

# ALGEBRAIC POINCARÉ COBORDISM

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## Introduction

The object of this paper is to give a reasonably leisurely account of the algebraic Poincaré cobordism theory of Ranicki [15], [16] and the further development due to Weiss [19], along with some of the applications to manifolds and vector bundles. It is a companion paper to Ranicki [17], which is an introduction to algebraic surgery using forms and formations.

Algebraic Poincaré cobordism is modelled on the bordism groups  $\Omega_*(X)$  of maps  $f : M \rightarrow X$  from manifolds to a space  $X$ . The Wall [18] surgery obstruction groups  $L_*(A)$  of a ring with involution  $A$  were expressed in [15] as the cobordism groups of  $A$ -module chain complexes  $C$  with a quadratic Poincaré duality

$$\psi : H^{n-*}(C) \cong H_*(C) ,$$

and the surgery obstruction  $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$  of an  $n$ -dimensional normal map  $(f, b) : M \rightarrow X$  was expressed as the cobordism class  $(C, \psi)$  of an  $n$ -dimensional f.g. free  $\mathbb{Z}[\pi_1(X)]$ -module chain complex  $C$  such that

$$H_*(C) = \ker(f_* : H_*(\widetilde{M}) \rightarrow H_*(\widetilde{X}))$$

together with an  $n$ -dimensional quadratic Poincaré duality  $\psi$ . The passage from the bundle map  $b : \nu_M \rightarrow \nu_X$  to  $\psi$  used an equivariant chain level version of the relationship established by Thom between the Wu classes of the normal bundle  $\nu_M$  of a manifold  $M$  and the action of the Steenrod algebra on the Thom class of  $\nu_M$ .

A chain bundle  $(C, \gamma)$  over a ring with involution  $A$  is an  $A$ -module chain complex  $C$  together with a Tate  $\mathbb{Z}_2$ -hypercohomology class  $\gamma \in \hat{H}^0(\mathbb{Z}_2; C^* \otimes C^*)$ . The  $L$ -groups  $L^*(C, \gamma)$  of [19] are the cobordism groups of symmetric Poincaré complexes over  $A$  with a chain bundle map to  $(C, \gamma)$ , which are related to the quadratic  $L$ -groups by an exact sequence of abelian groups

$$\cdots \rightarrow L_n(A) \rightarrow L^n(C, \gamma) \rightarrow Q_n(C, \gamma) \rightarrow L_{n-1}(A) \rightarrow \cdots$$

with the  $Q$ -groups  $Q_*(C, \gamma)$  defined purely homologically. The surgery obstruction groups  $L_*(A)$  of [18] and the symmetric  $L$ -groups  $L^*(A)$  of Mishchenko [13] are

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particular examples of the generalized  $L$ -groups  $L^*(C, \gamma)$ . The main novelty of this paper is an explicit formula obtained in §7 for the addition of elements in  $Q_*(C, \gamma)$ . The Wu classes  $v_*(\nu) \in H^*(X; \mathbb{Z}_2)$  of a  $(k-1)$ -spherical fibration  $\nu$  over a space  $X$  (e.g. the sphere bundle of a  $k$ -plane bundle) determine a chain bundle  $(C(\tilde{X}), \gamma(\nu))$  over  $\mathbb{Z}[\pi_1(X)]$ , with  $\tilde{X}$  the universal cover of  $X$ , and with a morphism

$$\pi_{n+k}(T(\nu)) \rightarrow Q_n(C(\tilde{X}), \gamma) .$$

For a  $k$ -plane bundle  $\nu$  the morphism factors through the flexible signature map of [19]

$$\Omega_n(X, \nu) = \pi_{n+k}(T(\nu)) \rightarrow L^n(C(\tilde{X}), \gamma(\nu)) .$$

with  $\Omega_n(X, \nu)$  the bordism group of normal maps  $(f : M \rightarrow X, b : \nu_M \rightarrow \nu)$  from  $n$ -dimensional manifolds.

In subsequent joint work with Frank Connolly a computation of  $Q_*(C, \gamma)$  will be used to compute the Cappell Unil-groups in certain special cases.

The titles of the sections are

1. Rings with involution
2. Chain complexes
3. Symmetric, quadratic and hyperquadratic structures
4. Algebraic Wu classes
5. Algebraic Poincaré complexes
6. Chain bundles
7. Normal complexes
8. Normal cobordism
9. Normal Wu classes
10. Forms
11. An example.

## §1. Rings with involution

In §1 we show how an involution  $a \mapsto \bar{a}$  on a ring  $A$  determines a duality involution functor

$$(\text{f.g. projective left } A\text{-modules}) \rightarrow (\text{f.g. projective left } A\text{-modules}) .$$

More generally, duality can be defined using an antistructure on  $A$  in the sense of Wall [18], and the  $L$ -theory results described in this paper all have versions for rings with antistructure.

Let  $A$  be an associative ring with 1, together with an involution

$$\bar{\phantom{x}} : A \rightarrow A ; a \mapsto \bar{a} ,$$

that is a function satisfying

$$\overline{a+b} = \bar{a} + \bar{b} , \bar{\bar{a}} = a , \overline{ab} = \bar{b} . \bar{a} , \bar{1} = 1 \in A (a, b \in A) .$$

In the topological applications  $A = \mathbb{Z}[\pi]$  is a group ring, for some group  $\pi$  equipped with a morphism  $w : \pi \rightarrow \mathbb{Z}_2 = \{+1, -1\}$ , and the involution is defined by

$$\bar{\phantom{x}} : A \rightarrow A ; \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g w(g) g^{-1} .$$

We take  $A$ -modules to be left  $A$ -modules, unless a right  $A$ -action is expressly specified. Given an  $A$ -module  $M$  there is defined a right  $A$ -module  $M^t$  with the same additive group and

$$M^t \times A \rightarrow M^t ; (x, a) \mapsto \bar{a}x .$$

The *dual* of an  $A$ -module  $M$  is the  $A$ -module with additive group

$$M^* = \text{Hom}_A(M, A)$$

and  $A$  acting by

$$A \times M^* \rightarrow M^* ; (a, f) \mapsto (x \mapsto f(x)\bar{a}) .$$

The *dual* of an  $A$ -module morphism  $f \in \text{Hom}_A(M, N)$  is the  $A$ -module morphism

$$f^* : N^* \rightarrow M^* ; g \mapsto (x \mapsto g(f(x))) .$$

For a f.g. (finitely generated) projective  $A$ -module  $M$  the natural  $A$ -module isomorphism

$$M \rightarrow M^{**} ; x \mapsto (f \mapsto \overline{f(x)})$$

will be used to identify

$$M^{**} = M .$$

## §2. Chain complexes

In order to adequately deal with the quadratic nature of the function  $C \rightarrow C^t \otimes_A C$  sending an  $A$ -module chain complex  $C$  to the  $\mathbb{Z}$ -module chain complex  $C^t \otimes_A C$  it is necessary to use the equivalence of Dold [7] and Kan [8] (cf. May [10, §22]) between positive  $\mathbb{Z}$ -module chain complexes and simplicial  $\mathbb{Z}$ -modules. This is now recalled, along with some other properties of chain complexes that we shall require.

An  $A$ -module chain complex

$$C : \cdots \rightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \rightarrow \cdots \quad (r \in \mathbb{Z})$$

is *n-dimensional* if each  $C_r$  ( $0 \leq r \leq n$ ) is a f.g. projective  $A$ -module and  $C_r = 0$  for  $r < 0$  and  $r > n$ . By an abuse of terminology chain complexes of the chain homotopy type of an  $n$ -dimensional chain complex will also be called *n-dimensional*.

The *suspension* of an  $A$ -module chain complex  $C$  is the  $A$ -module chain complex defined by

$$d_{SC} = d_C : SC_r = C_{r-1} \rightarrow SC_{r-1} = C_{r-2} .$$

If  $C$  is  $n$ -dimensional then  $SC$  is  $(n+1)$ -dimensional.

Given an  $A$ -module chain complex  $C$  let

$$C^r = (C_r)^* \quad (r \in \mathbb{Z}) .$$

The *dual A-module chain complex*  $C^{-*}$  is defined by

$$d_{C^{-*}} = (d_C)^* : (C^{-*})_r = C^{-r} \rightarrow (C^{-*})_{r-1} = C^{-r+1} .$$

The *n-dual A-module chain complex*  $C^{n-*}$  is defined by

$$d_{C^{n-*}} = (-1)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \rightarrow (C^{n-*})_{r-1} = C^{n-r+1} .$$

The *n-fold suspension* of the dual  $S^n C^{-*}$  is related to the *n-dual*  $C^{n-*}$  by the isomorphism

$$S^n C^{-*} \rightarrow C^{n-*} ; x \mapsto (-1)^{r(r-1)/2} x \ (x \in C^{n-r}) .$$

In particular,  $C^{0-*}$  is isomorphic to (but not identical to)  $C^{-*}$ .

A *chain map up to sign* between *A-module chain complexes*

$$f : C \rightarrow D$$

is a collection of *A-module morphisms*

$$\{f \in \text{Hom}_A(C_r, D_r) \mid r \in \mathbb{Z}\}$$

such that

$$d_D f = \pm f d_C : C_r \rightarrow D_{r-1} \ (r \in \mathbb{Z}) .$$

If the sign is always  $+$  this is just a *chain map*  $f : C \rightarrow D$ , as usual.

Given *A-module chain complexes*  $C, D$  let  $C^t \otimes_A D$ ,  $\text{Hom}_A(C, D)$  be the  $\mathbb{Z}$ -module chain complexes defined by

$$(C^t \otimes_A D)_n = \sum_{p+q=n} C_p \otimes_A D_q ,$$

$$d_{C^t \otimes_A D}(x \otimes y) = x \otimes d_D(y) + (-1)^q d_C(x) \otimes y ,$$

$$\text{Hom}_A(C, D)_n = \sum_{q-p=n} \text{Hom}_A(C_p, D_q) ,$$

$$d_{\text{Hom}_A(C, D)}(f)(x) = d_D(f(x)) + (-1)^q f(d_C(x)) .$$

A cycle  $f \in \text{Hom}_A(C, D)_n$  is a chain map up to sign  $f : S^n C \rightarrow D$ , and

$$H_n(\text{Hom}_A(C, D)) = H_0(\text{Hom}_A(S^n C, D))$$

is the  $\mathbb{Z}$ -module of chain homotopy classes of chain maps  $S^n C \rightarrow D$ .

For finite-dimensional  $C$  the *slant isomorphism* of  $\mathbb{Z}$ -module chain complexes

$$C^t \otimes_A D \rightarrow \text{Hom}_A(C^{-*}, D) ; x \otimes y \mapsto (f \mapsto \overline{f(x)} \cdot y)$$

will be used to identify

$$C^t \otimes_A D = \text{Hom}_A(C^{-*}, D) .$$

A cycle

$$f \in (C^t \otimes_A D)_n = \text{Hom}_A(C^{-*}, D)_n$$

is a chain map  $f : C^{n-*} \rightarrow D$ . Thus

$$H_n(C^t \otimes_A D) = H_n(\text{Hom}_A(C^{-*}, D)) = H_0(\text{Hom}_A(C^{n-*}, D))$$

is the  $\mathbb{Z}$ -module of chain homotopy classes of chain maps  $C^{n-*} \rightarrow D$ .

The *algebraic mapping cone*  $C(f)$  of an  $A$ -module chain map  $f : C \rightarrow D$  is the  $A$ -module chain complex defined as usual by

$$d_{C(f)} = \begin{pmatrix} d_D & (-1)^{r-1}f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \rightarrow C(f)_{r-1} = D_{r-1} \oplus C_{r-2} .$$

The relative homology  $A$ -modules

$$H_n(f) = H_n(C(f))$$

are such that there is defined an exact sequence

$$\cdots \rightarrow H_n(C) \xrightarrow{f_*} H_n(D) \rightarrow H_n(f) \rightarrow H_{n-1}(C) \rightarrow \cdots .$$

Let  $C(\Delta^n)$  denote the cellular chain complex of the standard  $n$ -simplex  $\Delta^n$  with the standard cell structure consisting of  $\binom{n+1}{r+1}$   $r$ -cells ( $0 \leq r \leq n$ ).

Given a  $\mathbb{Z}$ -module chain complex  $C$  let  $K(C)$  denote the simplicial  $\mathbb{Z}$ -module defined by the Dold-Kan construction, with one  $n$ -simplex for each chain map  $C(\Delta^n) \rightarrow C$  and the evident face and degeneracy maps  $d_i, s_i$ , such that

$$\pi_n(K(C)) = H_n(C) \quad (n \geq 0) .$$

Given a chain  $y \in C_n$  and cycles  $x_i \in \ker(d : C_{n-1} \rightarrow C_{n-2})$  ( $0 \leq i \leq n$ ) such that

$$dy = \sum_{i=0}^n (-1)^i x_i \in C_{n-1}$$

let  $(y; x_0, \dots, x_n)$  denote the  $n$ -simplex of  $K(C)$  defined by the chain map

$$f : C(\Delta^n) \rightarrow C$$

with

$$f : C(\Delta^n)_n = \mathbb{Z} \rightarrow C_n ; 1 \mapsto y ,$$

$$d_i f : C(\Delta^{n-1})_{n-1} = \mathbb{Z} \rightarrow C_{n-1} ; 1 \mapsto x_i \quad (0 \leq i \leq n) .$$

The chain  $x \in C_n$  is identified with the  $n$ -simplex  $(x; dx, 0, \dots, 0) \in K(C)^{(n)}$ .

Given  $\mathbb{Z}$ -module chain complexes  $C, D$  and a simplicial map

$$f : K(C) \rightarrow K(D)$$

(which need not preserve the  $\mathbb{Z}$ -module structure) there is defined a function

$$f : C_n \rightarrow D_n ; x \mapsto f(x) = f(x; dx, 0, \dots, 0)$$

such that

$$df(x) = f(dx) \in D_{n-1} \quad (x \in C_n) .$$

In general,  $f : C_n \rightarrow D_n$  is only additive on the level of homology, with

$$f(x + x') - f(x) - f(x') = d[x, x']_f \in D_n \quad (x, x' \in \ker(d : C_n \rightarrow C_{n-1}))$$

where the function

$$[\ , \ ]_f : \ker(d : C_n \rightarrow C_{n-1}) \times \ker(d : C_n \rightarrow C_{n-1}) \rightarrow D_{n+1}$$

is defined by

$$[x, x']_f = f(0; x + x', x, -x', 0, \dots, 0) - f(0; x, x, 0, \dots, 0) - f(0; x', 0, -x', 0, \dots, 0) .$$

The induced functions

$$f_* : H_n(C) \rightarrow H_n(D) ; x \mapsto f(x)$$

are thus morphisms of abelian groups, which fit into an exact sequence

$$\cdots \rightarrow H_{n+1}(f) \rightarrow H_n(C) \xrightarrow{f_*} H_n(D) \rightarrow H_n(f) \rightarrow H_{n-1}(C) \rightarrow \cdots ,$$

with the relative group  $H_n(f)$  ( $= \pi_n(f)$ ) the set of equivalence classes of pairs  $(x, y) \in C_n \times D_{n+1}$  such that

$$dx = 0 \in C_{n-1} , f(x) = dy \in D_n ,$$

subject to the equivalence relation

$$(x, y) \sim (x', y') \text{ if there exist } (u, v) \in C_{n+1} \times D_{n+2} \text{ such that}$$

$$x - x' = du \in C_n , y - y' = f(u; x, x', 0, \dots, 0) + dv \in D_{n+1} ,$$

and addition by

$$(x, y) + (x', y') = (x + x', y + y' + [x, x']_f) \in H_n(f) .$$

If  $f : K(C) \rightarrow K(D)$  does preserve the  $\mathbb{Z}$ -module structure (so that  $[ , ]_f = 0$ ) then  $f$  is essentially just a chain map  $f : C \rightarrow D$ , and the relative homology groups  $H_*(f)$  are just the homology groups  $H_*(C(f))$  of the algebraic mapping cone  $C(f)$ , as usual.

