HOMOTOPY THEORY AND BORDISM THEORY

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Warning

These notes are likely to contain many minor and major mistakes. Please bring them to the attention of the author, gentle reader. In particular, problems may contain false claims, misprints, and so on.

1. Fiber Bundles

Let B be a space. The category $\mathcal{S}_B$ of spaces over B is defined as follows. Objects are continuous maps $p : X \to B$ with arbitrary X, and a morphism from $p : X \to B$ to $q : Y \to B$ is a continuous map $f : X \to Y$ such that $qf = p$.
A space over B is called *trivial* if it is isomorphic in the category $\mathcal{S}_B$ to an object of the form

$$F \times B \xrightarrow{\text{projection}} B$$

(projection from a product to the second factor). *Note*: It can be difficult to decide whether or not a space over B is trivial. Try the following:

Exercise 1.1. Let $GL(n, \mathbb{R})$ be the general linear group of $\mathbb{R}^n$, consisting of all $n \times n$ matrices with nonvanishing determinant. Let B be the space of positive definite symmetric $n \times n$ matrices. Show that

$$GL(n, \mathbb{R}) \longrightarrow B \quad ; \quad A \mapsto A^T A$$

is a trivial space over B.

A more subtle property that spaces over B may or may not have is *local triviality*. Say that $p : E \to B$ is *locally trivial* if every $b \in B$ has a neighbourhood $U \subset B$ such that

$$\text{restriction of } p : p^{-1}(U) \longrightarrow U$$

is a trivial space over U.

Definition 1.2. A locally trivial space over B is also called a *fiber bundle* over B (and B is the *base space*, whereas E is the *total space*).

Example 1.3. The *Hopf fiber bundle* is a fiber bundle with total space $\mathbb{S}^3$ and base space $\mathbb{S}^2$. Think of $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$, and use stereographic projection to identify $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$, the one-point compactification of $\mathbb{C}$. Then define

$$p : \mathbb{S}^3 \longrightarrow \mathbb{C} \cup \{\infty\} \quad ; \quad (z_1, z_2) \mapsto z_1/z_2 \, .$$

Here z_1/z_2 means ∞ if $z_2 = 0$, because then $z_1 \neq 0$ since $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ for all $(z_1, z_2) \in \mathbb{S}^3$.

We check that p is a fiber bundle: First let $U = \mathbb{C} \subset (\mathbb{C} \cup \{\infty\})$. Then $p^{-1}(U)$ is the set of all $(z_1, z_2) \in \mathbb{S}^3$ such that $z_2 \neq 0$. The map

$$p^{-1}(U) \longrightarrow \mathbb{S}^1 \times U \quad ; \quad (z_1, z_2) \mapsto (z_2/|z_2|, z_1/z_2)$$

is an isomorphism of spaces over U. (Think of $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$.) The inverse isomorphism is given by

$$(w_1, w_2) \mapsto (w_1 w_2, w_1)/\|(w_1 w_2, w_1)\| .$$

We conclude that p is trivial over U. Next, let $V = \mathbb{C} \cup \{\infty\} \smallsetminus \{0\}$. Then we have a commutative diagram

$$\begin{tikzcd}
p^{-1}(U) \arrow[r] \arrow[d, "p"] & p^{-1}(V) \arrow[d, "p"] \\
U \arrow[r] & V
\end{tikzcd}$$

where the upper horizontal arrow is $(z_1, z_2) \mapsto (z_2, z_1)$ and the lower horizontal arrow is $z \mapsto z^{-1}$. Both horizontal arrows are homeomorphisms, so if the left-hand vertical arrow is a trivial space over U, then the right-hand arrow must be a trivial space over V. In conclusion, p is trivial over U and over V, and this is enough since the union of U and V is all of $\mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$. $\square$

Exercise 1.4. Is the Hopf fiber bundle trivial?

Exercise 1.5. Use Hamilton's quaternions and Cayley's octonions to produce fiber bundles

$$p : \mathbb{S}^7 \to \mathbb{S}^4 , \qquad q : \mathbb{S}^{15} \to \mathbb{S}^8$$

analogous to the Hopf bundle. *Info*: The quaternions $\mathbb{H}$ form a real 4-dimensional vector space with basis $\{1, i, j, k\}$ and associative (bilinear) multiplication given by

$$ij = k = -ji, \quad , jk = i = -kj, \quad ki = j = -ik$$

(and 1 acts as two-sided unit). Note that every nonzero element in $\mathbb{H}$ is invertible. The *conjugate* of a quaternion $x = a + bi + cj + dk$ is $\bar{x} := a - bi - cj - dk$ (where a, b, c, d are real). Using conjugation on $\mathbb{H}$, define the *Cayley algebra* $\mathbb{K}$ to be $\mathbb{H}^2$, with multiplication given by

$$(x, y)(u, v) = (xu - \bar{v}y, vx + y\bar{u}).$$

This is nonassociative (and noncommutative), but the essential thing here is that every nonzero element is invertible.

Exercise 1.6. Let E be the Moebius strip, $E = \mathbb{S}^1 \times \mathbb{R}/\sim$, where $\sim$ means: identify (z, r) with $(-z, -r)$. As before, think of $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. Show that

$$p : E \to \mathbb{S}^1 \quad ; \quad [(z, r)] \mapsto z^2$$

is a fiber bundle. Is it a trivial fiber bundle ? (And why z^2 in the formula for p ?)

Exercise 1.7. Show that the "usual" map $\mathbb{S}^n \to \mathbb{R}P^n$ is a fiber bundle. Is it trivial ?

Exercise 1.8. Let $GR_{2,2}$ be the space of 2-dimensional linear subspaces of $\mathbb{R}^4$. (Topologize this by identifying it with the set of 4×4-matrices A satisfying $A^2 = A$, $A^T = A$, $\mathrm{tr}(A) = 2$.) Let $O(4)$ be the orthogonal group of $\mathbb{R}^4$, and define

$$p : O(4) \to GR_{2,2}$$

by sending $C \in O(4)$ to the linear subspace spanned by the first two columns of C. Show that p is a fiber bundle.

The *fiber* of a fiber bundle $p : E \to B$ over a point $b \in B$ is the space $p^{-1}(b)$. The examples above suggest that it is in some sense "independent" of b. (In example 1.3, all fibers are homeomorphic to $\mathbb{S}^1$; in exercise 1.6, they are all homeomorphic to $\mathbb{R}$; what about exercise 1.7 and exercise 1.8 ?) In general, suppose that B is path-connected and let b and c be arbitrary points in B. Then:

Proposition 1.9. $\quad p^{-1}(b) \cong p^{-1}(c)$.

Proof. Choose a path $\omega : [0,1] \to B$ with $\omega(0) = b$ and $\omega(1) = c$. For sufficiently large n, each $\omega([i/n,(i+1)/n])$ (for $0 \le i < n$) will be contained in an open set $U_i \subset B$ such that p is trivial over U_i. Then

$$p^{-1}(\omega(i/n)) \cong p^{-1}(\omega((i+1)/n)) \qquad 0 \le i < n. \quad \square$$

Exercise 1.10. Let $E = \mathbb{C} \times \mathbb{S}^1/\sim$, where $\sim$ means: identify (z, w) with $(-z, -w)$. Define $p : E \to \mathbb{C}$ by $[(z, w)] \mapsto z^2$. Is p a fiber bundle ? If so, what are its fibers ?

Proposition 1.9 has a vast generalization, for which we need some more language. Let $p : E \to B$ be a fiber bundle B, and let $f : A \to B$ be any continuous map, with arbitrary A. We let

$$f^*E = \{(a, e) \in A \times E \mid f(a) = p(b)\}$$

(a subspace of $A \times E$, with the product topology). Now we have a space over A, given by

$$f^*p : f^*E \longrightarrow A \quad ; \quad (a, e) \mapsto a\,.$$

This is called the *pullback* (of p, under f).

Proposition 1.11. *The map f^*p is also a fiber bundle (with base space A).*

Proof. If p is trivial, then it is easy to see that f^*p is also trivial. If $\{U_i\}$ is an open cover of B such that p is trivial over each U_i, then one finds again that f^*p is trivial over each open set $f^{-1}(U_i) \subset A$. Since the $f^{-1}(U_i)$ cover A, this completes the proof. $\square$

Exercise 1.12. Let p be the fiber bundle in exercise 1.6. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be given by $f(z) = z^n$ for some fixed $n \in \mathbb{Z}$. For which values of n is f^*p a trivial fiber bundle?

Exercise 1.13. Adopt the notation of proposition 1.11.

