XXIII.—Amphicheiral Knots. By Mary Gertrude Haseman, Ph.D. Communicated by Dr. C. G. Knott, General Secretary. (With One Plate.)

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§ 1. The Intrinsic Symbol of an Amphicheiral.

The intrinsic symbol* of an amphicheiral knot is based on the idea of the sequence of the crossings; it replaces each letter of the alphabetical symbol by a number equal to one-half of the number of crossings intervening before the next occurrence of that letter as the knot is traversed in a definite direction. Hence it is seen that two knots, which have the same intrinsic symbol, are identical. Since the same number of pairs of crossings must elapse before the next occurrence of corresponding crossings of two identical knots when the knots are traversed in a given direction, it is seen that the converse is true also. It may be necessary to consider the complementary intrinsic symbol in order to detect identical knots. For example, the two knots:

(1) 10 5 5 9 9 9 9 3 4 4 4 4 8 8 10 5 5 9 9 9 3 4 4 4 4 8 8
    a g b l e m d h i j f k g b h u e i j f f k a l e m d u e,

(2) 3 5 5 9 9 9 9 1 0 4 4 4 4 8 8 3 5 5 9 9 9 9 1 0 4 4 4 4 8 8
    a g b l e m d a e i j f k g b h u e i j f k b l e m d u i,

are found to be identical since the intrinsic symbol of (1) coincides with the complementary symbol of (2). That they are identical may be verified by the fact that their compartment symbol is

![Diagram](https://via.placeholder.com/150)

By an interchange of the two crossings, a and h in (2), it is seen that the two symbols may be made to coincide. So two amphicheiral knots are identical when their intrinsic symbols agree except for an interchange of complementary numbers.

The intrinsic symbol of an amphicheiral knot of the first order offers certain points of interest. An amphicheiral centre of an amphicheiral knot of order 1 is

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defined as a mid-point of a lap of the thread so located that corresponding crossings occur at equal arachal distances when the knot is transversed along this thread in opposite directions from the point. Let the amphicheiral centre \( \Phi_1 \) be the mid-point of the lap of thread between the two corresponding crossings \( p \) and \( q \) of a knot with \( n \) crossings, and denote by \( a_i, \beta_i \) the number of pairs of crossings which elapse before the next occurrence of \( p, q \) respectively as the knot is traversed from \( p \) to \( \Phi_1 \) through \( q \). Hence \( n - 1 - a_i, n - 1 - \beta_i \) will be the number of pairs of crossings which elapse before the next occurrence of the crossings \( p, q \) respectively as the knot is traversed from \( q \) to \( \Phi_1 \) through \( p \). By the definition of an amphicheiral centre, \( a_i = n - 1 - \beta_i \) or \( a_i + \beta_i = n - 1 \). Similarly \( a_j + \beta_j = n - 1 \), where \( a_i \) and \( \beta_i \) are two crossings of the intrinsic symbol at equal arachal distances from \( \Phi_1 \). Thus an amphicheiral knot of the first order, as well as its pairs of amphicheiral centres, may be detected very easily from its intrinsic symbol. For example, the intrinsic symbol

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possible, however, to construct on the sphere one of the ambichiral forms of
the knot given in fig. 2 by means of a curve $C_1$ in contact with a great circle
at two diametrically opposite points, $P_1$, $P_2$, and of a curve $C_2$ which is obtained
as a reflection of $C_1$ in the plane $\pi_1$ of the great circle, followed by a second
reflection in a plane $\pi_2$ passed through the points $P_1$, $P_2$ perpendicularly to the
plane $\pi_1$. To secure this construction in the plane suppose $c_1$, which is the
projection of the curve $C_1$ in the plane $\pi_1$, to be the broken curve in fig. 4,
with contacts at the diametral points $p_1$, $p_2$. The curve $c_2$, represented by the
dotted curve, is the same curve as $c_1$, but drawn on the outside of the circle and
reflected in the line $p_1 p_2$. Now, imagine the curve $c_2$ to be rotated through an angle
of $\frac{\pi}{4}$ to the right or left: the resulting knot, where the contacts are regarded as
crossings, is found to be the knot shown in fig. 3.

The foregoing construction led me to seek for a similar construction in the plane
of the ambichirals of the first order with any number of crossings. Let the curve

\[ c_1 \]

\[ c_2 \]

\[ c_3 \]

\[ c_4 \]

$c_1$ have $\kappa$ contacts, $2\kappa$ intersections with the circle and $\sigma$ self sections; and denote by
$c_2$ the same curve on the outside of the circle but reflected in a diameter passing
through one of the contacts. Now, imagine $c_2$ rotated through angles $\frac{\pi}{2}$, $\frac{\pi}{\kappa}$, etc., and
the resulting curve is found to be an ambichiral of order 1 with $2(\kappa + 2\lambda + \sigma)$ crossings.
The curve $c_2'$, obtained by the reflection in the plane $\pi_2$ of the great circle,
ensures the desired correspondence of compartments; the reflection of the curve $c_2'$
in the plane $\pi_2$ does not alter the number of compartments, nor the number of laps of thread bounding each compartment; instead it interchanges corresponding
adjacent compartments. For instance, suppose the two regions $\rho_1, \rho_2$ by the first
reflection to go into the two adjacent regions $\rho_1', \rho_2'$ respectively. If, now, by the
second reflection $\rho_1$ becomes adjacent to $\rho_2'$, then $\rho_2$ must become adjacent to $\rho_1'$ by
the same reflection. Since rotation through an angle merely adds one crossing to
each of the regions which are in the relation of $\rho_1$, $\rho_2'$, then the desired correspondence
of compartments remains unaltered. Further corresponding crossings occur at
equal areal distances from the mid-points of the lap of thread common to two
corresponding compartments. Hence the knot is an ambichiral of the first order.
It is possible to have the two curves $C_1$ and $C_2$ in their various positions intersect in $\sigma'$ pairs of points, although it is not always possible to make corresponding laps of the thread intersect without introducing extra crossings. By this process I have succeeded in constructing all of the amphicheirals of order 1 with four, six, eight, and ten crossings (Nos. 1–21 in the Plate *), and find the tenfold knot No. 21 in the Plate, omitted by Tait in his census (Trans. Roy. Soc. Edin., vol. xxxii, Plate LXXIX).