### §3. Symmetric, quadratic and hyperquadratic structures

An  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{array} \right.$  structure on an  $A$ -module chain complex  $C$  is a cycle representing an element of the  $\left\{ \begin{array}{l} \mathbb{Z}_2\text{-hypercohomology} \\ \mathbb{Z}_2\text{-hyperhomology} \\ \text{Tate } \mathbb{Z}_2\text{-hypercohomology} \end{array} \right.$  group

$$\left\{ \begin{array}{l} H^n(\mathbb{Z}_2; C^t \otimes_A C) \\ H_n(\mathbb{Z}_2; C^t \otimes_A C) \\ \hat{H}^n(\mathbb{Z}_2; C^t \otimes_A C) \end{array} \right. \text{ in the sense of Cartan and Eilenberg [6].}$$

Given an  $A$ -module chain complex  $C$  let the generator  $T \in \mathbb{Z}_2$  act on  $C^t \otimes_A C$  by the *transposition involution*

$$T : C^t \otimes_A C \rightarrow C^t \otimes_A C ; x \otimes y \mapsto (-1)^{pq} y \otimes x \ (x \in C_p, y \in C_q) .$$

For finite-dimensional  $C$  use the slant isomorphism to identify

$$C^t \otimes_A C = \text{Hom}_A(C^{-*}, C) .$$

Under this identification the transposition involution corresponds to the duality involution on  $\text{Hom}_A(C^{-*}, C)$

$$T : \text{Hom}_A(C^{-*}, C) \rightarrow \text{Hom}_A(C^{-*}, C) ; \phi \mapsto (-1)^{pq} \phi^* \ (\phi \in \text{Hom}_A(C^p, C_q)) .$$

A cycle  $\phi \in \text{Hom}_A(C^{-*}, C)_n = (C^t \otimes_A C)_n$  is a chain map  $\phi : C^{n-*} \rightarrow C$ , and  $H_n(\text{Hom}_A(C^{-*}, C))$  is the  $\mathbb{Z}$ -module of chain homotopy classes of  $A$ -module chain maps  $C^{n-*} \rightarrow C$ . Let  $W$  be the standard free  $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of  $\mathbb{Z}$

$$W : \cdots \rightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \rightarrow 0$$

and let  $\widehat{W}$  be the complete resolution

$$\widehat{W} : \cdots \rightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \rightarrow \cdots$$

The  $\begin{cases} \mathbb{Z}_2\text{-hypercohomology} \\ \mathbb{Z}_2\text{-hyperhomology} \\ \text{Tate } \mathbb{Z}_2\text{-hypercohomology} \end{cases}$  groups of a  $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex  $C$  are defined by

$$\begin{cases} H^n(\mathbb{Z}_2; C) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C)) \\ H_n(\mathbb{Z}_2; C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C) \\ \widehat{H}^n(\mathbb{Z}_2; C) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C)) \end{cases}.$$

The evident short exact sequence of  $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$0 \rightarrow SW^{-*} \rightarrow \widehat{W} \rightarrow W \rightarrow 0$$

induces a long exact sequence of abelian groups

$$\cdots \rightarrow H_n(\mathbb{Z}_2; C) \xrightarrow{1+T} H^n(\mathbb{Z}_2; C) \xrightarrow{J} \widehat{H}^n(\mathbb{Z}_2; C) \xrightarrow{H} H_{n-1}(\mathbb{Z}_2; C) \rightarrow \cdots$$

An element  $\begin{cases} \phi \in H^n(\mathbb{Z}_2; C) \\ \psi \in H_n(\mathbb{Z}_2; C) \\ \theta \in \widehat{H}^n(\mathbb{Z}_2; C) \end{cases}$  is represented by an  $n$ -cycle of  $\begin{cases} \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C) \\ W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \\ \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C) \end{cases}$

which is just a collection of chains of  $C$   $\begin{cases} \{\phi_s \in C_{n+s} \mid s \geq 0\} \\ \{\psi_s \in C_{n-s} \mid s \geq 0\} \\ \{\theta_s \in C_{n+s} \mid s \in \mathbb{Z}\} \end{cases}$  such that

$$\begin{cases} d_C(\phi_s) + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T \phi_{s-1}) = 0 \in C_{n+s-1} \quad (s \geq 0, \phi_{-1} = 0) \\ d_C(\psi_s) + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1} T \psi_{s+1}) = 0 \in C_{n-s-1} \quad (s \geq 0) \\ d_C(\theta_s) + (-1)^{n+s-1}(\theta_{s-1} + (-1)^s T \theta_{s-1}) = 0 \in C_{n+s-1} \quad (s \in \mathbb{Z}) \end{cases}$$

with

$$1+T : H_n(\mathbb{Z}_2; C) \rightarrow H^n(\mathbb{Z}_2; C) ;$$

$$\psi = \{\psi_s \mid s \geq 0\} \mapsto \{(1+T)\psi_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases}\},$$

$$J : H^n(\mathbb{Z}_2; C) \rightarrow \widehat{H}^n(\mathbb{Z}_2; C) ;$$

$$\phi = \{\phi_s \mid s \geq 0\} \mapsto \{J\phi_s = \begin{cases} \phi_s & \text{if } s \geq 0 \\ 0 & \text{if } s \leq -1 \end{cases}\},$$

$$H : \widehat{H}^n(\mathbb{Z}_2; C) \rightarrow H_{n-1}(\mathbb{Z}_2; C) ;$$

$$\theta = \{\theta_s \mid s \in \mathbb{Z}\} \mapsto H\theta = \{H\theta_s = \theta_{-s-1} \mid s \geq 0\}.$$

Given an  $A$ -module chain complex  $C$  use the action of  $T \in \mathbb{Z}_2$  on  $C^t \otimes_A C$  by the transposition involution to define the  $\mathbb{Z}$ -module chain complex

$$\begin{aligned} W^\% C &= \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_A C) \\ W_\% C &= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes_A C) \\ \widehat{W}^\% C &= \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C^t \otimes_A C) . \end{aligned}$$

We shall be mainly concerned with finite-dimensional  $C$ , using the slant isomorphism to identify

$$C^t \otimes_A C = \text{Hom}_A(C^{-*}, C)$$

and

$$\begin{aligned} W^\% C &= \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Hom}_A(C^{-*}, C)) \\ W_\% C &= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \text{Hom}_A(C^{-*}, C) \\ \widehat{W}^\% C &= \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, \text{Hom}_A(C^{-*}, C)) . \end{aligned}$$

An  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{array} \right.$  structure on a finite-dimensional  $A$ -module

chain complex  $C$  is a cycle  $\left\{ \begin{array}{l} \phi \in (W^\% C)_n \\ \psi \in (W_\% C)_n \\ \theta \in (\widehat{W}^\% C)_n , \end{array} \right.$  which is just a collection of  $A$ -module

morphisms

$$\left\{ \begin{array}{l} \{\phi_s \in \text{Hom}_A(C^{n-r+s}, C_r) \mid r \in \mathbb{Z}, s \geq 0\} \\ \{\psi_s \in \text{Hom}_A(C^{n-r-s}, C_r) \mid r \in \mathbb{Z}, s \geq 0\} \\ \{\theta_s \in \text{Hom}_A(C^{n-r+s}, C_r) \mid r \in \mathbb{Z}, s \in \mathbb{Z}\} \end{array} \right.$$

such that

$$\left\{ \begin{array}{l} d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1} (\phi_{s-1} + (-1)^s T \phi_{s-1}) = 0 \\ \quad \quad \quad : C^{n-r+s-1} \rightarrow C_r \ (s \geq 0, \phi_{-1} = 0) \\ d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1} (\psi_{s+1} + (-1)^{s+1} T \psi_{s+1}) = 0 \\ \quad \quad \quad : C^{n-r-s-1} \rightarrow C_r \ (s \geq 0) \\ d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s-1} (\theta_{s-1} + (-1)^s T \theta_{s-1}) = 0 \\ \quad \quad \quad : C^{n-r+s-1} \rightarrow C_r \ (s \in \mathbb{Z}) . \end{array} \right.$$

An equivalence  $\left\{ \begin{array}{l} \xi : \phi \rightarrow \phi' \\ \chi : \psi \rightarrow \psi' \\ \nu : \theta \rightarrow \theta' \end{array} \right.$  of  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{array} \right.$  structures

on  $C$  is a chain  $\left\{ \begin{array}{l} \xi \in (W^\% C)_{n+1} \\ \chi \in (W_\% C)_{n+1} \\ \nu \in (\widehat{W}^\% C)_{n+1} \end{array} \right.$  such that

$$\left\{ \begin{array}{l} \phi' - \phi = d(\xi) \in (W^\% C)_n \\ \psi' - \psi = d(\chi) \in (W_\% C)_n \\ \theta' - \theta = d(\nu) \in (\widehat{W}^\% C)_n . \end{array} \right.$$



The  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$  structure group  $\begin{cases} Q^n(C) \\ Q_n(C) \\ \widehat{Q}^n(C) \end{cases}$  of a chain complex

$C$  is the abelian group of equivalence classes of  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$  structures on  $C$ , that is

$$\begin{cases} Q^n(C) = H^n(\mathbb{Z}_2; C^t \otimes_A C) = H_n(W^\% C) \\ Q_n(C) = H_n(\mathbb{Z}_2; C^t \otimes_A C) = H_n(W_\% C) \\ \widehat{Q}^n(C) = \widehat{H}^n(\mathbb{Z}_2; C^t \otimes_A C) = H_n(\widehat{W}^\% C) . \end{cases}$$

The  $Q$ -groups are related by a long exact sequence

$$\cdots \rightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \rightarrow \cdots$$

involving the morphisms induced in homology by the  $\mathbb{Z}$ -module chain maps

$$1 + T : W_\% C \rightarrow W^\% C , \quad J : W^\% C \rightarrow \widehat{W}^\% C , \quad H : \widehat{W}^\% C \rightarrow S(W_\% C)$$

defined by

$$1 + T : (W_\% C)_n \rightarrow (W^\% C)_n ;$$

$$\{\psi_s \in (C^t \otimes_A C)_{n-s} \mid s \geq 0\} \mapsto \{((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases} \} ,$$

$$J : (W^\% C)_n \rightarrow (\widehat{W}^\% C)_n ;$$

$$\{\phi_s \in (C^t \otimes_A C)_{n+s} \mid s \geq 0\} \mapsto \{(J\phi)_s = \begin{cases} \phi_s & \text{if } s \geq 0 \\ 0 & \text{if } s \leq -1 \end{cases} \} ,$$

$$H : (\widehat{W}^\% C)_n \rightarrow (W_\% C)_{n-1} ;$$

$$\{\theta_s \in (C^t \otimes_A C)_{n+s} \mid s \in \mathbb{Z}\} \mapsto \{(H\theta)_s = \theta_{s-1} \mid s \geq 0\} .$$

An  $n$ -dimensional symmetric structure  $\phi \in (W^\% C)_n$  is equivalent to the symmetrization  $(1+T)\psi$  of an  $n$ -dimensional quadratic structure  $\psi \in (W_\% C)_n$  if and only if the  $n$ -dimensional hyperquadratic structure  $J(\phi) \in (\widehat{W}^\% C)_n$  is equivalent to 0. An  $A$ -module chain map  $f : C \rightarrow D$  induces a  $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map

$$f^t \otimes_A f : C^t \otimes_A C \rightarrow D^t \otimes_A D ; \quad x \otimes y \mapsto f(x) \otimes f(y)$$

and hence  $\mathbb{Z}$ -module chain maps

$$f^\% : W^\% C \rightarrow W^\% D ,$$

$$f_\% : W_\% C \rightarrow W_\% D ,$$

$$\widehat{f}^\% : \widehat{W}^\% C \rightarrow \widehat{W}^\% D .$$

An  $A$ -module chain homotopy

$$g : f \simeq f' : C \rightarrow D$$

determines  $\mathbb{Z}$ -module chain homotopies

$$\begin{aligned} (g; f, f')^{\%} &: f^{\%} \simeq f'^{\%} : W^{\%}C \rightarrow W^{\%}D \\ (g; f, f')_{\%} &: f_{\%} \simeq f'_{\%} : W_{\%}C \rightarrow W_{\%}D \\ (\widehat{g; f, f'})^{\%} &: \widehat{f}^{\%} \simeq \widehat{f}'^{\%} : \widehat{W}^{\%}C \rightarrow \widehat{W}^{\%}D \end{aligned}$$

with

$$\begin{aligned} (g; f, f')^{\%} : (W^{\%}C)_n &= \sum_{s=0}^{\infty} (C^t \otimes_A C)_{n+s} \\ &\rightarrow (W^{\%}D)_{n+1} = \sum_{s=0}^{\infty} \sum_q D_{n-q+s+1}^t \otimes_A D_q ; \\ \sum_{s=0}^{\infty} \phi_s &\mapsto \sum_{s=0}^{\infty} ((f^t \otimes_A g + g^t \otimes_A f')(\phi_s) + (-1)^{q+s-1} (g^t \otimes_A g)(T\phi_{s-1})) \end{aligned}$$

and similarly for  $(g; f, f')_{\%}$ ,  $(\widehat{g; f, f'})^{\%}$ . Thus the induced morphisms in the  $Q$ -groups

$$\begin{aligned} f^{\%} &: Q^n(C) \rightarrow Q^n(D) \\ f_{\%} &: Q_n(C) \rightarrow Q_n(D) \\ \widehat{f}^{\%} &: \widehat{Q}^n(C) \rightarrow \widehat{Q}^n(D) \end{aligned}$$

depend only on the chain homotopy class of  $f$ , and are isomorphisms if  $f$  is a chain equivalence. For finite-dimensional  $C, D$  the slant isomorphisms are used to identify  $f^t \otimes_A f : C^t \otimes_A C \rightarrow D^t \otimes_A D$  with

$$\mathrm{Hom}_A(f^*, f) : \mathrm{Hom}_A(C^{-*}, C) \rightarrow \mathrm{Hom}_A(D^*, D) ; \theta \mapsto f\theta f^* ,$$

and similarly for  $f^{\%}, f_{\%}, \widehat{f}^{\%}$  and  $(g; f, f')^{\%}, (g; f, f')_{\%}, (\widehat{g; f, f'})^{\%}$ .

Although all the  $Q$ -groups are chain homotopy invariant, only the hyper-quadratic  $Q$ -groups  $\widehat{Q}^*(C)$  are additive. The sum of  $A$ -module chain maps  $f, g : C \rightarrow D$  is an  $A$ -module chain map  $f + g : C \rightarrow D$  such that

$$\begin{aligned} (f + g)^{\%} - f^{\%} - g^{\%} &: Q^n(C) \rightarrow H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \rightarrow Q^n(D) , \\ (f + g)_{\%} - f_{\%} - g_{\%} &: Q_n(C) \rightarrow H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \rightarrow Q_n(D) , \\ (\widehat{f + g})^{\%} - \widehat{f}^{\%} - \widehat{g}^{\%} &= 0 : \widehat{Q}^n(C) \rightarrow \widehat{Q}^n(D) \end{aligned}$$

with

$$\begin{aligned} Q^n(C) &\rightarrow H_n(C^t \otimes_A C) ; \phi = \{\phi_s \mid s \geq 0\} \mapsto \phi_0 , \\ Q_n(C) &\rightarrow H_n(C^t \otimes_A C) ; \psi = \{\psi_s \mid s \geq 0\} \mapsto (1 + T)\psi_0 , \\ H_n(D^t \otimes_A D) &\rightarrow Q^n(D) ; \theta \mapsto \{\phi_s = \begin{cases} (1 + T)\theta & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases} \} , \\ H_n(D^t \otimes_A D) &\rightarrow Q_n(D) ; \theta \mapsto \{\psi_s = \begin{cases} \theta & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases} \} . \end{aligned}$$

Given a finite-dimensional  $A$ -module chain complex  $C$  and  $n \geq 0$  define the  $n$ -fold suspension chain isomorphism

$$\begin{aligned} S^n &: S^n(\widehat{W}^\% C) \rightarrow \widehat{W}^\%(S^n C) ; \\ \theta &= \{\theta_s \in \text{Hom}_A(C^r, C_{m-r+s}) \mid s \in \mathbb{Z}\} \\ \mapsto S^n \theta &= \{(S^n \theta)_s = \theta_{s-n} \in \text{Hom}_A(C^r, C_{m-n+r+s}) \mid s \in \mathbb{Z}\} . \end{aligned}$$

For any (finite-dimensional)  $A$ -module chain complexes  $C, D$  there is defined a simplicial map

$$I : K(\text{Hom}_A(C, D)) \rightarrow K(\text{Hom}_{\mathbb{Z}}(\widehat{W}^\% C, \widehat{W}^\% D))$$

sending a cycle  $f \in \text{Hom}_A(C, D)_n$  (= a chain map up to sign  $f : S^n C \rightarrow D$ ) to the  $\mathbb{Z}$ -module chain map up to sign

$$I(f) = \widehat{f}^\% S^n : S^n(\widehat{W}^\% C) \xrightarrow{S^n} \widehat{W}^\% S^n C \xrightarrow{\widehat{f}^\%} \widehat{W}^\% D .$$

An  $n$ -simplex  $(g; f, f', 0, \dots, 0) \in K(\text{Hom}_A(C, D))^{(n)}$  (= an  $A$ -module chain homotopy up to sign  $g : f \simeq f' : S^n C \rightarrow D$ ) is sent to the  $\mathbb{Z}$ -module chain homotopy up to sign

$$I(g; f, f') = (g; f, f')^\% S^n : I(f) \simeq I(f') : S^n(\widehat{W}^\% C) \rightarrow \widehat{W}^\% D .$$

The failure of  $I$  to be linear on chain maps up to sign  $f : S^n C \rightarrow D$  is given by the chain homotopy up to sign

$$[f, f'] : (\widehat{f + f'})^\% \simeq \widehat{f}^\% + \widehat{f'}^\% : \widehat{W}^\%(S^n C) \rightarrow \widehat{W}^\% D$$

defined by

$$\begin{aligned} [f, f'] &: (S^n \widehat{W}^\% C)_m \rightarrow (\widehat{W}^\% D)_{m+n+1} ; \\ \theta &= \{\theta_s \in \text{Hom}_A(C^r, C_{m-r+s}) \mid s \in \mathbb{Z}\} \\ \mapsto [f, f']\theta &= \{T^{n+1} f \theta_{s-n+1} f'^* \in \text{Hom}_A(D^r, D_{m-n+r+s+1}) \mid s \in \mathbb{Z}\} . \end{aligned}$$

#### §4. Algebraic Wu classes

The algebraic Wu classes are the fundamental invariants of a duality structure on a chain complex  $C$ , which are obtained by an algebraic analogue of the Steenrod squares in the cohomology groups of a topological space. In the topological applications the algebraic Wu classes are closely related to the topological Wu classes, as explained in Ranicki [15].

Let  $S^r A$  ( $r \in \mathbb{Z}$ ) denote the  $A$ -module chain complex

$$S^r A : \dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots$$

concentrated in degree  $r$ . For any  $A$ -module chain complex  $C$  there are defined natural isomorphisms

$$H_0(\text{Hom}_A(C, S^r A)) \rightarrow H^r(C) ; (f : C_r \rightarrow A) \mapsto f^*(1) .$$

An element  $f \in H^r(C)$  is just a chain homotopy class of chain maps  $f : C \rightarrow S^r A$ . The Wu classes of a quadratic structure on  $C$  are the invariants of the equivalence class defined by sending an element  $f \in H^r(C)$  to the induced equivalence class

of quadratic structures on  $S^r A$ . The quadratic structure groups of the elementary complexes  $S^r A$  are identified with subquotients of the ground ring  $A$ .

An  $A$ -group  $M$  is an abelian group together with an  $A$ -action

$$A \times M \rightarrow M ; (a, x) \rightarrow ax$$

such that

$$a(x + y) = ax + ay, a(bx) = (ab)x, 1x = x \ (x, y \in M, a, b \in M) .$$

An  $A$ -module is an  $A$ -group  $M$  such that also

$$(a + b)x = ax + bx \in M .$$

An  $A$ -morphism of  $A$ -groups is a morphism of abelian groups

$$f : M \rightarrow N$$

such that

$$f(ax) = af(x) \in N \ (x \in M, a \in A) .$$

The set of  $A$ -group morphisms  $f : M \rightarrow N$  defines an abelian group  $\text{Hom}_A(M, N)$ , with addition by

$$(f + g)(x) = f(x) + g(x) \in N .$$

For  $A$ -modules  $M, N$  the  $A$ -morphisms  $f : M \rightarrow N$  coincide with  $A$ -module morphisms.