- Check that the fiber of f^*p over $a \in A$ is homeomorphic (canonically) to the fiber of p over $f(a)$.
- Prove: If $A = B$ and $f = \mathrm{id}$, then f^*p and p are (canonically) isomorphic as fiber bundles with base space B.
- Prove: For a continuous map $g : A' \to A$, arbitrary A', the fiber bundles $g^*(f^*p)$ and $(fg)^*p$ with base space A' are (canonically) isomorphic.

Theorem 1.14. *Let $p : E \to B$ be a fiber bundle, where B is paracompact (details below). Let f and g be continuous maps from A to B. If f and g are homotopic, then f^*p and g^*p are isomorphic fiber bundles over A.*

You can see from the technical assumption (B paracompact) that the proof is going to be difficult. Let's postpone it. If you don't know what paracompact means, please accept that any *metrizable* space is paracompact.

Corollary 1.15. Homotopy lifting property alias covering homotopy theorem: *With $p : E \to B$ as in theorem 1.14, let $f : X \to E$ be any continuous map (arbitrary X). Let $h : X \times [0,1] \to B$ be continuous, and suppose that $h(x,0) = pf(x)$ for all $x \in X$. Then there exists a continuous $\bar{h} : X \times [0,1] \to E$ such that $\bar{h}(x,0) = f(x)$ and $p\bar{h}(x,t) = h(x,t)$ for all $x \in X$ and $t \in [0,1]$.* Picture:

$$\begin{tikzcd} X \times 0 \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\bar{h}"] & B . \end{tikzcd}$$

Proof. First, observe that the maps

$$(x,t) \mapsto h(x,t) \qquad \text{and} \qquad (x,t) \mapsto h(x,0)$$

are homotopic. Call the second of these u. By theorem 1.14, the fiber bundles h^*p and u^*p with base space $X \times [0,1]$ are isomorphic. Translating into "everyday" language what the existence of such an isomorphism means, we find: there exists a family of homeomorphisms

$$J_{x,t} : \text{fiber of } p \text{ over } h(x,t) \longrightarrow \text{fiber of } p \text{ over } h(x,0) \ ,$$

depending continuously on $x \in X$ and $t \in [0,1]$. (Make sense of that if you feel it is necessary; in any case remember the first item of exercise 1.13.) Now define $\bar{h}(x,t)$ to be $J_{x,t}^{-1} J_{x,0} f(x)$. Note that this belongs to the fiber of p over $h(x,t)$, as it should. $\square$

Example 1.16. We shall show that the Hopf map $p : \mathbb{S}^3 \to \mathbb{S}^2$ is not nullhomotopic (i.e., not homotopic to a constant map). Suppose if possible that it is nullhomotopic. Let $h : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^2$ be a nullhomotopy (so that $h(x,0) = p(x)$ and $h(x,1) = *$, where $*$ is your favorite point in $\mathbb{S}^2$). Since p

is a fiber bundle, corollary 1.15 tells us that there exists a "lifted homotopy" $\bar{h} : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that $p\bar{h} = h$ and $h(x,0) = x$ for all $x \in \mathbb{S}^3$. Note that $\bar{h}$ is a homotopy from the identity map of $\mathbb{S}^3$ to a map whose image is contained in a fiber of p. Thus the identity map of $\mathbb{S}^3$ is homotopic to a non-surjective map; but this is nonsensical (why?). $\square$

Exercise 1.17. Note that the graded homomorphism

$$p_* : H_*(\mathbb{S}^3; \mathbb{Z}) \longrightarrow H_*(\mathbb{S}^2; \mathbb{Z})$$

induced by p looks exactly like the graded homomorphism induced by a constant map, because it has no other choice. This is one of the strange things about the Hopf map: it is not nullhomotopic, but (co)homology does not "detect" anything about it. And yet — it does. Recall that the mapping cone of a continuous map $f : A \to X$ is the identification space

$$\mathrm{cone}(f) = (A \times [0,1]) \amalg X / \sim$$

where $\sim$ identifies $(a,1)$ with $f(a)$ and $(a,0)$ with $(b,0)$ for all $a,b \in A$. Prove the following:

- If f is nullhomotopic, then
$$H^k(\mathrm{cone}(f); \mathbb{Z}) \cong H^k(X; \mathbb{Z}) \oplus H^k(\Sigma A; \mathbb{Z}) \qquad k > 0$$
and the isomorphism is compatible with the cup product.
- For $f = p$ =Hopf map, the mapping cone of p is homeomorphic to the complex projective plane $\mathbb{C}P^2$. There exists an element in $H^*(\mathbb{C}P^2; \mathbb{Z})$ whose cup square is nonzero.

Exercise 1.18. Let $p : M \to N$ be a fiber bundle, where M and N are compact manifolds (without boundary). Show that p is not nullhomotopic.

Exercise 1.19. Let $p : \mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf fiber bundle. Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be any continuous map. Show: if the degree of f is nonzero, then f^*p is a nontrivial fiber bundle; if the degree of f is $\neq \pm 1$, then f^*p and p are non-isomorphic fiber bundles over $\mathbb{S}^3$.

2. Homotopy invariance of pullbacks of fiber bundles

This is about the proof of theorem 1.14 and there is a lot of do-it-yourself to it. If you don't like the do-it-yourself part, see Husemoller's book *Fibre Bundles*, §4.9.

Exercise 2.1. Let $p : E \to Y \times [0,1]$ be a *trivial* fiber bundle, with fibers homeomorphic to some space F. Prove that any homeomorphism

$$\iota_0 : p^{-1}(Y \times \{0\}) \to F \times (Y \times \{0\})$$

over $(Y \times \{0\})$ extends to a homeomorphism

$$\iota : E \to F \times (Y \times [0,1])$$

over $Y \times [0,1]$.

Exercise 2.2. Let $p : E \to Y \times [0,1]$ be any fiber bundle, and suppose there exists an open cover $\{U_i\}$ of $[0,1]$ such that p is trivial over each open set $Y \times U_i$. Show that p is trivial, using exercise 2.1.

Exercise 2.3. (Notation of theorem 1.14): Suppose that A in theorem 1.14 is compact Hausdorff, and let $h : A \times [0,1] \to B$ be a homotopy from f to g. Prove: each $x \in A$ has a neighbourhood U such that h^*p is trivial over $U \times [0,1]$. (Use exercise 2.2.) Consequently, there exists a finite open cover $\{U_1, U_2, \dots, U_n\}$ of A such that h^*p is trivial over $U_i \times [0,1]$ for $1 \le i \le n$.

Definition 2.4. A *partition of unity* on a space Y is a collection $\{\phi_i \mid i \in \mathcal{J}\}$ of functions $\phi_i : Y \longrightarrow [0,1]$ such that

- each $y \in Y$ has a neighbourhood in Y where all but finitely many ϕ_i vanish;
- $\sum_i \phi_i \equiv 1$.

The partition of unity is *subordinate* to an open cover $\mathcal{U}$ of Y if the *carrier* of ϕ_i is contained in one of the open sets belonging to $\mathcal{U}$, for each $i \in \mathcal{J}$. (The carrier of ϕ_i is the closure of the complement of $\phi_i^{-1}(0)$ in Y.)

Exercise 2.5. Prove that for each open cover of a compact Hausdorff space, there exists a partition of unity subordinate to the open cover. (You will need Tietze-Urysohn or something equivalent.)

In the situation of exercise 2.3, choose a partition of unity on A subordinate to the open cover $\{U_1, U_2, \dots, U_n\}$. You can easily arrange it in such a way that it takes the form $\{\phi_1, \phi_2, \dots, \phi_n\}$ where the carrier of ϕ_i is contained in U_i. For $1 \le k \le n$, define

$$f_k : A \to B \quad ; \quad a \mapsto h\left(a\,, \sum_{i=1}^{k} \phi_i(a)\right).$$

Notice that $f_1 = f$ and $f_n = g$.

Exercise 2.6. Prove that $f_k^*p \cong f_{k+1}^*p$ for $1 \le k < n$, and therefore $f^*p \cong g^*p$ as required in theorem 1.14. (Do this first for $k = 1$.)

Unfortunately, when we replace the assumption: *A is compact Hausdorff* by the assumption: *B is paracompact*, which we have not used so far but which we should have used, then things are a little more unpleasant. We need more vocabulary.

Definition 2.7.

- An open cover of a space is *numerable* if there exists a partition of unity subordinate to it.
- A Hausdorff space is *paracompact* if every open cover of the space is numerable.

