11. *If two braids have the same ends and if the expressions* (5) *are the same for both braids then they are s-isotopic.*

PROOF: For $n = 1$ our theorem is trivial since the two strings are 1-homotopic. Let it be proved for braids with $n - 1$ strings. If B and C are two n-braids for which the assumption holds, let B' and C' be the braids resulting from cancellation of the n^{th} string. The expressions (5) for B' and C' are obtained by putting $t_n = 1$. So (5) is also the same for B' and C'. Therefore B' and C' are s-isotopic. Extend the s-isotopy to the whole space and apply this mapping to the braid B. It carries B into an s-isotopic braid D having again the same expressions (5). D and C have now the first $n - 1$ strings in common so that Theorem 10 shows that they are s-isotopic. This completes the proof.

THEOREM 12. *Isotopy and s-isotopy imply each other.*

The proof follows from Theorems 6, 9 and 11.

THEOREM 13. *The expressions* (5) *do not depend on the special braid-coordinates used. They depend even only on the class and give together with the ends a full system of invariants of the class that determines the class completely.*

The proof is now obvious. It is to be remarked, however, that the expressions depend on the selection of generators t_i and t_i'. We must now develop methods that allow the actual computations of these invariants and reveal the structure of our groupoid.

To do so we have first to change our notation slightly. Select in a plane n points X_i, a ray and paths l_i. Up to this point the numbering was considered unessential. Now we get a natural arrangement of our points by starting with our ray and going around Q in the positive sense of rotation (in a neighborhood of Q). The first path that we meet shall be called l_1, the next l_2 and so on. The points X_i are now numbered precisely as the paths leading to them. The very same pattern is now used for the upper planes as well as for the lower ones. The points X_i are now used as lower and upper ends of braids. We restrict ourselves to the investigation of braids whose lower ends are a subset of the X_i and whose upper ends are another subset. These subsets may or may not be the whole set, no restriction being put upon them. If B is such a braid and X_i, X_j the lower respectively upper end of one of its strings we write:

$$j = B(i).$$

Thus B maps a certain subset of the numbers $1, 2, \cdots n$ onto another subset. The numbering of the generators t_i and t_i' so far was connected with the numbering of the string. Now we change that and attach to them the subscript of the point around which they run. We also drop the accent on the t_i' and write uniformly t_i for all the generators in the different planes. This leads to the following situation:

We have a group F before us with the n free generators t_i. For the Poincaré group of a braid with the reference point in an upper or a lower plane, not all the generators are used; the Poincaré group of such a plane is therefore considered a subgroup of F generated by a subset of the t_i. Braid projection of a lower unto an upper plane will provide us with an isomorphic mapping of the group in the lower plane onto the group in the upper one. If T is an element of the group in the lower plane then its image under braid projection shall be denoted by $B(T)$. In this new notation (5) takes on the form:

$$B(t_i) = S_j^{-1} t_j S_j \quad \text{where} \quad j = B(i), \tag{11}$$

and where the numbering of all generators has been changed according to our new convention. (11) alone already gives us the isomorphic mapping and in this form contains also the information about the upper and lower ends of the strings of the braid. In case all the points are used for the lower ends, it will be an automorphism of F. Otherwise it maps a subgroup of F onto another subgroup.

Let A and B be two composable braids and form the composed braid in such a way that in AB the part B corresponds to negative, the part A to positive z. Returning for a moment to the interpretation of our projection in terms of the generators τ_i which allow to express projection in terms of homotopies we see:

$$AB(T) = A(B(T)).$$

We project namely a lower plane of AB first onto the plane $z = 0$ and the result onto an upper plane. Making use of the fact that (11) completely determines the class we see:

THEOREM 14. *The groupoid of braid classes whose lower and upper ends are subsets of the X_i is isomorphic to the groupoid of mappings in F indicated by* (11).

Consequently we express the braid class B in form of a substitution

$$B = \begin{pmatrix} t_i \\ S_j^{-1} t_j S_i \end{pmatrix} \tag{12}$$

where t_i runs of course only through certain of the generators. It is convenient to consider also more general substitutions

$$B = \begin{pmatrix} t_i \\ T_i \end{pmatrix}$$

in the free group F where certain t_i are mapped onto power products regardless of whether the substitution is derived from a braid or not. If the substitution is derived from a braid class then we say briefly that it is a braid. Also in this general case we denote by $B(T)$ the result of applying the substitution B onto the power product T.

The braid substitutions have one special property that we must derive. Draw a huge circle in a lower plane starting at the reference point of the Poincaré group and running around the braid. It is well known from the theory of the homotopies in a punctured plane that this element of the Poincaré group is homotopic to the product of the generators t_i (of course only those that we need for our braid) taken in the natural arrangement of the subscripts according to their size. Braid projection onto an upper plane carries the circle into a similar circle starting at P. This proves:

THEOREM 15. *If B is a braid then*

$$B\left(\prod t_i\right) = \prod t_j \tag{13}$$

both products taken in the natural arrangement of their subscripts.

