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Embeddings and Immersions

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Translated by
Kiki Hudson

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by Masahisa Adachi

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ABSTRACT. This book provides an introduction to the theory of embeddings and immersions of smooth manifolds and then gives applications of Gromov's theorems to foliations and complex structures on open manifolds.

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Preface to the English Edition

For the convenience of readers of this English edition I have replaced the original Japanese references with the appropriate references in English or French. I have also replaced some other references that are hard to obtain with those that are more readily available.

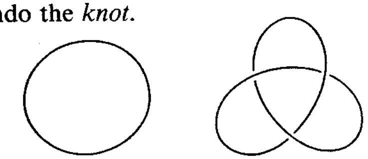
I wish to sincerely thank Professors Kobayashi and Nomizu for their advice. I am very grateful to Dr. Kiki Hudson, who has provided an excellent translation and pointed out misprints in the original edition.

Masahisa Adachi
October 28, 1992

Preface

Among closed surfaces the torus $T^2 = S^1 \times S^1$ can be thought of as sitting in three-dimensional Euclidean space $\mathbf{R}^3$, but the Klein bottle K^2 cannot be realized there. This observation naturally leads us to the question 'can a general n-dimensional manifold M^n be smoothly embedded in Euclidean space $\mathbf{R}^p$?'.

Further, it is possible to embed the circle S^1 in three-dimensional Euclidean space $\mathbf{R}^3$, but there is more than one way to do so. For example, we cannot move one of the two embeddings below to the other via an isotopy; that is, we cannot undo the *knot*.



This is generalized to the problem 'are two given embeddings $f, g : M^n \to \mathbf{R}^p$ isotopic?' Since the concept of topology was first established, these problems have been in its mainstream, and major contributions to solutions have come from H. Whitney and A. Haefliger. Still further research and development can be expected in this field.

In particular, the problem of classifying embeddings of the circle S^1 in three-dimensional Euclidean space $\mathbf{R}^3$ or the three-dimensional sphere S^3 through isotopies—a bit different from the isotopies mentioned in the previous paragraph —forms a field in topology called the *theory of knots*, which even today generates many research activities.

The problem of classifying immersions by regular homotopies is slightly easier than that of classifying embeddings by isotopies. Here is an example. In three-dimensional Euclidean space $\mathbf{R}^3$, is it possible to turn the sphere S^2 inside out smoothly allowing self-intersections? Think about it for a minute. It hardly seems likely, but a classification theorem for immersions shows that it can be done.

This classification theorem, the so-called Smale-Hirsch theorem, has been generalized step by step by A. Phillips, M. Gromov, A. Haefliger, and so on to the present stage where it now offers us a tool for finding solutions (or their candidates) to partial differential inequalities or partial differential equations of certain types. It also provides us with a method for eliminating singularites of certain C^∞ maps. There are further applications of these methods as well.

The aim of this book is to give an introduction to this theory in modern topology and its applications. In accordance with the principle of this series we have tried to make the first three chapters easy enough to understand at the level of lower-division mathematics.

In this book, unless otherwise stated, embeddings and immersions will be viewed in the C^∞ category. We first explain in detail the classification of regular closed curves in the plane by regular homotopies; this will serve as an intuitive preparation for the contents of the book.

In Chapter I, we give a summary of basic concepts about C^r manifolds and C^r maps which will be used in Chapter II and beyond.

The discussions in Chapter II evolve around Whitney's theorems. This chapter also serves as a prelude to Chapter VII. We develop Chapter III around the Smale-Hirsch theorem which is generalized to Gromov's theorem.

In Chapter IV we examine the convex integration theory due to Gromov which is another application of the Smale-Hirsch theorem.

In Chapter V we discuss an application of Gromov's theorem, namely, a classification theorem for foliations of open manifolds. In Chapter VI we study complex structures on open manifolds as an application of Gromov's theorem and Gromov's convex integration theory.

We study Haefliger's embedding theorem in Chapter VII, which is a continuation of Chapter II.

Finally, as references we give a list of books and papers we have either used, adapted, or quoted from directly, and also books and papers basic to embeddings and immersions.

The author thanks Kazuhiko Fukui, Shigeo Kawai, and Goo Ishikawa for their valuable help in writing this book.

We are deeply indebted to Professor Itiro Tamura who encouraged us to write this book and gave us valuable advice concerning the first draft.

Last but not least our deepest gratitude goes to Mr. Hideo Arai of Iwanami Shoten Publishers, without whose help this book would never have been realized.

Masahisa Adachi
May 1983

CHAPTER 0

Regular Closed Curves in the Plane

In this chapter we consider closed curves in the plane $\mathbf{R}^2$, whose tangent lines move continuously. To each closed curve we assign the "rotation number" γ, which is the angle the tangent line makes going around the curve once ($\gamma = \pm 2\pi$ for a closed curve). Our aim in this chapter is to show the following:

> Two closed curves with the same rotation number can be deformed from one to the other.

This chapter is based on Whitney [C20].

§1. Regular closed curves

We first define closed regular curves.

Let $I = [0, 1]$. Consider $f = (f_1, f_2)$, where f_1 and f_2 are C^1 functions (f is called a C^1 map). We say that f is a *parametrized regular closed curve* if it satisfies the following:

(i) $f(0) = f(1)$, $f'(0) = f'(1)$,
(ii) $f'(t) \neq 0$, for each $t \in I$.

The condition (i) shows that the curve is closed and (ii) says that f is regular in some sense with respect to the parameter t. See Figure 0.1.

To the above $f : I \to \mathbf{R}^2$ there corresponds a C^1 function $\tilde{f}$

$$\tilde{f} : (-\infty, \infty) \longrightarrow \mathbf{R}^2,$$

such that

(iii) $\tilde{f}(t) = f(t)$, $t \in I$,
(iv) $\tilde{f}(t+1) = \tilde{f}(t)$,
(v) $\tilde{f}'(t) \neq 0$.

Conversely to such an $\tilde{f}$ there corresponds an f as above. We say that $\tilde{f}$ is a *lift* of f.

