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## ON THE CONNECTIVITY RING OF AN ABSTRACT SPACE

By J. W. ALEXANDER

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### I. Introduction

1. The k-dimensional connectivity numbers of an ordinary, smooth n-space may either be obtained geometrically by calculating the maximal number of k-cycles of the space that are independent modulo the bounding cycles or analytically by calculating the maximal number of exact, k-dimensional symbolic differential forms

$$w = \sum A_{i_1 \cdots i_k} dx_{i_1} \cdots dx_{i_k}$$

(integrands of multiple integrals) that are independent, in the large, modulo the derived forms. The geometrical method of approach has been extended to compact metric spaces by Vietoris<sup>1</sup> and to still more general spaces by Čech.<sup>2</sup> Moreover, this branch of the theory has been very greatly perfected by the introduction of Pontrjagin's cycles with real coefficients reduced modulo 1.

Now, if we use Pontrjagin's cycles, the  $k^{\rm th}$  connectivity group of a compact, metric space becomes a compact, metric group. Moreover, by a theorem of Pontrjagin,<sup>3</sup> every such group may be identified with the character group of a countable, discrete group. This immediately suggests the advisability of regarding the discrete group, rather than its equivalent (though more complicated) metric character group, as the  $k^{\rm th}$  invariant of the space, and of looking for a revised theoretical treatment leading simply and directly to this group. We give such a treatment below, based on a suitable combinatory adaptation of the second, or analytic, method of approach.

One decided advantage of taking the discrete groups rather than their metric character groups as the fundamental connectivity groups of the space is that we can then define the *product*<sup>4</sup> (as distinguished from the sum) of two elements of the same or of different groups. The combined groups of all dimensionalities (or, more precisely, their direct sum) will thus become a *connectivity ring*, as distinguished from a set of isolated connectivity groups.

2. A rough sketch of the present theory was given by the author in two

<sup>&</sup>lt;sup>1</sup> L. Vietoris: "Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen," Math. Ann. 97 (1927), pp. 454-472.

<sup>&</sup>lt;sup>2</sup> E. Čech: "Théorie générale de l'homologie dans un espace quelconque," Fund. Math. 19 (1932), pp. 149-183.

<sup>&</sup>lt;sup>3</sup> L. Pontrjagin: "The theory of topological commutative groups," these Annals of Mathematics, 35 (1934), pp. 361-388.

<sup>4</sup> The bracket product of §11.

notes which appeared in the Proceedings of the National Academy,5 and in a paper read at the First International Topological Conference in Moscow.<sup>6</sup> Essentially the same theory was also developed, quite independently, by A. Kolmogoroff and likewise presented by him at the Moscow Congress. The definition of the product of two group elements (the analogue of the product of two symbolic differential forms) as originally given by Kolmogoroff and the present author has recently been modified by Čech,7 so that the revised product is a  $p^{th}$  part of the original one, where p is a constant depending on the dimensionalities of the two factors. This modification is an essential improvement, as may be seen, for example, when the theory is applied to manifolds.8 Nor is it in any way trivial, for in the theory as originally developed it is not clear, a priori, that the product of two elements has a  $p^{th}$  part. We have adopted Čech's modified definition of the product in this paper, although the change has necessitated a complete re-casting of all our proofs. The present paper deals only with the most fundamental definitions and theorems. Further developments of the theory will be given subsequently.

## II. SYMBOLIC COMPLEXES AND SPACES

3. We start with an arbitrary set of entities  $v_i$  called (symbolic) vertices. Any set of n+1 distinct vertices,  $(n=0,1,2,\cdots)$  will be called a (symbolic) n-simplex, or simplex of dimensionality n. Moreover, for the sake of uniformity, the null set (containing no vertices at all) will be called a simplex of dimensionality -1. The null set will be the only simplex of negative dimensionality.

