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FINITE *H*-SPACES AND LIE GROUPS

J.F. ADAMS

University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

The definition of an H-space goes back to Serre, in his thesis [7]. The given structure of an H-space comprises three things: a topological space X, a base-point $e \in X$, and a "product map" $\mu : X \times X \to X$. These, of course, may be required to satisfy various axioms. For present purposes we don't need to know the axioms; we do need to know some examples.

In the first class of examples, the space X is a topological group G; the point e is the unit element in G; and the map μ is given by the product in the group, $\mu(g,h) = gh$. The topological groups of most important to us here are the Lie groups.

In the second class of examples, X is a loop-space ΩY. That is, one starts from a space Y with base-point y_0; and one forms the space ΩY of continuous functions ω: $[0,1], 0, 1 \to Y, y_0, y_0$. These functions are called loops, and one gives the set of loops the compact-open topology. The base-point e is the loop constant at y_0; and one defines the product $\mu\,(\omega, \omega')$ of two loops in the usual way, that is,

$$\mu(\omega,\omega')(t) = \begin{cases} \omega(2t) & (0 \le t \le \frac{1}{2}), \\ \omega'(2t-1) & (\frac{1}{2} \le t \le 1). \end{cases}$$

Loop-spaces are of course the sort of function-space which Serre exploited with such success; and one may say that at this point he was proving some basic lemmas about their topology, by analogy with the known case of topological groups.

It was realised quite early that the second class of examples essentially contains the first. More precisely, let G be a topological group; then under mild restrictions we can form its classifying space BG, and the loop-space ΩBG gives us back G up to equivalence.

There are certainly H-spaces which are not loop-spaces; for example, let X be the unit sphere in the space of Cayley numbers, with μ defined by the multiplication of Cayley numbers. However, we understand such phenomena well enough; in this lecture I mostly want to talk about H-spaces which are loop-spaces ΩY, or as we usually say, H-spaces with classifying spaces Y.

Among such, it has always seemed that the Lie groups are distinguished by their finiteness properties. For example, a compact Lie group G is a finite complex, and a general loop-space ΩY is not even equivalent to a finite complex. In the theory of finite H-spaces, one tends to assume that X is an H-space which is equivalent to some finite complex.

In this subject, like any other, people prove theorems and construct counter-examples. The theorems tend to say that finite H-spaces behave in some way like Lie groups; the counter-examples show that there are finite H-spaces which are not Lie groups.

In 1963 [11] I.M. James suggested that one should look for such counterexamples among sphere bundles over spheres. This suggestion was apparently forgotten for a time. However, the celebrated counterexample of Hilton and Roitberg [3] is of this nature. We now have a good understanding of the range of counterexamples which have been constructed, owing to the advent of the method of localisation [9].

As a representative theorem, I quote the fine result of Hubbuck [4] that a finite H-space which is homotopy-commutative is actually equivalent to a torus T^n. I certainly do not want to disparage this result in any way when I say that unfortunately, as in other parts of group-theory, the general case is more difficult than the abelian case.

We should therefore ask, what are the fundamental statements about the topology of Lie groups which we should try to carry over to finite H-spaces?

To begin with, the Borel theorem carries over. If we exclude a finite number of primes p, then the mod p cohomology of the classifying space Y is a polynomial algebra, on generators whose number and degrees do not depend on p. More formally, there is an integer l, the *rank*, and integers $(2d_1, 2d_2, \ldots, 2d_l)$, the *type*, so that for $p \geq p_0(Y)$ we have

$$H^*(Y;\, F_p) \cong F_p[y_1, y_2, \ldots, y_l]$$

with y_i of degree $2d_i$.

In the classical case, when we start from a compact connected Lie group G, we have a maximal torus T and a Weyl group W. Then a classical result says that

$$H^*(BG;\, F_p) \to H^*(BT;\, F_p)^W$$

is iso for $p \geq p_1(G)$. Here $H^*(BT;\, F_p)^W$ means the subalgebra of elements in $H^*(BT;\, F_p)$ which are invariant under W.