(This is not exactly the standard definition of paracompactness, but here it is very useful.) In particular, we can choose an open cover $\{U_i \mid i \in \mathcal{J}\}$ of B in theorem 1.14 such that p is trivial over each U_i. Then, since B is paracompact, the open cover is numerable. It follows easily that the open cover $\{h^{-1}(U_i)\}$ of $A\times[0,1]$, with h as in exercise 2.3, is numerable (although $A\times[0,1]$ need not be paracompact). Then you can use the proposition below (quoted from Dold's book, *Lectures on Algebraic Topology*), and exercise 2.2, to produce a *numerable* open cover of A such that h^*p is trivial over $U_i\times[0,1]$ for each U_i in the open cover. This open cover may be incurably infinite, but you can still use the idea of exercise 2.6 to complete the proof. Choose a well-ordering on the indexing set for the open cover.

Proposition 2.8. *If A is a space and $\mathcal{W}$ is a numerable open covering of $A\times[0,1]$, then there exist a numerable covering $\mathcal{U}$ of A and a function $r:\mathcal{U}\to\mathbb{Z}$, with values $r(U)>1$, such that every set*

$$U\times\left[\frac{i-1}{r(U)},\frac{i+1}{r(U)}\right]\qquad \text{where } U\in\mathcal{U},\quad i\in\mathbb{Z},\quad 0<i<r(U)$$

is contained in some $W\in\mathcal{W}$.

3. Fibrations

Definition 3.1. A map $p:E\to B$ of spaces is a *fibration* if it has the homotopy lifting property (which we have already encountered in corollary 1.15): given maps $f:X\to E$ and $h:X\times[0,1]\to B$ where $h(x,0)=pf(x)$ for all $x\in X$, there exists a map $\bar{h}:X\times[0,1]\to E$ such that $p\bar{h}=h$ and $\bar{h}(x,0)=f(x)$.

Informally: Think of h as a one-parameter family (alias homotopy) of continuous maps,

$$(h_t)_{t\in[0,1]}.$$

The hypothesis is that such a homotopy can always be lifted to a homotopy $(\bar{h}_t)$ between maps from X to E ("lifted" means $p\bar{h}_t=h_t,\forall t$), with an "initial" lift $\bar{h}_0$ which can be prescribed arbitrarily and must be prescribed somehow.
We have seen that fiber bundles with paracompact base space are fibrations. Fibrations are more ubiquitous in homotopy theory than fiber bundles.

Example 3.2. For any B, let PB be the space of paths in B (the space of continuous maps from $[0,1]$ to B, with the compact-open topology; details below). For $\omega\in PB$, let $q(\omega)=\omega(1)$. Then $q:PB\to B$ is a fibration. To see that q is a fibration, suppose given a homotopy $(h_t:X\to B)$ and a lift $f:X\to PB$, so that $qf=h_0$. For $x\in X$ and $t\in[0,1]$, define $\bar{h}_t(x)$ to be the path

$$s\mapsto\begin{cases} f(x)(s+st) & 0\le s+st\le 1\\ h_{s+st-1}(x) & 1\le s+st\le 1+t\,.\end{cases}$$

Then $\bar{h}_t$ lifts h_t for all $t\in[0,1]$, and $\bar{h}_0$ agrees with f.

Definition 3.3. (related to the example 3.2): Let X and Y be spaces, where X is locally compact. It is customary to equip the space $C(X,Y)$ of continuous maps from X to Y with the *compact-open* topology, which is generated by the following subsets $W_{K,U}$ where $K \subset X$ is compact and $U \subset Y$ is open:

$$W_{K,U} = \{f : X \to Y \mid f(K) \subset U\}.$$

("Generated" means: a subset of $C(X,Y)$ is open iff it is a union of finite intersections of subsets of the form $W_{K,U}$.)

Exercise 3.4. Imitating example 3.2, show that

$$PB \longrightarrow B \times B \quad ; \quad \omega \mapsto (\omega(0), \omega(1))$$

is a fibration.

Definition 3.5. The *pullback* of a fibration $p : E \to B$ under a continuous map $g : A \to B$ is $g^*p : g^*E \longrightarrow A$, where

$$g^*E = \{(a,e) \in A \times E \mid g(a) = p(e)\}$$

and g^*p is defined by $(a,e) \mapsto a$.

Proposition 3.6. *The pullback g^*p is also a fibration.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} g^*E & \xrightarrow{\hat{g}} & E \\ {\scriptstyle g^*p}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & B \end{array}$$

where $\hat{g}(a,e) := e$. Now suppose that we have

$$\begin{array}{ll} (h_t : X \longrightarrow A) & \text{(a homotopy)} \\ f : X \longrightarrow g^*E & \text{(a lift of } h_0). \end{array}$$

Then

$$\begin{array}{ll} (gh_t : X \longrightarrow B) & \text{is a homotopy} \\ \hat{g}f : X \longrightarrow E & \text{lifts } gh_0 \,. \end{array}$$

Since $p : E \to B$ is a fibration, there exists a homotopy $(w_t : X \to E)$ such that w_t lifts gh_t, for all t, and such that $w_0 = \hat{g}f$. Then $(\bar{h}_t : X \to g^*E)$ given by

$$\bar{h}_t(x) := (h_t(x), w_t(x))$$

is the required homotopy lifting (h_t) (and let's not forget to check that $\bar{h}_0 = f$). ☐

Corollary 3.7. *Any (continuous) map $f : A \to B$ can be written as a composition $f = f^\sharp\iota$,*

$$A \xrightarrow{\iota} A^\sharp \xrightarrow{f^\sharp} B\,,$$

where ι is a homotopy equivalence and $f^\sharp$ is a fibration.

Proof. We let $A^\sharp$ be the pullback of

$$PB \longrightarrow B \times B$$

(as in exercise 3.4) under $f \times \mathrm{id} : A \times B \to B \times B$. Then $A^\sharp$ is the total space of a fibration $A^\sharp \to A \times B$, and we compose this with the projection $A \times B \to B$ which is also a fibration. (Why?) This gives us

$$f^\sharp : A^\sharp \to B$$

which is a fibration because a composition of two fibrations is a fibration (see exercise below). Now note that $A^\sharp$ is the space of triples $(\omega, (a, b))$ where $a \in A$ and $b \in B$ and ω is a path in B starting at $f(a)$ and ending at b. Further, $f^\sharp$ sends this triple to b. We can define $\iota : A \to A^\sharp$ by

$$a \mapsto (\omega_{f(a)}, (a, f(a)))$$

where $\omega_{f(a)}$ denotes the constant path with constant value $f(a)$. It is not hard to show that ι is a homotopy equivalence (in fact, the inclusion of a strong deformation retract). Also, $f^\sharp\iota$ equals f. □

Exercise 3.8. Show that a composition of two fibrations is again a fibration.

Next, we want to investigate whether proposition 1.9 and the stronger theorem 1.14 have analogs for fibrations. We proceed in very small steps.

Exercise 3.9. Let $E = \{(x, y) \in [0, 1]^2 \mid 0 \le y \le x \le 1\}$. Let $p : E \to [0, 1]$ be the projection $(x, y) \mapsto x$. Show that p is a fibration. (It is not a fiber bundle—why not?)

Definition 3.10. (Informal) For a space B and subspace A, say that B *deforms into* A if there exists a homotopy $(h_t : B \to B)$, where $0 \le t \le 1$, such that $h_0 = \mathrm{id}_B$, $h_1(B) \subset A$, and $h_t(A) \subset A$ for all t. In this case, the inclusion $A \hookrightarrow B$ is a homotopy equivalence. (Why?)

Example 3.11. Let $p : E \to B$ be a fibration. Let A be a subspace of B such that B deforms into A. Then $E = p^{-1}(B)$ deforms into $p^{-1}(A)$. *Proof*: Let $h = (h_t)$ be the deformation (of B into A). The homotopy lifting property supplies a diagonal arrow $\bar{h}$ in the commutative diagram

$$\begin{tikzcd}
E \times 0 \arrow[r, "\cong"] \arrow[d, hook] & E \arrow[d, "p"] \\
E \times [0,1] \arrow[r, "h\cdot(p\times\mathrm{id})"'] \arrow[ur, dashed, "\bar{h}"] & B\,.
\end{tikzcd}$$

(where $p \times \mathrm{id}$ is a map from $E \times [0, 1]$ to $B \times [0, 1]$.) Then $\bar{h} = (\bar{h}_t)$ is the required deformation. Note that $\bar{h}_t$ "covers" h_t in the sense that $p\bar{h}_t = h_t p$. We will use this below.

