I thank you for the privilege of giving this lecture. Prof. Toda has published about 70 papers on topology. It would seem to follow that some of these papers will get less than a minute of our time each. In fact, I can hardly attempt an exhaustive presentation of Toda's work. It will be better for all of us if I concentrate on a few key points. Let me begin by taking you back in time.

I first met Prof. Toda in 1955, at a meeting in Oxford which Henry Whitehead had organised and called the Young Topologists' Conference. You will quickly calculate that both Prof. Toda and I were younger then. Toda had spent the previous year in France and his English was not so fluent as it became later. Still, he could write on the blackboard. He wrote on the blackboard a large table.

\[ 2\Pi_{n+r}(S^n) \]

And he proceeded to fill the table in, beginning with the groups we already knew, and proceeding to those we didn't know. Where there was something noteworthy he pointed to it, turned to the audience and smiled politely; and when he got to a group \( \mathbb{Z}_2 \), the audience clapped a little, for this was a new record for 2-torsion. Finally he indicated the Hopf invariant \( H : \pi_{31}(S^{16}) \to \mathbb{Z} \). He wasn't the
only person interested in that invariant, because IIopf was sitting in the front row. IIopf wasn’t a young topologist even then, but Henry had made a few exceptions. Toda communicated that
\[ H^{-1}\{1\} = \emptyset. \]
Hopf became most interested. He wanted to know the status of this assertion. Was this a theorem Toda had proved, or was it a conjecture or the hypothesis of something to follow?

John Moore, who was sitting beside IIopf, tried to reassure him. “That’s solid.” Unfortunately IIopf understood British English but not American English, so further interpreters were required. But in the end, everyone was happy with this theorem.

The methods by which Toda had done these calculations, and proved this result, were basically those set out in his book, “Composition Methods in Homotopy Groups of Spheres,” Princeton Univ. Press 1962.

First, we have the methods of suspension-theory. Let me use the language of localisation, although our present understanding of localisation is more recent. Toda’s best contribution to it was in 1971. Then at the prime 2, we have a fibering whose fiber, total space and base are (up to weak equivalence)
\[ S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}. \]
Here \( \Omega X \) is the loop-space on \( X \), as introduced by Serre. By taking the exact homotopy sequence of such a fibering, we get an exact sequence
\[ \cdots \to \pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \to \pi_{r+1}(S^{2n+1}) \to \pi_{r-1}(S^n) \]
in which the first arrow is the Freudenthal suspension \( E \). So much is due to James.

At an odd prime \( p \), matters depend on the parity of \( n \). For the even case we have
\[ \Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1} \]
so the case of an even-dimensional sphere is reduced to the case of odd spheres. We now wish to study the double suspension
\[ E^2 : \pi_r(S^{2n-1}) \to \pi_{r+2}(S^{2n+1}). \]
Toda studied it in two steps: first, at \( p \) there is a fibering

\[
X \to \Omega S^{2n+1} \to \Omega S^{2pn+1}
\]

where \( X = S^{2n} \cup e^{4n} \cup e^{6n} \cup \cdots \cup e^{2(p-1)n} \) is the first part of a CW-model for \( \Omega S^{2n+1} \).

Here I should point out that Toda had possessed a CW-model for the loops on a suspension \( \Omega SX \) since 1953, which is two years before James' work appeared in print. At all events, there is at \( p \) a fibering

\[
S^{2n-1} \to \Omega X \to \Omega S^{2pn-1}.
\]

It seems fair to say that Toda's insight into such function-spaces and the fiberings in which they take part remained unequalled for twenty years. The work of Cohen, Moore and Neisendorfer didn't start to appear till 1979.

The methods of suspension-theory allow one to calculate \( \pi_{n+r}(S^n) \) by double induction over \( r \) and \( n \), provided one can keep a firm control on all the elements involved. For example, suppose one has an exact sequence

\[
\cdots A \overset{i}{\to} B \overset{j}{\to} C \overset{k}{\to} D \cdots
\]

and one has calculated the groups \( B, C \); then one wishes to calculate the homomorphism \( j \) between them; clearly one can only hope to determine \( j(b) \) if one knows exactly what \( b \) is. If \( j(b) = 0 \), there exists \( a \) such that \( i(a) = b \). If we have a constructive reason why \( j(b) = 0 \), we should have a construction for \( a \) as needed.