For  $\epsilon = \pm 1$  let the generator  $T \in \mathbb{Z}_2$  act on  $A$  by the  $\epsilon$ -involution

$$T_\epsilon : A \rightarrow A ; a \mapsto \epsilon \bar{a} .$$

$$\text{Define the } \begin{cases} \mathbb{Z}_2\text{-cohomology} \\ \mathbb{Z}_2\text{-homology} \\ \text{Tate } \mathbb{Z}_2\text{-cohomology} \end{cases} \quad A\text{-groups} \quad \begin{cases} H^*(\mathbb{Z}_2; A, \epsilon) \\ H_*(\mathbb{Z}_2; A, \epsilon) \\ \widehat{H}^*(\mathbb{Z}_2; A, \epsilon) \end{cases} \quad \text{by}$$

$$\begin{cases} H^r(\mathbb{Z}_2; A, \epsilon) = \begin{cases} \ker(1 - T_\epsilon : A \rightarrow A) & \text{if } r = 0 \\ \widehat{H}^r(\mathbb{Z}_2; A, \epsilon) & \text{if } r \geq 1 \\ 0 & \text{if } r < 0 \end{cases} \\ H_r(\mathbb{Z}_2; A, \epsilon) = \begin{cases} \text{coker}(1 - T_\epsilon : A \rightarrow A) & \text{if } r = 0 \\ \widehat{H}^{r+1}(\mathbb{Z}_2; A, \epsilon) & \text{if } r \geq 1 \\ 0 & \text{if } r < 0 \end{cases} \\ \widehat{H}^r(\mathbb{Z}_2; A, \epsilon) = \ker(1 - (-1)^r T_\epsilon : A \rightarrow A) / \text{im}(1 + (-1)^r T_\epsilon : A \rightarrow A) \ (r \in \mathbb{Z}) . \end{cases}$$

The  $A$ -action

$$A \times \widehat{H}^r(\mathbb{Z}_2; A, \epsilon) \rightarrow \widehat{H}^r(\mathbb{Z}_2; A, \epsilon) ; (a, x) \mapsto ax\bar{a}$$

defines an  $A$ -module structure on  $\widehat{H}^r(\mathbb{Z}_2; A, \epsilon)$ . The  $A$ -actions

$$A \times H^0(\mathbb{Z}_2; A, \epsilon) \rightarrow H^0(\mathbb{Z}_2; A, \epsilon) ; (a, x) \mapsto ax\bar{a}$$

$$A \times H_0(\mathbb{Z}_2; A, \epsilon) \rightarrow H_0(\mathbb{Z}_2; A, \epsilon) ; (a, x) \mapsto ax\bar{a}$$

are not linear in  $A$ , and so do not define  $A$ -module structures.

For  $\epsilon = +1$  the groups  $\begin{cases} H^*(\mathbb{Z}_2; A, \epsilon) \\ H_*(\mathbb{Z}_2; A, \epsilon) \\ \widehat{H}^*(\mathbb{Z}The A, \epsilon) \end{cases}$  are denoted by  $\begin{cases} H^*(\mathbb{Z}_2; A) \\ H_*(\mathbb{Z}_2; A) \\ \widehat{H}^*(\mathbb{Z}_2; A). \end{cases}$

The natural  $\mathbb{Z}$ -module isomorphisms

$$\begin{aligned} Q^n(S^r A) &\rightarrow H^{2r-n}(\mathbb{Z}_2; A, (-1)^r) ; \phi \mapsto \phi_{2r-n}(1)(1) \\ Q_n(S^r A) &\rightarrow H_{n-2r}(\mathbb{Z}_2; A, (-1)^r) ; \psi \mapsto \psi_{n-2r}(1)(1) \\ \widehat{Q}^n(S^r A) &\rightarrow \widehat{H}^{r-n}(\mathbb{Z}_2; A, (-1)^r) ; \theta \mapsto \theta_{2r-n}(1)(1) \end{aligned}$$

will be used as identifications.

The *Wu classes* of a  $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$  structure  $\begin{cases} \phi \in (W^\% C)_n \\ \psi \in (W_\% C)_n \\ \theta \in (\widehat{W}^\% C)_n \end{cases}$  are the invariants of the equivalence class of structures defined by the *A*-morphisms

$$\begin{cases} v_r(\phi) : H^{n-r}(C) = H_0(\text{Hom}_A(C, S^{n-r} A)) \\ \quad \rightarrow Q^n(S^{n-r} A) = H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) ; f \mapsto (f \otimes f)(\phi_{n-2r}) \\ v^r(\psi) : H^{n-r}(C) = H_0(\text{Hom}_A(C, S^{n-r} A)) \\ \quad \rightarrow Q_n(S^{n-r} A) = H_{2r-n}(\mathbb{Z}_2; A, (-1)^{n-r}) ; f \mapsto (f \otimes f)(\psi_{2r-n}) \\ \widehat{v}_r(\theta) : H^{n-r}(C) = H_0(\text{Hom}_A(C, S^{n-r} A)) \\ \quad \rightarrow \widehat{Q}^n(S^{n-r} A) = \widehat{H}^r(\mathbb{Z}_2; A) ; f \mapsto (f \otimes f)(\theta_{n-2r}) . \end{cases}$$

## §5. Algebraic Poincaré complexes

An algebraic Poincaré complex is a chain complex with Poincaré duality, such as arises from a compact  $n$ -manifold or a normal map.

An  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  (Poincaré) complex over  $A$   $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  is an  $n$ -dimensional  $A$ -module chain complex  $C$  together with an  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  structure  $\begin{cases} \phi \in (W^\% C) \\ \psi \in (W_\% C)_n \end{cases}$  (such that  $\begin{cases} \phi_0 \\ (1+T)\psi_0 \end{cases} : C^{n-*} \rightarrow C$  is a chain equivalence).

$$\begin{aligned} \text{An } (n+1)\text{-dimensional } \begin{cases} \text{symmetric} \\ \text{quadratic} \\ (\text{symmetric, quadratic}) \end{cases} \text{ (Poincaré) pair over } A \\ \left( f : C \rightarrow D , \begin{cases} (\delta\phi, \phi) \\ (\delta\psi, \psi) \\ (\delta\phi, \psi) \end{cases} \right) \end{aligned}$$

consists of an  $n$ -dimensional  $A$ -module chain complex  $C$ , an  $(n+1)$ -dimensional  $A$ -module chain complex  $D$ , a chain map  $f : C \rightarrow D$  and a cycle

$$\begin{cases} (\delta\phi, \phi) \in C(f^\% : W^\% C \rightarrow W^\% D)_{n+1} = (W^\% D)_{n+1} \oplus (W^\% C)_n \\ (\delta\psi, \psi) \in C(f_\% : W_\% C \rightarrow W_\% D)_{n+1} = (W_\% D)_{n+1} \oplus (W_\% C)_n \\ (\delta\phi, \psi) \in C((1+T)f_\% : W_\% C \rightarrow W^\% D)_{n+1} = (W^\% D)_{n+1} \oplus (W_\% C)_n \end{cases}$$

(such that the  $A$ -module chain map  $D^{n+1-*} \rightarrow C(f)$  defined by

$$\begin{cases} (\delta\phi, \phi)_0 = \begin{pmatrix} \delta\phi_0 \\ \phi_0 f^* \end{pmatrix} \\ (1+T)(\delta\psi, \psi)_0 = \begin{pmatrix} (1+T)\delta\psi_0 \\ (1+T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \rightarrow C(f)_r = D_r \oplus C_{r-1} \\ (\delta\phi, (1+T)\psi)_0 = \begin{pmatrix} \delta\phi_0 \\ (1+T)\psi_0 f^* \end{pmatrix} \end{cases}$$

is a chain equivalence). The *boundary* of the pair is the  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{quadratic} \end{cases}$

(Poincaré) complex  $\begin{cases} (C, \phi) \\ (C, \psi) \\ (C, \psi). \end{cases}$

A *homotopy equivalence* of  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complexes

$$\begin{cases} (f, \chi) : (C, \phi) \rightarrow (C', \phi') \\ (f, \xi) : (C, \psi) \rightarrow (C', \psi') \end{cases}$$

is a chain equivalence  $f : C \rightarrow C'$  together with an equivalence of  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  structures on  $C'$   $\begin{cases} \chi : f^{\%}(\phi) \rightarrow \phi' \\ \xi : f^{\%}(\psi) \rightarrow \psi' \end{cases}$ . There is a similar notion of homotopy equivalence for pairs.

An  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complex  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  is *connected* if

$$\begin{cases} H_0(\phi_0 : C^{n-*} \rightarrow C) = 0 \\ H_0((1+T)\psi_0 : C^{n-*} \rightarrow C) = 0. \end{cases}$$

It was shown in Ranicki [15] that there is a natural one-one correspondence between the homotopy equivalence classes of connected  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complexes over  $A$  and the homotopy equivalence classes of  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré pairs over  $A$ . A connected  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complex  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  determines the  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré pair

$$(i_C : \partial C \rightarrow C^{n-*}, \begin{cases} (0, \partial\phi) \\ (0, \partial\psi) \end{cases})$$

defined by

$$\begin{aligned}
i_C &= (0 \ 1) : \partial C_r = C_{r+1} \oplus C^{n-r} \rightarrow C^{n-r}, \\
d_{\partial C} &= \begin{cases} \begin{pmatrix} d_C & (-1)^r \phi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} : \\ \begin{pmatrix} d_C & (-1)^r (1+T)\psi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} : \end{cases} \\
\partial C_r &= C_{r+1} \oplus C^{n-r} \rightarrow \partial C_{r-1} = C_r \oplus C^{n-r+1}, \\
\begin{cases} \partial \phi_0 = \begin{pmatrix} (-1)^{n-r-1} T \phi_1 & (-1)^{r(n-r-1)} \\ 1 & 0 \end{pmatrix} : \\ \partial \psi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \end{cases} \\
\partial C^{n-r-1} &= C^{n-r} \oplus C_{r+1} \rightarrow \partial C_r = C_{r+1} \oplus C^{n-r}, \\
\begin{cases} \partial \phi_s = \begin{pmatrix} (-1)^{n-r+s-1} T \phi_{s+1} & 0 \\ 0 & 0 \end{pmatrix} : \\ \partial C^{n-r+s-1} = C^{n-r+s} \oplus C_{r-s+1} \rightarrow \partial C_r = C_{r+1} \oplus C^{n-r} \ (s \geq 1), \\ \partial \psi_s = \begin{pmatrix} (-1)^{n-r-s} T \psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} : \\ \partial C^{n-r-s-1} = C^{n-r-s} \oplus C_{r+s+1} \rightarrow \partial C_r = C_{r+1} \oplus C^{n-r} \ (s \geq 1). \end{cases}
\end{aligned}$$

The  $(n-1)$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré complex

$$\begin{cases} \partial(C, \phi) = (\partial C, \partial \phi) \\ \partial(C, \psi) = (\partial C, \partial \psi) \end{cases}$$

is the *boundary* of the connected  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complex  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$ .

The connected complex  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  is a Poincaré complex if and only if the bound-

ary  $\begin{cases} \partial(C, \phi) \\ \partial(C, \psi) \end{cases}$  is contractible (= homotopy equivalent to 0). A Poincaré complex

$\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  is the boundary of an  $(n+1)$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré pair

$\begin{cases} (f : C \rightarrow D, (\delta \phi, \phi)) \\ (f : C \rightarrow D, (\delta \psi, \psi)) \end{cases}$  if and only if it is homotopy equivalent to the boundary of

a connected  $(n+1)$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complex.

The  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré complexes  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}, \begin{cases} (C', \phi') \\ (C', \psi') \end{cases}$  are

*cobordant* if  $\begin{cases} (C, \phi) \oplus (C', -\phi') \\ (C, \psi) \oplus (C', -\psi') \end{cases}$  is the boundary of an  $(n+1)$ -dimensional

$\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré pair  $\begin{cases} ((f, f') : C \oplus C' \rightarrow D, (\delta \phi, \phi \oplus -\phi')) \\ ((f, f') : C \oplus C' \rightarrow D, (\delta \psi, \psi \oplus -\psi')) \end{cases}$ . Homotopy equivalent Poincaré complexes are cobordant.

The  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \end{array} \right. L\text{-groups} \left\{ \begin{array}{l} L^n(A) \\ L_n(A) \end{array} \right. (n \geq 0)$  are the cobordism groups of  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \end{array} \right.$  Poincaré complexes over  $A$ . The quadratic  $L$ -groups  $L_*(A)$  are 4-periodic, with isomorphisms

$$L_n(A) \rightarrow L_{n+4}(A) ; (C, \psi) \mapsto (S^2 C, \psi) \quad (n \geq 0) ,$$

and are just the surgery obstruction groups of Wall [18]. The symmetric  $L$ -groups  $L^*(A)$  were introduced by Mishchenko [13]. The corresponding maps in the symmetric  $L$ -groups

$$L^n(A) \rightarrow L^{n+4}(A) ; (C, \phi) \mapsto (S^2 C, \phi) \quad (n \geq 0)$$

are not isomorphisms in general. The symmetric and quadratic  $L$ -groups are related by an exact sequence

$$\cdots \rightarrow L_n(A) \xrightarrow{1+T} L^n(A) \rightarrow \widehat{L}^n(A) \rightarrow L_{n-1}(A) \rightarrow \cdots$$

with

$$1 + T : L_n(A) \rightarrow L^n(A) ; (C, \psi) \mapsto (C, (1 + T)\psi)$$

and  $\widehat{L}^n(A)$  the relative cobordism group of  $n$ -dimensional (symmetric, quadratic) Poincaré pairs over  $A$ . The relative  $L$ -groups  $\widehat{L}^*(A)$  are 8-torsion, so that the symmetrization maps  $1 + T : L_n(A) \rightarrow L^n(A)$  are isomorphisms modulo 8-torsion. If  $\widehat{H}^*(\mathbb{Z}_2; A) = 0$  (e.g. if  $1/2 \in A$ ) then  $\widehat{L}^*(A) = 0$  and the symmetrization maps are isomorphisms.

The *symmetric construction* of Ranicki [15] is the natural chain map

$$\phi_X = 1 \otimes \Delta : C(X) \rightarrow W^{\%}C(\widetilde{X}) = \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} C(\widetilde{X}))$$

induced by an Alexander-Whitney-Steenrod diagonal chain approximation  $\Delta$ , for any space  $X$  and any regular cover  $\widetilde{X}$ , with  $\pi$  the group of covering translations. For  $\widetilde{X} = X$  the mod 2 reduction of the composite

$$H_n(X) \xrightarrow{\phi_X} Q^n(C(X)) \xrightarrow{v_r} \text{Hom}_{\mathbb{Z}}(H^{n-r}(X), Q^n(S^{n-r}\mathbb{Z}))$$

is given by the  $r$ th Steenrod square

$$v_r(\phi_X(x))(y) = \langle Sq^r(y), x \rangle \in \mathbb{Z}_2 .$$

The symmetric signature of Mishchenko [13] is defined for any  $n$ -dimensional geometric Poincaré complex  $X$  to be the symmetric Poincaré cobordism class

$$\sigma^*(X) = (C(\widetilde{X}), \phi_X([X])) \in L^n(\mathbb{Z}[\pi_1(X)]) .$$

The symmetric  $L$ -groups of  $\mathbb{Z}$  are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 \text{ (deRham invariant)} & \text{if } n \equiv 1 \pmod{4} \\ 0 & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 3 \pmod{4} . \end{cases}$$



The *quadratic construction* of [15] associates to any stable  $\pi$ -equivariant map  $F : \Sigma^\infty \tilde{X}_+ \rightarrow \Sigma^\infty \tilde{Y}_+$  a natural chain map

$$\psi_F : C(X) \rightarrow W_{\%} C(\tilde{Y}) = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C(\tilde{Y}) \otimes_{\mathbb{Z}[\pi]} C(\tilde{Y}))$$

such that

$$(1 + T)\psi_F = F^{\%}\phi_X - \phi_Y F_* : C(X) \rightarrow W^{\%} C(\tilde{Y})$$

with  $\tilde{X}$  a regular cover of  $X$  with group of covering translations  $\pi$ ,  $\tilde{X}_+ = \tilde{X} \cup \{\text{pt.}\}$ , and similarly for  $Y$ . For  $\tilde{X} = X$ ,  $\tilde{Y} = Y$ ,  $\pi = \{1\}$  the mod 2 reduction of the composite

$$H_n(X) \xrightarrow{\psi_F} Q_n(C(Y)) \xrightarrow{v^r} \text{Hom}_{\mathbb{Z}}(H^{n-r}(Y), Q_n(S^{n-r}\mathbb{Z}))$$

is given by the  $(r+1)$ th functional Steenrod square

$$v^r(\psi_F(x))(y) = \langle Sq_{(\Sigma^\infty y)_F}^{r+1}(\Sigma^\infty \iota), \Sigma^\infty x \rangle \in \mathbb{Z}_2$$

with  $\iota \in H^{n-r}(K(\mathbb{Z}_2, n-r); \mathbb{Z}_2) = \mathbb{Z}_2$  the generator.

The Wall [18] surgery obstruction of an  $n$ -dimensional normal map  $(f, b) : M \rightarrow X$  was expressed in [15] as the quadratic Poincaré cobordism class

$$\sigma_*(f, b) = (C(f^!), e_{\%}\psi_F([X])) \in L_n(\mathbb{Z}[\pi_1(X)])$$

with  $F : \Sigma^\infty \tilde{X}_+ \rightarrow \Sigma^\infty \tilde{M}_+$  a  $\pi_1(X)$ -equivariant  $S$ -dual of  $T(\tilde{b}) : T(\nu_{\tilde{M}}) \rightarrow T(\nu_{\tilde{X}})$  inducing the Umkehr chain map

$$f^! : C(\tilde{X}) \simeq C(\tilde{X})^{n-*} \xrightarrow{f^*} C(\tilde{M})^{n-*} \simeq C(\tilde{M})$$

and  $e : C(\tilde{M}) \rightarrow C(f^!)$  the inclusion in the algebraic mapping cone. The quadratic  $L$ -groups of  $\mathbb{Z}$  are given by

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature/8)} & \text{if } n \equiv 0 \pmod{4} \\ 0 & \text{if } n \equiv 1 \pmod{4} \\ \mathbb{Z}_2 \text{ (Arf invariant)} & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 3 \pmod{4} . \end{cases}$$

## §6. Chain bundles

A *bundle* over a finite-dimensional  $A$ -module chain complex  $C$  is a 0-dimensional hyperquadratic structure on  $C^{0-*}$ , that is a cycle

$$\gamma \in (\widehat{W}^{\%} C^{0-*})_0 ,$$

as defined by a collection of  $A$ -module morphisms

$$\{\gamma_s \in \text{Hom}_A(C_{r-s}, C^{-r}) \mid r, s \in \mathbb{Z}\}$$

such that

$$(-1)^{r+1} d_C^* \gamma_s + (-1)^s \gamma_s d_C + (-1)^{s-1} (\gamma_{s-1} + (-1)^s T \gamma_{s-1}) = 0 : C_{r-s+1} \rightarrow C^{-r} .$$

An *equivalence* of bundles over  $C$

$$\chi : \gamma \rightarrow \gamma'$$

is an equivalence of hyperquadratic structures, as defined by a collection of  $A$ -module morphisms

$$\{\chi_s \in \text{Hom}_A(C_{r-s-1}, C^{-r}) \mid r, s \in \mathbb{Z}\}$$

such that

$$\gamma'_s - \gamma_s = (-1)^{r+1} d_C^* \chi_s + (-1)^s \chi_s d_C + (-1)^s (\chi_{s-1} + (-1)^s T \chi_{s-1}) : C_{r-s} \rightarrow C^{-r}.$$

Thus

$$\widehat{Q}^0(C^{0-*}) = H_0(\widehat{W}^{\%} C^{0-*})$$

is the abelian group of equivalence classes of bundles over  $C$ .