A simplex S will be a component of a simplex T if every vertex of S is a vertex of T. If S is a component of T the vertices of T, but not of S, will determine a component S' of T called the complement of S with respect to T. Two simplexes will be incident if either is a component of the other. The null set will, of course, be a component of every simplex S and, therefore, incident to every S; a simplex S will be a component of itself and, therefore, incident to itself.

Now, let S be any simplex of dimensionality greater than zero, and let the vertices of S be  $x_0, x_1, \dots, x_n$ ,  $(n \ge 1)$ . Then the various possible permutations of the vertices  $x_i$  may be divided into two classes such that two permutations belong to the same or to different classes according as they differ by an even or by an odd number of inversions. We shall call a definite, though arbitrary, one of the two classes the positive class and the other the negative class

<sup>&</sup>lt;sup>5</sup> "On the chains of a complex and their duals," and "On the ring of a compact metric space," Proc. Nat. Acad. 21 (1935), pp. 509-512.

<sup>&</sup>lt;sup>6</sup> September 1935.

<sup>7 &</sup>quot;Multiplications on a complex" by E. Čech in the present number of these Annals, pp. 681-697. A similar modification was also suggested independently by H. Whitney in a letter to L. Zippin. The definition of the product announced by the author in the second of the two *Proceedings* notes referred to above is not the significant one, as was noticed by him while the note was in press. His revised definition was equivalent to Kolmogoroff's.

<sup>8</sup> Cf. Čech, loc. cit.

thereby orienting the simplex. The simplex can have either of two opposite orientations, corresponding to the two possible ways of naming the permutation classes. If s denotes an oriented simplex, then -s will denote the same simplex with the opposite orientation. An oriented simplex will often be denoted by a positive permutation of its vertices,  $s = x_0, x_1, \dots, x_n$ . We shall not orient the simplex of dimensionality -1 nor the simplexes of dimensionality 0.

- 4. A (symbolic) complex will be any set of simplexes such that there is at least one simplex in the set and such that every component of every simplex of the set is in the set. The set need not be finite nor even countable; it may contain simplexes of all dimensionalities. Every complex will contain the null set. A complex  $\Pi'$  will be called a subcomplex of a complex  $\Pi$  if every simplex of  $\Pi'$  is a simplex of  $\Pi$ .
- 5. A (symbolic) space will be any set of complexes  $\Pi_{\sigma}$  such that to each pair of complexes  $\Pi_{\sigma_1}$ , and  $\Pi_{\sigma_2}$  of the set there corresponds at least one complex  $\Pi_{\sigma_3}$ of the set which is a common subcomplex of both  $\Pi_{\sigma_1}$  and  $\Pi_{\sigma_2}$ . To illustrate the topological significance of this definition we shall show the connection between a symbolic space, as here defined, and an ordinary abstract space in the sense of classical point set theory. Consider an arbitrary point set  $\Sigma$ . A covering  $[\sigma]$  of  $\Sigma$  will be any set of subsets  $\sigma$  of  $\Sigma$  such that each point of  $\Sigma$  belongs to at least one of the subsets  $\sigma$ . A covering  $[\sigma_1]$  will be a refinement of a covering  $[\sigma]$ if every element  $\sigma_1$  of  $[\sigma_1]$  is a subset of at least one element  $\sigma$  of  $[\sigma]$ . An abstract space will consist, essentially, of a point set  $\Sigma$  and of a set of coverings  $[\sigma]$  of  $\Sigma$  such that to each pair of coverings  $[\sigma_1]$  and  $[\sigma_2]$  of the set there corresponds at least one covering  $[\sigma_3]$  which is a common refinement of both  $[\sigma_1]$  and (We shall get specific types of spaces by making specific assumptions about the coverings.) Now, every abstract space determines a definite symbolic space such that: (a) the points of the abstract space are the underlying vertices of the symbolic space; (b) each covering  $[\sigma]$  of the abstract space determines a complex  $\Pi_{\sigma}$  of the symbolic space consisting of all simplexes such that their vertices belong to the same element of the covering  $[\sigma]$ . Since two coverings  $[\sigma_1]$  and  $[\sigma_2]$  always have a common refinement, two complexes  $\Pi_{\sigma_1}$  and  $\Pi_{\sigma_2}$ will always have a common subcomplex  $\Pi_{\sigma_2}$ . Therefore, the complexes  $\Pi_{\sigma}$  will determine a symbolic space, as required.