So secondly, we have methods of explicit geometric construction. In the homotopy groups of spheres we have the following primary operations: addition and subtraction; composition; Whitehead product. It was proved by Hilton that these exhaust the primary operations.

In fact, these operations suffice to explain the main point of Toda's proof about \( \pi_{31}(S^{16}) \). Barratt and Hilton had already proved that, in the stable homotopy groups of spheres, composition is anticommutative. More precisely, suppose given maps

\[
S^{a+b} \overset{f}{\to} S^b, \quad S^{c+d} \overset{g}{\to} S^d.
\]
Then you can form the smash product

\[ S^{a+b} \wedge S^{c+d} \xrightarrow{f \wedge g} S^b \wedge S^d \]

and write it in the two equivalent ways

\[ (1 \wedge g)(f \wedge 1) = (f \wedge 1)(1 \wedge g) \]

or

\[ (E^b g)(E^{c+d} f) = (-1)^{ac}(E^d f)(E^{a+b} g). \]

Suppose you suspend once less, and form

\[ (E_{b-1}^b g)(E^{c+d-1} f) - (-1)^{ac}(E^{d-1} f)(E^{a+b-1} g). \]

Then you have something which vanishes if you suspend it once more. There is a reason for this; you can write it in terms of the Whitehead product. Also the answer is zero if either \( f \) or \( g \) is a suspension, so you expect the result to depend only on the Hopf invariants of \( f \) and \( g \), and that’s what you find:

\[ = \pm \left[ \iota_{b+d-1}, \iota_{d-b-1} \right] E^{*} H f \ E^{*} H g. \]

It’s a theorem of Toda that this equality holds (under dimensional restrictions removed by Barratt). In particular, in \( \pi_{29}(S^{15}) \) we get

\[ [\iota_{15}, \iota_{15}] = 2\sigma_{15} \sigma_{22} \neq 0, \]

proving the result about the Hopf invariant on \( \pi_{31}(S^{16}) \). Of course, the same formula is used to identify other Whitehead products you need for \( \pi_{n+14}(S^n) \), \( n = 2, \ldots, 15 \).

Still, when primary operations are not enough you have to turn to secondary ones. The most celebrated secondary operation in homotopy theory is one which
Toda introduced; he tried to call it the toric constructon, but everyone else calls it the Toda bracket. Suppose given 4 spaces and 3 maps

\[ W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \]

and suppose \( hg \simeq 0 \), \( gf \simeq 0 \), so that we can extend these composites over cones:

\[ C_+ W \xrightarrow{f} \bigcup W \xrightarrow{g} X \xrightarrow{h} Z \]

Then we have \( hgf = 0 \) for two different reasons; we get a map \( SW \to Z \) by regarding \( SW \) as the union of the positive and negative cones. Taking account of the choices, we get a double coset in

\[ h[SW, Y]/[SW, Z]/[SX, Z]Sf. \]

These operations have good properties, and are important in Toda's calculations, and indeed in everyone else's calculations. But I will put this aside for now.

If we add the method of lifting homotopy groups, then we have mentioned the most important ingredients in Toda's book, but we shall have to hurry to mention some of what came later.

Next, about extended powers. Let \( X \) be a space with base-point (e.g. a finite CW-complex) and let \( G \) be a subgroup of \( \Sigma_n \) (e.g. \( Z_p \subset \Sigma_p \)). Then we can form

\[ ep(X) = D(X) = \frac{(EG) \times G \bigwedge^n X}{n} \]

If you have a preferred CW-model for \( EG \) (as with \( C = Z_p \)) you write

\[ ep^r(X) = \frac{(EG)^r \times G \bigwedge^n X}{n} \]
Toda used these extended powers to prove the following result among others. Let $y \in \pi_i S^t$, $py = 0$. Then $\alpha_1 y^p = 0$. The proof was, in principle, simple. If $py = 0$, then $y$ extends to a map

$$M = S^m \cup_p e^{m+1} \xrightarrow{f} S^{m-1}. $$

This yields

$$ep^{p-1}(M) \xrightarrow{ep^{p-1}(f)} ep^{p-1}(S^{m-1}) \xrightarrow{r} S^{pm-1} \xrightarrow{y^p} S^{pm-p^t}.$$ 

Now $ep^{p-1}(S^{m-1})$ is easy to analyse and has a retraction on $S^{pm-p^t}$. So the composite is $y^p$. But we can also look at the complex $ep^{p-1}(M)$ and find that $i\alpha_1 = 0$ here: so $y^p \alpha_1 = 0$.