A *chain bundle* over  $A$   $(C, \gamma)$  is a finite-dimensional  $A$ -module chain complex  $C$  together with a bundle  $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$ .

Given a chain bundle  $(C, \gamma)$  over  $A$  and an  $A$ -module chain map  $f : B \rightarrow C$  define the *pullback* chain bundle  $(B, f^* \gamma)$  using the image of  $\gamma$  under the  $\mathbb{Z}$ -module chain map

$$\widehat{f}^* : \widehat{W}^{\%} C^{0-*} \rightarrow \widehat{W}^{\%} B^{0-*}$$

induced by the dual  $A$ -module chain map  $f^* : C^{0-*} \rightarrow B^{0-*}$ . The equivalence class of the pullback bundle  $f^* \gamma$  depends only on the chain homotopy class of the chain map  $f$ , by the chain homotopy invariance of the  $Q$ -groups.

A *map* of chain bundles over  $A$

$$(f, \chi) : (C, \gamma) \rightarrow (C', \gamma')$$

is a chain map  $f : C \rightarrow C'$  together with an equivalence of bundles over  $C$

$$\chi : \gamma \rightarrow f^* \gamma'.$$

The *composite* of chain bundle maps

$$(f, \chi) : (C, \gamma) \rightarrow (C', \gamma'), (f', \chi') : (C', \gamma') \rightarrow (C'', \gamma'')$$

is the chain bundle map

$$(f', \chi')(f, \chi) = (f' f, \chi + \widehat{f'^*}^{\%}(\chi')) : (C, \gamma) \rightarrow (C'', \gamma'').$$

A *homotopy of chain bundle maps*

$$(g, \eta) : (f, \chi) \simeq (f', \chi') : (C, \gamma) \rightarrow (C', \gamma')$$

is a chain homotopy

$$g : f \simeq f' : C \rightarrow C'$$

together with an equivalence of 1-dimensional hyperquadratic structures on  $C^{0-*}$

$$\eta : \chi - \chi' + (g^*; f^*, f'^*)^{\%}(\gamma') \rightarrow 0.$$

A map of chain bundles  $(f, \chi) : (C, \gamma) \rightarrow (C', \gamma')$  is an *equivalence* if there exists a homotopy inverse. This happens precisely when  $f : C \rightarrow C'$  is a chain equivalence, in which case any chain homotopy inverse

$$f' = f^{-1} : C' \rightarrow C$$

can be used to define a homotopy inverse

$$(f', \chi') : (C', \gamma') \rightarrow (C, \gamma).$$

Given a chain bundle  $(B, \beta)$  over  $A$  and a finite-dimensional  $A$ -module chain complex  $C$  use the pullback construction to define abelian group morphisms

$$I_\beta : H_n(B^t \otimes_A C) \rightarrow \widehat{Q}^n(C) ; f \mapsto \widehat{f}^\% (S^n \beta) ,$$

using the slant isomorphism

$$B^t \otimes_A C \rightarrow \text{Hom}_A(B^{-*}, C) ; x \otimes y \mapsto (f \mapsto \overline{f(x)} \cdot y)$$

to identify a cycle  $f \in (B^t \otimes_A C)_n$  with a chain map  $f : B^{-*} \rightarrow C$ . Weiss [19] developed an algebraic analogue of the representation theorem of Brown [3] to obtain for any ring with involution  $A$  the existence of a directed system  $\{(B(r), \beta(r)) \mid r \geq 0\}$  of chain bundles over  $A$  and chain bundle maps

$$(B(r), \beta(r)) \rightarrow (B(r+1), \beta(r+1))$$

such that the abelian group morphisms

$$\varinjlim_r I_{\beta_r} : \varinjlim_r H_n(B(r)^t \otimes_A C) \rightarrow \widehat{Q}^n(C)$$

are isomorphisms for any finite-dimensional  $A$ -module chain complex  $C$ . In general, the direct limit  $A$ -module chain complex

$$B(\infty) = \varinjlim_r B(r)$$

is not finite-dimensional. As in [19] we shall ignore this inconvenience and treat  $B(\infty)$  as if it were finite-dimensional, so that there is defined the *universal chain bundle* over  $A$

$$(B(\infty), \beta(\infty)) = \varinjlim_r (B(r), \beta(r)) ,$$

with the universal property that for any finite-dimensional  $A$ -module chain complex  $C$  the abelian group morphisms

$$I_{\beta(\infty)} : H_n(B(\infty)^t \otimes_A C) \rightarrow \widehat{Q}^n(C)$$

are isomorphisms. In particular, there are defined isomorphisms

$$H_0(\text{Hom}_A(C, B(\infty))) \rightarrow \widehat{Q}^0(C^{0-*}) ; f \mapsto f^*(\beta(\infty))$$

for any finite-dimensional  $C$ . Thus every chain bundle  $(C, \gamma)$  has a classifying map

$$(f, \chi) : (C, \gamma) \rightarrow (B(\infty), \beta(\infty))$$

and the equivalence classes of bundles  $\gamma \in (\widehat{W}^\% C^{0-*})_0$  over  $C$  are in one-one correspondence with the chain homotopy classes of chain maps  $f : C \rightarrow B(\infty)$ .

The *Wu classes* of a chain bundle  $(C, \gamma)$  are the Wu classes of  $\gamma$ , the  $A$ -module morphisms

$$\widehat{v}_r(\gamma) : H_r(C) \rightarrow \widehat{H}^r(\mathbb{Z}_2; A) ; x \mapsto \langle \gamma_{-2r}, x \otimes x \rangle .$$

The universal chain bundle  $(B(\infty), \beta(\infty))$  is characterized by the property that the Wu classes define  $A$ -module isomorphisms

$$\widehat{v}_r(\gamma) : H_r(C) \rightarrow \widehat{H}^r(\mathbb{Z}_2; A) \quad (r \geq 0) .$$

For example, if  $A = \mathbb{Z}$  the chain bundle  $(B(\infty), \beta(\infty))$  defined by

$$\begin{aligned} d_{B(\infty)} &= \begin{cases} 2 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} : B(\infty)_r = \mathbb{Z} \rightarrow B(\infty)_{r-1} = \mathbb{Z}, \\ \beta(\infty)_s &= \begin{cases} 1 & \text{if } 2r = s \\ 0 & \text{otherwise} \end{cases} : B(\infty)_{r-s} = \mathbb{Z} \rightarrow B(\infty)^{-r} = \mathbb{Z} \end{aligned}$$

is universal, with the Wu classes defining isomorphisms

$$\widehat{v}_r(\beta(\infty)) : H_r(B(\infty)) \rightarrow \widehat{H}^r(\mathbb{Z}_2; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases}.$$

The symmetric and quadratic constructions of Ranicki [15] were extended in Weiss [19] and Ranicki [16, 2.5] : a spherical fibration  $\nu : X \rightarrow BG(k)$  determines a chain bundle  $(C(\tilde{X}), \gamma)$  over  $\mathbb{Z}[\pi_1(X)]$ , and there is defined a natural transformation of exact sequences from the certain exact sequence of Whitehead [20]

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Gamma_{n+k+1}(T(\nu)) & \longrightarrow & \pi_{n+k}(T(\nu)) & \xrightarrow{h} & \dot{H}_{n+k}(T(\nu)) & \longrightarrow \Gamma_{n+k}(T(\nu)) \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow \phi_X U & \\ \cdots \longrightarrow & \widehat{Q}^{n+1}(C(\tilde{X})) & \longrightarrow & Q_n(C(\tilde{X}), \gamma) & \longrightarrow & Q^n(C(\tilde{X})) & \xrightarrow{J_\gamma} \widehat{Q}^n(C(\tilde{X})) \longrightarrow \cdots \end{array}$$

with  $h$  the Hurewicz map and  $U : \dot{H}_{n+k}(T(\nu)) \rightarrow H_n(X)$  the Thom isomorphism. The topological Wu classes of  $\nu$  are the algebraic Wu classes of the induced chain bundle  $(C(X; \mathbb{Z}_2), 1 \otimes \gamma)$  over  $\mathbb{Z}_2$

$$v_*(\nu) = \widehat{v}_*(1 \otimes \gamma) \in H^*(X; \mathbb{Z}_2) = \text{Hom}_{\mathbb{Z}_2}(H_*(X; \mathbb{Z}_2), \mathbb{Z}_2).$$

## §7. Normal complexes

An  $n$ -dimensional normal space  $(X, \nu_X, \rho_X)$  (Quinn [14]) is a finite  $n$ -dimensional  $CW$  complex  $X$  together with a  $(k-1)$ -spherical fibration  $\nu_X : X \rightarrow BG(k)$  and a map  $\rho_X : S^{n+k} \rightarrow T(\nu_X)$  to the Thom space of  $\nu_X$ . An  $n$ -dimensional geometric Poincaré complex  $X$  has a unique equivalence class of normal structures  $(\nu_X, \rho_X)$ , with  $\nu_X$  the Spivak normal fibration and  $\rho_X$  representing the fundamental class  $[X] \in H_n(X)$ . A normal complex is the algebraic analogue of a normal space, consisting of a symmetric complex with normal chain bundle.

A *normal structure*  $(\gamma, \theta)$  on an  $n$ -dimensional symmetric complex  $(C, \phi)$  is a bundle  $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$  together with an equivalence of  $n$ -dimensional hyperquadratic structures on  $C$

$$\theta : J(\phi) \rightarrow (\widehat{\phi}_0)^{\%}(S^n \gamma),$$

as defined by a chain  $\theta \in (\widehat{W}^{\%} C)_{n+1}$  such that

$$J(\phi) - (\widehat{\phi}_0)^{\%}(S^n \gamma) = d\theta \in (\widehat{W}^{\%} C)_n.$$

The Wu classes of  $\phi$  and  $\gamma$  are then related by a commutative diagram

$$\begin{array}{ccc} H^{n-r}(C) & \xrightarrow{\phi_0} & H_r(C) \\ v_r(\phi) \downarrow & & \downarrow \widehat{v}_r(\gamma) \\ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) & \xrightarrow{J} & \widehat{H}^r(\mathbb{Z}_2; A) . \end{array}$$

An *equivalence* of  $n$ -dimensional normal structures on  $(C, \phi)$

$$(\chi, \eta) : (\gamma, \theta) \rightarrow (\gamma', \theta')$$

is an equivalence of bundles  $\chi : \gamma \rightarrow \gamma'$  together with an equivalence of  $(n+1)$ -dimensional hyperquadratic structures on  $C$

$$\eta : \theta - \theta' + (\widehat{\phi}_0)^\% (S^n \chi) \rightarrow 0 .$$

An  $n$ -dimensional symmetric Poincaré complex  $(C, \phi \in (W^\% C)_n)$  has a unique equivalence class of normal structures  $(\gamma, \theta)$ , with the equivalence class of bundles  $[\gamma] \in \widehat{Q}^0(C^{0-*})$  the image of the equivalence class of symmetric structures  $[\phi] \in Q^n(C)$  under the composite

$$Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{((\phi_0)^\%)^{-1}} \widehat{Q}^n(C^{n-*}) \xrightarrow{(S^n)^{-1}} \widehat{Q}^0(C^{0-*}) .$$

If  $(\gamma, \theta)$ ,  $(\gamma', \theta')$  are two such normal structures on  $(C, \phi)$  there exists an equivalence of bundles  $\chi : \gamma \rightarrow \gamma'$ . As  $\phi_0 : C^{n-*} \rightarrow C$  is a chain equivalence the cycle

$$\theta - \theta' + (\phi_0)^\% (S^n \chi) \in (\widehat{W}^\% C)_{n+1}$$

is such that there exist a cycle  $\lambda \in \widehat{W}^\% C^{n-*})_{n+1}$  and a chain  $\mu \in (\widehat{W}^\% C)_{n+2}$  such that

$$\theta - \theta' + (\phi_0)^\% (S^n \chi) = (\phi_0)^\% (\lambda) + d\mu \in (\widehat{W}^\% C)_{n+1} .$$

There is now defined an equivalence of normal structures on  $(C, \phi)$

$$(\chi - (S^n)^{-1}(\lambda), \mu) : (\gamma, \theta) \rightarrow (\gamma', \theta') .$$

An  $n$ -dimensional normal (Poincaré) complex over  $A$   $(C, \phi, \gamma, \theta)$  is an  $n$ -dimensional symmetric (Poincaré) complex  $(C, \phi)$  together with a normal structure  $(\gamma, \theta)$ . Symmetric Poincaré complexes are regarded as normal Poincaré complexes by choosing a normal structure in the unique equivalence class.

An  $n$ -dimensional normal complex  $(C, \phi, \gamma, \theta)$  is *connected* if the  $n$ -dimensional symmetric complex  $(C, \phi)$  is connected, that is

$$H_0(\phi_0 : C^{n-*} \rightarrow C) = 0 .$$

The correspondence described in §5 between the homotopy equivalence classes of connected  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \end{array} \right.$  complexes and those of  $n$ -dimensional  $\left\{ \begin{array}{l} \text{symmetric} \\ \text{quadratic} \end{array} \right.$  Poincaré pairs has the following generalization to connected normal complexes and (symmetric, quadratic) Poincaré pairs.

A connected  $n$ -dimensional normal complex  $(C, \phi, \gamma, \theta)$  determines the  $n$ -dimensional (symmetric, quadratic) Poincaré pair

$$(i_C : \partial C \rightarrow C^{n-*}, (\delta\phi, \psi))$$

defined by

$$\begin{aligned} i_C &= (0 \ 1) : \partial C_r = C_{r+1} \oplus C^{n-r} \rightarrow C^{n-r}, \\ d_{\partial C} &= \begin{pmatrix} d_C & (-1)^r \phi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} : \\ \partial C_r &= C_{r+1} \oplus C^{n-r} \rightarrow \partial C_{r-1} = C_r \oplus C^{n-r+1}, \\ \psi_0 &= \begin{pmatrix} \chi_0 & 0 \\ 1 + \gamma_{-n} \phi_0^* & \gamma_{-n-1}^* \end{pmatrix} : \\ \partial C^r &= C^{r+1} \oplus C_{n-r} \rightarrow \partial C_{n-r-1} = C_{n-r} \oplus C^{r+1}, \\ \psi_s &= \begin{pmatrix} \chi_{-s} & 0 \\ \gamma_{-n-s} \phi_0^* & \gamma_{-n-s-1}^* \end{pmatrix} : \\ \partial C^r &= C^{r+1} \oplus C_{n-r} \rightarrow \partial C_{n-r-s-1} = C_{n-r-s} \oplus C^{r+s+1} \ (s \geq 1), \\ \delta\phi_s &= \gamma_{-n-s} : C_r \rightarrow C^{n-r+s} \ (s \geq 0). \end{aligned}$$

The  $(n-1)$ -dimensional quadratic Poincaré complex

$$\partial(C, \phi, \gamma, \theta) = (\partial C, \psi)$$

is the *quadratic boundary* of the connected  $n$ -dimensional normal complex  $(C, \phi, \gamma, \theta)$ .

(Compare with the definition in §6 of the boundary  $(n-1)$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  Poincaré complex  $\begin{cases} \partial(C, \phi) \\ \partial(C, \psi) \end{cases}$  of a connected  $n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  complex  $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$ ). Conversely, given an  $n$ -dimensional (symmetric, quadratic) Poincaré pair  $(f : C \rightarrow D, (\delta\phi, \psi))$  there is defined a connected  $n$ -dimensional normal complex  $(C(f), \phi, \gamma, \theta)$  with the symmetric structure

$$\begin{aligned} \phi_s &= \begin{cases} \begin{pmatrix} \delta\phi_0 & 0 \\ (1+T)\psi_0 f^* & 0 \end{pmatrix} & \text{if } s = 0 \\ \begin{pmatrix} \delta\phi_1 & 0 \\ 0 & (1+T)\psi_0 \end{pmatrix} & \text{if } s = 1 \\ \begin{pmatrix} \delta\phi_s & 0 \\ 0 & 0 \end{pmatrix} & \text{if } s \geq 2 \end{cases} \\ &: C(f)^r = D^r \oplus C^{r-1} \rightarrow C(f)_{n-r+s} = D_{n-r+s} \oplus C_{n-r+s-1}. \end{aligned}$$

The normal structure  $(\gamma, \chi)$  is determined up to equivalence by the Poincaré duality, with  $\gamma \in \hat{Q}^0(D^{-*})$  the image of  $(\delta\phi/(1+T)\psi) \in Q^n(C(f))$  under the composite

$$Q^n(C(f)) \xrightarrow{((\delta\phi_0, (1+T)\psi_0)^{\%})^{-1}} Q^n(D^{n-*}) \xrightarrow{J} \hat{Q}^n(D^{n-*}) \xrightarrow{S^{-n}} \hat{Q}^0(D^{-*}).$$

The composite isomorphism

$$\widehat{Q}^0(C(f)^{0-*}) \xrightarrow{S^n} \widehat{Q}^n(C(f)^{n-*}) \xrightarrow{(\delta\phi_0, (1+T)\psi_0)^\%} \widehat{Q}^n(D)$$

sends the equivalence class  $[\gamma] \in \widehat{Q}^0(C(f)^{0-*})$  to the element  $\alpha \in \widehat{Q}^n(D)$  represented by

$$\alpha_s = \begin{cases} \delta\phi_s & \text{if } s \geq 0 \\ f\psi_{-s-1}f^* & \text{if } s \leq -1 \end{cases} : D^r \rightarrow D_{n-r+s}.$$

There is thus a natural one-one correspondence between the homotopy equivalence classes of connected  $n$ -dimensional normal complexes over  $A$  and the homotopy equivalence classes of  $n$ -dimensional (symmetric, quadratic) Poincaré pairs over  $A$ . In §8 below this correspondence will be used to identify the cobordism group  $\widehat{L}^n(A)$  of  $n$ -dimensional (symmetric, quadratic) Poincaré pairs over  $A$  with the cobordism group of  $n$ -dimensional normal complexes over  $A$ .