DEFINITION 0.1. Let f and g be parametrized regular closed curves. We say that f and g are *equivalent* and write $f \sim g$ if there exists a C^1 function $\eta : (-\infty, \infty) \to (-\infty, \infty)$ such that

$$\eta'(t) > 0, \quad \text{for each } t \in \mathbf{R}, \qquad \eta(t+1) = \eta(t) + 1, \qquad \tilde{g}(t) = \tilde{f} \circ \eta(t).$$

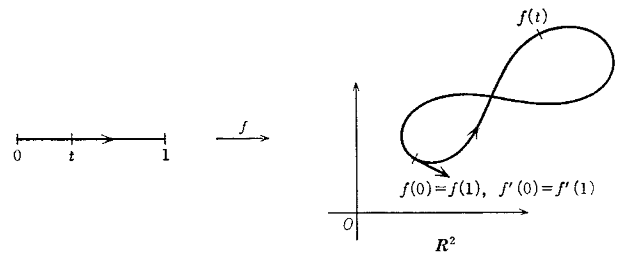


FIGURE 0.1

Clearly, $\sim$ is an equivalence relation; hence, $\sim$ divides parametrized regular closed curves into equivalence classes which are called *regular closed curves* or simply *curves*. If $f \sim g$, then we have $f(I) = g(I)$.

PROPOSITION 0.1. *Let C be a regular closed curve. Then there exists an element g in C such that $\|g'(t)\|$ is constant, where $\| \ \|$ is a norm on $\mathbf{R}^2$.*

PROOF. Let $f \in C$ and let $\tilde{f}$ be a lift of f. Set

$$L(t) = \int_0^t \|\tilde{f}'(s)\|\, ds, \qquad L = L(1).$$

Then $L = L(C)$ is the length of the curve C. Since $\tilde{f}'(t) \neq 0$, $L(t)$ is of class C^1 and monotone increasing. Hence, we can solve $s = 1/L \cdot L(t)$ for t, say $t = \eta(s)$. $\eta'(s)$ is continuous and positive. Since $\tilde{f}$ is periodic we have

$$L(t+1) - L(t) = \int_t^{t+1} \|\tilde{f}'(s)\|\, ds = \int_0^1 \|\tilde{f}'(s)\|\, ds.$$

Therefore, $\eta(s+1) = \eta(s) + 1$. Thus, if we set

$$\tilde{g}(t) = \tilde{f} \circ \eta(t),$$

we see that $\tilde{g}$ is a lift of some element g of C. Further we have

$$\tilde{g}'(t) = \tilde{f}' \circ \eta(t) \cdot \frac{L}{L'(\eta(t))}, \qquad \|\tilde{g}'(t)\| = L.$$

So $\|g'(t)\|$ is constant. □

PROPOSITION 0.2. *Let C be a regular closed curve, and let g be an element of C defined as above. Suppose h is an element of C such that $\|h'(t)\|$ is a constant k. Then*

(i) $k = L$ *and*
(ii) $\tilde{h}(t) = \tilde{g}(t+a)$ *for some constant a.*

In other words, two elements of C with each $\|h'(t)\|$ constant differ by some rotation of the circle S^1.

PROOF. Since $h \sim g$ there exists $\eta : (-\infty, \infty) \to (-\infty, \infty)$ such that $\tilde{h}(t) = \tilde{g}(\eta(t))$. But

$$h'(t) = g'(\eta(t))\eta'(t),$$

and so $k = L \cdot \eta'(t)$. Hence,

$$1 = \eta(1) - \eta(0) = \int_0^1 \eta'(t)\, dt = \int_0^1 \frac{k}{L}\, dt = \frac{k}{L}.$$

Hence, we get $k = L$. It follows that $\eta'(t) = 1$ and that $\eta(t) = t + a$. □

DEFINITION 0.2. Let f_0 and f_1 be parametrised regular closed curves. We say that f_0 is a *deformation* of f_1 or that f_0 and f_1 are *regularly homotopic*, and write $f_0 \underset{r}{\simeq} f_1$ if the following holds: For some continuous map $F : I \times I \to \mathbf{R}^2$

(i) $F(t, 0) = f_0$, $F(t, 1) = f_1(t)$, and
(ii) if we set $f_u(t) = F(t, u)$, then $f_u : I \to \mathbf{R}^2$ is a parametrized regular curve for each $u \in I$. Here we say that F or the $\{f_u\}$ is a *regular homotopy*.

We see that the relation $\underset{r}{\simeq}$ of being regularly homotopic is an equivalence relation. See Figure 0.2.

PROPOSITION 0.3. *Let C be a regular closed curve, and let f_0, $f_1 \in C$. Then f_0 is a deformation of f_1 in C; that is, there exists a regular homotopy $f_u \in C$, $u \in I$, connecting f_0 and f_1.*

PROOF. That f_0 and f_1 are equivalent implies that $\tilde{f}_1(t) = \tilde{f}_0 \circ \tilde{\eta}(t)$ for some function η as in Definition 0.1.

Set

$$\eta_u(t) = u\eta(t) + (1-u)t, \qquad 0 \le u \le 1,$$
$$\tilde{f}_u(t) = \tilde{f}_0 \circ \eta_u(t), \qquad \tilde{f}_0 \text{ is a lift of } f_0.$$

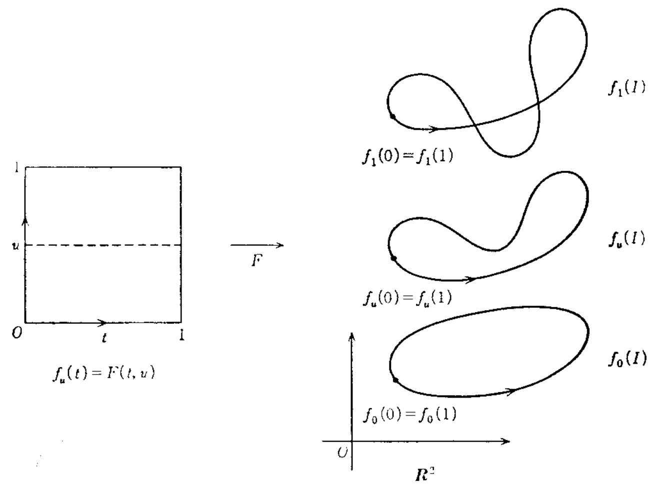


FIGURE 0.2

Here we have $\eta_0(t) = t$, $\eta_1(t) = \eta(t)$. Hence, $\tilde{f}_1$ is a lift of f_1, and we have

$$\eta_u(t+1) = u[\eta(t)+1] + (1-u)(t+1) = \eta_u(t) + 1,$$

$$\frac{d\eta_u(t)}{dt} = u\frac{d\eta(t)}{dt} + (1-u) > 0, \qquad 0 \le u \le 1.$$

Therefore, each f_u is a parametric regular closed curve, and so we have the proposition. □

By virtue of Proposition 0.3 the expression " a regular closed curve C is a deformation of a regular closed curve C' " makes sense.

§2. Regular homotopies

We have the following basic

LEMMA 0.1. *Let $g : I \to \mathbf{R}^2$ be a continuous map and suppose $g(t) \neq 0$ for each $t \in I$. For $p \in \mathbf{R}^2$,*

$$f(t) = p + \int_0^t g(s)\,ds$$

is a parametrized regular curve if and only if

$$g(0) = g(1), \qquad \int_0^1 g(s)\,ds = 0.$$

The lemma is obvious.