Likewise the knot No. 22 in the Plate has been omitted from my census (see Trans. Roy. Soc. Edin., vol. liii, pp. 253–4) of the amphicheirals with twelve crossings. It is possible that this method will reveal other omissions. It is to be noted that the maximum number of contacts were used in the constructions of these amphicheirals,

but I cannot say whether this is necessary in the construction of the knots with a greater number of crossings.

§ 3. SKEW AMPHICHEIRALS.

If, however, the curve $c_2'$, obtained by a single reflection in the plane $\pi_1$, is used instead of the curve $C_2$, there results upon rotation a knot which can be distorted into an amphicheiral of the first order—that is to say, it belongs to Tait's second class. When the curve $C_1$ is symmetrical about the line $\rho \rho_2$, the resulting knot is an amphicheiral of the first class of order 1, since then the curve $c_2'$ is identical with the curve $C_2$.

In this construction there arise certain knots, called by me skew amphicheirals of the second order, which exhibit the amphicheiral symmetry in spite of the fact that they belong, by the above statement, to the second class of order 1. The intrinsic symbol of all such knots, as I have found, classes them with the amphicheirals of the first class of order 2, although corresponding regions are not opposite on the sphere. An example of a skew amphicheiral is shown in fig. 5; it is found to be identical with the knot in fig. 3. It is to be noted that in the knot shown in fig. 3 the curve $c_2'$ possesses symmetry of such a nature that its relation to curve $c_1$ is the same whether $c_2'$ be rotated to the right or left.

* The numbers at the lower right-hand corners are Tait's numbers.
Another very good example (see fig. 6) of a skew amphicheiral is given by
the symbol

\[ a e b a e l d e e b f r g k h g i r j i t k h l d \]
\[ m q n m o z p o q n r j o a w j u f e v w t x p \]

Because of the relation of the curve \( c' \) to the curve \( c \), the crossing \( a \) may correspond to either \( g \) or \( s \); likewise the crossing \( m \) may correspond to either \( g \) or \( s \). Therefore we may expect not only the numbers in the last half of the sequence to be a repetition of those in the first half, but also the first half to consist of a repetition of a certain group of numbers; the number of repetitions will probably depend on the number of contacts.

The only skew amphicheirals, which I have found, may be obtained as the unsymmetrical distortions of an amphicheiral of the first class of order 1, and in view of the fact that the curve for knots with 4, 6, 8, 10, 12, 14 crossings seems to lead always to a knot which can be distorted into an amphicheiral of the first class of order 1, I am of the opinion that they do not constitute a distinct class, although it may be that, they will form another class in the case of the knots with a greater number of crossings.

If, therefore, an amphicheiral knot is defined as one whose primary and secondary symbols are identical—that is to say, one whose intrinsic symbol belongs to one of the two arrangements mentioned on p. 598—it is seen that Tait’s classification given in *Trans. Roy. Soc. Edin.*, vol. xxxii, p. 499, is sufficient provided that it be admitted that an amphicheiral knot can belong to the first class of one order and to the second class of the other order.


As has been shown by Tait, *Trans. Roy. Soc. Edin.*, vol. xxxii, there are no amphicheiral knots of order 2 with 4, 6, 8, or 10 crossings. There are two knots* of the second order with twelve crossings, both of which may be constructed on models involving one pair of contacts, although they appear among the knots which required a greater number of contacts. In a consideration of the maximum number of contacts necessary to construct the amphicheirals with \( n \) crossings I was led to construct the amphicheirals of the second order with fourteen crossings, of which there are ten in number, as shown in Nos. 23–32 in the Plate. Nine of these were constructed on models with one pair of contacts, whereas the tenth one, No. 32, required two or more pairs of contacts. Hence it will be necessary to pass to the amphicheirals with a greater number of crossings in order to determine the maximum number of contacts required.

An interesting amphicheiral is the form which is obtained by the single distor-

tion $D_1$ of the amphicheiral of the second order with fourteen crossings, No. 25 in the Plate. Its intrinsic symbol

$$11 \ 10 \ 3 \ 3 \ 11 \ 10 \ 4 \ 3 \ 1 \ 10 \ 12 \ 4 \ 3 \ 10 \ 9 \ 1 \ 3 \ 3 \ 12 \ 10 \ 9 \ 3 \ 2 \ 10 \ 10 \ 3 \ 2$$

shows that it is an amphicheiral of the first class of order 1 with one pair of amphicheiral centres. This is an example of Tait's supposed third class (Trans. Roy. Soc. Edin., vol. xxxii, p. 499), which has the property of being changed into its own perversion by a single distortion, but, contrary to his idea, it must belong to the first class of order 1 and to the second class of order 2.

It is of interest to note that all of those amphicheirals which belong simultaneously to the first class of orders 1 and 2 exhibit two pairs of amphicheiral centres with the exception of one, which has fourteen pairs.
MARY G. HASEMAN: AMPHICHEIRAL KNOTS.