Let  $(B, \beta)$  be a chain bundle over  $A$ . A *normal*  $(B, \beta)$ -structure  $(\gamma, \theta, f, \chi)$  on an  $n$ -dimensional symmetric complex  $(C, \phi)$  over  $A$  is a normal structure  $(\gamma, \theta)$  on  $(C, \phi)$  together with a chain bundle map

$$(f, \chi) : (C, \gamma) \rightarrow (B, \beta).$$

There are also the corresponding relative notions of *normal*  $(B, \beta)$ -structure on symmetric and (symmetric, quadratic) pairs. For the universal chain bundle  $(B(\infty), \beta(\infty))$  over  $A$  a normal  $(B(\infty), \beta(\infty))$ -structure  $(\gamma, \theta, f, \chi)$  on a symmetric complex  $(C, \phi)$  is to all intents and purposes the same as a normal structure  $(\gamma, \theta)$ .

A normal  $(0,0)$ -structure  $(\gamma, \theta, 0, \chi)$  on an  $n$ -dimensional symmetric complex  $(C, \phi)$  determines an equivalence to 0 of the hyperquadratic structure  $J(\phi) \in (\widehat{W}^\%C)_n$

$$\xi = \theta + \phi_0^\%(S^n\chi) : J(\phi) \rightarrow 0.$$

Such an equivalence  $\xi : J(\phi) \rightarrow 0$  consists of a quadratic structure  $\psi \in (W_\%C)_n$  and an equivalence of symmetric structures

$$\eta : (1+T)\psi \rightarrow \phi,$$

with

$$\begin{aligned} \psi_s &= \xi_{-s-1} \in \text{Hom}_A(C^*, C)_{n-s} \quad (s \geq 0), \\ \eta_s &= \xi_s \in \text{Hom}_A(C^*, C)_{n+s+1} \quad (s \geq 0). \end{aligned}$$

Thus a normal  $(0,0)$ -structure on a symmetric complex  $(C, \phi)$  is to all intents and purposes an equivalence of the symmetric structure  $\phi$  to  $(1+T)\psi$  for some quadratic structure  $\psi$  on  $C$ .

An  $n$ -dimensional  $(B, \beta)$ -normal (Poincaré) complex  $(C, \phi, \gamma, \theta, f, \chi)$  is an  $n$ -dimensional symmetric (Poincaré) complex  $(C, \phi)$  together with a normal  $(B, \beta)$ -structure  $(\gamma, \theta, \phi, \chi)$ .

In §8 below the cobordism group  $\widehat{L}\langle B, \beta \rangle^n(A)$  of  $n$ -dimensional  $(B, \beta)$ -normal complexes over  $A$  will be identified with the twisted quadratic group  $Q_n(B, \beta)$  (introduced by Weiss [19]) of equivalence classes of pairs  $(\phi, \theta)$  such that  $(B, \phi, \beta, \theta, 1, 0)$  is an  $n$ -dimensional  $(B, \beta)$ -normal complex.

An  $n$ -dimensional symmetric structure  $(\phi, \theta)$  on a chain bundle  $(C, \gamma)$  is an  $n$ -dimensional symmetric structure  $\phi \in (W^{\%}C)_n$  together with an equivalence of  $n$ -dimensional hyperquadratic structures on  $C$

$$\theta : J(\phi) \rightarrow (\phi_0)^{\%}(S^n \gamma) ,$$

as defined by a chain  $\theta \in (\widehat{W}^{\%}C)_{n+1}$  such that

$$J(\phi) - (\phi_0)^{\%}(S^n \gamma) = d(\theta) \in (\widehat{W}^{\%}C)_n .$$

Thus  $(C, \phi)$  is an  $n$ -dimensional symmetric complex with normal structure  $(\gamma, \theta)$ .

An *equivalence* of  $n$ -dimensional symmetric structures on  $(C, \gamma)$

$$(\xi, \eta) : (\phi, \theta) \rightarrow (\phi', \theta')$$

is defined by an equivalence of symmetric structures  $\xi : \phi \rightarrow \phi'$  together with an equivalence of hyperquadratic structures on  $C$

$$\eta : \theta - \theta' + J(\xi) + (\xi_0; \phi_0, \phi'_0)^{\%}(S^n \gamma) \rightarrow 0 ,$$

as defined by chains  $\xi \in (W^{\%}C)_{n+1}$ ,  $\eta \in (\widehat{W}^{\%}C)_{n+2}$  such that

$$\phi' - \phi = d(\xi) \in (W^{\%}C)_n C^{-*}$$

$$\theta' - \theta + J(\xi) + (\xi_0; \phi_0, \phi'_0)^{\%}(S^n \gamma) = d(\eta) \in (\widehat{W}^{\%}C)_{n+1} .$$

The *twisted quadratic  $Q$ -group*  $Q_n(C, \gamma)$  is the abelian group of equivalence classes of  $n$ -dimensional symmetric structures on a chain bundle  $(C, \gamma)$ , with addition by

$$(\phi, \theta) + (\phi', \theta') = (\phi + \phi', \theta + \theta' + [\phi_0, \phi'_0](S^n \gamma)) \in Q_n(C, \gamma) .$$

The twisted quadratic  $Q$ -groups  $Q_*(C, \gamma)$  fit into an exact sequence of abelian groups

$$\dots \rightarrow \widehat{Q}^{n+1}(C) \xrightarrow{H_\gamma} Q_n(C, \gamma) \xrightarrow{N_\gamma} Q^n(C) \xrightarrow{J_\gamma} \widehat{Q}^n(C) \rightarrow \dots$$

with the morphisms

$$H_\gamma : \widehat{Q}^{n+1}(C) \rightarrow Q_n(C, \gamma) ; \theta \mapsto (0, \theta) ,$$

$$N_\gamma : Q_n(C, \gamma) \rightarrow Q^n(C) ; (\phi, \theta) \mapsto \phi ,$$

$$J_\gamma : Q^n(C) \rightarrow \widehat{Q}^n(C) ; \phi \mapsto J(\phi) - (\phi_0)^{\%}(S^n \gamma)$$

induced by simplicial maps. In the untwisted case  $\gamma = 0$  there is defined an isomorphism of exact sequences

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \widehat{Q}^{n+1}(C) & \xrightarrow{H} & Q_n(C) & \xrightarrow{1+T} & Q^n(C) & \xrightarrow{J} & \widehat{Q}^n(C) & \longrightarrow & \dots \\ & & \parallel & & \downarrow \cong & & \parallel & & \parallel & & \\ \dots & \longrightarrow & \widehat{Q}^{n+1}(C) & \xrightarrow{H_0} & Q_n(C, 0) & \xrightarrow{N_0} & Q^n(C) & \xrightarrow{J_0} & \widehat{Q}^n(C) & \longrightarrow & \dots \end{array}$$

with

$$Q_n(C) \rightarrow Q_n(C, 0) ; \psi \mapsto ((1+T)\psi, \theta) , \theta_s = \begin{cases} \psi_{-s-1} & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 0. \end{cases}$$



The twisted quadratic groups  $Q_*(C, \gamma)$  are covariant in  $(C, \gamma)$ . Given a map of chain bundles  $(f, \chi) : (C, \gamma) \rightarrow (C', \gamma')$  and an  $n$ -dimensional symmetric structure  $(\phi, \theta)$  on  $(C, \gamma)$  define an  $n$ -dimensional symmetric structure on  $(C', \gamma')$

$$(f, \chi)_{\%}(\phi, \theta) = (f^{\%}(\phi), \widehat{f}^{\%}(\theta) + (f\phi_0)^{\%}(S^n\chi)) .$$

The resulting morphisms of the twisted quadratic  $Q$ -groups

$$(f, \chi)_{\%} : Q_n(C, \gamma) \rightarrow Q_n(C', \gamma')$$

depend only on the homotopy class of  $(f, \chi)$ . There is defined a morphism of exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \widehat{Q}^{n+1}(C) & \xrightarrow{H_{\gamma}} & Q_n(C, \gamma) & \xrightarrow{N_{\gamma}} & Q^n(C) & \xrightarrow{J_{\gamma}} & \widehat{Q}^n(C) & \longrightarrow & \cdots \\ & & \widehat{f}^{\%} \downarrow & & (f, \chi)_{\%} \downarrow & & f^{\%} \downarrow & & \widehat{f}^{\%} \downarrow & & \\ \cdots & \longrightarrow & \widehat{Q}^{n+1}(C') & \xrightarrow{H_{\gamma'}} & Q_n(C', \gamma') & \xrightarrow{N_{\gamma'}} & Q^n(C') & \xrightarrow{J_{\gamma'}} & \widehat{Q}^n(C') & \longrightarrow & \cdots \end{array}$$

which is an isomorphism if  $(f, \chi)$  is an equivalence.

The *characteristic element* of an  $n$ -dimensional  $(B, \beta)$ -normal complex  $(C, \phi, \gamma, \theta, f, \chi)$  is defined by

$$(f, \chi)_{\%}(\phi, \theta) \in Q_n(B, \beta) .$$

In §8 the cobordism class of a  $(B, \beta)$ -normal complex will be identified with the characteristic element.

$$\begin{aligned} \text{A map of } n\text{-dimensional } & \left\{ \begin{array}{l} \text{normal} \\ (B, \beta)\text{-normal} \end{array} \right. \text{ complexes} \\ & \left\{ \begin{array}{l} (f, \xi, \chi, \eta) : (C, \phi, \gamma, \theta) \rightarrow (C', \phi', \gamma', \theta') \\ (f, \xi, \chi, \eta, h, \mu) : (C, \phi, \gamma, \theta, g, \lambda) \rightarrow (C', \phi', \gamma', \theta', g', \lambda') \end{array} \right. \end{aligned}$$

consists of

- (i) a chain map  $f : C \rightarrow C'$ ,
- (ii) an equivalence  $\xi : f^{\%}(\phi) \rightarrow \phi'$  of  $n$ -dimensional symmetric structures on  $C'$ ,
- (iii) an equivalence  $\chi : \gamma \rightarrow f^*\gamma'$  of bundles on  $C$ ,
- (iv) an equivalence of  $(n+1)$ -dimensional hyperquadratic structures on  $C'$

$$\eta : J(\xi) + \theta' - \widehat{f}^{\%}(\theta) + (\xi_0; f^{\%}\phi_0, \phi'_0)^{\%}(S^n\gamma') + (f\phi_0)^{\%}(S^n\chi) \rightarrow 0 ,$$

and in the  $(B, \beta)$ -normal case also

- (v) a homotopy of bundle maps

$$(h, \mu) : (g, \lambda) \simeq (g', \lambda')(f, \chi) : (C, \gamma) \rightarrow (B, \beta) .$$

Note that  $(C, \phi, \gamma, \theta, g, \lambda)$  and  $(C', \phi', \gamma', \theta', g', \lambda')$  have the same characteristic element

$$(g, \lambda)_{\%}(\phi, \theta) = (g', \lambda')_{\%}(\phi', \theta') \in Q_n(B, \beta) .$$

It is convenient for computational purposes to describe the behaviour of the twisted quadratic groups under direct sum. The direct sum of chain bundles  $(C, \gamma)$ ,  $(C', \gamma')$  is the chain bundle

$$(C, \gamma) \oplus (C', \gamma') = (C \oplus C', \gamma \oplus \gamma') .$$

Let

$$\begin{aligned} i &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C \rightarrow C \oplus C' , \quad i' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : C' \rightarrow C \oplus C' , \\ j &= (1 \ 0) : C \oplus C' \rightarrow C , \quad j' = (0 \ 1) : C \oplus C' \rightarrow C' . \end{aligned}$$

The twisted quadratic groups of the direct sum are such that there is defined a long exact sequence

$$\begin{aligned} \cdots \rightarrow Q_n(C, \gamma) \oplus Q_n(C', \gamma') &\xrightarrow{i_*} Q_n(C \oplus C', \gamma \oplus \gamma') \xrightarrow{j_*} H_n(C^t \otimes_A C') \\ &\xrightarrow{k_*} Q_{n-1}(C, \gamma) \oplus Q_{n-1}(C', \gamma') \xrightarrow{i_*} Q_{n-1}(C \oplus C', \gamma \oplus \gamma') \rightarrow \cdots \end{aligned}$$

with

$$\begin{aligned} i_* &= (i_{\%} \ i'_{\%}) : Q_n(C, \gamma) \oplus Q_n(C', \gamma') \rightarrow Q_n(C \oplus C', \gamma \oplus \gamma') , \\ j_* &: Q_n(C \oplus C', \gamma \oplus \gamma') \rightarrow H_n(C^t \otimes_A C') ; (\phi, \theta) \mapsto (j \otimes j')\phi_0 , \\ k_* &: H_n(C^t \otimes_A C') \rightarrow Q_{n-1}(C, \gamma) \oplus Q_{n-1}(C', \gamma') ; \\ &(f : C^{n-*} \rightarrow C') \mapsto ((0, \widehat{f}^{\%}(S^n \gamma')), (0, -\widehat{f}^{*\%}(S^n \gamma')) . \end{aligned}$$

For  $\gamma = 0$  and  $\gamma' = 0$  the long exact sequence collapses into split exact sequences of the untwisted quadratic  $Q$ -groups

$$0 \rightarrow Q_n(C) \oplus Q_n(C') \rightarrow Q_n(C \oplus C') \rightarrow H_n(C^t \otimes_A C') \rightarrow 0 .$$

## §8. Normal cobordism

Given a  $k$ -plane vector bundle  $\nu : X \rightarrow BO(k)$  over a space  $X$  let  $\Omega_n(X, \nu)$  ( $n \geq 0$ ) denote the bordism groups of bundle maps

$$(f, b) : (M^n, \nu_M) \rightarrow (X, \nu)$$

with  $M^n$  a smooth closed  $n$ -manifold and  $\nu_M : M \rightarrow BO(k)$  the normal bundle of an embedding  $M^n \subset S^{n+k}$  (Lashof [9]). The Thom space of  $\nu_M$  is given by

$$T(\nu_M) = E(\nu_M)/\partial E(\nu_M)$$

with  $E(\nu_M)$  the tubular neighbourhood of  $M^n$  in  $S^{n+k}$ , so that there is defined a collapse map

$$\rho_M : S^{n+k} \rightarrow S^{n+k}/(S^{n+k} \setminus E(\nu_M)) = E(\nu_M)/\partial E(\nu_M) = T(\nu_M) .$$

The Pontrjagin-Thom isomorphism

$$\begin{aligned} \Omega_n(X, \nu) &\rightarrow \pi_{n+k}(T(\nu)) ; \\ (f : M^n \rightarrow X, b : \nu_M \rightarrow \nu) &\mapsto (T(b)(\rho_M) : S^{n+k} \xrightarrow{\rho_M} T(\nu_M) \xrightarrow{T(b)} T(\nu)) \end{aligned}$$

has inverse

$$\begin{aligned} \pi_{n+k}(T(\nu)) &\rightarrow \Omega_n(X, \nu) ; \\ (\rho : S^{n+k} \rightarrow T(\nu)) &\mapsto (f = \rho| : M^n = \rho^{-1}(X) \rightarrow X, b : \nu_M \rightarrow \nu) , \end{aligned}$$

using smooth transversality to choose a representative  $\rho$  transverse regular at the zero section  $X \subset T(\nu)$ .

Given a  $(k-1)$ -spherical fibration  $\nu : X \rightarrow BG(k)$  over a space  $X$  let  $\Omega_n^N(X, \nu)$  (resp.  $\Omega_n^P(X, \nu)$ ) denote the bordism group of fibration maps

$$(f, b) : (M^n, \nu_M) \rightarrow (X, \nu)$$

with  $(M^n, \nu_M : M \rightarrow BG(k), \rho_M : S^{n+k} \rightarrow T(\nu_M))$  an  $n$ -dimensional normal space (resp. geometric Poincaré complex with Spivak normal structure). According to the theory of Quinn [14] there is a geometric theory of transversality for normal spaces, so that by analogy with the Pontrjagin-Thom isomorphism for smooth bordism there is defined an isomorphism

$$\begin{aligned} \Omega_n^N(X, \nu) &\rightarrow \pi_{n+k}(T(\nu)) ; \\ (f : M^n \rightarrow X, b : \nu_M \rightarrow \nu) &\mapsto (T(b)(\rho_M) : S^{n+k} \xrightarrow{\rho_M} T(\nu_M) \xrightarrow{T(b)} T(\nu)) , \end{aligned}$$

with inverse

$$\begin{aligned} \pi_{n+k}(T(\nu)) &\rightarrow \Omega_n^N(X, \nu) ; \\ (\rho : S^{n+k} \rightarrow T(\nu)) &\mapsto (f = \rho| : M^n = \rho^{-1}(X) \rightarrow X, b : \nu_M \rightarrow \nu) . \end{aligned}$$

The geometric Poincaré and normal bordism groups for  $n \geq 5$  are related by the Levitt-Jones-Quinn exact sequence

$$\cdots \rightarrow L_n(\mathbb{Z}[\pi_1(X)]) \rightarrow \Omega_n^P(X, \nu) \rightarrow \Omega_n^N(X, \nu) \rightarrow L_{n-1}(\mathbb{Z}[\pi_1(X)]) \rightarrow \cdots .$$

If  $\nu : X \rightarrow BG(k)$  admits a  $TOP$  reduction  $\tilde{\nu} : X \rightarrow BTOP(k)$  the forgetful maps from manifold to normal space bordism  $\Omega_n(X, \nu) \rightarrow \Omega_n^N(X, \nu)$  are isomorphisms, and

$$\Omega_n^P(X, \nu) = L_n(\mathbb{Z}[\pi_1(X)]) \oplus \Omega_n^N(X, \nu) .$$

A map of  $n$ -dimensional normal spaces

$$(f, b, c) : (M^n, \nu_M, \rho_M) \rightarrow (X^n, \nu_X, \rho_X)$$

is defined by a map of fibrations  $(f, b) : (M, \nu_M) \rightarrow (X, \nu_X)$  together with a homotopy

$$c : T(b)\rho_M \simeq \rho_X : S^{n+k} \rightarrow T(\nu_X) .$$

The mapping cylinder of  $f$

$$M(f) = M \times [0, 1] \cup X / \{(x, 1) = f(x) \mid x \in M\}$$

defines a cobordism  $(M(f); M, X)$  of normal spaces, identifying

$$M = M \times \{0\} \subset M(f) .$$

If  $M^n$  and  $X^n$  are Poincaré complexes the corresponding element of the relative bordism group is just the surgery obstruction

$$(M(f); M \cup -X) = \sigma_*(f, b) \in \Omega_{n+1}^{N,P}(X, \nu_X) = L_n(\mathbb{Z}[\pi_1(X)]) .$$

Ignoring questions of finite-dimensionality (or assuming that  $X$  is a finite  $n$ -dimensional  $CW$  complex) it is therefore possible to define the inverse isomorphism to  $\Omega_n^N(X, \nu) \rightarrow \pi_{n+k}(T(\nu))$  by

$$\pi_{n+k}(T(\nu)) \rightarrow \Omega_n^N(X, \nu) ; \rho \mapsto (X, \nu, \rho) ,$$

without an appeal to the transversality of normal spaces. The group  $\pi_{n+k}(T(\nu))$  consists of the equivalence classes of normal structures  $(\nu_X : X \rightarrow BG(k), \rho_X : S^{n+k} \rightarrow T(\nu_X))$  on  $X$  with  $\nu_X = \nu$ .

