EDITED BY JOAN P. HUTCHINSON AND STAN WAGON

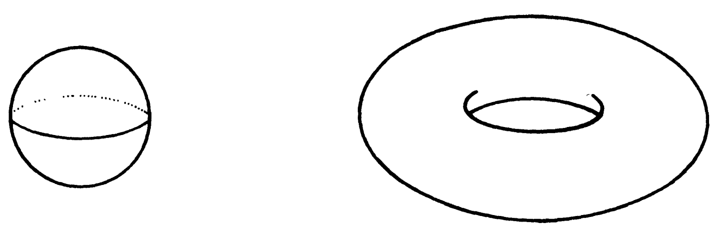
The Classification of Surfaces

PETER ANDREWS
Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada N2L 3C5

1. Introduction. The classification of surfaces is one of the truly beautiful theorems of modern mathematics. There are many proofs in the literature already, most of which are quite elementary. This note presents yet another proof, one that involves some very simple graph theory, an area increasingly popular with, and familiar to, students today. It also involves rudimentary surgery and thus could serve as an introduction to some of the methods of modern differential topology.

A surface is a compact two-manifold without boundary. For those readers for whom this is not very helpful, [**7**, Chap. 1] and [**4**, Chap. 1, 2] both give a very leisurely, well-illustrated explanation.

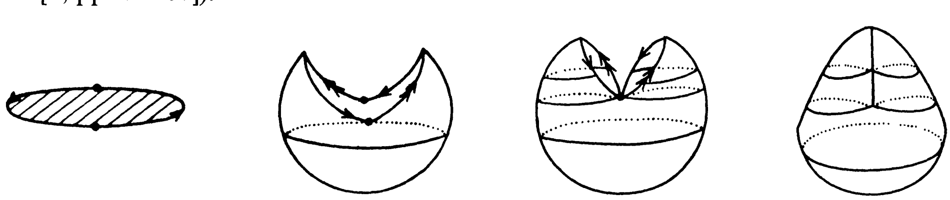
Perhaps the most familiar surfaces are the sphere and the torus (FIG. 1).



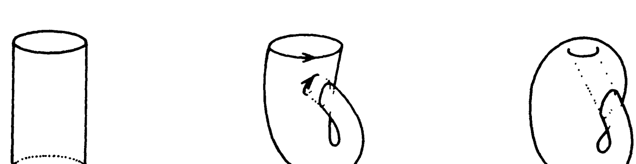
sphere torus

FIG. 1.

Two more exotic surfaces are the projective plane and the Klein bottle (FIG. 2; see also [**6**, pp. 64–67]).



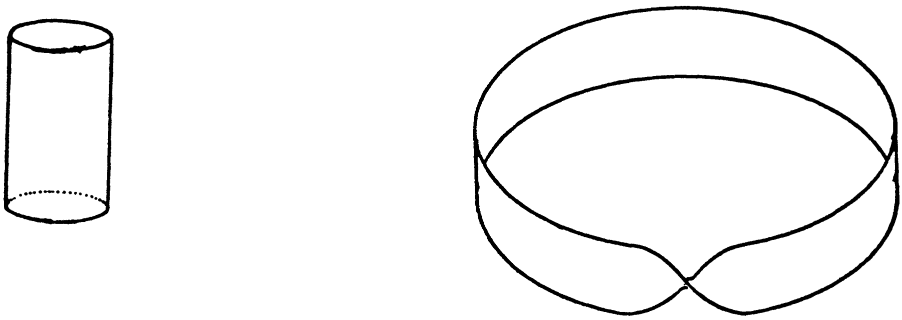
projective plane



Klein bottle

FIG. 2.

Most mathematicians are familiar with the Möbius strip (FIG. 3).