Definition 3.12. Let $p : X \to B$ and $q : Y \to B$ be spaces over B, and let $f, g : X \to Y$ be maps over B (i.e., $qf = p$, $qg = p$). Say that f and g are *homotopic over B* if there exists a homotopy (h_t) from f to g such that each h_t is a map over B (i.e., $qh_t = p$ for $0 \le t \le 1$). A map $e : X \to Y$ over B is a *homotopy equivalence over B* if there exists another map $d : Y \to X$ over B such that both de and ed are homotopic over B to the appropriate identity maps. If such a homotopy equivalence exists, we say that $p : X \to B$ and $q : Y \to B$ are *homotopy equivalent as spaces over B*.

Lemma 3.13. *Take a space over B, say $X \to B$, and a subspace $Y \subset X$. Suppose that X deforms into Y, and suppose the deformation $\{h_t\}$ can be so chosen that each h_t is a map over B. Then the inclusion $Y \hookrightarrow X$ is a homotopy equivalence over B.*

Proof. Easy. □

Lemma 3.14. *Let $q : D \to A \times [0,1]$ be a fibration. Let $D_0 = q^{-1}(A \times \{0\})$. Then D and D_0 are homotopy equivalent as spaces over A, where we use the following reference maps to A:*

$$D \xrightarrow{\ q\ } A \times [0,1] \xrightarrow{\text{projection}} A$$
$$D_0 \xrightarrow[q]{\quad} A \times \{0\} \cong A\ .$$

Proof. First deform $A \times [0,1]$ into $A \times \{0\}$ using the deformation given by $h_t(a,s) = (a, ts)$. Then cover this by a deformation $(\bar{h}_t)$ of D into D_0, as in the proof of example 3.11. Finally note that each $\bar{h}_t$ is a map over A, and apply lemma 3.13. □

Corollary 3.15. *Let $p : E \to B$ be a fibration, and let $f, g : A \to B$ be homotopic maps. Then*

$$f^*p : f^*E \to A\,, \qquad g^*p : g^*E \to A$$

are homotopy equivalent as spaces over A.

Note: For comparison, in theorem 1.14, the conclusion was *homeomorphic as spaces over A*.

Proof. Let $h : A \times [0,1] \to B$ be a homotopy from f to g. Let

$$\iota_0 : A \to A \times [0,1]\ , \qquad \iota_1 : A \to A \times [0,1]$$

be given by $\iota_0(a) = (a, 0)$ and $\iota_1(a) = (a, 1)$. With a view to lemma 3.14 write $D := h^*E$, and $q = h^*p$. Then D_0, as a space over A, is canonically isomorphic to f^*E with the reference map f^*p to A. By lemma 3.14, this is homotopy equivalent over A to D. By symmetry, g^*E with reference map g^*p to A is also homotopy equivalent over A to D. □

Corollary 3.16. *Let $p : E \to B$ be a fibration, where B is path connected. For arbitrary $x, y \in B$, we have*

$$p^{-1}(x) \simeq p^{-1}(y)$$

(all fibers of p have the same homotopy type).

Exercise 3.17. Explain why corollary 3.15 implies corollary 3.16.

Exercise 3.18. Let B be a connected CW-space with base point, and let $f : * \hookrightarrow B$ be the inclusion of the base point. This is usually not a fibration, but we can convert it into a fibration using corollary 3.7. What we get is

$$p : E \longrightarrow B$$

where E is the space of paths $\omega : [0,1] \to B$ with $\omega(0) = *$, and p is the evaluation map sending ω to $\omega(1)$. It is easy to see directly that E is contractible (and that p is a fibration). The fiber of p over a point $b \in$ is then the space of paths in B with initial point $*$ and endpoint b. Again, it is easy to see directly that fibers at different points of B are homotopy equivalent. The fiber over the base point is the space of all paths in B with initial point $*$ and endpoint $*$. This is called the *loop space* of B, denoted by ΩB.

Keeping this notation, specialize to $B = \mathbb{S}^n$ for some $n > 0$. Cover $\mathbb{S}^n$ by two open sets U, V, where U is $\mathbb{S}^n$ minus the base point, and V is $\mathbb{S}^n$ minus the antipode of the base point.

- Prove that
$$p^{-1}(U) \simeq \Omega\mathbb{S}^n \simeq p^{-1}(V)\,, \qquad p^{-1}(U \cap V) \simeq \mathbb{S}^{n-1} \times \Omega\mathbb{S}^n.$$
(Construct these homotopy equivalences in such a way that the inclusion of $p^{-1}(U \cap V)$ in $p^{-1}(U)$ "corresponds" to the projection from $\mathbb{S}^{n-1} \times \Omega\mathbb{S}^n$ to $\Omega\mathbb{S}^n$, if possible.)
- Prove that
$$H_i(\mathbb{S}^{n-1} \times \Omega\mathbb{S}^n) \cong H_i(\Omega\mathbb{S}^n) \oplus H_{i-n+1}(\Omega\mathbb{S}^n)$$
(homology with coefficients $\mathbb{Z}$). Here you must use the Künneth theorem.
- Use all this to calculate the homology groups of $\Omega\mathbb{S}^n$. (Hint: E is a union of two open subsets, $E = p^{-1}(U) \cup p^{-1}(V)$. In this situation you have a Mayer-Vietoris sequence.)

4. Spectral Sequences

Spectral sequences were invented by Jean Leray (late 1940's), and it is said that Jean-Pierre Serre made them prominent. They are not as bad as you have been told. They usually arise in connection with a filtration of a space by subspaces, or a filtration of a chain complex by chain subcomplexes. Let's focus on chain complexes (of abelian groups) for simplicity. A *filtration* of a chain complex C is an ascending sequence of chain subcomplexes

$$\dots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

with the properties

$$\bigcup_s C(s) = C\ , \qquad C(s) = 0 \text{ for some } s$$

(usually $C(s)$ is zero for $s < 0$). The task is, roughly speaking, to express the homology groups of C in terms of the homology groups of the subquotients $C(s)/C(s-1)$. That is what spectral sequences are good for.

Notation 4.1. $C(s,t) := C(s)/C(t)$ for $t \leq s$.

More precisely, we are dealing with two families of abelian groups. The first of these consists of the groups

$$\mathcal{E}^1_{s,t} := H_{s+t}C(s, s-1) \qquad s,t \in \mathbb{Z}$$

and we pretend that we know it. The second family consists of the groups

$$\mathcal{E}^\infty_{s,t} := \frac{\mathrm{im}\big(H_{s+t}C(s) \to H_{s+t}C\big)}{\mathrm{im}\big(H_{s+t}C(s-1) \to H_{s+t}C\big)}$$

(where the arrows are induced by inclusion). These are subquotients of the homology groups of C. We pretend that we want to know them. If we did, we would know H_*C up to "extension problems". To repeat—the task is

express (all) the groups $\mathcal{E}^\infty_{s,t}$ in terms of (all) the groups $\mathcal{E}^1_{s,t}$.

We shall introduce further families of abelian groups denoted

$$\mathcal{E}^2_{s,t}\,,\ \mathcal{E}^3_{s,t}\,,\ \mathcal{E}^4_{s,t}\,,\ \dots$$

and depending on $s,t \in \mathbb{Z}$. They will serve as stepping stones.

Notation 4.2. $\mathcal{E}^r_{s,t}$ is the group of those elements in $H_{s+t}C(s,s-1)$ which come from $H_{s+t}C(s,s-r)$, modulo the group of those which go to zero in $H_{s+t}C(s+r-1,s-1)$.

Exercise 4.3. How do we know that elements in $H_{s+t}(C(s,s-1)$ which go to zero in $H_{s+t}C(s+r-1,s-1)$ come from $H_{s+t}C(s,s-r)$?

Exercise 4.4. Write $C(\infty) := C$, $C(-\infty) = 0$. Prove that $\mathcal{E}^\infty_{s,t}$ has a description very similar to that of $\mathcal{E}^r_{s,t}$ just given, namely: *The group of all elements in $H_{s+t}C(s,s-1)$ which come from $H_{s+t}C(s,-\infty)$, modulo the subgroup of those which go to zero in $H_{s+t}C(\infty,s-1)$.*

Exercise 4.4 suggests that some kind of "convergence" takes place: $\mathcal{E}^r_{s,t}$ goes to $\mathcal{E}^\infty_{s,t}$ as $r \to \infty$. We return to this point below.