A result of Adams and Wilkerson [2] carries this over to finite H-spaces, in the following way. Suppose that for a particular prime p we have an isomorphism

$$H^*(Y;\, F_p) \cong F_p[y_1, y_2, \ldots, y_l],$$

with y_i of degree $2d_i$, as above; and suppose also that p does not divide $d_1 d_2 \cdots d_l$. It is not assumed that there is any particular geometric relation between ΩY and a torus. Nevertheless, the proof constructs something like a Weyl group, namely a finite subgroup W_p of $\mathrm{GL}(l, Z_p^\wedge)$ which is generated by generalised reflections; and since $\mathrm{GL}(l, Z_p^\wedge)$ acts on $H^*(BT^l;\, F_p)$, of course the subgroup W_p does so. Then the result gives an isomorphism

$$H^*(Y;\, F_p) \xrightarrow{\cong} H^*(BT^l;\, F_p)^{W_p},$$

and this isomorphism preserves Steenrod operations. The point of all this is that the p-adic reflection groups are classified by known results of algebra; so one comes down to a list of 37 cases, some of which are infinite families.

We may accept this as reasonably satisfactory for the primes p which are sufficiently large, in the sense that $p \geq p_0(Y)$ and p does not divide $d_1 d_2 \cdots d_l$. The next problem is, what further information can one get using the small primes? One can convince oneself that one cannot get a good answer using Steenrod operations, and one had better turn to K-theory. I will not offer reasons, since I think that you will believe it. In any case, we glimpse the prospect of running the Adams–Wilkerson programme over again, but using K-theory instead of ordinary cohomology $H^*(\ ;F_p)$. The counterexamples still force us to work one prime at a time; you might think of using $K^*(\ ;F_p)$, but that seems to be a bad idea; it is probably best to use $K^*(\ ;Z_p^\wedge)$. So when I write $K^*(\ )$, I mean $K^*(\ ;Z_p^\wedge)$.

In the classical case we have a result

$$K(BG) \xrightarrow{\cong} K(BT)^W$$

without any restriction on the prime p. So we glimpse the possibility of doing away with the condition $p \geq p_0(Y)$ which was essential to the Borel theorem.

We can now envisage a programme in two steps. The first step should belong to topology: we should assume given a finite H-space in the sense that $X \simeq \Omega Y$ and X is equivalent to a finite complex; and we should deduce that $K(Y)$ has good algebraic properties. The second step should belong to algebra: we should assume given an algebraic object R with the same structure that $K(Y)$ has, and with the good properties proved for $K(Y)$ in the first step; and we should deduce an isomorphism

$$R \xrightarrow{\cong} K(BT)^{W_p}.$$

There is something known about the first step. Namely, if $p > 2$ and $\pi_1(X) = 0$, then

$$K(Y) \cong Z_p^\wedge[[y_1, y_2, \ldots, y_l]],$$

just as in the classical case. This follows from work of J.P. Lin [5, 6]. I have the impression that there is a great deal of information implicit in this result; unfortunately, we don't yet know how to get it out.

Next we must face the question: how much structure must we consider on $K(Y)$, and how many good properties of it must we prove and use? I take it as read that our algebraic objects R will be algebras over $Z_p^\wedge$, with operations λ^i, and complete for the topology defined by powers of the augmentation ideal. We have to deal with things a bit less obvious.

When we were using $H^*(Y;F_p)$, we made essential use of the grading and the "unstable axiom" on Steenrod operations. It is orthodox belief that when you use K-theory, you substitute the filtration on $K(Y)$ for the grading on $H^*(Y;F_p)$. We believe that we have to use a filtration on $K(Y)$; the only question is, which filtration? A priori one can think of two which make sense to an algebraist: Atiyah's γ-filtration, and the rational filtration. Here I take Atiyah's γ-filtration as known, but I should say something about the rational filtration. To a topologist, I say that $y \in K(Y)$ has rational filtration $\geq 2n$ if $\mathrm{ch}_r\, y = 0$ in $H^{2r}(Y;\, Q_{\hat{p}})$ for $r<n$; and then I tell you that you can make sense of it for an algebraist too.