Following Weiss [19] we shall now identify the algebraic normal bordism groups  $\hat{L}\langle B, \beta \rangle^n(A)$  with the twisted quadratic groups  $Q_n(B, \beta)$ , the algebraic analogues of the homotopy groups of the Thom space  $\pi_{n+k}(T(\nu))$ .

A *cobordism* of  $n$ -dimensional normal complexes  $(C, \phi, \gamma, \theta)$ ,  $(C', \phi', \gamma', \theta')$  is defined by an  $(n+1)$ -dimensional symmetric pair

$$((f, f') : C \oplus C' \rightarrow D, (\delta\phi, \phi \oplus -\phi'))$$

together with bundle maps

$$(f, \zeta) : (C, \gamma) \rightarrow (D, \delta\gamma) , (f', \zeta') : (C', \gamma') \rightarrow (D, \delta\gamma)$$

and an equivalence of hyperquadratic structures on  $D$

$$\delta\theta : J(\delta\phi) - (\delta\phi_0; f\phi_0 f^*, f'\phi'_0 f'^*)^{\%}(S^n \delta\gamma) + f; (\phi'_0)^{\%}(S^n \zeta') \rightarrow 0 .$$

Similarly for the cobordism of  $(B, \beta)$ -normal complexes.

The *symmetric*  $(B, \beta)$ -*structure*  $L$ -*groups* of  $A$   $L\langle B, \beta \rangle^n(A)$  ( $n \geq 0$ ) of Weiss [19] are the cobordism groups of  $n$ -dimensional  $(B, \beta)$ -normal Poincaré complexes over  $A$   $(C, \phi, \gamma, \theta, f, \chi)$ . For the  $\begin{cases} \text{universal} \\ \text{zero} \end{cases}$  chain bundle  $\begin{cases} (B(\infty), \beta(\infty)) \\ (0, 0) \end{cases}$  over  $A$

these are just the  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$   $L$ -groups

$$\begin{cases} L\langle B(\infty), \beta(\infty) \rangle^n(A) = L^n(A) \\ L\langle 0, 0 \rangle^n(A) = L_n(A) . \end{cases}$$

The *symmetric*  $(B, \beta)$ -*structure*  $\hat{L}$ -*groups*  $\hat{L}\langle B, \beta \rangle^n(A)$  ( $n \geq 0$ ) are the cobordism groups of  $n$ -dimensional  $(B, \beta)$ -normal complexes over  $A$ . For the  $\begin{cases} \text{universal} \\ \text{zero} \end{cases}$

chain bundle  $\begin{cases} (B(\infty), \beta(\infty)) \\ (0, 0) \end{cases}$  over  $A$  these are just the

$$\begin{cases} \hat{L}\langle B(\infty), \beta(\infty) \rangle^n(A) = \hat{L}^n(A) \\ \hat{L}\langle 0, 0 \rangle^n(A) = 0 . \end{cases}$$

Algebraic surgery was used in Ranicki [15] to prove that every  $n$ -dimensional quadratic Poincaré complex  $(C, \psi)$  is cobordant to a highly-connected complex  $(C', \psi')$ , with

$$H_r(C') = 0 \quad (2r \leq n-2) .$$

The boundary of an  $n$ -dimensional normal complex  $(C, \phi, \gamma, \theta)$  is an  $(n-1)$ -dimensional quadratic Poincaré complex  $(\partial C, \psi)$ . Glueing on to  $(C, \phi, \gamma, \theta)$  the trace of

the surgery making  $(\partial C, \psi)$  highly-connected there is obtained an  $n$ -dimensional normal complex  $(C', \phi', \gamma', \theta')$  which is cobordant to  $(C, \phi, \gamma, \theta)$  and which has a highly-connected boundary, with

$$H_r(\partial C') = H_{r+1}(\phi'_0 : C'^{m-*} \rightarrow C') = 0 \quad (2r \leq n-3) .$$

In particular, this shows that every normal complex is cobordant to a connected complex. Thus  $\widehat{L}\langle B, \beta \rangle^n(A)$  is also the cobordism group of connected  $n$ -dimensional  $(B, \beta)$ -normal complexes over  $A$ . The one-one correspondence established in §7 between connected  $n$ -dimensional normal complexes and  $n$ -dimensional (symmetric, quadratic) Poincaré pairs generalizes to a one-one correspondence between connected  $n$ -dimensional  $(B, \beta)$ -normal complexes over  $A$  and  $n$ -dimensional (symmetric, quadratic)  $(B, \beta)$ -normal Poincaré pairs over  $A$ , for any chain bundle  $(B, \beta)$  over  $A$ . It follows that  $\widehat{L}\langle B, \beta \rangle^n(A)$  can be identified with the cobordism group of  $n$ -dimensional (symmetric, quadratic)  $(B, \beta)$ -normal Poincaré pairs, and that there is defined an exact sequence

$$\cdots \rightarrow L_n(A) \rightarrow L\langle B, \beta \rangle^n(A) \rightarrow \widehat{L}\langle B, \beta \rangle^n(A) \xrightarrow{\partial} L_{n-1}(A) \rightarrow \cdots ,$$

with  $\partial$  defined by the quadratic boundary

$$\partial : \widehat{L}\langle B, \beta \rangle^n(A) \rightarrow L_{n-1}(A) ; (C, \phi, \gamma, \theta, f, \chi) \mapsto \partial(C, \phi, \gamma, \theta) .$$

A map of  $n$ -dimensional normal complexes

$$(f, \xi, \chi, \eta) : (C, \phi, \gamma, \theta) \rightarrow (C', \phi', \gamma', \theta')$$

determines an  $(n+1)$ -dimensional symmetric pair  $((f \ 1) : C \oplus C' \rightarrow C', (\xi, \phi \oplus -\phi'))$ , bundle maps

$$(f, \chi) : (C, \gamma) \rightarrow (C', \gamma') , (1, 0) : (C', \gamma') \rightarrow (C', \gamma')$$

and an equivalence of hyperquadratic structures on  $C'$

$$\eta : J(\xi) - (\xi_0; f\phi_0 f^*, \phi'_0)^\% (S^n \gamma') + f^\% (\theta - (\phi_0)^\% (S^n \gamma)) - \theta' \rightarrow 0 ,$$

defining a cobordism between  $(C, \phi, \gamma, \theta)$  and  $(C', \phi', \gamma', \theta')$  by analogy with the mapping cylinder construction of geometric normal bordisms. Similarly for maps of  $(B, \beta)$ -normal complexes. It follows that the abelian group morphisms

$$\begin{aligned} \widehat{L}\langle B, \beta \rangle^n(A) &\rightarrow Q_n(B, \beta) ; (C, \phi, \gamma, \theta, f, \chi) \mapsto (f, \chi)^\% (\phi, \theta) , \\ Q_n(B, \beta) &\rightarrow \widehat{L}\langle B, \beta \rangle^n(A) ; (\phi, \theta) \mapsto (B, \phi, \beta, \theta, 1, 0) \end{aligned}$$

are inverse isomorphisms.

For example, if  $A = \mathbb{Z}$  and  $(B(\infty), \beta(\infty))$  is the universal chain bundle over  $\mathbb{Z}$  (as constructed at the end of §6) then

$$L\langle B(\infty), \beta(\infty) \rangle^n(\mathbb{Z}) = L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{4} \\ 0 & \text{if } n \equiv 2, 3 \pmod{4} \end{cases},$$

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 1, 3 \pmod{4} \end{cases},$$

$$\widehat{L}\langle B(\infty), \beta(\infty) \rangle^n(\mathbb{Z}) = Q_n(B(\infty), \beta(\infty)) = \begin{cases} \mathbb{Z}_8 & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 & \text{if } n \equiv 1, 3 \pmod{4} \\ 0 & \text{if } n \equiv 2 \pmod{4} \end{cases}.$$

A spherical fibration  $\nu : X \rightarrow BG(k)$  determines a chain bundle  $(C(\tilde{X}), \gamma)$  over  $\mathbb{Z}[\pi_1(X)]$  ([15], [19]) and there is defined a natural transformation of exact sequences from the Levitt-Jones-Quinn Poincaré bordism sequence

$$\begin{array}{ccccccc} \cdots \rightarrow L_n(\mathbb{Z}[\pi_1(X)]) & \longrightarrow & \Omega_n^P(X, \nu) & \longrightarrow & \pi_{n+k}(T(\nu)) & \longrightarrow & L_{n-1}(\mathbb{Z}[\pi_1(X)]) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow L_n(\mathbb{Z}[\pi_1(X)]) & \longrightarrow & L^n(C(\tilde{X}), \gamma) & \longrightarrow & Q_n(C(\tilde{X}), \gamma) & \longrightarrow & L_{n-1}(\mathbb{Z}[\pi_1(X)]) \rightarrow \cdots \end{array}$$

with  $\Omega_n^P(X, \nu) \rightarrow L^n(C(\tilde{X}), \gamma)$  a generalized symmetric signature map.

### §9. Normal Wu classes

The Wu classes of the symmetric structure  $\phi$  and the bundles  $\beta, \gamma$  in an  $n$ -dimensional  $(B, \beta)$ -normal complex  $(C, \phi, \gamma, \theta, f, \chi)$  are related by a commutative diagram

$$\begin{array}{ccccc} H^{n-r}(C) & \xrightarrow{\phi_0} & H_r(C) & \xrightarrow{f_*} & H_r(B) \\ v_r(\phi) \downarrow & & \widehat{v}_r(\gamma) \downarrow & \swarrow \widehat{v}_r(\beta) & \\ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) & \xrightarrow{J} & \widehat{H}^r(\mathbb{Z}_2; A) & & \end{array}$$

For any chain bundle  $(B, \beta)$  and any chain complex  $C$  we shall now define symmetric  $(B, \beta)$ -structure groups  $Q\langle B, \beta \rangle^n(C)$  ( $n \geq 0$ ) to fit into an exact sequence

$$\cdots \rightarrow Q\langle B, \beta \rangle^n(C) \rightarrow Q^n(C) \oplus H_n(B^t \otimes_A C) \rightarrow \widehat{Q}^n(C) \rightarrow Q\langle B, \beta \rangle^{n-1}(C) \rightarrow \cdots.$$

The Wu classes  $v_r(\phi)$  of a symmetric complex  $(C, \phi)$  will then be refined to the normal Wu classes of a  $(B, \beta)$ -normal complex  $(C, \phi, \gamma, \theta, f, \chi)$

$$v_r = v_r(\phi, \gamma, \theta, f, \chi) : H^{n-r}(C) \rightarrow Q\langle B, \beta \rangle^n(S^{n-r}A),$$

with

$$v_r(\phi) : H^{n-r}(C) \xrightarrow{v_r} Q\langle B, \beta \rangle^n(S^{n-r}A) \rightarrow Q^n(S^{n-r}A) = H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}).$$

In §11 below the normal Wu classes will be used to define a  $\mathbb{Z}_4$ -valued quadratic function on  $H^n(C)$  for a  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over  $\mathbb{Z}_2$  with normal  $(v_{n+1} = 0)$ -structure, as required to define the  $\mathbb{Z}_8$ -valued invariant of Brown [4].

Let  $(B, \beta)$  be a chain bundle over  $A$ , and let  $C$  be a finite-dimensional  $A$ -module chain complex. An  $n$ -dimensional symmetric  $(B, \beta)$ -structure on  $C$   $(\phi, \theta, f)$  is defined by an  $n$ -dimensional symmetric structure  $\phi \in (W\%C)_n$  together with a chain  $\theta \in (\widehat{W}\%C)_{n+1}$  and a chain map  $f : B^{n-*} \rightarrow C$  such that

$$J(\phi) - \widehat{f}\%(S^n\beta) = d\theta \in (\widehat{W}\%C)_n .$$

An  $n$ -dimensional  $(B, \beta)$ -normal structure  $(\phi, \gamma, \theta, g, \chi)$  on  $C$  determines the  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $(\phi, \theta + (\phi_0)\%(S^n\chi), \phi_0g^*)$  on  $C$ . Conversely, if  $f^* : C^{n-*} \rightarrow B$  is a composite

$$f^* : C^{n-*} \xrightarrow{\phi_0} C \xrightarrow{g} B$$

(as is always the case up to chain homotopy if  $(C, \phi)$  is a Poincaré complex) the symmetric  $(B, \beta)$ -structure  $(\phi, \theta, f)$  determines the  $n$ -dimensional  $(B, \beta)$ -normal structure  $(\phi, g^*\gamma, \theta, g, 0)$ .

An  $n$ -dimensional symmetric  $(B, \beta)$ -structure (Poincaré) complex over  $A$   $(C, \phi, \theta, f)$  is an  $n$ -dimensional  $A$ -module chain complex  $C$  together with an  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $(\phi, \theta, f)$  (such that  $\phi_0 : C^{n-*} \rightarrow C$  is a chain equivalence). As for symmetric (Poincaré) pairs there is also the analogous notion of symmetric  $(B, \beta)$ -structure (Poincaré) pair. There is essentially no difference between symmetric  $(B, \beta)$ -structure Poincaré complexes and  $(B, \beta)$ -normal Poincaré complexes, so that the  $L$ -groups  $L\langle B, \beta \rangle^n(A)$  ( $n \geq 0$ ) can also be regarded as the cobordism groups of  $n$ -dimensional symmetric  $(B, \beta)$ -structure Poincaré complexes over  $A$ .

An equivalence of  $n$ -dimensional symmetric  $(B, \beta)$ -structures on  $C$

$$(\xi, \eta, g) : (\phi, \theta, f) \rightarrow (\phi', \theta', f')$$

is defined by an equivalence of symmetric structures  $\xi : \phi \rightarrow \phi'$  together with a chain  $\eta \in (\widehat{W}\%C)_{n+2}$  and a chain homotopy  $g : f \simeq f' : B^{n-*} \rightarrow C$  such that

$$J(\xi) - (g; f, f')\%(S^n\beta) - \theta' + \theta = d\eta \in (\widehat{W}\%C)_{n+1} .$$

The  $n$ -dimensional symmetric  $(B, \beta)$ -structure group of  $C$   $Q\langle B, \beta \rangle^n(C)$  is the abelian group of equivalence classes of  $n$ -dimensional symmetric  $(B, \beta)$ -symmetric structures on  $C$ , with addition by

$$(\phi, \theta, f) + (\phi', \theta', f') = (\phi + \phi', \theta + \theta' + [f, f'](S^n\beta), f + f') \in Q\langle B, \beta \rangle^n(C) .$$

There is also a more economical description of  $Q\langle B, \beta \rangle^n(C)$  as the abelian group of equivalence classes of pairs  $(\psi, f)$  defined by an  $n$ -dimensional quadratic structure  $\psi \in (W\%C)_n$  and a chain map  $f : B^{n-*} \rightarrow C$  such that

$$f\%H(S^n\beta) = d\psi \in (W\%C)_{n-1} ,$$

so that up to signs

$$f\beta_{-n-s-1}f^* = d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* \in \text{Hom}_A(C^{-*}, C)_{n-s} \ (s \geq 0) ,$$

subject to the equivalence relation

$$\begin{aligned} (\psi, f) \sim (\psi', f') \text{ if there exist a chain homotopy } g : f \simeq f' : B^{n-*} \rightarrow C \\ \text{and an equivalence of quadratic structures} \\ \chi : \psi' - \psi \rightarrow (g; f, f')_{\%} H(S^n \beta) , \end{aligned}$$

with addition by

$$(\psi, f) + (\psi', f') = (\psi + \psi' + H([f, f'](S^n \beta)), f + f') .$$

The pair  $(\psi, f)$  determines the triple  $(\phi, \theta, f)$  with

$$\begin{aligned} \phi_s &= \begin{cases} f\beta_{s-n}f^* & \text{if } s \geq 1 \\ f\beta_{-n}f^* + (1+T)\psi_0 & \text{if } s = 0 , \end{cases} \\ \theta_s &= \begin{cases} 0 & \text{if } s \geq 0 \\ \psi_{-s-1} & \text{if } s \leq -1 . \end{cases} \end{aligned}$$

Conversely, a triple  $(\phi, \theta, f)$  determines the pair  $(\psi, f)$  with

$$\psi_s = \theta_{-s-1} \ (s \geq 0) .$$

Given an  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $(\phi, \theta, f)$  on  $C$ , a chain bundle map  $(g, \chi) : (B, \beta) \rightarrow (B', \beta')$  and a chain map  $h : C \rightarrow C'$  define the *pushforward*  $n$ -dimensional symmetric  $(B', \beta')$ -structure on  $C'$

$$\langle g, \chi \rangle (h)_{\%} (\phi, \theta, f) = (h_{\%}(\phi), h_{\%}(\theta + S^n(\widehat{f}_{\%}\chi)), hfg^*) .$$

Thus the groups  $Q\langle B, \beta \rangle^*(C)$  are covariant in both  $(B, \beta)$  and  $C$ , with pushforward abelian group morphisms

$$\begin{aligned} \langle g, \chi \rangle (h)_{\%} : Q\langle B, \beta \rangle^n(C) &\rightarrow Q\langle B', \beta' \rangle^n(C') ; \\ (\phi, \theta, f) &\mapsto (h_{\%}(\phi), \widehat{h}_{\%}(\theta + \widehat{f}_{\%}(S^n \chi)), hfg^*) \end{aligned}$$

depending only on the homotopy classes of  $(g, \chi)$  and  $h$ .