Select a subscript $i < n$ and put $X_\nu(z) = X_\nu = \text{const}$ for $\nu \neq i, i+1$. We connect now X_i and X_{i+1} by a broken line starting at X_i and running then parallel to l_i until it comes near the ray; then it runs parallel to l_{i+1} until it comes close to X_{i+1} with which it is then connected. If $X(t)$ $(0 \leqq t \leqq 1)$ is the parametric representation of this line we put

$$X_i(z) = \begin{cases} X_i & \text{for} \quad z \leqq 0 \\ X(z) & \text{for} \quad 0 \leqq z \leqq 1 \\ X_{i+1} & \text{for} \quad z \geqq 1. \end{cases}$$

Then we draw a similar parallel curve between X_{i+1} and X_i running farther out and not intersecting the previous one but at the ends. The string $X_{i+1}(z)$ is explained in a similar fashion than $X_i(z)$ but it has X_{i+1} as lower and X_i as upper end. That this braid carries t_ν into itself if $\nu \not\equiv i, i+1$ is seen by using ordinary projection which shows that τ_ν is homotopic to t_ν. t_i is mapped into a transform of t_{i+1}. To find the transformer we go back to (10). As parallel curve we use one that will under ordinary projection become the parts of $l_i(\epsilon)$ and $l_{i+1}(\epsilon)$ up to the ray. Consider now the right side of (10). Instead of l_i we have to write l_{i+1} in our new notation. The path projects by ordinary projection still into a homotopic path. But this homotopic path is now obviously 1. So t_i is carried into t_{i+1}. The image of t_{i+1} can now be found by a simpler method, namely, by Theorem 15. Since the product of the t_i must remain fixed we find by a simple computation that t_{i+1} is mapped into $t_{i+1}^{-1}t_i t_{i+1}$. The class of this particular braid shall be called σ_i and the corresponding substitution is:

$$\sigma_i = \begin{pmatrix} t_i, & t_{i+1} \\ t_{i+1}, & t_{i+1}^{-1}\, t_i\, t_{i+1} \end{pmatrix} \tag{14}$$

with the understanding that the generators that are not mentioned are left unchanged. For σ_i^{-1} we have to compute the inverse substitution and an easy computation gives:

$$\sigma_i^{-1} = \begin{pmatrix} t_i, & t_{i+1} \\ t_i\, t_{i+1}\, t_i^{-1}, & t_i \end{pmatrix}. \tag{15}$$

To check whether a given braid is σ_i it suffices, however, to check the following properties: dropping the i^{th} and $i+1^{\text{st}}$ string we must obtain a unit. In it the $i+1^{\text{st}}$ string must correspond to the unit homotopy. After reinserting it the i^{th} string must have unit homotopy (always using the simpler formula (9) rather then (10)). According to our theory these checks already determine the class.

From now on the nature of our proofs will be mostly algebraic.

Consider a rather general substitution B that maps each generator t_i onto a transform $Q_i^{-1}t_kQ_i = T_i$ (in reduced form) of some generator t_k. When is B a braid? The answer is given by the condition of Theorem 15, namely that

$$T_1T_2 \cdots T_n = t_1t_2 \cdots t_n . \tag{16}$$

The necessity is obvious. To prove the sufficiency we assume each Q_i written as a product of terms t_ν^ϵ, $\epsilon = \pm 1$. The number of terms shall be called the length of Q_i and the sum of all these lengths the length of B. If the length of B is 0 then (16) can hold only if each $T_i = t_i$ or $B = 1$ which is the unit braid. So we may assume our contention proved for all braids with smaller length than B. (16) can hold only if some cancellations take place on the left side. Since each T_i is already reduced these cancellations must take place between adjacent factors of the left side. Two cases are conceivable:

1) In a cancellation between two neighbors the middle terms are never affected. Carrying them out in (16) there will be a residue R_i left from each T_i and this residue must contain the middle term of T_i. We obtain:

$$R_1R_2 \cdots R_n = t_1t_2 \cdots t_n$$

and no further cancellation is possible. This proves $R_i = t_i =$ middle term. The terms on the left side of the middle term of T_1 never could be cancelled at all since no factor is on their left. So $Q_1 = 1$. Now there is no further chance for Q_2^{-1} to be cancelled, so it must be 1 too. This shows $T_i = t_i$ so that this case is settled.

2) Or else there are two neighbors T_i and $T_{i+1} = Q_{i+1}^{-1}t_sQ_{i+1}$ such that one or both of the middle terms are reached in a cancellation. They cannot be affected at the same time since their positive exponent prevents it. Now two alternatives are forced upon us:

a) t_k is affected first. Consider the product $T_iT_{i+1}T_i^{-1}$. Because of the special form of the T_r a cancellation is now possible on both sides of T_{i+1}. Carry it out, term by term, on both sides until the middle term of T_i and T_i^{-1} is reached and stop the cancellation at that point even if it is possible to go on. More than half of T_i and T_i^{-1} will have been absorbed, the middle term of T_{i+1} will not yet be reached and T_{i+1} will have lost as many factors as the other two. A few remnants from these factors will remain but they will be shorter than the loss. It is of course very easy to write this down formally. What is important is, that the length of this product is shorter than that of T_{i+1}. Consider now the substitution $B\sigma_i^{-1}$. We find:

$$B\sigma_i^{-1} = \begin{pmatrix} t_\nu, & t_i, & t_{i+1} \\ T_\nu, & T_iT_{i+1}T_i^{-1}, & T_i \end{pmatrix} \qquad \nu \neq i, i+1.$$

The product of the second line is $T_1T_2 \cdots T_n = t_1t_2 \cdots t_n$. It has still the general form of B but is shorter. So this substitution A is a braid. Since $B = A\,\sigma_i$, we see that B is a braid.

b) t_s is affected first. Then we find that $T_{i+1}^{-1}T_iT_{i+1}$ is shorter than T_i so that

$$B\sigma_i = \begin{pmatrix} t_\nu, & t_i, & t_{i+1} \\ T_\nu, & T_{i+1}, & T_{i+1}^{-1}T_iT_{i+1} \end{pmatrix} \qquad \nu \neq i, i+1$$

is a braid. This proves that $B = A\sigma_i^{-1}$ is a braid.