DEFINITION 0.3. For a parametrized closed regular curve $f : I \to \mathbf{R}^2$ we define the rotation number $\gamma(f) \in \mathbf{R}$ of f as follows: the map

$$f^* : I \longrightarrow S^1 \subset \mathbf{R}^2, \qquad f^*(t) = \frac{f'(t)}{\|f'(t)\|}$$

defines naturally the continuous map $\hat{f}^* : S^1 \to S^1$. Now define

$$\gamma(f) = 2\pi \cdot \deg(\hat{f}^*),$$

where $\deg(\hat{f}^*)$ is the *degree* of $\hat{f}^*$ (1).

---

(1) In general the degree $\deg(h)$ of a continuous map $h : S^1 \to S^1$ is an integer which represents the number of the times $h(S^1)$ wraps around S^1 inclusive of the sign of $h(S^1)$. The following is a more precise definition. Notice that the fundamental group of the circle S^1, $\pi_1(S^1)$, is isomorphic to $\mathbf{Z}$. Let s be the generator of this group. On the other hand h defines the homomorphism

$$h_* : \pi_1(S^1) \longrightarrow \pi_1(S^1)$$

and the image of s by h_* is $n \cdot s$, $n \in \mathbf{Z}$. We define the *degree* or the *mapping degree* $\deg(h)$ of h to be this n.

PROPOSITION 0.4. *Let f, g be parametrized regular closed curves. If f and g are regularly homotopic then $\gamma(f) = \gamma(g)$.*

PROOF. Let $f_u : I \to \mathbf{R}^2$ be a regular homotopy connecting f and g, say $f_0 = f$ and $f_1 = g$. Then f^* and g^* are homotopic through f_u^*, and so $\hat{f}^*$ and $\hat{g}^*$ are homotopic through $\hat{f}_u^*$. Hence, $\deg(\hat{f}^*) = \deg(\hat{g}^*)$. □

DEFINITION 0.4. Let C be a regular closed curve. We define the *rotation number* $\gamma(C)$ of C by $\gamma(C) = \gamma(f)$, $f \in C$.

By Proposition 0.3 the above definition does not depend on the choice of f.

THEOREM 0.1. *Regular closed curves C_0 and C_1 are regularly homotopic if and only if $\gamma(C_0) = \gamma(C_1)$.*

This theorem is known as the *Whitney-Graustein theorem.*

COROLLARY 0.1. *The family of the regular homotopy classes of regular closed curves in the plane is in one-to-one correspondence with the set $\mathbf{Z}$ of the integers by the map $C \mapsto \gamma(C)/2\pi$.*

PROOF OF THEOREM 0.1. The 'only if' part is evident by Proposition 0.4. To prove the 'if' part set $\gamma(C_0) = \gamma(C_1) = \gamma$. Choose $g_0 \in C_0$ and $f_1 \in C_1$ such that

$$\|g_0'(t)\| = L(C_0) = L_0, \qquad \|f_1'(t)\| = L(C_1) = L_1$$

(cf. Proposition 0.1). Define g_u by

$$g_u(t) = g_0(0) + \left[u \cdot \frac{L_1}{L_0} + (1-u)\right]\{g_0(t) - g_0(0)\}.$$

Then the family $\{g_u\}$ is a homotopy connecting g_0 and g_1. Further as $g_u'(t) \neq 0$, for each $t \in I$, the $\{g_u\}$ is actually a regular homotopy connecting g_0 and g_1. Set $f_0 = g_1$. We then have $\|f_0'(t)\| = \|g_1'(t)\| = L_1$.

We want to show that f_0 is regularly homotopic to f_1. Let K be the circle in $\mathbf{R}^2$ centered at the origin of radius L_1. Then f_0', $f_1' : I \to K \subset \mathbf{R}^2$. If $\hat{f}_0'$, $\hat{f}_1' : S^1 \to K$ are the natural maps corresponding to f_0' and f_1', we have $\deg(\hat{f}_0') = \deg(\hat{f}_1') = \gamma/2\pi$. Hence, $\hat{f}_0'$ and $\hat{f}_1'$ are homotopic. Now define

$$\theta : \mathbf{R} \longrightarrow K$$

by

$$\theta(t) = (L_1 \cos t,\, L_1 \sin t).$$

(i) For $\gamma \neq 0$ we have $\theta(0) = (L_1, 0)$. Without loss of generality we may assume that $f_0'(0) = f_1'(0) = \theta(0)$. As $f_i'(t) \in K$, $i = 0, 1$, denoting by $F_i(t)$ the argument of $f_i'(t)$, we have the following

$$F_i : I \longrightarrow \mathbf{R}, \qquad i = 0, 1,$$
$$f_i'(t) = \theta \circ F_i(t), \qquad F_i(0) = 0.$$

Then by the definition of and the assumption on γ we see that

$$F_i(1) = \gamma, \qquad i = 0, 1.$$

Now set

$$\begin{cases} F_u(t) = uF_1(t) + (1-u)F_0(t), & 0 \le u \le 1. \\ h_u(t) = \theta \circ F_u(t), \end{cases}$$

Then the $\{h_u\}$ is a homotopy connecting f_0' and f_1'. Set

$$\begin{cases} \varphi_u(t) = h_u(t) - \int_0^1 h_u(s)\,ds, \\ f_u(t) = f_0(0) + u[f_1(0) - f_0(0)] + \int_0^t \varphi_u(s)\,ds. \end{cases}$$

Evidently $\int_0^1 \varphi_u(t)\,dt = 0$, and so $f_u(0) = f_u(1)$. Moreover, we have that $f_u'(t) = \varphi_u(t)$. Since $F_u(0) = 0$, $F_u(1) = \gamma$, and γ is an integral multiple of 2π, we get

$$\begin{aligned} f_u'(1) - f_u'(0) &= \theta \circ F_u(1) - \theta \circ F_u(0) \\ &= \theta(\gamma) - \theta(0) = 0; \end{aligned}$$

therefore, $f_u'(1) = f_u'(0)$, for each $u \in I$.

Next we show that $f_u'(t) \neq 0$, $u \in [0, 1]$. We have

$$f_u'(t) = h_u(t) - \int_0^1 h_u(s)\,ds, \qquad h_u(t) \in K.$$

If $\gamma \neq 0$, then $\int_0^1 h_u(s)\,ds$ lies in the interior of K, because by Schwarz's inequality

$$\left\| \int_0^1 h_u(s)\,ds \right\|^2 \le \int_0^1 \|h_u(s)\|^2\,ds.$$

But $h_u(s)$ is not a constant number, and hence the above inequality must be a strict inequality. Moreover, $\|h_u(s)\|^2 = L_1$ implies that

$$\left\| \int_0^1 h_s(s)\,ds \right\|^2 < L_1.$$

Hence, $f_u'(t) \neq 0$. Thus, we have shown that f_u is a regular closed curve, and so the $\{f_u\}$ is a regular homotopy connecting f_0 and f_1.