You might conjecture that the two filtrations are the same for good spaces like BG. They are not. Take $p=3$ and take G to be the exceptional Lie group F_4. If you look in Tits' tables [10] you will find that the first two non-trivial irreducible representations are (say) α of degree 26 and β of degree 52. β is the adjoint representation, and α is the one whose highest weight is a short root. Form

$$y = (\beta - 52) - 3(\alpha - 26) = \beta - 3\alpha + 26.$$

Then y has rational filtration 8, and $3y$ has γ-filtration 8, but y has γ-filtration 4.

So we have to choose. To guide our choice, we recall that we are interested in embedding $R = K(Y)$ in $K(BT)$; so evidently the filtration we want is the one that comes by pulling back the unique filtration on $K(BT)$, if such an embedding is possible. This describes the rational filtration; so we are committed to using the rational filtration.

Next it should be reasonable to study the relation between the operations λ^ior Ψ^k and the rational filtration. After all, we have one precedent to go on, and this is an unpublished proof by me that if

$$H^*(Y;\, Z_{\hat{p}}) = Z_{\hat{p}}[y], \quad y \in H^4,$$

then

$$K(Y) \cong K(BSU(2))$$

where the isomorphism preserves the operations. The nature of this proof is as follows. Let E_0 mean the associated graded, using the rational filtration. Then the hypothesis shows that there is an isomorphism

$$E_0K(Y) \cong E_0K(BSU(2)).$$

But if there is one isomorphism then there is more than one: at least one which commutes with some operations and at least one which doesn't. It is essential to pick an isomorphism which does commute with the operations before one tries to lift it to an isomorphism $K(Y) \cong K(BSU(2))$.

Therefore, it should be reasonable to consider associated operations on the associated graded – which comes back to the question of studying the relation between the operations λ^i or Ψ^k and the rational filtration. Investigation reveals the following situation.

Suppose $y \in K(Y)$ and y is of rational filtration $\geq 2n$, so that $\mathrm{ch}_m\, y = 0$ for $m \leq n$. Set $s = [r/(p-1)]$. Then an old theorem of mine [1] says that $p^s \mathrm{ch}_{n-r}\,(y)$ is integral, in the sense that it lies in the image of

$$H^{2n-2r}(Y;\, Z_{\hat{p}}) \to H^{2n-2r}(Y;\, Q_{\hat{p}}).$$

Now suppose that $K(Y)$ is torsion free, as happens in our applications; and imagine that we don't know about $H^*(Y; Z_p)$, but we make a new definition: an element $h \in H^{2n+2r}(Y; Q_{\hat{p}})$ is integral if there exists $y \in K(Y)$ such that ch $y = h +$ (higher terms). This will agree with the usual definition if $H^*(Y; Z_{\hat{p}})$ is torsion-free – and we have to assume Y is innocent until it is proved guilty. With this new definition, we get a different theorem: if y is of rational filtration $\geq 2n$, then $s!\, p^s\, \mathrm{ch}_{n-r}(y)$ is integral in the new sense. This is different from the old result if $s \geq p$, and apparently it cannot be improved without further assumptions. Let us therefore make a further definition, and say that $K(Y)$ "has good integrality" if $p^s\, \mathrm{ch}_{n-r}(y)$ is integral in the new sense (whenever y has rational filtration $\geq 2n$). If this holds, then the relation between operations and rational filtration will be such as we are used to for torsion-free spaces. All torsion-free spaces have good integrality, but maybe some other spaces do also.