An  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $(\phi, \theta, f)$  on  $C$  determines an  $n$ -dimensional symmetric structure  $\phi \in (W_{\%}C)_n$  on  $C$ , so that there is defined a forgetful map

$$s : Q\langle B, \beta \rangle^n(C) \rightarrow Q^n(C) ; (\phi, \theta, f) \mapsto \phi .$$

An  $n$ -dimensional quadratic structure  $\psi \in (W_{\%}C)_n$  on  $C$  determines an  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $((1+T)\psi, \theta, 0)$  on  $C$  for any  $(B, \beta)$ , with

$$\theta_s = \begin{cases} \psi_{-s-1} & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 0 . \end{cases}$$

Thus there are also defined forgetful maps

$$s : Q_n(C) \rightarrow Q\langle B, \beta \rangle^n(C) ; \psi \mapsto ((1+T)\psi, \theta, 0) ,$$

and  $sr = 1 + T : Q_n(C) \rightarrow Q^n(C)$ .

Let  $P\langle B, \beta \rangle^n(C)$  be the abelian group of equivalence classes of  $n$ -dimensional symmetric  $(B, \beta)$ -structures  $(\phi, \theta, f)$  with  $\phi = 0$ , to be denoted  $(\theta, f)$ , subject to the equivalence relation

$$(\theta, f) \sim (\theta', f') \text{ if there exists an equivalence of } (B, \beta)\text{-structures}$$



$$(0, \eta, g) : (0, \theta, f) \rightarrow (0, \theta', f').$$

The symmetric  $(B, \beta)$ -structure groups  $Q\langle B, \beta \rangle^*(C)$  and the groups  $P\langle B, \beta \rangle^*(C)$  are related by a commutative braid of exact sequences of abelian groups

$$\begin{array}{ccccc}
 & & 1 + T & & \\
 & \curvearrowright & & \curvearrowright & \\
 Q_n(C) & & Q^n(C) & & P\langle B, \beta \rangle^{n-1}(C) \\
 & \searrow & \nearrow J & \searrow & \\
 & Q\langle B, \beta \rangle^n(C) & & \widehat{Q}^n(C) & \\
 & \nearrow & \searrow & \nearrow H & \\
 P\langle B, \beta \rangle^n(C) & & H_n(B^t \otimes_A C) & & Q_{n-1}(C) \\
 & \curvearrowleft & & \curvearrowleft & 
 \end{array}$$

If  $(B, \beta)$  is the  $\begin{cases} \text{universal} \\ \text{zero} \end{cases}$  chain bundle  $\begin{cases} (B(\infty), \beta(\infty)) \\ (0, 0) \end{cases}$  the forgetful map

$$\begin{cases} Q\langle B(\infty), \beta(\infty) \rangle^n(C) \rightarrow Q^n(C) ; (\phi, \theta, f) \mapsto \phi \\ Q_n(C) \rightarrow Q\langle 0, 0 \rangle^n(C) ; \psi \mapsto ((1 + T)\psi, \theta) \end{cases}$$

is an isomorphism and

$$\begin{cases} P\langle B(\infty), \beta(\infty) \rangle^n(C) = 0 \\ P\langle 0, 0 \rangle^n(C) = \widehat{Q}^{n+1}(C) . \end{cases}$$

The  $Wu$  classes of an  $n$ -dimensional symmetric  $(B, \beta)$ -structure  $(\phi, \theta, f)$  on  $C$  are the  $A$ -morphisms

$$\begin{aligned} v_r(\phi, \theta, f) : H^{n-r}(C) &\rightarrow Q\langle B, \beta \rangle^n(S^{n-r}A) ; \\ (x : C \rightarrow S^{n-r}A) &\mapsto \langle 1, 0 \rangle(x)^\%(\phi, \theta, f) . \end{aligned}$$

Now

$$Q\langle B, \beta \rangle^n(S^{n-r}A) = \begin{cases} H_r(B) & \text{if } 2r < n \\ \{(a, b) \in A \oplus B_r \mid db = 0 \in B_{r-1}\} / \sim & \text{if } 2r = n \\ \{(a, b) \in A \oplus B_r \mid a + (-1)^r \bar{a} + \beta_{-2r}(b)(b) = 0 \in A, db = 0 \in B_{r-1}\} / \sim & \text{if } 2r > n \end{cases}$$

with the equivalence relation  $\sim$  defined by

$$\begin{aligned} (a, b) \sim (a', b') &\text{ if there exists } (x, y) \in A \oplus B_{r+1} \text{ such that} \\ &a' - a = x + (-1)^{r+1} \bar{x} + \beta_{-2r-2}(y)(y) \\ &\quad + \beta_{-2r-1}(y)(b) + \beta_{-2r-1}(b')(y) \in A, \\ &b' - b = dy \in B_r \end{aligned}$$

and addition by

$$(a, b) + (a', b') = (a + a' + \beta_{-2r}(b)(b'), b + b') .$$

The map to the symmetric  $Q$ -group is given by

$$Q\langle B, \beta \rangle^n(S^{n-r}A) \rightarrow Q^n(S^{n-r}A) = H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) ;$$

$$\begin{cases} b \mapsto \beta_{-2r}(b)(b) & \text{if } 2r < n \\ (a, b) \mapsto a + (-1)^r \bar{a} + \beta_{-2r}(b)(b) & \text{if } 2r = n \\ 0 & \text{if } 2r > n . \end{cases}$$

The Wu classes are given by

$$v_r(\phi, \theta, f) : H^{n-r}(C) \rightarrow Q\langle B, \beta \rangle^n(S^{n-r}A) ;$$

$$z \mapsto \begin{cases} f^*(z) & \text{if } 2r < n \\ (\theta_{n-2r-1}(z)(z), f^*(z)) & \text{if } 2r \geq n \end{cases} \quad (z \in C^{n-r}, d^*z = 0) .$$

## §10. Forms

In Ranicki [15] the even-dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$   $L$ -groups  $\begin{cases} L^{2n}(A) \\ L_{2n}(A) \end{cases} (n \geq 0)$  were related to the Witt groups  $\begin{cases} W^{(-1)^n}(A) \\ W_{(-1)^n}(A) \end{cases}$  of nonsingular  $\begin{cases} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{cases}$  forms over  $A$ . In particular, it was shown that

$$\begin{cases} L^0(A) = W^{+1}(A) \\ L_{2n}(A) = W_{(-1)^n}(A) \end{cases} \quad (n \geq 0) .$$

This relationship between  $L$ -groups and Witt groups will now be generalized to the even-dimensional symmetric  $(B, \beta)$ -structure  $L$ -groups  $L\langle B, \beta \rangle^{2n}(A)$  and the Witt groups  $W_{Q(n)}(A)$  of nonsingular  $Q(n)$ -quadratic forms over  $A$ , with  $(B, \beta)$  any chain bundle over  $A$  and

$$Q(n) = Q\langle B, \beta \rangle^{2n}(S^n A) .$$

Let  $\epsilon = \pm 1$ . An  $\epsilon$ -symmetric form over  $A$   $(M, \lambda)$  is a f.g. projective  $A$ -module  $M$  together with an element  $\lambda \in \text{Hom}_A(M, M^*)$  such that

$$\epsilon \lambda^* = \lambda : M \rightarrow M^* .$$

Equivalently, the form is defined by a pairing

$$\lambda : M \times M \rightarrow A ; (x, y) \rightarrow \lambda(x, y) = \lambda(x)(y)$$

such that

$$\begin{aligned} \lambda(ax, by) &= b\lambda(x, y)\bar{a} , \\ \lambda(x + x', y) &= \lambda(x, y) + \lambda(x', y) , \\ \epsilon \overline{\lambda(y, x)} &= \lambda(x, y) \quad (x, y \in M, a, b \in A) . \end{aligned}$$

Let  $Q(\epsilon)$  be an  $A$ -group together with  $A$ -morphisms

$$\begin{aligned} r : Q(\epsilon) &\rightarrow H^0(\mathbb{Z}_2; A, \epsilon) = \{a \in A \mid \epsilon \bar{a} = a\} , \\ s : H_0(\mathbb{Z}_2; A, \epsilon) &= A / \{b - \epsilon \bar{b} \mid b \in A\} \rightarrow Q(\epsilon) \end{aligned}$$

such that

$$rs = 1 + T_\epsilon : H_0(\mathbb{Z}_2; A, \epsilon) \rightarrow H^0(\mathbb{Z}_2; A, \epsilon) .$$

A  $Q(\epsilon)$ -quadratic form over  $A$   $(M, \lambda, \mu)$  is an  $\epsilon$ -symmetric form  $(M, \lambda)$  together with an  $A$ -morphism  $\mu : M \rightarrow Q(\epsilon)$  such that

$$\begin{aligned} r(\mu(x)) &= \lambda(x, x) \in H^0(\mathbb{Z}_2; A, \epsilon) , \\ \mu(x + y) - \mu(x) - \mu(y) &= s(\lambda(x, y)) \in Q(\epsilon) \ (x, y \in M) . \end{aligned}$$

There is an evident notion of *isomorphism* of  $Q(\epsilon)$ -quadratic forms.

A  $Q(\epsilon)$ -quadratic form  $(M, \lambda, \mu)$  is *nonsingular* if  $\lambda \in \text{Hom}_A(M, M^*)$  is an isomorphism of  $A$ -modules.

A nonsingular  $Q(\epsilon)$ -quadratic form  $(M, \lambda, \mu)$  is *hyperbolic* if there exists a direct summand  $L \subset M$  such that

(i) the inclusion  $j \in \text{Hom}_A(L, M)$  fits into an exact sequence

$$0 \rightarrow L \xrightarrow{j} M \xrightarrow{j^* \lambda} L^* \rightarrow 0 ,$$

(ii)  $\mu j = 0 : L \rightarrow Q(\epsilon)$  .

The  $Q(\epsilon)$ -quadratic Witt group of  $A$   $W_{Q(\epsilon)}(A)$  is the abelian group of equivalence classes of nonsingular  $Q(\epsilon)$ -quadratic forms  $(M, \lambda, \mu)$ , subject to the equivalence relation

$$\begin{aligned} (M, \lambda, \mu) \sim (M', \lambda', \mu') \text{ if there exists an isomorphism} \\ (M, \lambda, \mu) \oplus (N, \nu, \rho) \rightarrow (M', \lambda', \mu') \oplus (N', \nu', \rho') \\ \text{for some hyperbolic } Q(\epsilon)\text{-quadratic forms} \\ (N, \nu, \rho), (N', \nu', \rho') . \end{aligned}$$

For  $Q(\epsilon) = H^0(\mathbb{Z}_2; A, \epsilon)$ ,  $r = 1$ ,  $s = 1 + T_\epsilon$  a  $Q(\epsilon)$ -quadratic form  $(M, \lambda, \mu)$  may be identified with the  $\epsilon$ -symmetric form  $(M, \lambda)$ , since  $\lambda$  determines  $\mu$  by

$$\mu(x) = \lambda(x, x) \in H^0(\mathbb{Z}_2; A, \epsilon) \ (x \in M) .$$

The Witt group of  $\epsilon$ -symmetric forms  $W_{Q(\epsilon)}(A)$  is denoted by  $W^\epsilon(A)$ .

For  $Q(\epsilon) = H_0(\mathbb{Z}_2; A, \epsilon)$ ,  $r = 1 + T_\epsilon$ ,  $s = 1$  a  $Q(\epsilon)$ -quadratic form  $(M, \lambda, \mu)$  is just a  $\epsilon$ -quadratic form in the sense of Wall [18]. The Witt group of  $\epsilon$ -quadratic forms  $W_{Q(\epsilon)}(A)$  is denoted by  $W_\epsilon(A)$ .

For  $Q(\epsilon) = \text{im}(1 + T_\epsilon : H_0(\mathbb{Z}_2; A, \epsilon) \rightarrow H^0(\mathbb{Z}_2; A, \epsilon))$ ,  $r = \text{projection}$ ,  $s = \text{injection}$  a  $Q(\epsilon)$ -quadratic form  $(M, \lambda, \mu)$  is just an  $\epsilon$ -symmetric form  $(M, \lambda)$  for which there exists an  $\epsilon$ -quadratic form  $(M, \lambda, \mu : M \rightarrow H_0(\mathbb{Z}_2; A, \epsilon))$ . Such an  $\epsilon$ -symmetric form is *even*. The Witt group of even  $\epsilon$ -symmetric forms  $W_{Q(\epsilon)}(A)$  is denoted by  $W\langle v_0 \rangle^\epsilon(A)$ .

For  $\epsilon = +1$   $\begin{cases} \epsilon\text{-symmetric} \\ \epsilon\text{-quadratic} \end{cases}$  is abbreviated to  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  .

A  $2n$ -dimensional  $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$  (Poincaré) complex over  $A$   $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$  with  $H^n(C)$  a f.g. projective  $A$ -module determines a (nonsingular)  $\begin{cases} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{cases}$

$$\text{form over } A \begin{cases} (H^n(C), \phi_0, v_n(\phi)) \\ (H^n(C), (1+T)\psi_0, v^n(\psi)) \end{cases} \text{ with } \begin{cases} v_n(\phi) : H^n(C) \rightarrow H^0(\mathbb{Z}_2; A, (-1)^n) ; x \mapsto \phi_0(x)(x) \\ v^n(\psi) : H^n(C) \rightarrow H_0(\mathbb{Z}_2; A, (-1)^n) ; x \mapsto \psi_0(x)(x) . \end{cases}$$

$$\text{Conversely, a (nonsingular) } \begin{cases} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{cases} \text{ form } \begin{cases} (M, \lambda) \\ (M, \lambda, \mu) \end{cases} \text{ determines a } 2n\text{-dimensional } \begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases} \text{ (Poincaré) complex } \begin{cases} (C, \phi) \\ (C, \psi) \end{cases} \text{ such that}$$

$$\begin{cases} \phi_0 \\ (1+T)\psi_0 \end{cases} = \lambda : C^n = M \rightarrow C_n = M^* , C_r = 0 \ (r \neq n) , \\ v^n(\psi) = \mu : H^n(C) = M \rightarrow H_0(\mathbb{Z}_2; A, (-1)^n) .$$

The corresponding morphisms from the Witt groups to the  $L$ -groups

$$\begin{cases} W^{(-1)^n}(A) \rightarrow L^{2n}(A) ; (M, \lambda) \mapsto (C, \phi) \\ W_{(-1)^n}(A) \rightarrow L_{2n}(A) ; (M, \lambda, \mu) \mapsto (C, \psi) \end{cases}$$

were shown in Ranicki [15] to be isomorphisms for  $n = 0$  if  $A$  is any ring, and for all  $n \geq 0$  if  $A$  is  $\begin{cases} \text{a Dedekind} \\ \text{any} \end{cases}$  ring. For a Dedekind ring  $A$  the inverse isomorphism in symmetric  $L$ -theory is given by

$$L^{2n}(A) \rightarrow W^{(-1)^n}(A) ; (C, \phi) \mapsto (H^n(C)/(\text{torsion}), \phi_0) .$$

The inverse isomorphism in quadratic  $L$ -theory is given for any  $A$  by

$$L_{2n}(A) \rightarrow W_{(-1)^n}(A) ; \\ (C, \psi) \mapsto (\text{coker}\left(\begin{pmatrix} d^* & 0 \\ (1+T)\psi_0 & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \rightarrow C^n \oplus C_{n+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}) .$$

If  $A$  is a field this isomorphism can also be expressed as

$$(C, \psi) \mapsto (H^n(C), (1+T)\psi_0, v^n(\psi))$$

but this is not the case in general – see Milgram and Ranicki [12, p.406].

Given  $A$ -groups  $M, N$  and a symmetric bilinear pairing

$$\phi : N \times N \rightarrow M$$

such that

$$\phi(ay, ay') = a\phi(y, y') \in M \ (a \in A, y, y' \in N)$$

let  $M \times_\phi N$  be the  $A$ -group of pairs  $(x \in M, y \in N)$ , with addition by

$$(x, y) + (x', y') = (x + x' + \phi(y, y'), y + y') \in M \times_\phi N$$

and  $A$  acting by

$$A \times (M \times_\phi N) \rightarrow M \times_\phi N ; (a, (x, y)) \mapsto (ax, ay) .$$

There is then defined a short exact sequence of  $A$ -groups and  $A$ -morphisms

$$0 \rightarrow M \rightarrow M \times_\phi N \rightarrow N \rightarrow 0$$

with

$$M \rightarrow M \times_{\phi} N ; x \mapsto (x, 0)$$

$$M \times_{\phi} N \rightarrow N ; (x, y) \mapsto y .$$

Given a chain bundle  $(B, \beta)$  over  $A$  define the  $A$ -group

$$\begin{aligned} Q(n) &= Q(B, \beta)^{2n}(S^n A) \\ &= \{(a, b) \in A \oplus B_n \mid db = 0 \in B_{n-1}\} / \sim , \end{aligned}$$

where

$$\begin{aligned} (a, b) \sim (a', b') \text{ if there exist } (x, y) \in A \oplus B_{n+1} \text{ such that} \\ a' - a &= x + (-1)^{n+1} \overline{x} + \beta_{-2n-2}(y)(y) \\ &\quad + \beta_{-2n-1}(y)(b) + \beta_{-2n-1}(b')(y) , \\ b' - b &= dy , \end{aligned}$$

with addition by

$$(a, b) + (a', b') = (a + a' + \beta_{-2n}(b)(b'), b + b')$$

and  $A$ -action by

$$A \times Q(n) \rightarrow Q(n) ; (x, (a, b)) \mapsto x(a, b) = (xa\overline{x}, xb) .$$

The  $A$ -morphisms

$$\begin{aligned} r : Q_{2n}(S^n A) &= H_0(\mathbb{Z}_2; A, (-1)^n) \rightarrow Q(n) ; a \mapsto (a, 0) , \\ s : Q(n) &\rightarrow Q^{2n}(S^n A) = H^0(\mathbb{Z}_2; A, (-1)^n) ; \\ (a, b) &\mapsto a + (-1)^n \overline{a} + \beta_{-2n}(b)(b) \end{aligned}$$

are such that there is defined a commutative braid of exact sequences

$$\begin{array}{ccccccc} & & & 1+T & & & \\ & \curvearrowright & & \curvearrowright & & \curvearrowright & \\ H_{n+1}(B) & & Q_{2n}(S^n A) & & Q^{2n}(S^n A) & & 0 \\ & \searrow & \downarrow r & \nearrow s & \searrow J & \nearrow & \\ & \widehat{Q}^{2n+1}(S^n A) & & Q(n) & & \widehat{Q}^{2n}(S^n A) & \\ & \nearrow & \nwarrow & \nwarrow & \nearrow & \nwarrow & \\ 0 & & P(B, \beta)^{2n}(S^n A) & & H_n(B) & & 0 \\ & \curvearrowleft & \curvearrowleft & \curvearrowleft & \curvearrowleft & \curvearrowleft & \end{array}$$

with

$$Q(n) \rightarrow H_n(B) ; (a, b) \mapsto b .$$

If  $(B, \beta)$  is such that for all  $y \in B_{n+1}$  there exists  $x \in A$  such that

$$(\beta_{-2n-2} + \beta_{-2n-1}d)(y)(y) = x + (-1)^n \overline{x} \in A$$

(e.g. if  $d = 0 : B_{n+1} \rightarrow B_n$  and  $v_{n+1}(\beta) = 0 : H_{n+1}(B) \rightarrow \widehat{H}^{n+1}(\mathbb{Z}_2; A)$ ) then there is a natural identification of  $A$ -groups

$$Q(n) = Q_{2n}(S^n A) \times_{\beta_{-2n}} H_n(B)$$

with

$$\beta_{-2n} : H_n(B) \times H_n(B) \rightarrow Q_{2n}(S^n A) ; (b, b') \mapsto \beta_{-2n}(b)(b') .$$