Our proof also shows that B can be expressed as a powerproduct of the σ_i.

The proof would also have worked if the subscripts i ran through a subset of all indices only. No condition need be put on the t_k and Q_i. (15) has to be replaced by the condition that B leaves $\prod t_i$ (in the natural order) invariant. The braids σ_i have of course to be replaced by the corresponding braids for this subset of ends.

The most general case would be finally this. A subset of the t_i and mappings of the previous kind are given. B carries $\prod t_i$ into $\prod t_j$ where the t_j form another subset also in the natural order. To reduce this case to the previous one let B_1 be a braid having the X_i as lower ends and the X_j as upper ones. The substitution $B\,B_1^{-1}$ maps $\prod t_j$ onto itself and is therefore a braid. So B is a braid. Let (i) and (j) be subsets of the indices both equal in number. Our result shows that

$$B_{(i)(j)} = \begin{pmatrix} t_i \\ t_j \end{pmatrix} \text{ is a braid.} \quad (i, j \text{ natural arrangement}). \tag{17}$$

Our results may be expressed in the theorem:

THEOREM 16. *A substitution is a braid if, and only if, it has the general form of a transformation and if it satisfies the condition of Theorem* 15. *The full group of n-braids has the σ_i as generators. A general braid can be expressed as a product of a braid of the form* (17) *followed by generators like the σ_i but concerning the lower ends of the braid only.*

A simple computation of substitutions shows that the following relations hold between the σ_i:

$$\sigma_i\sigma_k = \sigma_k\sigma_i \qquad \text{if } |\,i - k\,| \geqq 2 \tag{18}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \tag{19}$$

In Z. I have shown that these relations form a full set of defining relations for the group. The method is geometric and can easily be made rigorous by means of the tools developed in this paper. However a more interesting proof shall be given in a paper by F. Bohnenblust which is essentially algebraic and leads deeper into the theory of the group. I shall therefore omit a proof here especially since no use shall be made of this fact in this paper. All we shall use is that these relations hold.

A simple operation is that of removing a string. What does it mean for the substitution? Let A be the braid and remove the string with X_r as lower and

X_m as upper end. That means that we have to cancel the column referring to the image of t_r. In addition to that we have to substitute everywhere in the second row $t_m = 1$ since that describes the shrinkage in the homotopy generators.

The inverse problem in a somewhat simplified form would be:

Given a braid in the form (12), assume r does not occur among the i and m not among the j. Form a new braid A by inserting one string with X_r and X_m as respective lower and upper ends. The simplification will consist in the special position which the new string is in, we shall namely get $A(t_r) = t_m$. To do it, enlarge the meaning of B by prescribing of a new substitution C that it shall have the same effect on the t_i as B and map t_r onto t_m. This substitution will in general not be a braid since the condition concerning the product will be violated. Define two new substitutions α and β by their effect on the t_i respectively t_j and t_r and t_r resp t_m:

$$\alpha(t_i) = \begin{cases} t_i & \text{if} \quad i < r \\ t_r^{-1} t_i t_r & \text{if} \quad i > r, \end{cases} \qquad \alpha(t_r) = t_r;$$

(20)

$$\beta(t_j) = \begin{cases} t_j & \text{if} \quad j < m \\ t_m t_j t_m^{-1} & \text{if} \quad j > m \end{cases}, \qquad \beta(t_m) = t_m.$$

Then $A = \beta C \alpha$ is the desired braid. We first prove that it is a braid by showing that A carries the product

$$\prod_{i<r} t_i \cdot t_r \cdot \prod_{i>r} t_i \quad \text{into} \quad \prod_{j<m} t_j \cdot t_m \cdot \prod_{j>m} t_j.$$

Indeed α carries it into

$$\prod_{i<r} t_i \cdot t_r \cdot t_r^{-1} \prod_{i>r} t_i \cdot t_r = \prod t_i \cdot t_r$$

C transforms it into

$$\prod t_j \cdot t_m = \prod_{j<m} t_j \cdot \prod_{j>m} t_j \cdot t_m$$

and β into

$$\prod_{j<m} t_j \cdot t_m \prod_{j>m} t_j \cdot t_m^{-1} \cdot t_m = \prod_{j<m} t_j \cdot t_m \cdot \prod_{j>m} t_j.$$

It is seen immediately that cancellation of the r^{th} string leads back to B. So A is the desired braid. Obviously $A(t_r) = t_m$.

We may now combine both operations. If A is a braid, we may first cancel the r^{th} string and then reinsert it with the same ends so that it maps t_r onto t_m. The new braid shall be denoted by $A^{(r)}$. Theorem 7 shows that A may be obtained from $A^{(r)}$ by multiplying it from the left by a uniquely determined m-pure braid (m and not r-pure because of our change of notation). That $A^{(r)}$ is uniquely determined by A follows from Theorem 8 since $A^{(r)}(t_r) = t_m$ describes the homotopy class of the r^{th} string. We may write:

$$A = U_m A^{(r)}, \quad U_m \ m\text{-pure}.$$

We know how to compute $A^{(r)}$ from A and shall see a little later how U_m can be computed.

The elements S_j in their dependency on the braid shall be denoted by $S_j(A)$. In order to have well defined elements before us, we must still make an agreement about the arbitrary power of t_j that still may be added as a left factor. We choose it in such a way that the sum of the exponents of t_j in S_j is 0.