(ii) Case $\gamma = 0$. Suppose we can change $F_u(t)$ so that for each $u \in [0, 1]$, $F_u(t)$ is not a constant map. Then $f_u'(t) \neq 0$ for each u and we have the proof. To make such a change take a point t_0 with $F_1(t_0) \neq 0$ and deform $F_0(t)$ to $F_1(t)$ in a sufficiently small neighborhood of t_0. Denoting by F_u the newly obtained deformation of F_0 to F_1 we repeat the above process. We then see that F_u is not a constant map for each u. □

Lemma 0.1 suggests a later development of our subject.

CHAPTER I

C^r Manifolds, C^r Maps, and Fiber Bundles

In this chapter we shall collect together the fundamental facts about C^r manifolds, C^r maps, and fiber bundles as well as other preparatory items necessary in the later chapters.

§1. C^∞ manifolds and C^∞ maps

Here we give a brief summary of C^∞ manifolds.

A. C^∞ manifolds. First we define a C^∞ manifold. Let $\mathbf{R}^n$ be n dimensional Euclidean space with a fixed coordinate system. Then a point x of $\mathbf{R}^n$ is represented by the n-tuple

$$x = (x_1, x_2, \ldots, x_n).$$

Consider a function defined on an open subset U of $\mathbf{R}^n$

$$f : U \longrightarrow \mathbf{R}^1.$$

Let r be a natural number or ∞. We say that f is *differentiable of class* C^r, f is of *class* C^r, or simply f is C^r if at each point x of U all partial derivatives of f of the form

$$\left.\frac{\partial^s f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_s}}\right|_x, \qquad \begin{cases} 1 \le i_1 \le \cdots \le i_s \le n, \\ 1 \le s \le r \end{cases}$$

exist and are continuous.

Consider a map $f : U \to \mathbf{R}^p$ from an open subset U of $\mathbf{R}^n$ to $\mathbf{R}^p$. Writing $f(x) = (f_1(x), \ldots, f_p(x)) \in \mathbf{R}^p$, we say that f is *differentiable of class* C^r if for each $1 \le i \le p$, $f_i : U \to \mathbf{R}$ is a differentiable function of class C^r. The definition of a C^∞ map is similar. A real analytic map f is sometimes called C^ω.

DEFINITION 1.1. A topological space M^n is called an n-dimensional *topological manifold* if it satisfies the following:

(i) M^n is a Hausdorff space.

(ii) For each point x of M^n there exists a neighborhood $U(x)$ which is homeomorphic to $\mathbf{R}^n$.

(iii) M^n satisfies the second axiom of countability.

Now we define a differentiable structure on a topological manifold.

Let M^n be a topological manifold of dimension n. By a C^∞ *coordinate system* or an C^∞ *atlas* for M^n we mean a family $\mathcal{S} = \{(V_j, \varphi_j) \mid j \in J\}$ of pairs (V_j, φ_j) of open sets V_j in M^n and homeomorphisms $\varphi_j : V_j \to \mathbf{R}^n$ of V_j in $\mathbf{R}^n$ satisfying the following:

(i) $M^n = \bigcup_{j\in J} V_j$,
(ii) If $V_i \cap V_j \neq \emptyset$, then the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(V_i \cap V_j) \longrightarrow \varphi_j(V_i \cap V_j)$$

from an open subset of $\mathbf{R}^n$ to an open subset of $\mathbf{R}^n$ is of class C^∞ (Figure 1.1).

The pair (V_j, φ_j) is a *chart* or *a system of local coordinates* and V_j is a *coordinate neighborhood.*

Two C^∞ atlases $\mathcal{S} = \{(V_j, \varphi_j) \mid j \in J\}$ and $\mathcal{S}' = \{(V'_k, \varphi'_k) \mid k \in K\}$ are *equivalent*, $\mathcal{S} \sim \mathcal{S}'$, if the combined family $\mathcal{S} \cup \mathcal{S}'$ of the two systems is also a C^∞ atlas for M^n. Evidently the relation $\sim$ is an equivalence relation. An equivalence class $\mathcal{D} = [\mathcal{S}]$ in M^n is a *differentiable structure* or a C^∞ *structure* for M^n, and the pair $(M^n, \mathcal{D})$ is a *differentiable* or C^∞ *manifold* with the *underlying topological manifold* M^n.

The above definition is known as Whitney's definition. More generally if the maps $\varphi_j \circ \varphi_i^{-1}$ in the definition of a C^∞ manifold are of class C^r, $0 \le r \le \omega$, we say that $(M^n, \mathcal{D})$ is a C^r *manifold.* A C^0 manifold is a topological manifold. Often a differentiable manifold is understood to be a C^1 manifold; however, in this book we agree for simplicity that a differentiable manifold is a C^∞ manifold, which is also called a *smooth manifold.*

Next we discuss orientations of a C^∞ manifold $(M^n, \mathcal{D})$. Let $\mathcal{D} = [\mathcal{S}]$, $\mathcal{S} = \{(V_j, \varphi_j) \mid j \in J\}$. For $x \in V_i \cap V_j$ let $a_{ji}(x)$ be the Jacobian matrix of $\varphi_j \circ \varphi_i^{-1}$ at $\varphi_i(x)$:

$$a_{ji}(x) = D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(x)}, \qquad x \in V_i \cap V_j.$$

Then it is easy to see that

$$a_{kj}(x) \cdot a_{ji}(x) = a_{ki}(x), \qquad x \in V_i \cap V_j \cap V_k.$$

If we set $k = i$, it follows that $a_{ji}(x)$ has an inverse. Hence $a_{ji}(x) \in \mathrm{GL}(n, \mathbf{R})$, where $\mathrm{GL}(n, \mathbf{R})$ denotes the general linear group of $\mathbf{R}^n$. Hence, we have a continuous map

$$a_{ji} : V_i \cap V_j \longrightarrow \mathrm{GL}(n, \mathbf{R}).$$

A differentiable atlas $\mathcal{S} = \{(V_j, \phi_j) \mid j \in J\}$ is *oriented* if for all i, j and all $x \in V_i \cap V_j$, the determinant $|a_{ij}(x)|$ is positive.