For any chain bundle  $(B, \beta)$  and any  $Q(n)$ -quadratic form  $(M, \lambda, \mu)$  there exist  $A$ -module morphisms

$$g : M \rightarrow B_n , \psi : M \rightarrow M^*$$

such that

$$\begin{aligned} dg &= 0 : M \rightarrow B_{n-1} , \\ \lambda - g^* \beta_{-2n} g &= \psi + (-1)^n \psi^* : M \rightarrow M^* , \\ \mu : M &\rightarrow Q(n) ; x \mapsto (\psi(x)(x), g(x)) . \end{aligned}$$

If  $(M, \lambda, \mu)$  is a nonsingular form there is thus defined a  $2n$ -dimensional symmetric  $(B, \beta)$ -structure Poincaré complex  $(C, \phi, \theta, f)$  with

$$\begin{aligned} \phi_0 &= \lambda : C^n = M \rightarrow C_n = M^* , \quad C_r = 0 \ (r \neq n) , \\ \theta_{-1} &= \psi : C^n = M \rightarrow C_n = M^* , \\ f &= g\lambda^{-1} : C_n = M^* \xrightarrow{\lambda^{-1}} M \xrightarrow{g} B_n , \\ v_n(\phi, \theta, f) &= \mu : H^n(C) = M \rightarrow Q(n) . \end{aligned}$$

The construction defines a morphism of abelian groups

$$W_{Q(n)}(A) \rightarrow L\langle B, \beta \rangle^{2n}(A) ; (M, \lambda, \mu) \mapsto (C, \phi, \theta, f) .$$

Conversely, if  $(C, \phi, \theta, f)$  is a  $2n$ -dimensional symmetric  $(B, \beta)$ -structure Poincaré complex such that  $H^n(C)$  is a f.g. projective  $A$ -module there is defined a nonsingular  $Q(n)$ -quadratic form  $(H^n(C), \phi_0, v_n(\phi, \theta, f))$ , with

$$v_n(\phi, \theta, f) : H^n(C) \rightarrow Q(n) ; x \mapsto (\theta_{-1}(x)(x), f(x)) .$$

It follows that for any ring  $A$  there is a natural identification of the 0-dimensional  $L$ -group with the Witt group

$$L\langle B, \beta \rangle^0(A) = W_{Q(0)}(A) .$$

For a field  $A$  the morphisms

$$W_{Q(n)}(A) \rightarrow L\langle B, \beta \rangle^{2n}(A) \ (n \geq 0)$$

are injections, which are split by

$$L\langle B, \beta \rangle^{2n}(A) \rightarrow W_{Q(n)}(A) ; (C, \phi, \theta, f) \mapsto (H^n(C), \phi_0, v_n(\phi, \theta, f)) \ (n \geq 0) .$$

For any ring with involution  $A$  let  $(B(\infty), \beta(\infty))$  be the universal chain bundle of Weiss [19] (cf. §6 above), with isomorphisms

$$\begin{aligned} \widehat{v}_m(\beta(\infty)) &: H_m(B(\infty)) \rightarrow \widehat{H}^m(\mathbb{Z}_2; A) , \\ L\langle B(\infty), \beta(\infty) \rangle^m(A) &\cong L^m(A) \end{aligned}$$

and an exact sequence

$$\cdots \rightarrow L_m(A) \xrightarrow{1+T} L^m(A) \rightarrow Q_m(B(\infty), \beta(\infty)) \xrightarrow{\partial} L_{m-1}(A) \rightarrow \cdots$$

The cokernel of the symmetrization map in the Witt groups

$$\text{coker}(1 + T : L_0(A) \rightarrow L^0(A)) = \text{im}(L^0(A) \rightarrow Q_0(B(\infty), \beta(\infty)))$$

was computed for noetherian  $A$  by Carlsson [5] in terms of 'Wu invariants' prior to the general theory of Weiss [19].

For  $n \geq 0$  let  $(B\langle n+1 \rangle, \beta\langle n+1 \rangle)$  be the  $(v_{n+1} = 0)$ -universal chain bundle over  $A$ , characterized up to equivalence by the properties

- (i)  $\hat{v}_r(\beta\langle n+1 \rangle) : H_r(B\langle n+1 \rangle) \rightarrow \hat{H}^r(\mathbb{Z}_2; A)$  is an isomorphism for  $r \neq n+1$ ,
- (ii)  $H_{n+1}(B\langle n+1 \rangle) = 0$ .

The  $(v_{n+1} = 0)$ -symmetric  $L$ -groups of  $A$  are defined by

$$L\langle v_{n+1} \rangle^m(A) = L\langle B\langle n+1 \rangle, \beta\langle n+1 \rangle \rangle^m(A) \quad (m \geq 0).$$

Define the  $A$ -group

$$Q\langle v_{n+1} \rangle = Q\langle B\langle n+1 \rangle, \beta\langle n+1 \rangle \rangle^{2n}(S^n A),$$

to fit into the commutative braid of exact sequences

In §11 below we shall make use of the surjections

$$L\langle v_{n+1} \rangle^{2n}(A) \rightarrow W_{Q\langle v_{n+1} \rangle}(A) ; (C, \phi, \theta, f) \mapsto (H^n(C), \phi_0, v_n(\phi, \theta, f)) \quad (n \geq 0)$$

defined for a field  $A$ .

### §11. An example

As an illustration of the exact sequence of §8

$$\cdots \rightarrow L_n(A) \rightarrow L\langle B, \beta \rangle^n(A) \rightarrow Q_n(B, \beta) \xrightarrow{\partial} L_{n-1}(A) \rightarrow \cdots$$

we compute the Witt groups  $L^0(A)$ ,  $L_0(A)$ ,  $L\langle v_1 \rangle^0(A)$  for  $A$  a perfect field of characteristic 2, without appealing to the theorem of Arf [1] on the classification of quadratic forms over such  $A$  (cf. Example 2.14 of Ranicki [16]).

For any field  $A$   $\left\{ \begin{array}{l} L^{2n}(A) \\ L_{2n}(A) \end{array} \right.$  is the Witt group of nonsingular  $\left\{ \begin{array}{l} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{array} \right.$  forms over  $A$ , and  $\left\{ \begin{array}{l} L^{2n+1}(A) = 0 \\ L_{2n+1}(A) = 0 \end{array} \right. \quad (n \geq 0)$  - see Ranicki [15] for details.

Let then  $A$  be a perfect field of characteristic 2, so that squaring defines an automorphism

$$A \rightarrow A ; a \mapsto a^2 .$$

Let  $A$  have the identity involution

$$\bar{\phantom{x}} : A \rightarrow A ; a \mapsto \bar{a} = a .$$

As an additive group

$$\widehat{H}^r(\mathbb{Z}_2; A) = A \quad (r \in \mathbb{Z})$$

with  $A$  acting by

$$A \times \widehat{H}^r(\mathbb{Z}_2; A) \rightarrow \widehat{H}^r(\mathbb{Z}_2; A) ; (a, x) \mapsto a^2 x ,$$

and there is defined an isomorphism of  $A$ -modules

$$A \rightarrow \widehat{H}^r(\mathbb{Z}_2; A) ; a \mapsto a^2 .$$

The chain bundle over  $A$   $(B(\infty), \beta(\infty))$  defined by

$$\begin{aligned} d_{B(\infty)} &= 0 : B(\infty)_r = A \rightarrow B(\infty)_{r-1} = A , \\ \beta(\infty)_s &= \begin{cases} 1 \\ 0 \end{cases} : B(\infty)_r = A \rightarrow B(\infty)^{-r-s} = A \text{ if } \begin{cases} s = -2r \\ s \neq -2r \end{cases} \end{aligned}$$

is universal. The twisted quadratic groups of  $(B(\infty), \beta(\infty))$  are given up to isomorphism by

$$\begin{aligned} Q_{2n}(B(\infty), \beta(\infty)) &= Q^\bullet(A) , \\ Q_{2n+1}(B(\infty), \beta(\infty)) &= Q_\bullet(A) , \end{aligned}$$

with the abelian groups  $Q^\bullet(A)$ ,  $Q_\bullet(A)$  defined by

$$\begin{aligned} Q^\bullet(A) &= \{a \in A \mid a + a^2 = 0\} = \mathbb{Z}_2 , \\ Q_\bullet(A) &= A / \{b + b^2 \mid b \in A\} , \end{aligned}$$

and isomorphisms defined by

$$\begin{aligned} Q_{2n}(B(\infty), \beta(\infty)) &\rightarrow Q^\bullet(A) ; (\phi, \theta) \mapsto \phi_0(1)(1) , \\ \phi_0 &: B(\infty)^n = A \rightarrow B(\infty)_n = A , \\ Q_{2n+1}(B(\infty), \beta(\infty)) &\rightarrow Q_\bullet(A) ; (\phi, \theta) \mapsto \theta_{-1}(1)(1) , \\ \theta_{-1} &: B(\infty)^{n+1} = A \rightarrow B(\infty)_{n+1} = A . \end{aligned}$$

A symmetric form over  $A$   $(M, \lambda)$  is even if and only if

$$\lambda(x, x) = 0 \in A \quad (x \in M) .$$



A nonsingular even symmetric form over  $A$   $(M, \lambda)$  is hyperbolic, since for any  $x \in M$  there exists  $y \in M$  such that  $\lambda(x)(y) = 1 \in A$ , so that a hyperbolic summand may be split off  $(M, \lambda)$

$$(M, \lambda) = (Ax \oplus Ay, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \oplus (M', \lambda') ,$$

with

$$\text{rank}_A M' = (\text{rank}_A M) - 2 .$$

Thus

$$L\langle v_0 \rangle^0(A) = W\langle v_0 \rangle(A) = 0$$

and the symmetrization maps

$$1 + T : L_{2n}(A) = L_0(A) \rightarrow L\langle v_0 \rangle^0(A) \rightarrow L^{2n}(A)$$

are zero. It is now immediate from the exact sequence

$$\cdots \rightarrow L_m(A) \xrightarrow{1+T} L^m(A) \rightarrow Q_m(B(\infty), \beta(\infty)) \xrightarrow{\partial} L_{m-1}(A) \rightarrow \cdots$$

that

$$L^{2n}(A) = Q^\bullet(A) , \quad L_{2n}(A) = Q_\bullet(A) .$$

In the symmetric case there is defined an isomorphism

$$L^{2n}(A) \rightarrow Q^\bullet(A) = \mathbb{Z}_2 ; (C, \phi) \mapsto \phi_0(v)(v) = \text{rank}_A H^n(C) ,$$

sending a  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  over  $A$  to the element  $\phi_0(v)(v) \in Q^\bullet(A)$ , with  $v \in H^n(C)$  the unique cohomology class such that

$$\phi_0(x)(v) = \phi_0(x)(x) \in \widehat{H}^n(\mathbb{Z}_2; A) \quad (x \in H^n(C)) .$$

The inverse isomorphism

$$Q^\bullet(A) = \mathbb{Z}_2 \rightarrow L^{2n}(A)$$

sends  $1 \in Q^\bullet(A)$  to the  $2n$ -dimensional symmetric Poincaré complex  $(C, \phi)$  defined by

$$\phi_0 = 1 : C^n = A \rightarrow C_n = A , \quad C_r = 0 \quad (r \neq 0) .$$

In the quadratic case there is defined an isomorphism

$$\partial : Q_\bullet(A) \rightarrow L_{2n}(A) ; a \mapsto (C, \psi)$$

with  $(C, \psi)$  the  $2n$ -dimensional quadratic Poincaré complex over  $A$  given by

$$\psi_0 = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} : C^n = A \oplus A \rightarrow C_n = A \oplus A , \quad C_r = 0 \quad (r \neq n) .$$

The inverse isomorphism  $L_{2n}(A) \rightarrow Q_\bullet(A)$  sends a  $2n$ -dimensional quadratic Poincaré complex  $(C, \psi)$  over  $A$  to the Arf invariant  $c \in Q_\bullet(A)$  of the nonsingular quadratic form  $(H^n(C), (1+T)\psi_0, v^n(\psi))$  over  $A$ , as defined by

$$c = \sum_{i=1}^m v^n(\psi)(x_{2i})v^n(\psi)(x_{2i+1}) \in Q_\bullet(A)$$

with  $\{x_i \mid 1 \leq i \leq m\}$  any basis for  $H^n(C)$  such that

$$(1+T)\psi_0(x_i, x_j) = \begin{cases} 1 & \text{if } (i, j) = (2r, 2r+1) \text{ or } (2r+1, 2r) \\ 0 & \text{otherwise} . \end{cases}$$

The chain bundle over  $A$  ( $B\langle v_{n+1} \rangle, \beta\langle v_{n+1} \rangle$ ) ( $n \geq 0$ ) defined by

$$B\langle v_{n+1} \rangle_r = \begin{cases} A & \text{if } r \neq n+1 \\ 0 & \text{if } r = n+1 \end{cases},$$

$$d = 0 : B\langle v_{n+1} \rangle_r \rightarrow B\langle v_{n+1} \rangle_{r-1},$$

$$\beta\langle v_{n+1} \rangle_s = \begin{cases} 1 & : B\langle v_{n+1} \rangle_r \rightarrow B\langle v_{n+1} \rangle^{-r-s} \text{ if } \begin{cases} s = -2r, r \neq n+1 \\ \text{otherwise} \end{cases} \\ 0 & \end{cases}$$

is ( $v_{n+1} = 0$ )-universal. Define symmetric bilinear pairings

$$\begin{aligned} \rho & : A \times A \rightarrow A ; (a, b) \mapsto ab, \\ \sigma & : Q^\bullet(A) \times Q^\bullet(A) \rightarrow A ; (1, 1) \mapsto 1 \end{aligned}$$

such that

$$\begin{aligned} Q\langle B\langle v_{n+1} \rangle, \beta\langle v_{n+1} \rangle \rangle^{2n}(S^n A) &= A \times_\rho A, \\ Q_{2n}(B\langle v_{n+1} \rangle, \beta\langle v_{n+1} \rangle) &= A \times_\sigma Q^\bullet(A), \\ Q_{2n+1}(B\langle v_{n+1} \rangle, \beta\langle v_{n+1} \rangle) &= 0. \end{aligned}$$

Let

$$\begin{aligned} Q\langle v_1 \rangle &= Q\langle B\langle v_1 \rangle, \beta\langle v_1 \rangle \rangle^0(A) \\ &= A \times_\rho A. \end{aligned}$$

Given a nonsingular  $Q\langle v_1 \rangle$ -quadratic form  $(M, \lambda, \mu)$  over  $A$  there exist  $v \in M$ ,  $\psi \in \text{Hom}_A(M, M^*)$  such that

$$\begin{aligned} \lambda(x, y) &= \lambda(x, v)\lambda(y, v) + \psi(x)(y) + \psi(y)(x) \in A \quad (x, y \in M) \\ \mu : M &\rightarrow Q\langle v_1 \rangle = A \times_\rho A ; x \mapsto (\psi(x)(x), \lambda(x, v)). \end{aligned}$$

The morphism

$$\begin{aligned} L\langle v_1 \rangle^0(A) &= W_{Q\langle v_1 \rangle}(A) \rightarrow Q_0(B\langle v_1 \rangle, \beta\langle v_1 \rangle) ; \\ (M, \lambda : M \times M &\rightarrow A, \mu : M \rightarrow Q\langle v_1 \rangle) \mapsto \mu(v) = (\psi(v)(v), \lambda(v, v)) \end{aligned}$$

fits into a short exact sequence

$$0 \rightarrow L_0(A) \rightarrow L\langle v_1 \rangle^0(A) \rightarrow Q_0(B\langle v_1 \rangle, \beta\langle v_1 \rangle) \rightarrow 0.$$

The injection

$$L_{2n}(A) \rightarrow L\langle v_{n+1} \rangle^{2n}(A) \rightarrow W_{Q\langle v_1 \rangle}(A) = L\langle v_1 \rangle^0(A)$$

sends the cobordism class of a  $2n$ -dimensional quadratic Poincaré complex over  $A$   $(C, \psi)$  to the Witt class of the nonsingular  $Q\langle v_1 \rangle$ -quadratic form  $(H^n(C), (1+T)\psi_0, iv^n(\psi))$ , with  $i$  the canonical injection

$$i : H_0(\mathbb{Z}_2; A, (-1)^n) = A \rightarrow Q\langle v_1 \rangle = A \times_\rho A ; a \mapsto (a, 0).$$

In the special case  $A = \mathbb{Z}_2$

$$\begin{aligned} L_{2n}(\mathbb{Z}_2) &= Q_\bullet(\mathbb{Z}_2) = \mathbb{Z}_2, \quad L^{2n}(\mathbb{Z}_2) = Q^\bullet(\mathbb{Z}_2) = \mathbb{Z}_2, \\ Q\langle v_{n+1} \rangle &= \mathbb{Z}_4, \quad Q_0(B\langle v_1 \rangle, \beta\langle v_1 \rangle) = \mathbb{Z}_4, \end{aligned}$$

with

$$L\langle v_{n+1} \rangle^{2n}(\mathbb{Z}_2) = W_{\mathbb{Z}_4}(\mathbb{Z}_2) = \mathbb{Z}_8$$

the Witt group of nonsingular  $\mathbb{Z}_4$ -valued quadratic forms over  $\mathbb{Z}_2$ . See Weiss [19, §11] for the the algebraic Poincaré bordism interpretation of the work of Browder [2] and Brown [4] on the Kervaire invariant and its generalization, which applies also to the work of Milgram [11].

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