Let now A and B be two composable braids. B maps t_i into $S_j(B)^{-1}t_jS_j(B)$. A maps this into $A(S_j(B))^{-1}\cdot S_k(A)^{-1}\cdot t_k\cdot S_k(A)\cdot A(S_j(B))$, where $k = A(j)$. Since the transformer is determined to within a power of t_k we get to within such a power

$$S_k(AB) = S_k(A)A(S_j(B)), \qquad k = A(j). \tag{21}$$

On the left side and in the first factor on the right the sum is 0; in $S_j(B)$ the sum of the exponents of t_j is 0. A maps it into a power product where the sum of the exponents of t_k is 0. So (21) is correct as it stands.

A rather elementary invariant can be derived from (21). Calling $H_j(A)$ the sum of all exponents in $S_j(A)$ we obtain:

$$H_k(AB) = H_k(A) + H_j(B) \qquad k = A(j). \tag{22}$$

Defining now the "twining number" $T(A)$ as the sum of all $H_k(A)$ we get:

$$T(AB) = T(A) + T(B). \tag{23}$$

Since $T(\sigma_i^\epsilon) = \epsilon$ for $\epsilon = \pm 1$ this invariant can in case of the full group of n-braids also be explained as the sum of all exponents of the σ_i in any expression of A by the σ_i. This allows us to determine the factor commutator group without making use of the fact that the system of relations (18), (19) is complete. Making all σ_i commutative (19) gives the equality of all the σ_i. In the factor commutator group a braid A shrinks therefore to $\sigma_1^{T(A)}$ So this group is infinite cyclic and $T(A)$ gives the position of A in it.

The homology class of a string is obtained from $S_j(A)$ by the substitution of $t_j = 1$. The result may be denoted by $\bar{S}_j(A)$. In order to obtain a formula similar to (21) we must see what effect that substitution has on the second term on the right of (21). A maps t_j onto a transform of t_k. After substituting $t_k = 1$ all terms coming from t_j will disappear. Hence we may substitute $t_j = 1$ in $S_j(B)$. In addition to that, we must also substitute $t_k = 1$ in the result wherever it appears from the rest of the substitution. The same effect is achieved if the braid A is replaced by one where the j^{th} string has been dropped. Let us denote this braid by A_{-j}. Then we have:

$$\bar{S}_k(AB) = \bar{S}_k(A)\cdot A_{-j}(\bar{S}_j(B)), \qquad k = A(j). \tag{24}$$

Consider the special case that A is k-pure. Then $k = j$, A_{-j} a unit. Hence:

$$\bar{S}_k(AB) = \bar{S}_k(A)\cdot\bar{S}_k(B); \text{ if } A \text{ is } k\text{-pure}. \tag{25}$$

A still more special case is obtained when both A and B are k-pure. Theorem 8 tells that the homotopy class of the k^{th} string together with the ends determines any k-pure braid completely. It also shows that every homotopy class is possible. (25) means therefore that $\bar{S}_k(A)$ gives an isomorphic mapping of the group of k-pure braids with given ends onto the free group of the generators t_i with $i \neq k$. If we denote by A_{ik} the k-pure braid that is mapped onto t_i then the A_{ik} are the generators of our group and the mapping $\bar{S}_k(A)$ means just a replacement of each A_{ik} by t_i. These braids satisfy $A_{ik} = A_{ki}$. We prove this by giving at the same time the full substitution of A_{ik}^{ϵ} for $i < k$ and ϵ any integer.

$$(26) \qquad A_{ik}^{\epsilon} = A_{ki}^{\epsilon} = \begin{pmatrix} t_r, & t_i, & t_r, & t_k \\ t_r, & t_i C_{ik}, & C_{ik}^{-1} t_r C_{ik}, & C_{ik}^{-1} t_k \\ r < i \text{ or } > k, & & i \leqq r \leqq k, & \end{pmatrix}$$

$$C_{ik} = (t_i^{-1} t_k^{-1})^{\epsilon} \cdot (t_i t_k)^{\epsilon}.$$

Writing out the critical terms we see that all are transformations of generators. The product property holds so they are braids. $\bar{S}_i(A_{ik}^{\epsilon}) = t_k^{\epsilon}$, $\bar{S}_k(A_{ik}^{\epsilon}) = t_i^{\epsilon}$ and the braid reduces to a unit if we drop either the i^{th} or the k^{th} string. This proves all our contentions.

It is convenient to introduce also the inverse mapping F_k to $\bar{S}_k$. It maps t_i onto A_{ik}. Let it also have a meaning for t_k whose image shall be 1.

Let A be a braid and assume $k = A(j)$. We can write $A = U \cdot A^{(j)}$ where U is k-pure. Making use of (25) we obtain $\bar{S}_k(A) = \bar{S}_k(U)\bar{S}_k(A^{(j)}) = \bar{S}_k(U)$ since $A^{(j)}$ maps by definition t_j onto t_k, whence $\bar{S}_k(A^{(j)}) = 1$. Now applying F_k gives $U = F_k(\bar{S}_k(A))$ or:

$$(27) \qquad A = F_k(\bar{S}_k(A)) \cdot A^{(j)}, \qquad A(j) = k.$$

This is the algebraic form of Theorem 7.

(27) also solves the general question: given a braid B; insert a new string with the given homotopy class $\bar{S}_k(A)$. We have learned to form a braid with the homotopy class 1, it is the braid $A^{(j)}$. If we substitute this and the given $\bar{S}_k(A)$ in (27) we obtain A expressed by the A_{ik} and $A^{(j)}$. Use now (26).