Our problem right now is: How can we make the "step" from $\mathcal{E}^r_{s,t}$ to $\mathcal{E}^{r+1}_{s,t}$? We can make it by introducing certain homomorphisms

$$d : \mathcal{E}^r_{s,t} \longrightarrow \mathcal{E}^r_{s-r,t+r-1}$$

(for all $r > 0$ and $s,t \in \mathbb{Z}$) and verifying two facts:

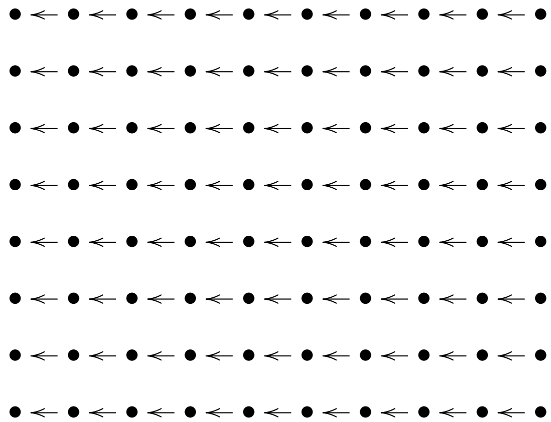
- the composition $\mathcal{E}^r_{s+r,\,t-r+1} \xrightarrow{\ d\ } \mathcal{E}^r_{s,t} \xrightarrow{\ d\ } \mathcal{E}^r_{s-r,\,t+r-1}$ is zero ;
- the quotient group

$$\frac{\ker\big(\mathcal{E}^r_{s,t} \xrightarrow{\ d\ } \mathcal{E}^r_{s-r,\,t+r-1}\big)}{\mathrm{im}\big(\mathcal{E}^r_{s+r,\,t-r+1} \xrightarrow{\ d\ } \mathcal{E}^r_{s,t}\big)}$$

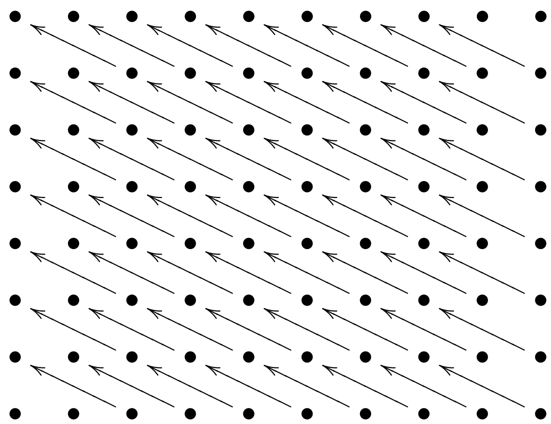
is isomorphic to $\mathcal{E}^{r+1}_{s,t}$.

Briefly: each $\mathcal{E}^r_{**}$ comes equipped with a *differential*, and $\mathcal{E}^{r+1}_{**}$ is simply the *homology* of the differential on $\mathcal{E}^r_{**}$. But this is too brief — one should also know the "direction" of the differential. Picture:

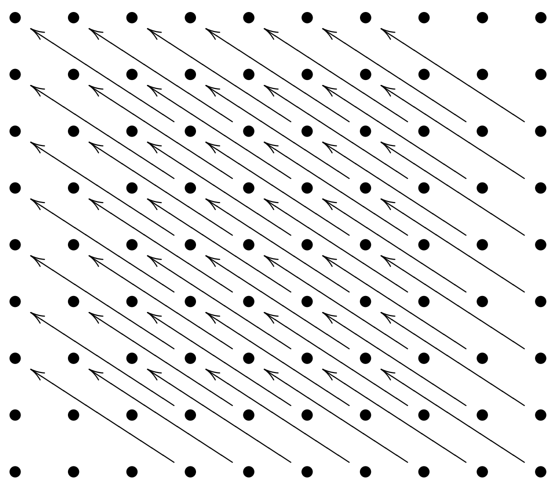
$\mathcal{E}^1_{**}$:



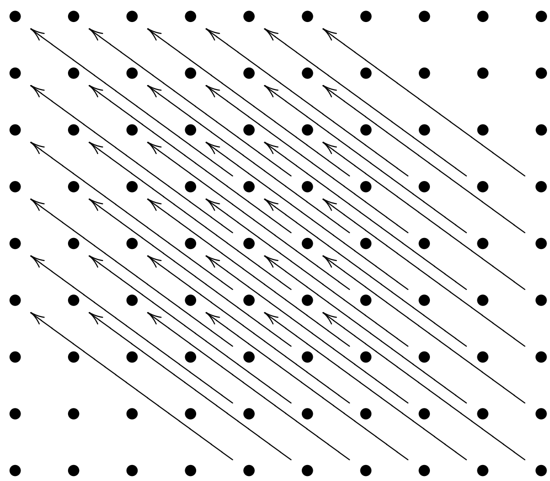
$\mathcal{E}^2_{**}$:



$\mathcal{E}^3_{**}$:



$\mathcal{E}^4_{**}$:



Definition 4.5. The differentials

$$d : \mathcal{E}^r_{s,t} \longrightarrow \mathcal{E}^r_{s-r,t+r-1}$$

are defined as follows. Each $x \in \mathcal{E}^r_{s,t}$ is the image of some

$$\bar{x} \in H_{s+t}C(s, s-r)\,.$$

We have the boundary operator

$$\partial : H_{s+t}C(s, s-r) \longrightarrow H_{s+t-1}C(s-r)\,.$$

Let $d(x)$ be the coset of $\partial(\bar{x})$ in $\mathcal{E}^r_{s-r,\,t+r-1}$.
To check that this is well defined, we need to show that $\partial(\bar{x}) \equiv 0$ in $\mathcal{E}^r_{s-r,\,t+r-1}$ if $x = 0$ in $\mathcal{E}^r_{s,t}$. If $x = 0$, then $\bar{x}$ goes to 0 in the homology group

$$H_{s+t}C(s+r-1, s-1).$$

(Remember notation 4.2.) From the commutative diagram (horizontal arrows induced by various inclusions)

$$\begin{array}{ccc} H_{s+t}C(s, s-r) & \longrightarrow & H_{s+t}C(s+r-1, s-1) \\ \partial\big\downarrow & & \partial\big\downarrow \\ H_{s+t-1}C(s-r) & \longrightarrow & H_{s+t-1}C(s-1) \end{array}$$

we conclude that $\partial(\bar{x})$ goes to zero in $H_{s+t-1}C(s-1)$. Then it also goes to zero in $H_{s+t}C(s-1, s-r-1)$, and then $\partial(\bar{x}) \equiv 0$ in in $\mathcal{E}^r_{s-r,\,t+r-1}$ by notation 4.2, as required.

Verification 4.6. It is easy to verify the first of the two "facts": $dd = 0$ (on each $\mathcal{E}^r_{**}$). To describe the isomorphism

$$\mathcal{E}^{r+1}_{s,t} \quad\cong\quad \frac{\ker\bigl(d\colon \mathcal{E}^r_{s,t} \to \mathcal{E}^r_{s-r,\,t+r-1}\bigr)}{\mathrm{im}\bigl(d\colon \mathcal{E}^r_{s+r,\,t-r+1} \to \mathcal{E}^r_{s,t}\bigr)}$$

we introduce some notation:

$$\begin{aligned}
Z^r_{s,t}: \quad &= \mathrm{im}\big(H_{s+t}C(s,s-r) \to H_{s+t}C(s,s-1)\big)\,, \\
B^r_{s,t}: \quad &= \ker\big(H_{s+t}C(s,s-1) \to H_{s+t}C(s+r-1,s-1)\big) \\
&= \mathrm{im}\big(\partial: H_{s+t+1}C(s+r-1,s) \to H_{s+t}C(s,s-1)\big)
\end{aligned}$$

for $r > 0$ (actually, allow $r = \infty$ also). Observe that

$$\dots \subset B^r_{s,t} \subset B^{r+1}_{s,t} \dots \subset B^\infty_{s,t} \subset Z^\infty_{s,t} \dots \subset Z^{r+1}_{s,t} \subset Z^r_{s,t} \subset \dots$$

$$\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}\,, \qquad \mathcal{E}^{r+1}_{s,t} = Z^{r+1}_{s,t}/B^{r+1}_{s,t}\,.$$

We shall check that

$$\ker\big(d\colon \mathcal{E}^r_{s,t} \to \mathcal{E}^r_{s-r,\,t+r-1}\big) = Z^{r+1}_{s,t}/B^r_{s,t}\,,$$

$$\mathrm{im}\big(d\colon \mathcal{E}^r_{s+r,\,t-r+1} \to \mathcal{E}^r_{s,t}\big) = B^{r+1}_{s,t}/B^r_{s,t}.$$