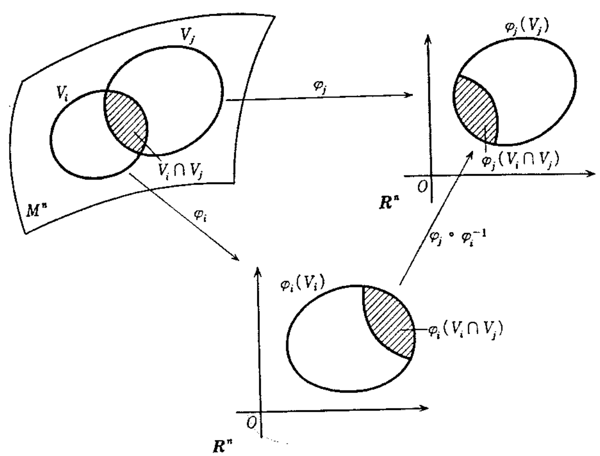


FIGURE 1.1

Let $\mathcal{S} = \{(V_j, \phi_j) \mid j \in J\}$ and $\mathcal{S}' = \{(V'_k, \phi'_k) \mid k \in K\}$ be oriented C^∞ atlases for M^n. For all $(j, k) \in J \times K$ and all $x \in V_j \cap V'_k$ with $V_j \cap V'_k \neq \emptyset$ the determinants of the Jacobian matrices of $\phi'_k \circ \phi_j^{-1}$ at $\phi_j(x)$ are either all positive or all negative, and we say that $\mathcal{S}$ and $\mathcal{S}'$ are *positively related* or *negatively related* accordingly. The oriented C^∞ atlases for M^n are divided into two classes according to the relation 'positively related'.

DEFINITION 1.2. An equivalence class of an oriented C^∞ atlas for M^n is called an *orientation* of M^n.

A C^∞ manifold $(M^n, \mathcal{D})$ is said to be *orientable* if it admits an oriented C^∞ atlas $\mathcal{S}$ such that $[\mathcal{S}] = \mathcal{D}$.

We say that an orientable manifold is *oriented* when we specify its orientation. The n dimensional sphere S^n, $n \ge 1$, is orientable.

We list some examples of differentiable manifolds. They will remind the reader that differentiable manifolds abound everywhere we look.

(1) n-dimensional Euclidian space is a C^∞ manifold.
(2) The n-dimensional sphere

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$$

with the relative topology as a subspace of $\mathbf{R}^{n+1}$ is a C^∞ manifold.

(3) Open submanifolds. Let $(M^n, \mathcal{D})$ be a C^∞ manifold and let U be an open subset of M^n. For an atlas $\mathcal{S} = \{(V_j, \varphi_j) \mid j \in J\}$,

$$\mathcal{S}_U = \{(V_j \cap U, \varphi_j | V_j \cap U) \mid j \in J\}$$

becomes an atlas of U. Set $\mathcal{D}_U = [\mathcal{S}_U]$ and say that $(U, \mathcal{D}_U)$ is an *open submanifold* of $(M^n, \mathcal{D})$. This definition does not depend on the choice of a representative $\mathcal{S}$.

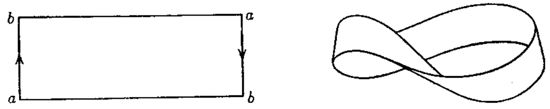


FIGURE 1.2

(4) Submanifolds. Let $(M^n, \mathscr{D})$ be a C^∞ manifold, and let A be a subset of M^n. Regard $\mathbf{R}^k$, $0 \le k \le n$, as a subspace of $\mathbf{R}^n$: $\mathbf{R}^k = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_{k+1} = \cdots = x_n = 0\}$. Now assume that we can choose a representative $\mathscr{S} = \{(V_j, \varphi_j) | j \in J\}$ of $\mathscr{D}$ such that for each j with $V_j \cap A \neq \emptyset$

$$\varphi_j | V_j \cap A : V_j \cap A \longrightarrow \mathbf{R}^k \subset \mathbf{R}^n$$

is a homeomorphism onto an open subset of $\mathbf{R}^k$. Then evidently A is a topological manifold, and $\mathscr{S}_A = \{(V_j \cap A, \varphi_j | V_j \cap A) | j \in J\}$ defines an atlas of A. We say that $(A, \mathscr{S}_A)$ is a *submanifold* of M^n.

REMARK. A submanifold in Example 4 is different from a "submanifold" as used in differential geometry. Our submanifolds are submanifolds in differential geometry, but the converse is not true.

(5) Product manifolds. Let $(M, \mathscr{D})$ and $(M', \mathscr{D}')$ be C^∞ manifolds of dimensions n and n' respectively. Set $\mathscr{D} = [\mathscr{S}]$, $\mathscr{D}' = [\mathscr{S}']$, $\mathscr{S} = \{(V_j, \varphi_j) | j \in J\}$, $\mathscr{S}' = \{(V'_k, \varphi'_k) | k \in K\}$. Clearly, $M \times M'$ is an $n + n'$ topological manifold. Further, the set

$$\mathscr{S} \times \mathscr{S}' = \{(V_j \times V'_k, \varphi_j \times \varphi'_k) | (j, k) \in J \times K\}$$

turns out to be an atlas for $M \times M'$. We say that $(M \times M', [\mathscr{S} \times \mathscr{S}'])$ is the *product manifold* of $(M, \mathscr{D})$ and $(M', \mathscr{D}')$. When there is no confusion we simply write $M \times M'$.

EXAMPLE. The *torus* $T^2 = S^1 \times S^1$ is the product of two copies of the circle S^1.

(6) The Möbius strip. We obtain a *Möbius strip* by twisting a strip of a tape and pasting the edges as shown in Figure 1.2. More precisely the Möbius strip M^2 is defined by

$$M^2 = [0, 1] \times [0, 1] / \sim,$$
$$(0, t) \sim (1, 1 - t), \qquad t \in [0, 1].$$

The interior $\overset{\circ}{M}{}^2$ of the Möbius strip is a two-dimensional C^∞ manifold. This manifold is not orientable.