A symbolic space will be more general than an abstract space, in the sense that not every symbolic space can be determined by an abstract space.

### III. SKEW SYMMETRIC FUNCTIONS ON A COMPLEX

6. Let R be an arbitrary ring (which need not necessarily be commutative.<sup>9</sup> An *m-function*,  $(m = 1, 2, \dots)$ , will be any skew-symmetrical function

 $<sup>^{9}</sup>$  It can, perhaps, be shown that the only essentially independent invariants of a complex or space are the ones obtained by taking R to be the ring of all integers. This question should be further investigated.

 $\varphi(x_0, \dots, x_m)$  of m+1 variable vertices  $x_i$ , such that the values of the function are numbers in the ring R. When the vertices  $x_i$  are all distinct they determine an oriented m-simplex  $s=x_0, x_1, \dots, x_m$ . The value of the function  $\varphi$  corresponding to the vertices of s will be called the value of  $\varphi$  on s. Since the function  $\varphi$  is skew symmetrical, every even permutation of the vertices  $x_i$  leaves its value unaltered and every odd permutation merely changes its sign. The function  $\varphi$  will therefore have a unique value on s, and its value on the oppositely oriented simplex -s will be the negative of its value on s. If two or more of the vertices  $x_i$  are identical the value of  $\varphi$  will, of course, be zero. A 0-function will simply be a function  $\varphi(x)$  of a single variable vertex x. It will have a definite value on each 0-simplex. For the sake of uniformity, we shall regard the elements of the ring R as functions of dimensionality -1 defined on the null set (the simplex of dimensionality -1).

An *m*-function  $(m \ge -1)$  will be said to vanish on a complex  $\Pi$  if it vanishes on every *m*-simplex of  $\Pi$ . Two *m*-functions will be said to be *identical* on  $\Pi$  if their difference vanishes on  $\Pi$ .

7. With every *m*-function  $\varphi$  there is associated an (m+1)-function  $\varphi'$  determined by the following relation<sup>10</sup>

(7:1) 
$$\varphi'(x_0, \dots, x_{m+1}) = \sum_{i=0}^{m+1} (-1)^i \varphi(x_0, \dots, \hat{x}_i, \dots, x_{m+1}).$$

We shall call the function  $\varphi'$  the derivative of the function  $\varphi$ . Its value on any oriented (m+1)-simplex  $s=x_0, \dots, x_{m+1}$  will clearly be the sum of the values of  $\varphi$  on the *m*-components of s, with due regard to the proper orientation of the latter. The proof that the function  $\varphi'$  is skew symmetrical and, therefore, actually an (m+1)-function follows, at once, from the obvious fact that when we permute any two consecutive vertices  $x_s$  and  $x_{s+1}$  we merely change the sign of the function.

8. Theorem 1. The second derivative  $\varphi''$  of  $\varphi$  (i.e., the derivative of the derivative) always vanishes identically.

For we have

$$\varphi''(x_0, \dots, x_{m+2}) = \sum_{0}^{m+2} (-1)^i \varphi'(x_0, \dots, \hat{x}_i, \dots, x_{m+2})$$

$$= \sum_{k < i} (-1)^{k+i} \varphi(x_0, \dots, \hat{x}_k, \dots, \hat{x}_i, \dots, x_{m+2})$$

$$+ \sum_{k > i} (-1)^{k+i-1} \varphi(x_0, \dots, \hat{x}_i, \dots, \hat{x}_k, \dots, x_{m+2}).$$

Moreover, the two sums in the last member cancel one another, since they only

$$\varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1}) = \varphi(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{m+1}).$$

<sup>&</sup>lt;sup>10</sup> The symbol  $\hat{x}_i$  will be used throughout to denote the absence of the variable  $x_i$ . Thus

differ in sign. (This may be seen by permuting the dummy indices k and i in either one of them.)