(These are equalities, not random isomorphisms, between subgroups of $\mathcal{E}^r_{s,t}$.) This is equivalent to saying that the map

$$\begin{array}{rcl}
u\colon Z^r_{s,t}/Z^{r+1}_{s,t} & \longrightarrow & B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1} \\
x + Z^{r+1}_{s,t} & \mapsto & \partial(\bar{x}) + B^r_{s-r,\,t+r-1}
\end{array}$$

(details as in definition 4.5) is well defined and isomorphic, for all s,t and $r > 0$. (To show *equivalent*, use the commutative diagram

$$\begin{tikzcd}
Z^r_{s,t}/B^r_{s,t} \arrow[r, "d"] \arrow[d, "p"] & Z^{r+1}_{s-r,t+r-1}/B^r_{s-r,t+r-1} \\
Z^r_{s,t}/Z^{r+1}_{s,t} \arrow[r, dotted, "u"] & B^{r+1}_{s-r,t+r-1}/B^r_{s-r,t+r-1} \arrow[u, "j"']
\end{tikzcd}$$

where p is a surjection and j is an injection.) The proof of the claim concerning u is a diagram chase: Use the diagram

$$\begin{tikzcd}
 & H_{s+t}C(s,s-r-1) \arrow[r, "j_1"] \arrow[d] & H_{s+t}C(s,s-1) \arrow[d, equal] \\
H_{s+t}C(s-1,s-r) \arrow[r] \arrow[d, "\partial_1"] & H_{s+t}C(s,s-r) \arrow[r, "j_2"] \arrow[d, "\partial_2"] & H_{s+t}C(s,s-1) \\
H_{s+t-1}C(s-r,s-r-1) \arrow[r, equal] & H_{s+t-1}C(s-r,s-r-1) &
\end{tikzcd}$$

with exact middle row and middle column; note that

$$Z^r_{s,t} = \mathrm{im}(j_2)\,, \quad Z^{r+1}_{s,t} = \mathrm{im}(j_1)\,,$$

$$B^{r+1}_{s-r,\,t+r-1} = \mathrm{im}(\partial_2)\,, \quad B^r_{s-r,\,t+r-1} = \mathrm{im}(\partial_1)\,. \quad \square$$

Remark 4.7. The assumption $C(s_0) = 0$ for some s_0 implies that $Z^r_{s,t}$ becomes stationary for fixed s, t and $r \to \infty$. Thus for sufficiently large r (where "sufficient" depends on s, t) the group $\mathcal{E}^{r+1}_{s,t}$ is a quotient of $\mathcal{E}^r_{s,t}$. Therefore

$$\mathcal{E}^\infty_{s,t} \cong \mathrm{colim}_r \mathcal{E}^r_{s,t}$$

(if colimits alias direct limits of systems of groups mean anything to you).

The following problem is important because it shows that what we have seen so far in this section is an enhanced version of the long exact homology sequence of a pair of chain complexes.

Exercise 4.8. Suppose that the filtration of C has only two *stages*; i.e., suppose $C(-1) = 0$ and $C(s) = C(1)$ for all $s \geq 1$. Then all we have is a chain complex $C(1)$ and a chain subcomplex $C(0) \subset C(1)$. What is $\mathcal{E}^1_{**}$, what is $\mathcal{E}^\infty_{**}$, what is the differential on $\mathcal{E}^1_{**}$, what is $\mathcal{E}^2_{**}$, what is the differential on $\mathcal{E}^2_{**}$, etc. ?

Of course, when people write papers and books on spectral sequences, they don't write about enhanced versions of the long exact homology sequence of a pair of chain complexes. What these people write about is

- cleverly designed filtrations of certain chain complexes (or similar objects) of general interest ;
- the meaning, interpretation etc. of the first "terms" of the spectral sequence (in practice $\mathcal{E}^1_{**}$ is very "big" and depends on certain choices, whereas $\mathcal{E}^2_{**}$ is not and does not);
- their experiences with the differentials in the spectral sequence.

The last item is the depressing aspect of spectral sequence theory. Quite often the differentials are kind enough to vanish on $\mathcal{E}^r_{**}$ for all $r \geq 2$, but if they do not the prospects are bleak.

The next example/exercise illustrates some of this (not the depressing aspect), and you should recognize it as something familiar.

Exercise 4.9. Let C be the singular chain complex of a CW-space X, and let $C(s)$ be the singular chain complex of the s-skeleton X^s. This defines a filtration on C. Describe the resulting spectral sequence in detail.

We now come to the first (and, for this course, last) serious example of a spectral sequence: the *Leray-Serre spectral sequence of a fibration.* Let $p : E \to B$ be a fibration, and assume that B is a simply connected CW-space. We make no special assumptions on the fibers. Let C be the *singular* chain complex of the total space E, and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton of the CW-space B. Then

$$0 = C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

and $C = \cup_s C(s)$ (prove this). This looks like a cleverly designed filtration of C. Let's find out what the E^1_{**} term of the associated spectral sequence is. This amounts to calculating the homology of $C(s, s-1)$ for all s.

To this end, choose a point z_i in each s-cell $V_i \subset B^s$. Let

$$U = B^s \setminus \{z_i\}\,, \quad V = \bigcup V_i\,, \quad \bar{U} = p^{-1}(U)\,, \quad \bar{V}_i = p^{-1}(V_i)\,, \quad \bar{V} = \bigcup \bar{V}_i\,.$$

Then $U \cup V = B^s$ and $\bar{U} \cup \bar{V} = p^{-1}(B^s)$. Since $\bar{U}$ and $\bar{V}$ are open in $p^{-1}(B^s)$, we have excision:

$$H_*(\bar{V}, \bar{U} \cap \bar{V}) \xrightarrow{\cong} H_*(\bar{U} \cup \bar{V}, \bar{U}) \cdot$$

Further, U deforms into B^{s-1} (see definition 3.10), and therefore $\bar{U}$ deforms into $p^{-1}(B^{s-1})$ according to example 3.11. Therefore

$$H_*C(s, s-1) = H_*(p^{-1}(B^s), p^{-1}(B^{s-1})) \xrightarrow{\cong} H_*(\bar{U} \cup \bar{V}, \bar{U})\,.$$

This shows

(a) $$H_*C(s, s-1) \cong H_*(\bar{V}, \bar{U} \cap \bar{V})\,.$$

Next, let F_i be the fiber of p over z_i. We find

(b) $$H_*(\bar{V}, \bar{U} \cap \bar{V}) \cong \bigoplus_i H_*(\bar{V}_i, \bar{V}_i \setminus F_i)\,.$$

Now each V_i is a cell, hence contractible, and so by corollary 3.15, each $p\colon \bar{V}_i \longrightarrow V_i$ is homotopy equivalent over V_i to a trivial fibration. This gives a (well defined) isomorphism

(c) $$H_*(\bar{V}_i, \bar{V}_i \setminus F_i) \cong H_*\left(V_i \times F_i, (V_i \setminus \{z_i\}) \times F_i\right) \cong H_{*-s}(F_i)$$

(the last isomorphism is essentially the suspension isomorphism). Combining (a), (b) and (c), we see that

(d) $$H_{s+t}C(s, s-1) \cong \bigoplus_i H_t(F_i)$$

where the direct sum is indexed by the set of s-cells of B. We can simplify this further in two ways.

Firstly, any choice of path $\omega : [0,1] \to B$ from z_i to z_j determines inclusions

$$F_i \hookrightarrow \omega^* E \hookleftarrow F_j$$

which are homotopy equivalences (say, by lemma 3.13). Then

$$H_*(F_i) \cong H_*(\omega^* E) \cong H_*(F_j)\,.$$

The isomorphism seems to depend on the choice of ω. But we are assuming that B is simply connected, so it does not. In more detail: if λ is another path connecting z_i with z_j, then λ is homotopic to ω with endpoints fixed, and by continuity, λ must give rise to the same isomorphism from $H_*(F_i)$ to $H_*(F_j)$. We see that all the $H_*(F_i)$ are canonically isomorphic, so we can do away with the indices altogether and write $H_*(F)$ for all of them without being too ambiguous.