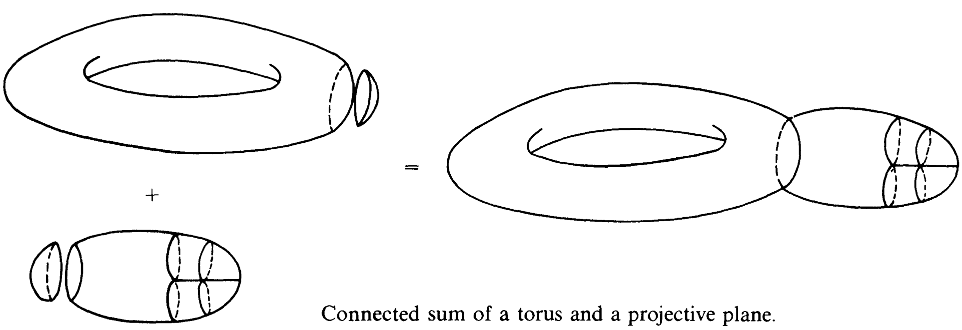


cylinder Möbius strip

FIG. 3.

This is not a surface but, like the cylinder, a surface-with-boundary. The boundary of the cylinder is, of course, two disjoint circles, while the boundary of the Möbius strip is just a single circle. The Möbius strip will be quite important in our proof because it is so closely related to the projective plane. In fact, if the interior of a small disc is removed from a projective plane the resulting surface-with-boundary is a Möbius strip (see [**5**, p. 33]).

Any two surfaces can be combined to produce a third by forming what is called their connected sum (see [**5**, p. 79]). A small open disc is removed from each surface, and the two resulting boundary circles are then glued together (FIG. 4).



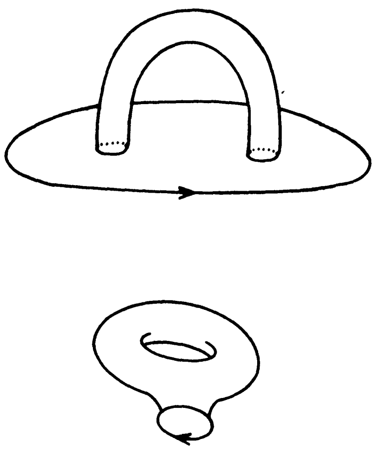
Connected sum of a torus and a projective plane.

FIG. 4.

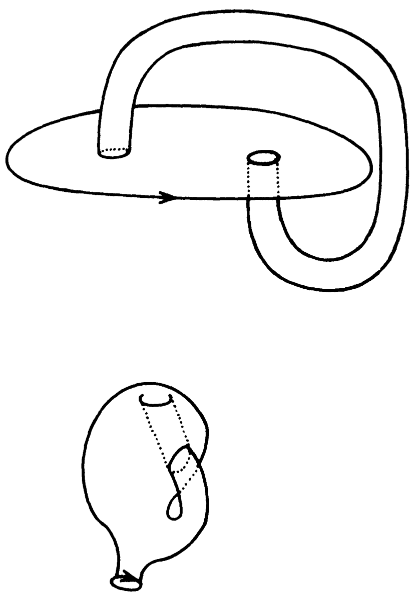
The simplest and weakest form of the classification theorem says that any surface can be built from the few examples described above using only the operation of the connected sum.

THEOREM. *Any surface is homeomorphic to the connected sum of a sphere, m tori, n projective planes, and r Klein bottles, where m, n, and r are greater than or equal to zero.*

The basic idea of this proof is the same as that of [**5**, pp. 88–89]. We attempt to simplify the surface by removing tori, projective planes, or Klein bottles. Begin by looking for a nonseparating simple closed curve γ on Σ, that is, a curve γ for which $\Sigma - \gamma$ is still connected. If W is a small open neighborhood of γ on Σ, then $\overline{W}$ (the closure of W) is either a cylinder or a Möbius strip. Consider $\Sigma - W$ the surface-with-boundary obtained by removing W from Σ. The boundary of $\Sigma - W$ will be one or two circles since $\text{bdry}(\Sigma - W) = \text{bdry}(\overline{W})$. Glue a disc (or discs) onto the boundary circle(s) of $\Sigma - W$ to get a new surface Σ'. In essence we go from Σ to Σ' by removing a cylinder (handle) or a Möbius strip. A little care must be taken here. The handle could have been attached nicely (FIG. 5(a)) or badly (FIG. 5(b)). It is a good exercise to show that the nice handle in FIG. 5(a) is homeomorphic to a torus with a disc removed and that the bad handle in FIG. 5(b) is homeomorphic to a Klein bottle with a disc removed.



(a) "nice" handle



(b) "bad" handle

FIG. 5.