An *m*-function will be said to be *exact* (with reference to a complex  $\Pi$ ) if its derivative vanishes on  $\Pi$ ; it will be said to be *derived* (with reference to  $\Pi$ ) if it is identical on  $\Pi$  with the derivative of some (m-1)-function  $\psi$ .

Corollary. Every derived function  $\varphi$  is exact.

Because

$$\varphi = \psi' \qquad (on \Pi)$$

implies

$$\varphi' = \psi'' = 0 \qquad (on \Pi).$$

9. We shall next define a law of composition according to which an m-function  $\varphi$  and an n-function  $\psi$  will combine to determine an (m+n)-function  $[\varphi\psi]$ , called the *bracket product* of  $\varphi$  and  $\psi$ . Let us arrange the underlying vertices  $v_i$  in a definite, though arbitrary, linear order  $\alpha$ .<sup>11</sup> Then, if the variables  $x_0, x_1, \dots, x_{m+n}$  represent distinct vertices in normal order with respect to  $\alpha$ , we shall assign to the product  $[\varphi\psi]$  the value

$$(9:1) [\varphi\psi](x_0, \cdots, x_{m+n}) = \varphi(x_0, \cdots, x_m) \psi(x_m, \cdots, x_{m+n}) ,$$

where the expression on the right is the ordinary product of two elements of the ring R. Since the function  $[\varphi\psi]$  is skew symmetric, its value will now be determined for arbitrary choices of the variables  $x_i$ . (It should be noticed that the functions  $\varphi$  and  $\psi$  in the right-hand member of (9:1) both involve the variable  $x_m$ .)

Since the right-hand member of (9:1) is an ordinary product in the ring R we can verify, at once, that bracket multiplication is distributive with respect to addition:

$$(9:2) \qquad [(\varphi_1 + \varphi_2)(\psi_1 + \psi_2)] = [\varphi_1\psi_1] + [\varphi_1\psi_2] + [\varphi_2\psi_1] + [\varphi_2\psi_2].$$

Of course, the bracket product is also associative though not, in general, commutative.

The function  $[\varphi\psi]$  depends essentially on the linear ordering  $\alpha$  of the vertices. When we wish to emphasize this dependence we shall write  $[\varphi\psi]_{\alpha}$  in place of  $[\varphi\psi]$ .

10. Theorem 2. If  $\varphi$  is any m-function and  $\psi$  any n-function, then the deriva-

<sup>&</sup>lt;sup>11</sup> The idea of defining the bracket product  $[\varphi\psi]$  with reference to a specific ordering of the vertices v, is due to Čech and Whitney. In the original definitions of Kolmogoroff and the present author, the vertices were not ordered, and the product consisted of a set of terms corresponding to various permutations of the vertices. An extraneous numerical factor was thus introduced.

tive of the bracket product  $[\varphi\psi]$  satisfies the relation

(10:1) 
$$[\varphi\psi]' = [\varphi'\psi] + (-1)^m [\varphi\psi'].$$

For we may write

$$[\varphi\psi]'(x_0, \dots, x_{m+n+1}) = \sum_{i=0}^{m+n+1} (-1)^i [\varphi\psi](x_0, \dots, \hat{x}_i, \dots, x_{m+n+1})$$

$$= \sum_{i=0}^{m} (-1)^i \varphi(x_0, \dots, \hat{x}_i, \dots, x_{m+1}) \psi(x_{m+1}, \dots, x_{m+n+1})$$

$$+ \sum_{m+1}^{m+n+1} (-1)^i \varphi(x_0, \dots, x_m) \psi(x_m, \dots, \hat{x}_i, \dots, x_{m+n+1}).$$

Moreover, if we add the sum  $(-1)^{m+1}\varphi(x_0, \dots, x_m)\psi(x_{m+1}, \dots, x_{m+n+1})$  to the first sum in the last expression, and substract it from the second, we obtain, at once, the desired relation (10:1).