Another application of (24) is this: let A be j-pure and assume of B that $B(t_j) = t_k$ hence $B^{-1}(t_k) = t_j$. Then $\bar{S}_k(B) = \bar{S}_j(B^{-1}) = 1$. This leads to $\bar{S}_k(BAB^{-1}) = B_{-j}(\bar{S}_j(A))$. BAB^{-1} is k-pure so we can apply F_k. This leads to:

$$BAB^{-1} = F_k(B_{-j}(\bar{S}_j(A))).$$

To get a still more general formula let B be now any braid. We first replace in the previous formula B by $B^{(j)}$. On the right side $B_{-j}^{(j)}$ appears. It is followed by the mapping F_k which anyhow maps t_k onto 1. So this braid may be replaced by B itself. Use now (27) on B. We obtain

$$(28) \qquad BAB^{-1} = F_k(\bar{S}_k(B)) \cdot F_k(B(\bar{S}_j(A))) \cdot (F_k(\bar{S}_k(B)))^{-1}, \quad B(j) = k.$$

B is here completely general so that we have before us the general transformation formula for a j-pure braid A. If A is given as a power product of the A_{ij} then $\bar{S}_j$ is a very trivial mapping and so is F_k. The right side is directly expressed as a power product of the A_{ik}. It is a very powerful formula that allows us to write down transformation formulas whose direct computation would be very painful.

As an application let $A = A_{ij}$ and $B = \sigma_r$. We give only the result of the computation which is now very easy:

$$(29) \qquad \sigma_r A_{ij} \sigma_r^{-1}$$

$$= \begin{cases} A_{ij} & \text{if} \quad r \neq i-1, i, j-1, j \\ A_{i+1,j} & \text{if} \quad r = i \\ A_{j,j-1} A_{i,j-1} A_{j,j-1}^{-1} & \text{if} \quad r = j-1 \end{cases} \text{ but } i \neq j-1; \qquad A_{j,j-1} \text{ for } i = j-1$$

$$\begin{cases} A_{ij}^{-1} A_{i-1,j} A_{ij} & \text{if} \quad r = i-1 \\ A_{i,j+1} & \text{if} \quad r = j \end{cases} \text{ but } i \neq j+1; \qquad A_{j,j+1} \text{ for } i = j+1.$$

It is to be remarked that the symmetry $A_{ij} = A_{ji}$ gives other expressions for the same transforms. We note the very special cases $r = i, j$. Since a very simple computation gives $A_{i,i+1} = \sigma_i^2$ we obtain for $i < j$ the following explicit expressions of the A_{ij} in terms of the generators σ_r:

$$(30 \qquad A_{ij} = \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{2} \sigma_{j-2} \cdots \sigma_i = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}.$$

As a second application we study the structure of the group I of n-braids with identity permutation. We fist prove:

LEMMA: *If A is an element of I that maps t_i onto itself for $i \leqq j$ then the S_k for $k > j$ do not depend on these t_i.*

PROOF: Dropping in the substitution A the first j columns will give a substitution that still satisfies the product condition, so it is an $(n - j)$-braid. This proves the lemma.

Now we use (27) for $j = k = 1$. On $A^{(1)}$ we use it for $k = 2$ and so on. Making use each time of the lemma we obtain a unique expression of A:

THEOREM 17. *The A_{ik} are generators of the group I. Every element can be expressed uniquely in the form:*

$$(31) \qquad A = U_1 U_2 \cdots U_{n-1}$$

where each U_j is a uniquely determined power product of the A_{ij} using only those with $i > j$. (Of course $A_{ij} = A_{ji}$).

The simple geometric meaning of this normal form shall be given later when we interpret our results in terms of projection.

What are the defining relations between these generators? Obviously those that permit the change of an arbitrary power product of them into the normal form. We must therefore find rules for interchanging factors of U_j with factors of U_i. For this purpose we derive all transformation rules for the expression $A_{rs}^{\epsilon}A_{ik}A_{rs}^{-\epsilon}$ ($\epsilon = \pm 1$). We use (28) with i instead of k for $B = A_{rs}^{\epsilon}$, $A = A_{ik}$. A simple computation yields:

THEOREM 18. *The braid $A_{rs}^{\epsilon}A_{ik}A_{rs}^{-\epsilon}$ ($\epsilon = \pm 1$) is i-pure. The following rules give its expression as i-pure braid:*

1) *If $i = r$ or s then it has already the desired form. Since the i-pure braids form a free group no other expression can be expected.*

2) *If all indices are different and if the pairs r, s and i, k do not separate each other we simply get A_{ik}.*

In all other cases we change, if necessary, first the names r, s in such a way that the arrangement i, r, s as compared with the natural arrangement of these three numbers is a permutation with the same sign as ϵ.

3) *If $k = r$ we get: $A_{is}^{-\epsilon}A_{ir}A_{is}^{\epsilon}$.*

4) *If $k = s$ we obtain:*

$$A_{is}^{-\epsilon}A_{ir}^{-\epsilon}\cdot A_{is}\cdot A_{ir}^{\epsilon}A_{is}^{\epsilon}.$$

5) *If finally the subscripts are all different and if the pairs r, s and i, k separate each other the result is:*

$$A_{is}^{-\epsilon}A_{ir}^{-\epsilon}A_{is}^{\epsilon}A_{ir}^{\epsilon}\cdot A_{ik}\cdot A_{ir}^{-\epsilon}A_{is}^{-\epsilon}A_{ir}^{\epsilon}A_{is}^{\epsilon}.$$

As defining relations the ones where i is the smallest index of all are sufficient. It also suffices to have only $\epsilon = +1$ since $\epsilon = -1$ is just only the inverse automorphism.