In this way, Toda obtained his second proof that $\alpha_1 \beta_1^p = 0$, which settled a problem and allowed him to push his calculations of the $p$ components of homotopy groups of spheres up to the range $(p^2 + 2p)(p - 1) - 3$.

It would seem that this proof influenced Nishida in his work on the nilpotence of the stable homotopy groups of spheres. For, after all, the number $n$ of factors does not have to be the same as $p$; if one increases $n$ one may hope to make $ep^n(M)$ more like an Eilenberg-MacLane space and so prove $i_y = 0$, $y^n y = 0$ provided $py = 0$.

And, as we know, the nilpotence theorem of Nishida influenced the nilpotence conjecture of Ravenel, which was the starting-point for the work of Hopkins. But I am getting out of historical order.

It appears from our account of the Toda bracket that it factors in the form

$$SW \xrightarrow{F} Y \cup_g CX \xrightarrow{H} Z.$$ 

That is, you can replace Toda brackets by ordinary composites, but even if $W, X, Y, Z$ are spheres you have to be willing to consider complexes more general than spheres. For example, let $M = S^n \cup_p e^{n+1}$, $p \geq 3$. Then there is a map

$$S^{2(p-1)} M \xrightarrow{A} M,$$
and we can iterate $A$; we have

$$
\begin{array}{ccccc}
S^{2(p-1)i}M & \xrightarrow{A} & S^{2i}M & \rightarrow & \cdots & \rightarrow & S^{2(p-1)i}M & \xrightarrow{A} & M \\
\uparrow & & & & & & & \downarrow & \\
S^{n+2(p-1)i} & & & & & & & S^{n+1} &
\end{array}
$$

and the composite is Toda's element $\alpha_i$.

Such a construction shows that there is something systematic going on, and one tries to repeat it. That is, one defines $V(0) = M$, $V(1) = M \cup_A CS^{2(p-1)}M$, obtains a map $B : S^{n} V(1) \to V(1)$ and gets Toda's family $\beta_j$. One would like to to say "and so on". The question is, whether one can. It is easy to lay down the criteria which a space of type $V(n)$ should satisfy: $H^*(V(n); Z_p)$ should be a certain module over the Steenrod algebra, or

$$BP^*(V(n)) = \frac{\pi_*(BP)}{(p, v_1, \ldots, v_n)}.$$  

Toda proved that $V(2)$ exists for $p \geq 5$, $V(3)$ exists for $p \geq 7$. These results are unsurpassed to this day: there is a Chinese manuscript claiming $V(4)$ exists for $p \geq 11$, but I don't accept the argument sketched. In another paper, Toda worked on the algebra of self-maps of $V(1)$ in particular and of $Z_p$-spaces in general. Toda's results of 1971 gained significance with the work of Miller, Ravenel and Wilson in 1977. They were working with the Novikov-Adams spectral sequence for computing stable homotopy groups of spheres using $BP$-theory. In the $E_2$-term, in other words, in Ext, they found algebraic periodicity phenomena, as if the spaces $V(n)$ and their maps existed; in other words, whether the $V(n)$ exist or not, the algebra is ready for them. More recently, it has been shown by Hopkins that any finite complex of a suitable sort will have a periodicity map with the good properties of $A, B, \ldots$. The defect in our knowledge is that we don't know finite complexes to which we should apply this theorem. The conservative guess would be

$$
\forall n \quad V(n) \quad \text{exists for } p \geq p_0(n) \\
\forall p \quad V(n) \quad \text{does not exist for } n \geq n_0(p).
$$

But this is far beyond our grasp. A lesser result saying that some other appropriate problem behaves like this guess would have philosophical interest.
For a long time, Toda has been accepted as the master of calculations in homotopy theory. A mathematician is entitled to credit if his ideas have carried on in the work of his pupils and of mathematicians in other lands who may never have met him: Toda has such credit. I have always benefitted from studying his work. I'm happy to pronounce congratulations and good wishes on his birthday.