Corollary 1. The bracket product  $[\varphi\psi]$  of two exact functions  $\varphi$  and  $\psi$  is always exact.

For the functions  $\varphi'$  and  $\psi'$  vanish on  $\Pi$ . Therefore, by (10:1), the function  $[\varphi\psi]'$  also vanishes on  $\Pi$ .

COROLLARY 2. The bracket product  $[\varphi\psi]$  of an exact function by a derived function or of a derived function by an exact function is a derived function.

**PROOF.** Suppose, for instance, that  $\varphi$  is exact and  $\psi$  derived. Then there exists a function  $\theta$  such that

$$\psi = \theta' \qquad (on \Pi) .$$

Moreover, we also have

$$\varphi' = 0 \qquad (on \Pi) ;$$

therefore, by (10:1)

$$[\varphi\theta]' = (-1)^m [\varphi\theta']$$
 (on  $\Pi$ );

therefore, finally

$$[(-1)^m \varphi \theta]' = [\varphi \theta'] = [\varphi \psi] \qquad (\text{on } \Pi) \ .$$

The case where  $\varphi$  is derived and  $\psi$  exact is treated in essentially the same manner.

### IV. THE CONNECTIVITY RING OF A COMPLEX

11. Let  $\Pi$  be an arbitrary complex. Then the set of all m-functions that are exact with respect to  $\Pi$  may be regarded as an abelian group  $G_E^m$  under the operation of addition. Moreover, the set of all m-functions that are derived with respect to  $\Pi$  may be regarded as a subgroup  $G_D^m$  of the group  $G_E^m$ , by the corollary to Theorem 1. We shall call the residue group  $G_E^m = G_E^m \mod G_D^m$  the  $m^{\text{th}}$  connectivity group of  $\Pi$ . Each element  $\Phi$  of the group  $G^m$  will be a class

of exact functions, such that two functions belong to the same class if, and only if, their difference is a derived function.

The groups  $G^m$  ( $m=0,1,\cdots$ ) are inter-related in the following manner. Let  $\Phi$  be an arbitrary element of the  $m^{\text{th}}$  group  $G^m$  and  $\Psi$  an arbitrary element of the  $n^{\text{th}}$  group  $G^n$ . Then the elements  $\Phi$  and  $\Psi$  together determine an element  $[\Phi\Psi]$  of the group  $G^{m+n}$  called the bracket product of  $\Phi$  and  $\Psi$ . The element  $[\Phi\Psi]$  will be the class containing the function  $[\varphi\psi]$ , where  $\varphi$  and  $\psi$  are arbitrary functions of the classes  $\Phi$  and  $\Psi$  respectively. It is easy to see that the class  $[\Phi\Psi]$  is independent of the choice of the functions  $\varphi$  and  $\psi$  within their respective classes. For the class  $\Phi$  is composed of exact functions of the type

$$\varphi = \varphi_0 + f,$$

where  $\varphi_0$  is an arbitrary element of the class and f an arbitrary derived function. Similarly, the class  $\Psi$  is composed of exact functions of the type

$$\psi = \psi_0 + a$$

where  $\psi_0$  is in the class and g a derived function. Now, by the distributive law of bracket multiplication, we have

(11:1) 
$$[\varphi\psi] = [\varphi_0\psi_0] + [\varphi_0g] + [f\psi_0] + [fg].$$

Moreover, by Corollary 2 of Theorem 2, the last three terms on the right are derived functions. Therefore, the functions  $[\varphi\psi]$  all belong to the same class as the functions  $[\varphi_0\psi_0]$ .

12. THEOREM 3. The bracket product  $[\Phi\Psi]$  is independent of the ordering  $\alpha$  of the vertices  $v_i$  and is, therefore, a function of the group elements  $\Phi$  and  $\Psi$  alone.