For braids whose permutation is not identity, a normal form is also easily obtained. Select to each of the $n!$ permutations π a braid B_π with this permutation. Any braid can then be written as a product like that in Theorem 17 followed by a B_π. This form is again unique.

The operation $A^{(j)}$ obviously satisfies:

$$(AB)^{(j)} = A^{(k)}B^{(j)}, \qquad k = B(j). \tag{32}$$

For the group I, it is therefore a homomorphic mapping and it suffices to know the result for the generators A_{ik}. $A_{ik}^{(r)} = 1$ for $r = i$ or k since A_{ik} is i- and k-pure. It is A_{ik} if $r < i$ or $> k$ as the substitution shows. If r is between i and k we apply (27) and Theorem 18. The result is:

$$A_{ik}^{(r)} = \begin{cases} 1 & \text{for} \quad r = i \text{ or } r = k \\ A_{ik} & \text{if} \quad r \text{ not between } i \text{ and } k \\ A_{ir}A_{ik}A_{ir}^{-1} = A_{kr}^{-1}A_{ki}A_{kr} & \text{if} \quad i < r < k. \end{cases} \tag{33}$$

We return now to the general group of n-braids. Let A be a braid that leaves t_j fixed if either $j < i$ or $j > k$. For the same reason as that in the previous lemma, we find that A maps t_i, $t_{i+1} \cdots t_k$ onto expressions depending on these variables alone. A can therefore be expressed in terms of the generators σ_i,

$\sigma_{i+1}, \cdots \sigma_{k-1}$. These braids form a subgroup denoted by G_{ik}. They behave as if they were braids of $k - i + 1$ strings (from i to k) only.

Put $c_{ik} = t_i t_{i+1} \cdots t_k$ and consider the following substitution C_{ik}:

$$C_{ik} = \begin{pmatrix} t_r & t_s \\ t_r & c_{ik}^{-1} t_s c_{ik} \\ r < i \text{ or } > k & i \leqq s \leqq k \end{pmatrix}. \tag{34}$$

It is a braid of G_{ik} since the product property holds. It is obviously commutative with any element of G_{ik} hence in the center of this group. If we drop the k^{th} string in C_{ik} and $C_{i,k-1}$ or the i^{th} in C_{ik} and $C_{i+1,k}$ we always obtain identical braids. $C_{i,k-1}$ leaves t_k and $C_{i+1,k}$ leaves t_i fixed. Hence:

$$C_{ik}^{(k)} = C_{i,k-1}, \qquad C_{ik}^{(i)} = C_{i+1,k} \qquad (C_{ii} \text{ means } 1). \tag{35}$$

Formula (27) gives:

$$C_{ik} = A_{ik}A_{i+1,k} \cdots A_{k-1,k}C_{i,k-1} = A_{i,i+1}A_{i,i+2} \cdots A_{ik}C_{i+1,k}. \tag{36}$$

This gives us the explicit expressions by the A_{ik}.

C_{ik} is closely related to a braid D_{ik} of G_{ik} which maps $t_i, t_{i+1} \cdots t_{k-1}$ onto t_{i+1}, $t_{i+2}, \cdots t_k$ but t_k onto $c_{ik}^{-1} t_1 c_{ik}$. We see that

$$C_{ik} = (D_{ik})^{k-i+1}. \tag{37}$$

The product $\sigma_i\sigma_{i+1} \cdots \sigma_{k-1}$ has the same effect on $t_i, t_{i+1}, \cdots t_{k-1}$ as D_{ik}. Because of the product property this suffices to establish equality. Hence:

$$C_{ik} = (\sigma_i\sigma_{i+1} \cdots \sigma_{k-1})^{k-i+1}. \tag{38}$$

THEOREM 19. *If $n = 2$ all braids are commutative. If $n \geqq 3$ let k be one of the numbers $\leqq n$. If B is commutative with every k-pure braid, then B is a power of $C_{1,n}$. This also determines of course the centers of the whole group, of I, G_{ik} and $I_{ik} = I \cap G_{ik}$.*

PROOF: If A is r-pure and $B(r) = s$, then BAB^{-1} is s-pure. Put $A = A_{ik}$. It is pure only for i and k. Since $BA_{ik}B^{-1} = A_{ik}$ we see that B can at most interchange i and k. If $n \geqq 3$ then i may be replaced by another index which shows that k remains fixed. Consequently i remains fixed too so B has identity as permutation.

We now make use of (28) where $j = k$. The braid B in the middle term on the right side came originally from B_{-k} which we introduce again. For A we take any k-pure braid so that the left side is A again. If we then apply the operation $\bar{S}_k$ to both sides we get:

$$\bar{S}_k(A) = \bar{S}_k(B) \cdot B_{-k}(\bar{S}_k(A)) \cdot (\bar{S}_k(B))^{-1}.$$

In this formula $\bar{S}_k(A)$ may be any power product T of the t_i with $i \neq k$. This shows:

$$B_{-k}(T) = d^{-1}Td, \text{ where } d = \bar{S}_k(B).$$

For $T = T_0 = \prod_{i \neq k} t_i$ we have on the other hand $B_{-k}(T_0) = T_0$. So T_0 is commutative with d. Since T_0 occurs in a generator system of the free group d is a power of T_0 say T_0^r. B_{-k} transforms all T with T_0^r. The same transformation is produced by $(C_{1,n})_{-k}^r$; B_{-k} is therefore this braid. Put now $C = B\, C_{1,n}^{-r}$. We find $C_{-k} = 1$ so C is k-pure. But C is still commutative with all k-pure braids. They form a free group with at least 2 generators whence $C = 1$ or $B = C_{1,n}^r$. This proves the contention.