For example, consider the exceptional Lie group G_2; then BG_2 has good integrality (at the prime 2), although it has 2-torsion. Similarly, $BPSU(p)$ has good integrality at the prime p, although it has p-torsion. I also checked on $R(T)^{W_p}$ for the first p-adic reflection group which is not a Weyl group and seemed interesting, namely type 12 of the list [8], with $p = 3$. Finally, while I was thinking of these things I received a letter to which I shall return at the end.

Well, one might be ready to frame a conjecture that if G is a compact connected Lie group, then BG has good integrality. No such luck. Take $p = 2$ and consider the group $G = \mathrm{Spin}\ (8)$. The rational filtration on $K(B\ \mathrm{Spin}\ 8)$ coincides with the usual filtration of the Atiyah–Hirzebruch spectral sequence. Let $y = \Delta^+ - \Delta^-$; calculation shows that

$$\mathrm{ch}\ y = \chi + \frac{1}{24} P_1 \chi + \cdots$$

where χ is the Euler class. This is just consistent with my old theorem, because $p^s = 4$ and $\frac{1}{4} P_1 \chi$ lies in the image of

$$H^{12}(B\ \mathrm{Spin}\ 8;\ Z_{\hat{2}}) \to H^{12}(B\ \mathrm{Spin}\ 8;\ Q_{\hat{2}}),$$

with mod 2 reduction $w_4 w_8$. However, it doesn't survive the Atiyah–Hirzebruch spectral sequence (because $\mathrm{Sq}^3\ w_4 w_8 = w_7 w_8$); so it is not integral in the new sense.

The only way forward, alas, is to change the definition of integrality again for our limited purpose. We appeal to the same argument as for the filtration. That is, we seek an embedding in $K(BT)$; so we pull back the definition of "integrality" from $K(BT)$ under the putative embedding, although we don't yet know it exists. We see that we have to add the following to our list of good properties of R.

"There exists a valuation v on $E_0 R$ such that $v(p) = 1$ and $v(y) \geq n$ if and only if there exist $z \in E_0 R$ and m such that $x^m = p^{mn} z$."

Clearly the second clause determines v uniquely if it exists at all, so this is indeed a property of the ring $E_0 R$. Moreover, it is a necessary condition for the existence of an embedding; to see this, you pull back the usual p-adic valuation on $E_0 K(BT) = H^*(BT;\ Z_{\hat{p}})$.

Similarly, we see that we have to add the following to our list of good properties of R.

"If $y\in R$ is of rational filtration $\geq 2n$, then $v(\mathrm{ch}_{n-r}\, y)\geq -s$ where $s=[r/(p-1)]$." This also is a necessary condition for the existence of an embedding; and so we cannot succeed unless we prove it for $K(Y)$. But both conditions look inconvenient to verify.

To begin with, let us try to define a valuation v on E_0R by

$$v(y)=\ \mathrm{Sup}\ \{q/m \mid y^m\in p^qE_0R\}.$$

Then, setting aside the question of whether v has the properties of a valuation, it looks like a struggle to prove that $v(y)$ is an integer. In fact, come to think of it, it looks like a struggle to prove that $v(y)$ is even finite. Meditating upon this, we arrive at one more necessary condition. We have to add the following to our list of good properties of R.

"E_0R is finitely generated as an algebra over $Z_p^{\wedge}$."

In fact, at this point it hits one that there are maybe half-a-dozen finiteness properties that one knows for the classifying space of a compact Lie group, but which one does not know for the classifying space of a finite H-space. This situation seems to call for further work.

After this unsatisfactory report, I imagine that you might like some entertainment. I recall a famous reference [12]. This purports to be a letter from a mathematician long since dead, saying how glad he is to see his results rediscovered independently after such a lapse of time, and giving some explicit formulae which his successors had not found. I have received a letter of a similar nature; let it speak for itself.

"Gentlemen,

Mathematicians may be divided into two classes; those who know and love Lie groups, and those who do not. Among the latter, one may observe and regret the prevalence of the following opinions concerning the compact exceptional simple Lie group of rank 8 and dimension 248, commonly called E_8.