We continue by looking for a nonseparating simple closed curve γ' on Σ' and repeating the process. This leaves two major questions: Will we ever reach a surface with no nonseparating curves? What can we say about such a surface? The answers are quite simple. After a finite number of these surgeries, we *will* obtain a surface with no nonseparating closed curves and such a surface is homeomorphic to a sphere. The verification of these statements is usually rather lengthy, though.

In this proof we use triangulated surfaces and some graph theory to describe a very concrete execution of the program just sketched. We show where to look for the non-separating curve γ and demonstrate easily that the procedure produces a sphere in a finite number of steps. For convenience, we use a slightly different technique for the surgery. Instead of removing an entire neighborhood of γ, we merely cut the surface open along γ. The surface-with-boundary created is homeomorphic to the one described above and so the ultimate effect is the same.

The idea of this proof came from Paul Melvin, who credits it originally to E. C. Zeeman. (The referee has pointed out that a similar version of this proof [also credited to Zeeman] appears in an excellent textbook by M. A. Armstrong [**1**]. Regrettably, this book does not seem to be very well known or readily available in North America. We hope that those readers who find this article useful will be able to examine Armstrong's text in its entirety. The author would also like to acknowledge financial support from Williams College, McMaster University, and Wilfrid Laurier University.)

2. Some Graph Theory. For purposes of this note, we assume that *all graphs are connected*. Recall that a *spanning tree* in a graph G is a subgraph of G that is a tree containing all the vertices of G. It is easy to see that every graph has a spanning tree.

Let $V(G)$ (resp., $E(G)$) denote the number of vertices (resp., edges) in a graph of G.

DEFINITION. The *cyclomatic number* of a graph G, denoted by $C(G)$, is defined by the equation

$$C(G) = 1 - V(G) + E(G).$$

The cyclomatic number measures the number of independent cycles in G. We need only one of its properties. Some deeper results about $C(G)$ may be found in [**2**, Chap. 2].

PROPOSITION. $C(G) \geqslant 0$. *Furthermore*, $C(G) = 0$ *if and only if* G *is a tree*.

Proof. The proof follows quite easily using induction on the number of edges.

3. Proof of the Theorem. As in most proofs of this theorem, we use combinatorial techniques and so assume that all surfaces are triangulated. It is a nontrival result that this introduces no loss of generality in dimension two (see, e.g., [**3**, §16.a]).

DEFINITION. A *surface* is a finite collection of triangles in some Euclidean space such that:

(i) the intersection of any two triangles is empty, a single vertex, or a single edge;
(ii) any edge is the intersection of exactly two triangles; and
(iii) the triangles incident with a given vertex can be ordered $\Delta_1, \Delta_2, \dots, \Delta_k$ so that Δ_i has exactly one edge in common with Δ_{i+1}, for $1 \leqslant i < k$, and Δ_k has exactly one edge in common with Δ_1.

For a surface Σ, let $\Delta_1, \Delta_2, \dots, \Delta_n$ be a list of all the triangles of Σ. The collection of vertices and edges of these triangles forms a graph that we also denote by Σ. Let B_i be the barycenter (centroid) of Δ_i. Form a new graph G_Σ whose vertices are the barycenters $B_1, B_2, \dots, B_n$. There will be an edge B_iB_j in G_Σ if and only if Δ_i and Δ_j have an edge e in common. In this case we say that B_iB_j crosses e.

Let T be any spanning tree of G_Σ and K_T be the subgraph of Σ consisting of all the vertices of Σ and those edges of Σ not crossed by T. It is easy to verify that K_T is connected since T is a tree.

It is perhaps helpful to think of T as lying along the edges of $b(\Sigma)$, the first barycentric subdivision of Σ. It is not, however, a subgraph of $b(\Sigma)$ because the vertices of T are just the barycenters of the triangles in Σ. We can use $b^2(\Sigma)$, the second barycentric subdivision of Σ, to produce neighborhoods $U \supset T$ and $V \supset K_T$ on the surface Σ satisfying:

(i) $U \cap V = \phi$,

(ii) $\overline{U} \cup \overline{V} = \Sigma$, and

(iii) bdry $U =$ bdry V.