Proof. We shall first consider the case where the number of underlying vertices  $v_i$  is finite, since the general case involves a slight added complication. When the number of vertices is finite we can pass from any linear arrangement  $\alpha$  to any other linear arrangement  $\beta$  by a finite sequence of simple steps, such that at each step we merely permute two consecutive vertices. The problem therefore reduces to showing that if the arrangement  $\beta$  differs from the arrangement  $\alpha$  by a single permutation of two consecutive vertices  $v_r$  and  $v_{r+1}$  of  $\alpha$  then the difference  $[\varphi\psi]_{\beta} - [\varphi\psi]_{\alpha}$  is identical on  $\Pi$  with a derived function  $\zeta$ . We shall actually construct an (m+n-1)-function  $\zeta$  such that its derivative satisfies the relation

(12:1) 
$$\zeta' = [\varphi \psi]_{\beta} - [\varphi \psi]_{\alpha} \qquad (\text{on } \Pi) .$$

Since the function  $\zeta$  must be skew symmetrical (when it involves more than one variable) it will be sufficient to define its value for the case where the vertices  $x_0, \dots, x_{m+n-1}$  are all distinct and in normal order with respect to  $\alpha$ .

Under these circumstances, we shall write

$$(12:2) \zeta(x_0, \cdots, x_{m+n-1}) = \begin{cases} (-1)^{m+1} \varphi(x_0, \cdots, x_{m-1}, x_m) \psi(x_{m-1}, x_m, \cdots, x_{m+n-1}) \\ & \text{or} \\ & 0, \end{cases}$$

where the upper determination is to hold when the two particular variables  $x_{m-1}$  and  $x_m$  have the values  $v_r$  and  $v_{r+1}$  respectively, while the lower one is to hold in all other cases. To complete the argument, we merely have to verify that the derivative  $\zeta'$  of  $\zeta$ ,

(12:3) 
$$\zeta'(x_0, \dots, x_{m+n}) = \sum_{i=0}^{m+n} (-1)^i \zeta(x_0, \dots, \hat{x}_i, \dots, x_{m+n}),$$

satisfies Relation (12:1). Here, again, we may, of course, assume that the vertices x, are distinct and in normal order with respect to  $\alpha$ .

For convenience, we shall consider four separate cases which will exhaust all the various possibilities.

Case 1. The two vertices  $v_r$  and  $v_{r+1}$  are not both present among the vertices  $x_i$ .

Then, by the defining relation (12:2) of  $\zeta$ , all terms in the right-hand member of (12:3) must vanish; therefore, we must have  $\zeta' = 0$ . Moreover, in this case, the vertices  $x_i$  are in normal order with respect to  $\beta$  as well as with respect to  $\alpha$ ; therefore, we have

$$[\varphi\psi]_{\beta} = [\varphi\psi]_{\alpha} = \varphi(x_0, \cdots, x_m) \psi(x_m, \cdots, x_{m+n}).$$

Thus, the difference  $[\varphi\psi]_{\beta} = [\varphi\psi]_{\alpha}$  vanishes, and Relation (12:1) is valid.

In the remaining three cases, two (necessarily consecutive) variables  $x_i$  and  $x_{i+1}$  will take on the values  $v_r$  and  $v_{r+1}$  respectively.

Case 2. 
$$x_{m-1} = v_r; x_m = v_{r+1}.$$

In this case, the first m+1 terms in the right-hand member of (12:3) are zero, since it is only for i > m that the  $(m-1)^{n}$  and  $m^{n}$  variables in  $\zeta(x_0, \dots, \hat{x}_i, \dots, x_{m+n})$  have the respective determinations  $v_r$  and  $v_{r+1}$ . We therefore have

$$\zeta'(x_0, \dots, x_{m+n})$$

$$= (-1)^{m+1} \varphi(x_0, \dots, x_{m-1}, x_m) \sum_{m+1}^{m+n} (-1)^i \psi(x_{m-1}, x_m, \dots, \hat{x}_i, \dots, x_{m+n}).$$

