which completes the induction for this case. In the remaining case where the point Q is on the curve k_0, the only difference is that an arc, l_1, of a curve of intersection in α_1, and not necessarily an entire curve, approaches the curve k_0 as α_1 approaches α_0. The necessary deformation of σ_1 is one such that the arc (or curve) l_1 shrinks to the point Q as α_1 approaches α_0. We perform a similarly modified deformation on σ_2 and complete the argument just as before, thereby proving the theorem.

A similar reduction may be applied for the case $p = 1$, but at some stage of the process the curve k_0 will be non-bounding. The side of σ containing the plane surface C bounded by k_0 will thus have to be tubular, that is to say, homeomorphic with the interior of an anchor ring. This is the theorem predicted by Tietze. For a general value of p, it is easy to show that the linear connectivity of either region bounded by σ is $(P_1 - 1) = P$, but the group of the region may be very complicated.

AN EXAMPLE OF A SIMPLY CONNECTED SURFACE BOUNDING A REGION WHICH IS NOT SIMPLY CONNECTED

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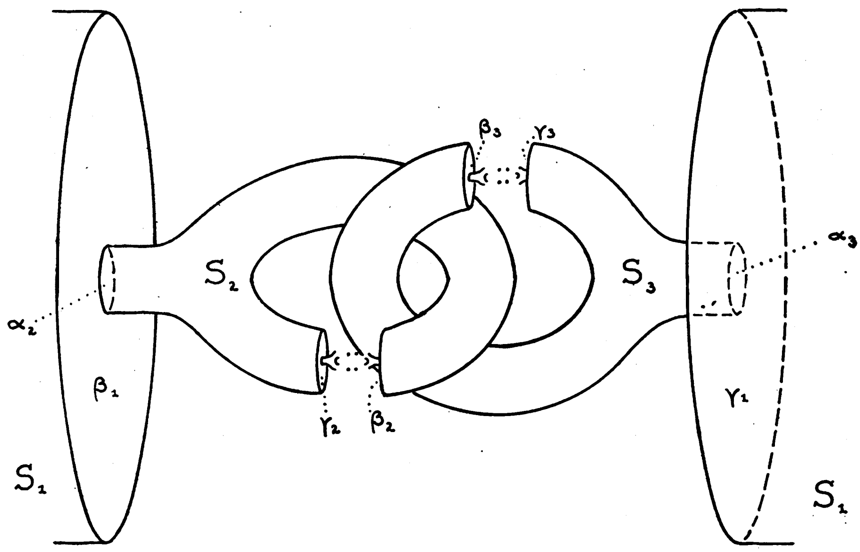
The following construction leads to a simplified example of a surface Σ of genus zero situated in spherical 3-space and such that its exterior is not a simply connected region. The surface Σ is obtained directly without the help of Antoine's inner limiting set.

The surface Σ will be the combination, modulo 2, of a denumerable infinity of simply connected surfaces S_i ($i = 1, 2, \ldots$), all precisely similar in shape, though their dimensions diminish to zero as i increases without bound. The shape of the surface S_i may perhaps be described most readily by referring to the accompanying figure in which the surfaces S_2 and S_3 are represented. By comparison with S_2 to which, by hypothesis, all the other surfaces S_i are similar, we see that the general surface S_i is roughly like a tube twisted into the shape of the letter C and terminating in a pair of circular 2-cells, β_i and γ_i. There is, however, a slight protuberance in the side of the tube terminating in a 2-cell α_i.

The position of the surfaces S_1, S_2, and S_3 with respect to one another is indicated in the figure, though only the two ends of S_1 terminating in α_1 and β_1 are shown. It will be noticed that the faces α_2 of S_2 and α_3 of S_3 are subfaces of the faces β_1 and γ_1 of S_1, respectively, and that the surfaces

S_2 and S_3 are hooked around one another, so to speak. When S_2 and S_3 are added modulo 2 to S_1 (which means that the points of α_2 and α_3 must be deleted from the combined surfaces), a simple closed surface, Σ_1, is obtained. The surface Σ_1 will be regarded as the first approximation of the desired surface Σ. The next approximating surface, Σ_2, is obtained by adjoining to Σ_1, modulo 2, the next four surfaces, S_4, S_5, S_6, S_7. The first two of these will be related to S_2 and the last two to S_3 in exactly the same way that the surfaces S_2 and S_3 are related to S_1; that is to say, a similarity transformation of the 3-space carrying S_1 into S_2 would carry S_2 and S_3 into S_4 and S_5, respectively, while one carrying S_1 into S_3 would carry S_2 and S_3 into S_6 and S_7, respectively. The third approximation Σ_3 is obtained by adjoining the next eight surfaces $S_8, \ldots, S_{15}$ in a similar manner, so that the pair S_8 and S_9 are attached to S_4 just as the pair S_2 and S_3 are



attached to S_1, and so on. The surface Σ is the limiting surface approached by the sequence $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$ It will be seen without difficulty that the interior of the limiting surface Σ is simply connected, and that the surface itself is of genus zero and without singularities, though a hasty glance at the surface might lead one to doubt this last statement. The exterior R of Σ is not simply connected, however, for a simple closed curve in R differing but little from the boundary of one of the cells γ_i cannot be deformed to a point within R. It is easily shown, in fact, that the group of R requires an infinite number of generators.

The points K of Σ which are not points of the approximating surfaces Σ_i form an inner limiting set of a much simpler type than the inner limiting set of Antoine, as was pointed out to me by Professor Veblen. For we may close down upon the points K by a system of spheres rather than by a complicated system of linking anchor rings.

This example shows that a proof of the generalized Schönfliess theorem announced by me two years ago, but never published, is erroneous.

REMARKS ON A POINT SET CONSTRUCTED BY ANTOINE

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From the consideration of a remarkable point set discovered by Antoine, the following two theorems may be derived:

Theorem 1. There exists a simple closed surface of genus 0 in 3-space such that the interior of the surface is not simply connected but has, on the contrary, an infinite group.

Theorem 2. There exists a simple closed curve in 3-space which is not knotted, inasmuch as it bounds a 2-cell without singularities, and yet such that its group[1] (as defined by Dehn) is not the same as the group of a circle in 3-space.

It follows without difficulty from the second theorem that if an isotopic deformation be defined in the customary manner (cf. for example, Veblen's Cambridge Colloquium Lectures), the group of a curve in 3-space is not an isotopic invariant. This suggests that a modified definition of isotopy might be advisable.

Antoine's point set is obtainable as follows. Within an anchor ring π in 3-space, we first construct a chain C or anchor rings π_i $(i = 1, 2, \ldots, s)$ such that each ring π_i is linked with its immediate predecessor and immediate successor, after the manner of links in an ordinary chain, and such, also, that the last ring π_s is linked with the first π_1, thereby making the chain closed. We further suppose that the chain C is constructed in such a way that it winds once around the axis of the ring π. Secondly, we make a similar construction within each of the anchor rings π_i, thereby obtaining chains C_i made up of rings π_{ij} $(j = 1, 2, \ldots, s)$, and repeat the process indefinitely, obtaining chains C_{ij} within π_{ij}, C_{ijk} within π_{ijk}, and so on. If we think of the system of rings within one of the rings π_i as the image of the system of rings within the ring π under a similarity transformation carrying the interior and boundary of π into the interior and boundary of π_i, it is clear that the diameters of the rings $\pi_{ijk}\ldots$ decrease towards zero as the number of subscripts to their symbols increases. The inner limiting set Σ determined by the infinite sequences of rings π^i, π_{ij}, π_{ijk}, $\ldots$ is the