For our next question we need a certain result about automorphisms of free groups. Let F be a free group with the generators t_i. Divide the subscripts into two classes p and q and in some other way into the classes g and h. We assume that there are at least two t_q and two t_h. Let x be a power product of the t_h that appears among a generator system of F and y a similar power product of the t_q. Since we assumed that there are at least two t_h, x will not be commutative with every t_h. Define now the automorphisms C and D by:

$$(39) \quad C(t_g) = t_g, \quad C(t_h) = x^{-a}t_hx^a; \quad D(t_p) = t_p, \quad D(t_q) = y^{-b}t_qy^b,$$

where a and b are positive integers. We ask for all automorphisms A that satisfy:

$$(40) \qquad D\,A = A\,C.$$

C leaves all t_g as well as x invariant. Their image T under A will satisfy (because of (40)):

$$(41) \qquad D(T) = T.$$

The power products $T = A(t_h)$ give $D(T) = AC(t_h) = A(x^{-a}t_hx^a)$ hence:

$$(42) \qquad D(T) = z^{-a}T\,z^a \text{ where } z = A(x).$$

The equations (42) (41) exhaust (40) and only the condition that A is an automorphism will have to be taken care of.

Denote by the letter P any power product of the t_p alone, by Q one of the t_q and by R one of the t_p and y. Any T can be written in the form:

$$(43) \qquad T = P_1Q_1P_2Q_2 \cdots$$

where P_1 may be absent. Then:

$$(44) \qquad D(T) = P_1y^{-b}Q_1y^bP_2 \cdots$$

Assume T satisfies (41). Each Q_i must be commutative with y^b. But y occurs in a generator system of F so Q_i is a power of y. Hence T is an R. We get

$$(45) \qquad A(t_g) = R_g, \qquad A(x) = z = R_0.$$

This takes care of (41). Assume now that T satisfies (42). We get:

$$(46) \qquad P_1y^{-b}Q_1y^bP_2y^{-b}Q_2 \cdots = z^{-a}P_1Q_1P_2Q_2 \cdots z^a.$$

If every Q is a power of y then T is an R and $D(T) = T$ must be commutative with z^a. But z is the image of a generator so is itself a generator. So T is a power of z. Since one t_h at least will not be commutative with x, its image T will not be commutative with z. So this case does not always happen.

Assume now that Q_i is the first of the Q in (46) that is not a power of y. Then the whole segments on the left of this factor in (46) must be equal since z on the right side is also an R. We obtain: (the earlier Q are powers of y)

$$P_1Q_1P_2 \cdots P_i y^{-b} = z^{-a}P_1Q_1 \cdots P_i \qquad \text{or}$$

$$z^a = R^{-1}y^bR.$$

Since y and z are generators this is only possible if $a = b$. $R^{-1}yR$ is also a generator and we get:

$$z = R^{-1}yR. \tag{47}$$

With this R put now $T = R^{-1}T_0R$. Because of $D(R) = R$ (42) gives:

$$D(T_0) = y^{-b}T_0y^b. \tag{48}$$

Writing now T_0 in the form (43) we get:

$$P_1y^{-b}Q_1y^bP_2 \cdots = y^{-b}P_1Q_1P_2 \cdots y^b.$$

The right side shows that P_1 must be absent. But also the presence of P_2 leads to a contradiction. T_0 is therefore a Q. Our results so far are:

$$A(t_g) = R_g\,, \; a = b, \quad A(x) = R^{-1}yR, \quad A(t_h) = R^{-1}Q_hR. \tag{49}$$

A maps the group generated by the t_g and x into the group of the t_p and y. A^{-1} satisfies $A^{-1}D^{-1} = C^{-1}A^{-1}$ where the roles of g and p are interchanged. It maps therefore the group of the t_p and y into the group of the t_g and x. The mappings are therefore one to one and this shows that the number of subscripts g is the same as that of the subscripts p.

We go now back to the braids and ask what automorphisms satisfy:

$$C_{ik}^aA = AC_{rs}^a\,, \qquad a > 0. \tag{50}$$

Our conditions are satisfied. The p are $< i$ or $> k$, the q satisfy $i \leqq q \leqq k$, the g are $< r$ or $> s$, the h satisfying $r \leqq h \leqq s$. $x = c_{rs}$ and $y = c_{ik}$ are generators. We must have $k - i = s - r$.

Assume a little more about the automorphism A namely that it maps each t_j onto a transform of another. Then:

$$A(t_g) = R_p^{-1}t_pR_p\,, \quad A(t_h) = R^{-1}Q_q^{-1}t_qQ_qR, \quad A(c_{rs}) = R^{-1}c_{ik}R \tag{51}$$

where each Q_q is a power product of the t_q, R_p and R power products of the t_p and of c_{ik}.