(7) Projective spaces. The *n-dimensional real projective space* $P_n(\mathbf{R}) = S^n / \sim$, $x \sim -x$, is an n-dimensional C^∞ manifold. We shall give a proof for the case $n = 2$. We may think of $P_2(\mathbf{R})$ as

$$P_2(\mathbf{R}) = \{[x_1, x_2, x_3] | \text{ not all } x_1, x_2, x_3 \text{ are zero, } x_i \in \mathbf{R}, i = 1, 2, 3\}.$$

Set

$$U_i = \{[x_1, x_2, x_3] | x_i \neq 0\}, \qquad i = 1, 2, 3,$$

where $[x_1, x_2, x_3]$ is the equivalence class containing (x_1, x_2, x_3) (the condition $x_i \neq 0$ does not depend on the choice of a representative). Then $\{U_1, U_2, U_3\}$ is an open cover of $P_2(\mathbf{R})$:

$$P_2(\mathbf{R}) = U_1 \cup U_2 \cup U_3.$$

In addition $P_2(\mathbf{R})$ evidently satisfies the axiom of second countability. Next we define $\varphi_i : U_i \to \mathbf{R}^2$, $i = 1, 2, 3$ by

$$\varphi_1[x_1, x_2, x_3] = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right),$$
$$\varphi_2[x_1, x_2, x_3] = \left(\frac{x_3}{x_2}, \frac{x_1}{x_2}\right),$$
$$\varphi_3[x_1, x_2, x_3] = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

Clearly the above definition does not depend on the choice of a representative (x_1, x_2, x_3) of a point of $P_2(\mathbf{R})$. It is also obvious that each φ_i is a topological map of U_i onto $\mathbf{R}^2$. Hence we see that $P_2(\mathbf{R})$ is a topological manifold. Next for a point $[x_1, x_2, x_3]$ in the intersection $U_{12} = U_1 \cap U_2$ of U_1 and U_2 put

$$\varphi_1[x_1, x_2, x_3] = (u_1, u_2),$$
$$\varphi_2[x_1, x_2, x_3] = (\bar{u}_1, \bar{u}_2).$$

Then we have the expressions

$$\bar{u}_1 = \frac{u_2}{u_1}, \qquad \bar{u}_2 = \frac{1}{u_1},$$
$$u_1 = \frac{1}{\bar{u}_2}, \qquad u_2 = \frac{\bar{u}_1}{\bar{u}_2}.$$

Since $[x_1, x_2, x_3] \in U_{12}$, we have $x_1 \neq 0$, $x_2 \neq 0$. Hence, $u_1 \neq 0$ and $\bar{u}_2 \neq 0$. Therefore, $\bar{u}_1$, $\bar{u}_2$ are C^∞ functions of (u_1, u_2), and u_1, u_2 are C^∞ functions of $(\bar{u}_1, \bar{u}_2)$. The same statement holds for points of $U_2 \cap U_3$ and $U_3 \cap U_1$. Hence the family

$$\mathscr{D} = \{(U_i, \varphi_i) | i = 1, 2, 3\}$$

defines a C^∞ structure on $P_2(\mathbf{R})$. Hence, $P_2(\mathbf{R})$ is a C^∞ manifold. We may consider a natural C^∞ structure on $P_n(\mathbf{R})$ (verbatim as for the case $n = 2$ above) and thus conclude that $P_n(\mathbf{R})$ is a C^∞ manifold.

B. Differentiable maps. Let $(M_1, \mathscr{D}_1)$ and $(M_2, \mathscr{D}_2)$ be C^∞ manifolds of dimensions m and n, respectively.

DEFINITION 1.3. Consider a map $f : M_1 \to M_2$ of M_1 into M_2. For a point x of M_1 choose a chart V_j about x from a representative $\{(V_j, \varphi_j) |$

$j \in J\}$ of $\mathscr{D}_1$ and a chart V'_k about $f(x)$ from a representative $\{(V'_k, \varphi'_k) \mid k \in K\}$ of $\mathscr{D}_2$. Then we have the map

$$\varphi'_k \circ f \circ \varphi_j^{-1} : \varphi_j(V_j) \longrightarrow \varphi'_k(f(V_j) \cap V'_k)$$

from $\mathbf{R}^m$ onto an open subset of $\mathbf{R}^n$. We say that f is *differentiable at* x if $\varphi'_k \circ f \circ \varphi_j^{-1}$ is infinitely (continuously) differentiable at $\varphi_j(x)$. The map f is *differentiable* if it is differentiable at each point of M_1. We also say that f is a C^∞ *map*. Likewise we define a C^r *map* for any natural number r, $0 \le r \le \omega$.

From the definition of C^∞ structures, it is easy to see that the above definition is independent of the choice of representatives of $\mathscr{D}_1$ and $\mathscr{D}_2$ as well as V_j and V'_k.

Let M_1, M_2 be C^∞ manifolds, and let $f : A \to M_2$ be a map of a subset A of M_1. We say that f is *differentiable in* A if we can extend f to a C^∞ map of an open neighborhood U of A.

DEFINITION 1.4. Let M_1 and M_2 be C^∞ manifolds. We say that a map $f : M_1 \to M_2$ is a *diffeomorphism* if f satisfies the following:

(i) f is a homeomorphism of M_1 onto M_2, and
(ii) f, f^{-1} are C^∞ maps.

DEFINITION 1.5. Let M_1 and M_2 be C^∞ manifolds. We say that M_1 and M_2 are *diffeomorphic* and write $M_1 \underset{d}{\approx} M_2$ if there exists a diffeomorphism $f : M_1 \to M_2$.

Evidently $\underset{d}{\approx}$ is an equivalence relation. In differential topology we identify two manifolds which are diffeomorphic. According to Klein, differential topology is a field of mathematics where one studies properties of differentiable manifolds invariant under diffeomorphisms; however, this is too narrow a definition for contemporary differential topology.

We next define the ranks of differentiable maps, immersions, and embeddings of C^∞ manifolds.

DEFINITION 1.6. Let M_1 and M_2 be differentiable manifolds. Let $f : M_1 \to M_2$ be a differentiable map. For x in M_1 choose charts (U_1, h_1) and (U_2, h_2) about x and $f(x)$. We define the *rank* of f at x to be the rank of the Jacobian matrix of the map

$$h_2 \circ f \circ h_1^{-1} : h_1(U_1 \cap f^{-1}(U_2)) \longrightarrow h_2(U_2)$$

at $h_1(x)$.

Evidently Definition 1.6 does not depend on the chioce of charts.

DEFINITION 1.7. Let M^n and V^p be differential manifolds of dimensions n and p. A differentiable map $f : M^n \to V^p$ is an *immersion* if the rank of f at each point x of M^n is n. An immersion f is an *embedding* if f is a homeomorphism of M^n in V^p. We say that f is a *submersion* if the rank of f at each point x of M^n is p.

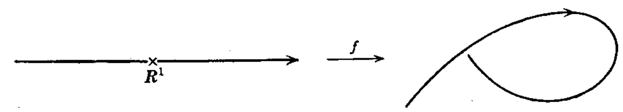


FIGURE 1.3

If $f : M^n \to V^p$ is an embedding, the image $f(M^n)$ is obviously a submanifold of V^p.

REMARK. An embedding f is an immersion but the converse is not true. Even when f is an immersion which is one-to-one into V^p, it may fail to be an embedding. Consider Figure 1.3.