On the other hand,

$$[\varphi\psi]_{\beta}(x_0, \dots, x_{m+n}) = -[\varphi\psi]_{\beta}(x_0, \dots, x_m, x_{m-1}, \dots, x_{m+n})$$

$$= -\varphi(x_0, \dots, x_m, x_{m-1}) \psi(x_{m-1}, x_{m+1}, \dots, x_{m+n})$$

$$= \varphi(x_0, \dots, x_m) \psi(x_{m-1}, x_{m+1}, \dots, x_{m+n}),$$

and

$$[\varphi\psi]_{\alpha}(x_0,\cdots,x_{m+n})=\varphi(x_0,\cdots,x_m)\psi(x_m,x_{m+1},\cdots,x_{m+n}).$$

Therefore

$$[\varphi\psi]_{\beta} - [\varphi\psi]_{\alpha}$$

$$= \varphi(x_{0}, \dots, x_{m}) \{ \psi(x_{m-1}, x_{m+1}, \dots, x_{m+n}) - \psi(x_{m}, x_{m+1}, \dots, x_{m+n}) \}$$

$$= (-1)^{m} \varphi(x_{0}, \dots, x_{m}) \sum_{m=1}^{m} (-1)^{i} \psi(x_{m-1}, \dots, \hat{x}_{i}, \dots, x_{m+n}) .$$

Now, if we substract this last expression from (12:4) we have

$$\zeta' - \{ [\varphi \psi]_{\beta} - [\varphi \psi]_{\alpha} \}$$

$$= (-1)^{m+1} \varphi(x_0, \dots, x_m) \sum_{m=1}^{m+n} (-1)^i \psi(x_{m-1}, \dots, \hat{x}_i, \dots, x_{m+n})$$

$$= (-1)^{m+1} \varphi(x_0, \dots, x_m) \psi'(x_{m-1}, \dots, x_{m+n}),$$

which vanishes because the derivative  $\psi'$  of the exact function  $\psi$  must be zero.

Case 3.  $x_m = v_r; x_{m+1} = v_{r+1}$ .

This case is essentially like the last. In the right-hand member of (12:3) all terms after the  $m^{th}$  are zero. We therefore have

$$\zeta'(x_0, \dots, x_{m+n})$$

$$= (-1)^{m+1} \sum_{i=0}^{m-1} (-1)^i \varphi(x_0, \dots, \hat{x}_i, \dots, x_{m+1}) \psi(x_m, x_{m+1}, \dots, x_{m+n}).$$

Moreover,

$$[\varphi\psi]_{\beta} - [\varphi\psi]_{\alpha} = (-1)^{m} \sum_{m}^{m+1} (-1)^{i} \varphi(x_{0}, \dots, \hat{x}_{i}, \dots, x_{m+1}) \psi(x_{m}, \dots, x_{m+n}).$$

Therefore

$$\zeta' - \{ [\varphi \psi]_{\beta} - [\varphi \psi]_{\alpha} \} = (-1)^{m+1} \varphi'(x_0, \dots, x_{m+1}) \psi(x_m, \dots, x_{m+n})$$

since  $\varphi$  is exact, and (12:1) is valid.

Case 4.  $x_i = v_r$ ;  $x_{i+1} = v_{r+1}$ , but  $i \neq m - 1$ , m.

This case is particularly simple. All terms in the right-hand member of (12:3) are zero; therefore,  $\zeta' = 0$ . Moreover, we have at once  $[\varphi\psi]_{\beta} = [\varphi\psi]_{\alpha}$ . Therefore (12:1) is again valid.