Split A into two substitutions:

$$B(t_g) = R_p^{-1}t_pR_p\,, \qquad B(t_{r+j}) = R^{-1}t_{i+j}R, \qquad 0 \leqq j \leqq k - i = s - r. \tag{52}$$

$$E(t_p) = t_p\,, \quad E(t_{i+j}) = Q_q^{-1}t_qQ_q \text{ where } r + j \text{ corresponds to } q. \tag{53}$$

A maps c_{rs} on one hand onto $R^{-1}c_{ik}R$; computed directly onto $\prod R^{-1}Q_q^{-1}t_qQ_qR$ whence $\prod Q_q^{-1}t_qQ_q = c_{ik}$. Now we see that E leaves c_{ik} fixed and therefore

$$A = E\,B. \tag{54}$$

We see at the same time that E is always a braid. The condition that A be an automorphism is therefore that B is one and that E is a braid. Should A be also a braid then B is one and conversely.

In case A is a braid the geometric significance of (52), (53) and (54) is obvious. B behaves as if the i^{th} up to k^{th} string were just only one strand (only the product of the t_h and of the t_q plays a role). A is obtained by weaving the pattern E into the i^{th} up to the k^{th} string, then, considering this partial braid as one string only, interweaving it with the other strings according to pattern B.

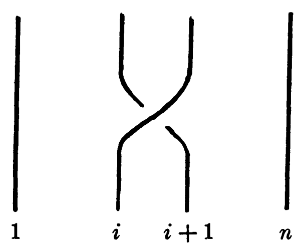

FIGURE 1

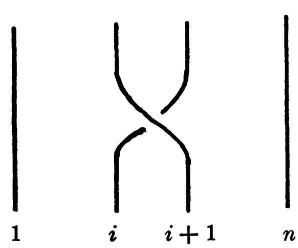

FIGURE 2

The question whether a given braid A can be considered as braid of braids amounts to checking relations of the type (50). It suffices of course to consider $a = 1$. Since we can decide whether or not they hold, this question is decided too.

THEOREM 20. *The number n is a group invariant of the group of braids with n strings.*

PROOF: Theorem 19 shows that $C_{1,n}$ is a generator of the center and therefore together with its reciprocal characterized by an inner property of the group. The number $T(C_{1,n}) = n(n - 1)$ is therefore also an invariant since it gives the position in the factor commutator group.

The structure of the group does not depend on the position of the ends. We may therefore put the ends in the special points with the coordinates $x = 0$, $y = i$, $(i = 1, 2, \cdots n)$. As ray for the Poincaré group we may select $x = 1$,

$y \leqq 0$ and as paths l_i the straight line segments form the beginning point of the ray to the ends. It is advisable to use as orientation of the plane the nega-

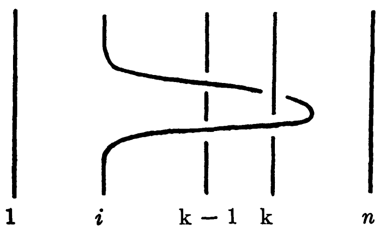

FIGURE 3

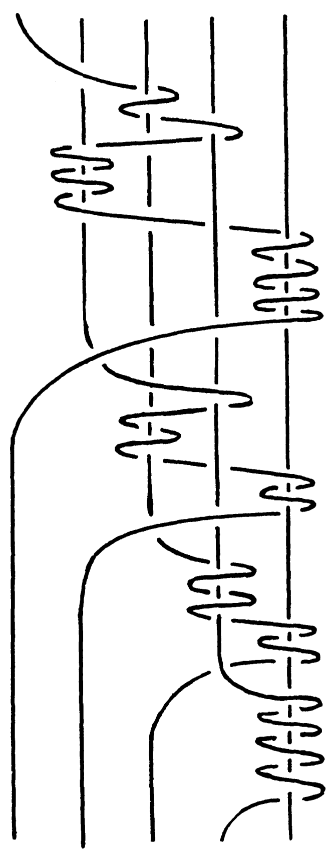

FIGURE 4

tive one, so the sense of rotation from the positive y-axis to the positive x-axis. This fixes all the necessary data and we are now in a position to interpret our results in the projection from the positive x-direction onto the yz-plane.

Theorem 4 shows us that any braid is isotopic to another one whose strings are broken lines. It leaves us so much freedom that we can assume in addition that the projection is free from any triple points and that no two double points occur at the same z-level.

Figures 1 and 2 show the generators σ_i and σ_i^{-1} in their projection. Corollary 3 to Theorem 2 shows indeed that they are reciprocal. The braid in Fig. 1 maps t_ν obviously onto itself if $\nu \neq i, i + 1$. It maps t_i onto t_{i+1} and, because of the product property, that is sufficient to establish its identity. Theorem 3 teaches how to read off from the projection of a braid its expression in terms of the generators σ_i.

Formula (38) shows that C_{ik} is simply the full twist of all the strings from the i^{th} to the k^{th} and that gives the geometric meaning of Theorem 19.

Formula (30) gives now the projection of the generator A_{ij}. It is indicated in Fig. 3.

The geometric meaning of the normal form of Theorem 17 and that mentioned after Theorem 18 is now also clear. Every braid is isotopic to another one whose pattern of projection is especially simple and is indicated in Fig. 4 for a special case. This pattern is unique. The braid in Fig. 4 has the expression:

$$A = A_{13}^{2}A_{14}^{-1}A_{12}^{-3}A_{14}A_{15}^{4}\cdot A_{24}^{-1}A_{23}^{2}A_{24}A_{25}^{-2}\cdot A_{34}^{3}A_{35}^{-2}\cdot A_{34}^{-1}\cdot A_{45}^{-4}.$$

Although it has been proved that every braid can be deformed into a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematics. He would therefore discourage such an experiment.

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