DEFINITION 1.8. Let M^n and V^p be C^∞ manifolds of dimensions n and p, and let $f : M^n \to V^p$ be a C^∞ map. A point y of V^p is a *regular value* of f if the rank of f at each point x in $f^{-1}(y)$ is p; otherwise, y is a *critical value*.

According to the above definition points not in the image under f are regular values.

PROPOSITION 1.1. *Let M^n and V^p be C^∞ manifolds of dimensions n and p, and let $f : M^n \to V^p$ be a C^∞ map. If y is a regular value of f, then either f^{-1} is the empty set or an $n-p$ dimensional submanifold of M^n.*

The proposition follows easily from the definitions of submanifolds and of regular values.

C. Tangent spaces and the differentials of C^∞ maps.

DEFINITION 1.9. Let M^n be a C^∞ manifold, and let x be a point of M^n. A C^∞ map $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = x$ of an open interval $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$ (ε is sufficiently small), into M^n is called a *curve* at x. Suppose c_1 and c_2 are curves at x. For a chart $(U_\alpha, \varphi_\alpha)$ about x, $\varphi_\alpha \circ c_1$ and $\varphi_\alpha \circ c_2$ are C^∞ maps of $(-\varepsilon, \varepsilon)$ into $\mathbf{R}^n$. We say that c_1 and c_2 are equivalent and write $c_1 \sim c_2$ if

$$\left.\frac{d(\varphi_\alpha \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\varphi_\alpha \circ c_2)}{dt}\right|_{t=0}$$

(Figure 1.4).

By virtue of the definition of C^∞ structures the above definition does not depend on the choice of a chart. It is also clear that $\sim$ is an equivalence relation. Therefore, we can divide the set C_x of curves at x in M^n by $\sim$. We represent the class containing the curve c by $[c]_x$.

DEFINITION 1.10. Let M^n be a C^∞ manifold and let $x \in M^n$. We say that the set of equivalence classes of curves at x in M^n

$$T_x(M^n) = C_x/\sim = \{[c]_x \mid c \text{ is a curve at } x\}$$

is the tangent space of M^n at x.

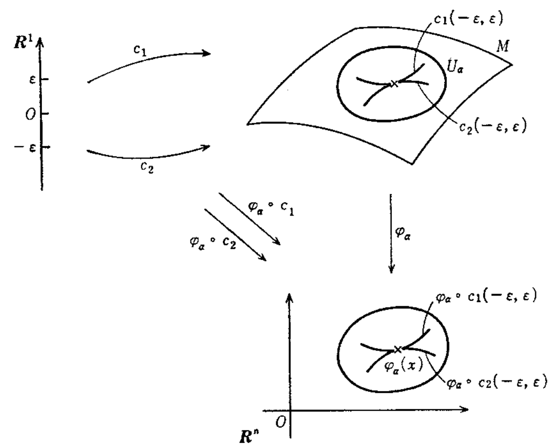


FIGURE 1.4

We define the operation of addition on $T_x(M^n)$ as follows.

DEFINITION 1.11. Let $[c_1]_x$ and $[c_2]_x$ be elements of $T_x(M^n)$, where $c_1, c_2 : (-\varepsilon, \varepsilon) \to M^n$ are C^∞ maps with $c_1(0) = c_2(0) = x$. For a chart $(U_\alpha, \varphi_\alpha)$ about x with $\varphi_\alpha(x) = 0$, we define the sum $\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}^n$ of the maps $\varphi_\alpha \circ c_1, \varphi_\alpha \circ c_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}^n$ by

$$(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)(t) = \varphi_\alpha \circ c_1(t) + \varphi_\alpha \circ c_2(t).$$

Hence, choosing a small enough ε' we have

$$(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)(-\varepsilon', \varepsilon') \subset \varphi_\alpha(U_\alpha).$$

Now we define $[c_1]_x + [c_2]_x$ by

$$[c_1]_x + [c_2]_x = [\varphi_\alpha^{-1}(\varphi_\alpha \circ c_1 + \varphi_\alpha \circ c_2)]_x.$$

By the definition of C^∞ atlases, the above definition depends on neither the choice of a chart $(U_\alpha, \varphi_\alpha)$ nor the choice of $\varepsilon > 0$ for the domain of the curve.

For an element $[c]_x$ of $T_x(M^n)$ and a real number λ we define the scalar product $\lambda[c]_x$ in the natural manner.

LEMMA 1.1. *The space $T_x(M^n)$ with the operations of sum and scalar multiplication as above is an n-dimensional vector space.*

PROOF. It is trivial that $T_x(M^n)$ is a vector space. Thus, we only need to show that the dimension of $T_x(M^n)$ is n. Choose a chart $(U_\alpha, \varphi_\alpha)$ about x and consider the following n curves at x:

$$u_i : (-\varepsilon, \varepsilon) \longrightarrow M^n, \qquad i = 1, 2, \dots, n,$$

$$u_i(t) = \varphi_\alpha^{-1}(\underbrace{0, \dots, 0}_{i-1 \text{ zeros}}, t, 0, \dots, 0),$$

where $(0, \dots, 0, t, 0, \dots, 0)$ denotes an element of $\mathbf{R}^n$ whose components, except the ith one which is t, are zero. Then clearly the equivalence class $[c]_x$ of a curve at x is expressed as a linear combination of the $[u_1]_x, \dots, [u_n]_x$ which are easily seen to be linearly independent. □

We next define the differential of a differentiable map.

LEMMA 1.2. *Let M and N be C^∞ manifolds, and let $c_1, c_2 : (-\varepsilon, \varepsilon) \to M$ be curves at x in M with $c_1 \sim c_2$. For a C^∞ map $f : M \to N$, then the curves $f \circ c_1$ and $f \circ c_2$ at $f(x)$ satisfy $f \circ c_1 \sim f \circ c_2$.*

PROOF. Let $(U_\alpha, \varphi_\alpha)$ be a chart about x, and let $(V_\lambda, \psi_\lambda)$ be a chart about $f(x)$. Then

$$\psi_\lambda \circ f \circ c_i = (\psi_\lambda \circ f \circ \varphi_\alpha)^{-1} \circ (\varphi_\alpha \circ c_i), \qquad i = 1, 2.$$

Further, by assumption

$$\left.\frac{d(\varphi_\alpha \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\varphi_\alpha \circ c_2)}{dt}\right|_{t=0},$$

and hence we get

$$\left.\frac{d(\psi_\lambda \circ f \circ c_1)}{dt}\right|_{t=0} = \left.\frac{d(\psi_\lambda \circ f \circ c_2)}{dt}\right|_{t=0}. \quad \square$$

DEFINITION 1.12. Let M and N be C^∞ manifolds and let $f : M \to N$ be a C^∞ map. For x in M define a map $(df)_x : T_x(M) \longrightarrow T_{f(x)}(N)$ by

$$(df)_x([c]_x) = [f \circ c]_{f(x)}.$$

By Lemma 1.2 this definition does not depend on the choice of a representative c of $[c]_x$. We say that $(df)_x$ is the *differential of f at x*.