Secondly, we obtain from (d) that

$$H_{s+t}C(s, s-1) \cong \bigoplus_i H_t(F_i) \cong \bigoplus_i H_t(F) \cong W_s(B) \otimes H_t(F)\,,$$

where $W_i(B)$ is the free abelian group generated by the s-cells of B. Note that $W_s(B)$ is the group of s-chains in the *cellular* chain complex of B. Summarizing:

Proposition 4.10. *The E^1_{**} term of the Leray-Serre spectral sequence is*

$$E^1_{s,t} \cong W_s(B) \otimes H_t(F)\,.$$

This is not the final result yet. What do the differentials on E^1_{**} look like? With the identifications of proposition 4.10, they take the form

$$d : W_s(B) \otimes H_t(F) \longrightarrow W_{s-1}(B) \otimes H_t(F)\,.$$

Going back to definition 4.5, you can easily see that they agree with

$$\partial \otimes \mathrm{id} : W_s(B) \otimes H_t(F) \longrightarrow W_{s-1}(B) \otimes H_t(F)$$

where ∂ is the "usual" boundary operator in the cellular chain complex $W_*(B)$. Then the resulting homology groups must be the homology groups of B with coefficients in the abelian group $H_t(F)$. This proves the main theorem of this section:

Theorem 4.11. *The $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence of the fibration $p : E \to B$ is*

$$\mathcal{E}^2_{s,t} \cong H_s(B; H_t(F))$$

and the spectral sequence converges to $H_*(E)$.

The last statement about "convergence" just means that $\mathcal{E}^\infty_{**}$ of the spectral sequence is a piecemeal version of $H_*(E)$, which we already know from the abstract theory.

Exercise 4.12. Investigate the Leray-Serre spectral sequences of the following fibrations or fiber bundles:

- the Hopf fiber bundles $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$, $\mathbb{S}^{15} \to \mathbb{S}^8$;
- for $n \geq 2$, the fibration $p : E \to \mathbb{S}^n$ where E is the space of paths ω in $\mathbb{S}^n$ such that $\omega(0)$ is the base point, and $p(\omega) = \omega(1)$.
- for $n \geq 2$, the fibration $p : E \longrightarrow \mathbb{S}^n \vee \mathbb{S}^n$ where E is the space of paths ω in $\mathbb{S}^n \vee \mathbb{S}^n$ such that $\omega(0)$ is the base point, and $p(\omega) = \omega(1)$.

You should be able to use the second of these items for a calculation of $H_*(\Omega\mathbb{S}^n)$ which is more elegant than the one coming from exercise 3.18. In the third item, you should also be able to calculate $H_*(\Omega(\mathbb{S}^n \vee \mathbb{S}^n))$. Conclude that the obvious inclusion $\Omega\mathbb{S}^n \vee \Omega\mathbb{S}^n \hookrightarrow \Omega(\mathbb{S}^n \vee \mathbb{S}^n)$ is *not* a homotopy equivalence.

Example 4.13. (See also Fuks-Fomenko-Gutenmacher, *Homotopic Topology*.) For $n > 0$, let's try to calculate the homology of $U(n)$ (the topological group of complex linear automorphisms of $\mathbb{C}^n$ preserving the standard hermitian inner product). First observe that the determinant is a continuous homomorphism

$$\det : U(n) \longrightarrow \mathbb{S}^1$$

with kernel $SU(n)$ (that's a definition). For $n = 1$ the determinant is also a *homeo*morphism, $U(1) \cong \mathbb{S}^1$, and for $n > 1$ the composition

$$U(1) \hookrightarrow U(n) \xrightarrow{\det} \mathbb{S}^1$$

is an isomorphism of topological groups. Thus $U(n)$ is a semidirect product, $U(n) \cong SU(n) \rtimes \mathbb{S}^1$, and therefore

$$U(n) \cong SU(n) \times \mathbb{S}^1 \qquad \text{in the category of spaces}$$

(but not in the category of topological groups). So it should be enough to calculate the homology of $SU(n)$ for $n > 1$. For this we observe that the evaluation map

$$p_n : SU(n) \longrightarrow \mathbb{S}^{2n-1} \quad ; \quad p(A) = Ae_1 \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$$

(where e_1 is the well-known standard basis vector) is a fiber bundle with fibers homeomorphic to $SU(n-1)$. (Proving this is about as hard as solving exercise 1.8. But it is clear that the fibers are as claimed: for $v \in \mathbb{S}^{2n-1}$, the fiber $p^{-1}(v)$ consists of all unitary $n \times n$ matrices of determinant 1 sending e_1 to v.) We now try to use our spectral sequence and induction. The fibers of the fibration p_2 are homeomorphic to $SU(1)$, which is a point, so

$$SU(2) \cong \mathbb{S}^3$$

which in particular calculates the homology. Next we have

$$p_3 : SU(3) \longrightarrow \mathbb{S}^5$$

with fibers homeomorphic to $SU(2) \cong \mathbb{S}^3$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(5,0)$, $(5,3)$. It follows immediately that the differentials on $\mathcal{E}^2_{**}$ as well as those on $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. are zero, so that

$$\mathcal{E}^2_{**} \cong \mathcal{E}^\infty_{**}$$

(the spectral sequence *collapses*). We conclude that

$$H_*(SU(3)) \cong H_*(\mathbb{S}^3 \times \mathbb{S}^5)$$

(but it is not claimed that $SU(3) \simeq \mathbb{S}^3 \times \mathbb{S}^5$). Next we have

$$p_4 : SU(4) \longrightarrow \mathbb{S}^7$$

with fibers homeomorphic to $SU(3)$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(0,5)$, $(0,8)$, $(7,0)$, $(7,3)$, $(7,5)$, $(7,8)$. Again you can easily convince yourself that none of the differentials on $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. has a chance to be nonzero. Therefore

$$H_*(SU(4)) \cong H_*(SU(3) \times \mathbb{S}^7) \cong H_*(\mathbb{S}^3 \times \mathbb{S}^5 \times \mathbb{S}^7)\,.$$

One might hope that this will go on forever. Let's try one more time: The $\mathcal{E}^2_{**}$ term of the spectral sequence for p_5 looks like

$$
\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

and we have a problem. Namely, there are two differentials in the spectral sequence which could be nonzero: they would be in the $\mathcal{E}^9_{**}$ term, from position $(9,0)$ to position $(0,8)$ and from position $(9,7)$ to position $(0,15)$. So our argument breaks down. All we know is that

$$H_*(U(n) \cong H_*(\mathbb{S}^1 \times \mathbb{S}^3 \times \cdots \times \mathbb{S}^{2n-1}) \qquad \text{for } n \leq 4\,.$$

For the cases $n > 4$, we need better equipment.

Exercise 4.14. Let $p : E \to \mathbb{S}^n$ be a fibration, where $n > 1$. Let F be the fiber of p over the base point. Show that there exists a long exact sequence of the form

$$\cdots \to H_{k-n+1}(F) \longrightarrow H_k(F) \xrightarrow{\ i\ } H_k(E) \longrightarrow H_{k-n}(F) \longrightarrow H_{k-1}(F) \to \dots\ .$$

This is called the *Wang sequence.*

5. Naturality properties of the Leray-Serre Spectral Sequence

In the previous section, we started with a filtered chain complex $C = \cup_s C(s)$, where $C(s) \subset C(s+1)$ for all s, and $C(s) = 0$ for some s. Then we constructed families of abelian groups $\mathcal{E}^1_{**}$, $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$ and differentials on each of these. (For more precision, I shall write $\mathcal{E}^r_{**}(C)$ instead of just $\mathcal{E}^r_{**}$.) It is obvious that all this depends functorially on the filtered chain complex C. Although it is obvious, it may be worth saying it in detail: Let $D = \cup_s D(s)$ be another filtered chain complex (so $D(s) \subset D(s+1)$ for all s, and $D(s) = 0$ for some s). Let $f : C \to D$ be a chain map taking $C(s)$ to $D(s)$, for all s. Then f induces maps

$$f^r_* : \mathcal{E}^r_{s,t}(C) \longrightarrow \mathcal{E}^r_{s,t}(D) \qquad \forall s, t \in \mathbb{Z}$$

for any $r > 0$, commuting with the differentials on $\mathcal{E}^r_{**}(C)$ and $\mathcal{E}^r_{**}(D)$. *Moreover*, if we make the identifications

$$\mathcal{E}^{r+1}_{**}(C) = H(\mathcal{E}^r_{**}(C)), \qquad \mathcal{E}^{r+1}_{**}(D) = H(\mathcal{E}^r_{**}(D))$$

where H means "homology", then f^{r+1}_* is simply the map of homology groups induced by f^r, and this holds for all $r > 0$. Briefly: f induces a *morphism of spectral sequences.*

Let's apply this observation to the situation where we have a commutative diagram of spaces and maps

$$\begin{array}{ccc} E & \xrightarrow{\bar{g}} & E_1 \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow \\ B & \xrightarrow{g} & B_1 \end{array}$$

where p and p_1 are fibrations, B and B_1 are simply connected CW-spaces, and g is cellular. We made the Leray-Serre spectral sequence of p using the filtration of the singular chain complex $C(E)$ by subcomplexes $C(p^{-1}(B^s))$. We would of course also make the Leray-Serre spectral sequence of p_1 by using the filtration of $C(E_1)$ by subcomplexes $C(p_1^{-1}(B_1^s))$. But $\bar{g}$ takes $p^{-1}(B^s)$ to $p_1^{-1}(B_1^s)$, for all s, so induces a map $C(E) \to C(E_1)$ respecting the filtrations. By the previous observation, this will lead to a morphism of spectral sequences. Call it $(g, \bar{g})_*$. What does it do to the $\mathcal{E}^1_{**}$ terms? There it will take the form

$$(g, \bar{g})_* : W_s(B) \otimes H_t(F) \longrightarrow W_s(B_1) \otimes H_t(F_1) \quad s, t \in \mathbb{Z}$$

(use proposition 4.10, write F for the fiber of p over some $x \in B$, and write F_1 for the fiber of p_1 over $g(x) \in B_1$).