(1) That he is remote and unapproachable, so that those who desire to make his acquaintance are well advised to undertake an arduous course of preparation with E_6 and E_7.

(2) That he is secretive; so that any useful fact about him is to be found, if at all, only at the end of a long, dark tunnel.

(3) That he holds world records for torsion.

Point (1) deserves the following comment. Any right-thinking mathematician who wishes to construct the root-system of E_6 does so as follows: first he constructs the root-system of E_8, and then inside it he locates the root-system of E_6. In this way he benefits from the great symmetry of the root-system of E_8, and its perspicuous nature. If this good precedent is not followed in other researches, one should consider whether to infer a lack of boldness in the investigator rather than a lack of cooperation from the subject-matter.

Since point (2) is equivalent to point (1), we may pass to point (3). And here we should first reject the defences offered by some who might otherwise pass as well-informed. For they appear to regard it as a venial blemish on an otherwise worthy character, comparable to holding world records for the drinking of beer. This will not do. Let us first consider the riotous profusion of torsion displayed by such groups as $PSU(n)$. It then becomes clear that one can award a title to E_8 only by restricting the competition to simply-connected groups. This is as if one were to award a title for drinking beer, having first fixed the rules so as to exclude all citizens of Heidelberg, Munich, Burton-on-Trent, and any other place where they actually brew or drink much of the stuff. In other words, it is contrary to natural justice.

In the second place, to consider the question at all reveals a certain preoccupation with ordinary cohomology. Any impartial observer must marvel at your obsession with this obscure and unhelpful invariant. The author, like all respectable Lie groups, is much concerned to present a decorous and seemly appearance to the eyes of K-theory; and taken in conjunction with other general theorems, this forces him to have a modest amount of torsion in ordinary cohomology. I shall seek some suitable person to inform you in an Appendix.

As a further argument against points (1) and (2), it is natural to release some small scrap of information which you would not otherwise possess. And this may also serve to guarantee the authenticity of this letter; for you must at least believe that it comes via some mathematician who would not mislead you about my views. You may then be expecting me to reveal, for example, $H^*(BE_8; F_5)$. I shall not oblige you. That could only encourage you in the low tastes that I have already condemned. Instead, I shall note the following possibility. It may happen that a space Y has its K-theory $K^*(Y)$ torsion-free and zero in odd degrees, but nevertheless a careful study of $K^*(Y)$ will reveal that Y must have torsion in its ordinary cohomology. Again, I shall seek some suitable person to inform you in an Appendix.

Be it therefore known and proclaimed among you, that my K-theory $K(E_8)$ and that of my classifying space $K(BE_8)$ cannot be criticised in this respect, at least at the prime 5. Their conduct is such as would be blameless and above reproach in the K-theory of a space without 5-torsion in its ordinary cohomology.

Given at our palace, etc, etc,

and signed

E_8."

Appendix 1. We have $\pi_3(E_8) = Z$. The generator is represented by a homomorphism $\theta: S^3 \to E_8$. Using the Hurewicz theorem, and so on, the induced map

$$(B\theta)^*: H^4(BS^3; Z) \leftarrow H^4(BE_8; Z)$$

must be iso. On the other hand, the representation rings $R(S^3)$ and $R(E_8)$ restrict so differently on their respective tori that the induced map of their associated graded objects can't be iso in degree 4; its image can be identified with 60 $H^4(BS^3; Z)$. Therefore, in the Atiyah–Hirzebruch spectral sequence for BE_8, the permanent cycles in degree 4 are 60 $H^4(BE_8; Z)$. So this Atiyah–Hirzebruch spectral sequence has non-zero differentials, and this can only happen if BE_8 has 2, 3 and 5-torsion in its ordinary cohomology.

Appendix 2. The discussion of "good integrality" in the body of the paper covers this point.

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