This completes the argument for the finite case. If the arrangement  $\beta$  differs from the arrangement  $\alpha$  by more than a single permutation of two consecutive vertices, we can pass from  $\alpha$  to  $\beta$  by a sequence of elementary steps, corresponding to each of which we can construct a function of the form of  $\zeta$ . Let

 $\chi(x_0, \dots, x_{m+n-1})$  be the sum of these functions. Then we shall, of course, have

(12:6) 
$$\chi' = [\varphi \psi]_{\beta} - [\varphi \psi]_{\alpha} \qquad (\text{on } \Pi) .$$

When the number of vertices is infinite, the argument must be slightly modified, since we cannot, in general, pass from an arrangement  $\alpha$  to an arrangement  $\beta$  by simple steps of the sort described above. Let us observe, however, that if we take any finite subset  $x_i$ ,  $(i=0,1,\cdots)$ , of the vertices  $v_i$  arranged in a sequence according to  $\alpha$ , we can rearrange them in a sequence according to  $\beta$  by permutations of consecutive vertices. Therefore, if the vertices  $x_i$  are the vertices of any finite subcomplex  $\Pi_0$  of  $\Pi$ , we can construct a function  $\chi$  such that a relation similar to (12:6) is valid at least on  $\Pi_0$ . The problem will thus be to determine the function  $\chi$  in such a way that it will be independent of  $\Pi_0$ , in which case Relation (12:6) will be valid over  $\Pi$  as a whole.

We notice that if  $x_i$   $(i = 0, 1, \dots k)$  is any finite sequence of vertices arranged according to  $\alpha$ , we can rearrange them according to  $\beta$  by a sequence of steps performed in precisely the following order. Let  $x_r$  be the first vertex such that the subsequence  $x_0, x_1, \dots x_r$  is not in normal order with respect to  $\beta$ . Then the two vertices  $x_{r-1}$  and  $x_r$  must clearly be in inverted order with respect The first step in the rearrangement will be to permute these two vertices. We shall then operate in a similar manner on the arrangement resulting from the first permutation, and so on. Since each permutation reduces by one the number of inversions with respect to  $\beta$ , the process will obviously come to an end after a finite number of steps, with the vertices arranged according to  $\beta$ . Now, the essential point to notice is the following. When we rearrange the sequence  $x_i$  according to the above rule, we simultaneously rearrange every subsequence of  $x_i$  according to precisely the same rule. For at the moment when we permute two vertices of the complete sequence according to rule, we either leave the arrangement of the subsequence unaltered or change it according to rule, depending on whether or not the two permuted vertices belong to the subsequence.

With the above in view, let us form the complex  $\Pi_{\bullet}$  consisting of any (m+n)-simplex s together with all its components of lower dimensionalities. On the complex  $\Pi_{\bullet}$  we shall define the function  $\chi$  in the manner indicated above. That is to say, we shall start with the vertices of  $\Pi_{\bullet}$  in normal order according to  $\alpha$ , rearrange them according to  $\beta$  by applying our rule, and construct the functions  $\zeta$  corresponding to the various steps in the transition. Moreover, we shall define the function  $\chi$  on  $\Pi_{\bullet}$  as the sum of the functions  $\zeta$  thus constructed. Now, we have only to notice that on any (m+n-1)-component

$$s_i = x_0 \cdots \hat{x}_i \cdots x_{n+m}$$

of s the value of the function  $\zeta$  corresponding to any given step is zero unless the two vertices that are permuted at this particular step both belong to  $s_i$ .

The value of  $\chi$  on  $s_i$  is, therefore, equal to the sum of the functions  $\zeta$  corresponding to the rearrangement of the vertices of  $s_i$  alone. In other words, the value of  $\chi$  on  $s_i$  is independent of the (m+n)-simplex s of which  $s_i$  is a component. Hence, the function  $\chi$  is uniquely determined.

### V. THE CONNECTIVITY RING OF A SPACE

13. An m-function  $\varphi$  will be said to be derived with reference to a symbolic space if it is derived with reference to any subcomplex  $\Pi_a$  of the space; it will be said to be exact with reference to the space if it is exact with reference to any complex  $\Pi_b$  of the space. With this understanding, the entire discussion, beginning with Theorem 3, will be applicable to symbolic spaces as well as to symbolic complexes. However, the ring of a symbolic space has a natural topology which we shall discuss in a subsequent paper. We shall also discuss the dual relation between the connectivity groups as here defined and the connectivity groups of Vietoris-Čech.

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