For example, U will be T together with the interiors of all the triangles and edges of $b^2(\Sigma)$ that intersect T. There is a similar description for V. We should remark here that $\overline{U}$ is homeomorphic to a two-dimensional disc since T is a tree.

FIGURE 6 shows a triangulation of the projective plane. (Note that there are identifications on the boundary of the hexagon.) One of the possible choices of T is drawn with heavy lines. The corresponding K_T is drawn with broken lines. The shaded region is U and the unshaded one is V.

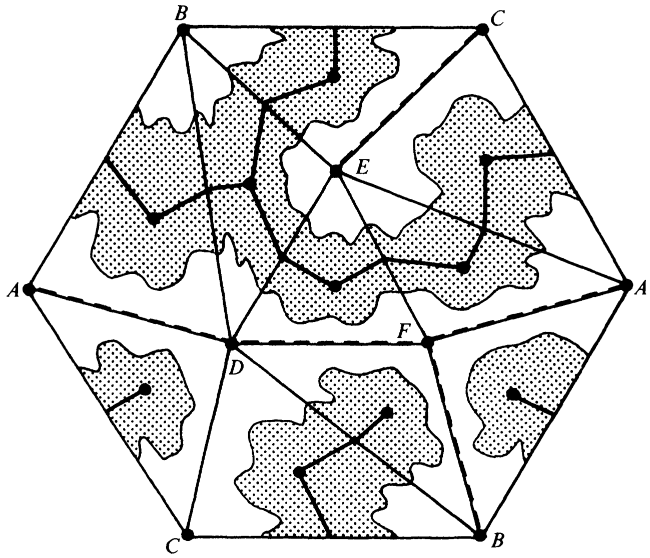


FIG. 6.

Now focus on the cyclomatic number $L = C(K_T)$. We consider two cases.

Case 1: $L = 0$. In this case K_T is a tree. Then $\overline{V}$ is a disc, as observed above. Thus Σ is the union of two discs, $\overline{U}$ and $\overline{V}$, joined by an identification of their common boundary. This means that Σ is homeomorphic to a sphere.

Case 2: $L \neq 0$. In this case, K_T is not a tree so there must be a cycle γ in K_T. This is the curve on which we do surgery.

LEMMA. *The cycle γ does not separate Σ.*

Proof. The tree T has a vertex in each triangle of Σ and does not cross γ. Thus any two points of $\Sigma - \gamma$ can be joined by a path in $\Sigma - \gamma$.

Instead of removing a whole neighborhood of γ, we just cut the surface open along γ to create a triangulated surface-with-boundary Σ_γ. Just as in our earlier discussion, the boundary of Σ_γ consists of one circle if γ has Möbius strip neighborhood and of two circles if γ has a cylinder neighborhood.

In the example of FIG. 6, $L = 1$ and essentially the only choice of γ is the cycle ADFA. The neighborhood of γ is shown in FIG. 7. It is obviously a Möbius strip.

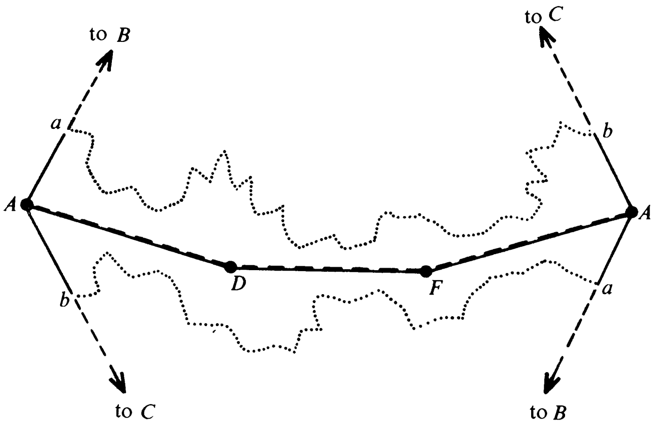


FIG. 7.

The next step is to cap off the boundary of Σ_γ with a suitably triangulated disc (or discs) as shown in FIG. 8. This creates a new surface Σ'. The heavy lines in FIGURE 8 show how the existing tree T can be extended into each new triangle to get a new tree T' that will be a spanning tree for $G_{\Sigma'}$. We then compute $L' = C(K_{T'})$ by simply counting the vertices and edges added to or lost from K_T to produce $K_{T'}$.