LEMMA 1.3. *Let M and N be C^∞ manifolds, let $f : M \to N$ be a C^∞ map, and let $x \in M$. Then*

(1) *The differential $(df)_x : T_x(M) \longrightarrow T_{f(x)}(N)$ of f at x is a linear map.*

(2) *The rank of f at x equals the rank of $(df)_x$.*

We leave the proof to the reader.

Thus, a C^∞ map $f : M \to N$ is an immersion if and only if the map $(df)_x : T_x(M) \to T_{f(x)}(N)$ is injective at each point x of M.

§2. Fiber bundles

This section contains a summary of the facts about fiber bundles which are needed throughout our book. The material presented here is based largely on the work of Steenrod [A7].

A. Examples of fiber bundles. In order to enhance the reader's understanding of the subject we first give several examples.

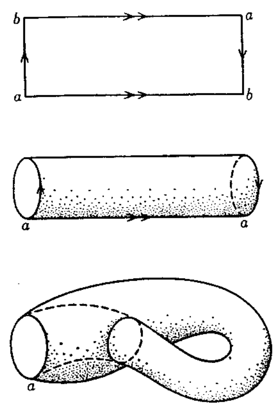


FIGURE 1.5

1. Product spaces Let X, Y be topological spaces, and set $B = X \times Y$. Define $p : B \to X$ by $p(x, y) = x$. Then p is a continuous map onto X and for each x in X, $p^{-1}(x) = Y_x$ is homeomorphic to Y. Fix a point y_0 of Y and define $f : X \to B$ by $f(x) = (x, y_0)$; f is continuous and $p \circ f(x) = x$.

2. The Möbius strip Recall that the Möbius strip is defined as follows:

$$M^2 = [0, 1] \times [0, 1]/\sim,$$
$$(0, t) \sim (1, 1 - t), \qquad t \in [0, 1].$$

Setting $X = S^1 = [0, 1]/\sim$, $Y = [0, 1]$, and $B = M^2$ we define $p : B \to X$ by $p([(s, t)]) = [s] \in S^1$. Then p is a continuous map onto X, and for a point x of X, $p^{-1}(x) = Y_x$ is homeomorphic to Y. Further, there exists a neighborhood $V(x)$ of x such that $p^{-1}(V(x))$ is homeomorphic to $V(x) \times Y$. In addition the map $f : X \to B$ defined by $f(x) = [(x, 1/2)]$ is continuous and satisfies $p \circ f(x) = x$.

3. The Klein bottle The *Klein bottle* is the surface K^2 which we obtain by pasting one pair of facing edges of the rectangular $I \times J$, $I = J = [0, 1]$, in the same direction and the other pair in the opposite direction (Figure 1.5). That is,

$$K^2 = I \times J/\sim, \qquad (0, t) \sim (1, 1 - t), \qquad t \in J,$$
$$(s, 0) \sim (s, 1), \qquad s \in I.$$

Put $B = K^2$, $X = S^1 = I/\sim$, and $Y = S^1 = J/\sim$. Define $p : B \to X$ by $p([(s, t)]) = [s] \in X$, which is continuous onto X. For each x of X $p^{-1}(x)$ is homeomorphic to $Y = S^1$. For a point x of X there exists a

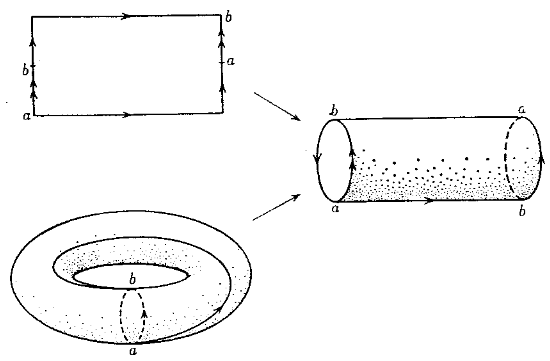


FIGURE 1.6

neighborhood $V(x)$ of x with $p^{-1}(V(x))$ homeomorphic to $V(x) \times Y$. The map $f : X \to B$ defined by $f(x) = [(x, 0)]$ is continuous, and $p \circ f(x) = x$.

4. Covering spaces Suppose B is a covering space of X and $p : B \to X$ is its covering map. Then p is evidently a continuous map onto X, and for each x in X the set $Y_x = p^{-1}(x)$ is discrete. In the case where X is arcwise connected, the Y_x are homeomorphic for all x in X. Further, for each x of X, there exists a neighborhood $V(x)$ of x and a homeomorphism between $p^{-1}(V(x))$ and $V(x) \times Y_x$ (B is a *covering space* of X, or (B, p, X) is a covering space, if (0) X is arcwise connected and locally arcwise connected, (i) $p : B \to X$ is a continuous surjection, (ii) for each $x \in X$ there exists an arcwise connected neighborhood V of x such that each connected component $\tilde{V}_\lambda$ is open in X and $p|\tilde{V}_\lambda : \tilde{V}_\lambda \to V$ is a homeomorphism onto V).

5. The twisted torus Consider $[0, 1] \times S^1$, and paste $\{0\} \times S^1$ and $\{1\} \times S^1$ with a 180° twist. The resulting surface T_W is the *twisted torus*:

$$T_W = [0, 1] \times S^1/\sim, \qquad (0, e^{2\pi i\theta}) \sim (1, e^{2\pi i(\theta+\pi)}).$$

Define $p : T_W \to S^1 = [0, 1]/\sim$ by $p([t, e^{2\pi i\theta}]) = [t]$. Then p is a continuous map onto X and for each point $[t]$ of S^1 there is a neighborhood V of $[t]$ in S^1 such that $p^{-1}(x)$ is homeomorphic to $V \times S^1$ (Figure 1.6).

B. The definition of a fiber bundle.

DEFINITION 1.13. Let G be a topological group, and let Y be a topological space. Suppose there is a continuous map $\eta : G \times Y \to Y$ satisying:

(i) For the unit e of G $\eta(e, y) = y$.
(ii) For all g_1, $g_2 \in G$ and $y \in Y$, $\eta(gg', y) = \eta(g, \eta(g', y))$.

Then we say that G is a *topological transformation group* of Y (with respect to η) and that G *acts* or *operates on* Y.