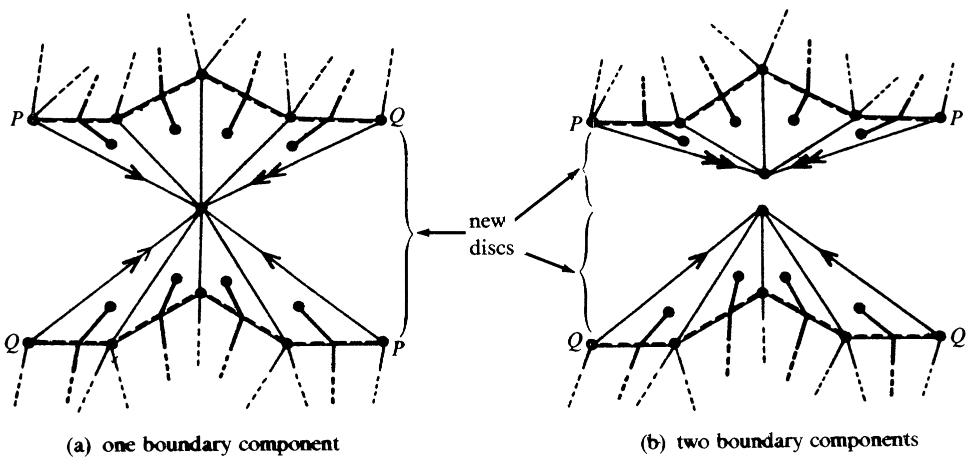


FIG. 8.

LEMMA. *If Σ_γ has one boundary circle, then $L' = L - 1$; if Σ_γ has two boundary circles, then $L' = L - 2$.*

The whole procedure can now be iterated, decreasing the value of L each time. In a finite number of steps, then, we arrive at the $L = 0$ case, which corresponds to a sphere. As outlined in Section 1, each surgery (the process of cutting along γ and capping off the boundary of Σ_γ) is equivalent to removing a torus, a projective plane or a Klein bottle. This completes the proof of the theorem.

REFERENCES

1. M. A. Armstrong, Basic Topology, McGraw-Hill (UK), London, 1979.
2. C. Berge, Graphs and Hypergraphs, North Holland, Amsterdam, 1976.
3. C. O. Christenson and W. L. Voxman, Aspects of Topology, Marcel Dekker, New York, 1977.
4. H. B. Griffiths, Surfaces, Cambridge University Press, Cambridge, 1976.
5. C. Kosniowski, A First Course in Algebraic Topology, Cambridge University Press, Cambridge, 1980.
6. J. Stillwell, Classical Topology and Combinatorial Group Theory, Springer-Verlag, New York, 1980.
7. J. R. Weeks, The Shape of Space, Marcel Dekker, New York, 1985.

Another Required Reading Program for Mathematics Majors

JAMES C. REBER
Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA 15705

I appreciated the article by Robert L. Brabenec, "A Required Reading Program for Mathematics Majors" [this MONTHLY, April 1987]. For precisely the reasons he presented we have introduced a reading program at Indiana University of Pennsylvania. The differences between programs, and respective advantages and disadvantages might be worth noting.

Our program is tied to individual courses required for all majors, and uses readings on reserve in the library. Each instructor teaching a course in the program requires the readings for that course, specifies how the requirement is met (typically with a short reaction paper), and determines how the assignment will be graded. General readings are used in courses populated primarily with mathematics majors; readings explicitly tied to the course are used in courses that include many non-mathematics majors.

One disadvantage of this approach: though encouraged to complete the assignment early in the semester, students often complete it amid the pressure of other work.

Advantages: students minoring in mathematics or taking just a few courses complete some readings; faculty are more likely to integrate the readings into the course; there is no departmentwide bookkeeping; and the program is easy to start since it can be phased in several courses at a time.

The explicit readings we use might be instructive, though I agree with Brabenec's comment, "The specific content of what our students are reading is not nearly as important as the fact that they are reading."

Calculus I. Read two of three essays on the nature of mathematics from *Mathematics: People/Problems/Results*: "Math and Creativity" by Alfred Adler; "The Meaning of Math" by Morris Kline; "Math as Creative Art" by Paul Halmos.