Proposition 5.1. *On $\mathcal{E}^1_{**}$ terms, $(g, \bar{g})_*$ is the tensor product of the chain map*

$$W_*(B) \longrightarrow W_*(B_1)$$

induced by g with the homomorphism

$$H_*(F) \longrightarrow H_*(F_1)$$

induced by the restriction of $\bar{g}$ to F.

Corollary 5.2. *On E^2_{**} terms,*

$$(g,\bar{g})_* : H_s(B;H_t(F)) \longrightarrow H_s(B_1;H_t(F_1)) \qquad (s,t \in \mathbb{Z})$$

agrees with the homomorphism induced by $g : B \to B_1$ and the homomorphism of coefficient groups $H_t(F) \to H_t(F_1)$ induced by the restriction of $\bar{g}$ to F.

Corollary 5.3. *If $g : B \to B_1$ is a homotopy equivalence, and the restriction of $\bar{g}$ to F is a homotopy equivalence $F \to F_1$, then $(g,\bar{g})_*$ is an isomorphism of the $\mathcal{E}^r_{**}$ terms for any $r \geq 2$.*

The proof of proposition 5.1 is by inspection, and the proof of corollary 5.3 is by induction on r starting with $r = 2$.
One consequence of corollary 5.3 that one should certainly be aware of is that the Leray-Serre spectral sequence of a fibration, from the $\mathcal{E}^2_{**}$ term onwards, does not depend on the CW-structure of the base space chosen. In fact, it is enough to assume that the base space is homotopy equivalent to a CW-space. To see why, let $q : D \to A$ be a fibration, where A is simply connected and homotopy equivalent to a CW-space. How can we set up a Leray-Serre spectral sequence for calculating $H_*(D)$? We can choose a homotopy equivalence $e_0 : A_0 \to A$, where A_0 is an honest CW-space. Then we have the "usual" commutative square

$$\begin{array}{ccc} e_0^*D & \xrightarrow{\bar{e}_0} & D \\ {\scriptstyle e_0^*q}\downarrow & & \downarrow{\scriptstyle q} \\ A_0 & \xrightarrow{e_0} & A \end{array}$$

and it turns out that $\bar{e}_0$ is also a homotopy equivalence, like e_0 (exercises 5.4 and 5.5 below). Then

$$H_*(e_0^*D) \cong H_*(D)$$

and for $H_*(e_0^*D)$ we have the Leray-Serre spectral of the fibration e_0^*q. Forgetting about its $\mathcal{E}^1_{**}$ term, we declare this to be the Leray-Serre spectral sequence of the fibration q also. It certainly converges to the right thing. We need to check that it is sufficiently well defined. Before we do so, let's observe that the $\mathcal{E}^2_{**}$ term is

$$\mathcal{E}^2_{s,t} \cong H_s(A_0;H_t(F)) \cong H_s(A;H_t(F))$$

where F is any fiber of e_0^*q (homeomorphic to some fiber of q, automatically). It follows that the $\mathcal{E}^2_{**}$ term at least is well defined (independent of the choices A_0 and e_0). Suppose now that $e_1 : A_1 \to A$ is another homotopy equivalence from a CW-space to A. Then we can find a third homotopy equivalence $e_I : A_I \to A$ from a CW-space to A, and *cellular* maps $j_0 : A_0 \to A_I$,

$j_1 : A_1 \to A_I$ making the diagram

$$\begin{array}{ccccc} A_0 & \longrightarrow & A_I & \longleftarrow & A_1 \\ {\scriptstyle e_0}\big\downarrow & & {\scriptstyle e_I}\big\downarrow & & {\scriptstyle e_1}\big\downarrow \\ A & \xrightarrow{=} & A & \xrightarrow{=} & A \end{array}$$

commutative (exercise 5.6 below). As above, the map j_0 gives rise to a morphism from the spectral sequence of e_0^*q to the spectral sequence of e_1^*q which is an isomorphism on $\mathcal{E}^2_{**}$ terms. The same can be said of j_1. It follows that the spectral sequences of e_0^*q and e_1^*q are isomorphic from the $\mathcal{E}^2_{**}$ term onwards. The isomorphism that we constructed extends the identification of $\mathcal{E}^2_{**}$ terms that we found previously: all $\mathcal{E}^2_{**}$ terms in sight look like

$$E^2_{s,t} \cong H_s(A; H_t(F))\,.$$

(It follows that the isomorphism we constructed is unique: it does not depend on the choice of A_I, e_I, j_0, j_1, because it does not depend on anything as far as $\mathcal{E}^2_{**}$ terms go.)

Exercise 5.4. The *mapping cylinder* of a continuous map $f : A \to B$ is the space

$$Z = \big(A \times [0,1]\big) \amalg B/\!\sim$$

where $\sim$ identifies $(a, 1)$ with $f(a)$ for all $a \in A$. Prove that Z deforms into the subspace B (see definition 3.10). Prove that Z also deforms into the subspace $A \cong A \times \{0\}$ if f is a homotopy equivalence.

Exercise 5.5. Let $q : D \to A$ be a fibration, and let $e_0 : A_0 \to A$ be a homotopy equivalence. Prove that the projection map $(x, y) \mapsto y$ from $e_0^*D \subset A_0 \times D$ to D is a homotopy equivalence. (You may want to use exercise 5.4. Try to reduce to the situation where e_0 is the inclusion of a subspace A_0 such that A deforms into A_0.)

Exercise 5.6. Let $e_0 : A_0 \to A$ and $e_1 : A_1 \to A$ be homotopy equivalences, where A_0 and A_1 are CW-spaces. Show that there exists another homotopy equivalence $e_I : A_I \to A$, and maps $j_0 : A_0 \to A$, $j_1 : A_1 \to A$, such that A_I is a CW-space, j_0, j_1 are cellular, and $e_I j_0 = e_0$, $e_I j_1 = e_1$. (See the commutative diagram just above).

Back to filtrations and spectral sequences: Let C be a chain complex with filtration

$$\dots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

such that $C = \cup_s C(s)$. This time asssume

$$C(s) = 0 \text{ for some } s\ , \qquad C(t) = C \text{ for some } t\ ,$$

and this time let's see what the filtration tells us about the *co*homology of C. This will only require minor changes. Thus let $D = \hom(C, \mathbb{Z})$, and grade it by giving degree $-n$ to the elements in $\hom(C_n, \mathbb{Z})$. This is unusual, but it means that the differential in D lowers degree by one as usual. Let $D(s) := \hom(C/C(-s-1),\mathbb{Z})$ (same grading conventions). Then

$$\dots D(-2) \subset D(-1) \subset D(0) \subset D(1) \subset D(2) \subset D(3) \subset \dots$$

and $D(s) = D$ for some s, $D(t) = 0$ for some t. So we are in the same situation as before, and we can make a spectral sequence converging to $H_*(D) = H^{-*}(C)$, with $\mathcal{E}^1_{**}$ term

$$\mathcal{E}^1_{s,t} = H_{s+t}D(s, s-1)\,.$$

Now throw in another assumption, namely: the inclusion homomorphisms from $C(s)_n$ to $C(s+1)_n$ are *split* injective in each degree n (for all s). Then there is a short exact sequence of chain complexes

$$D(s-1) \longrightarrow D(s) \longrightarrow \hom(C(-s,-s-1),\mathbb{Z})\ ,$$

the second arrow given by evaluation of homomorphisms to $\mathbb{Z}$ on $C(-s)$. Therefore

$$\mathcal{E}^1_{s,t} = H^{-s-t}C(-s,-s-1).$$

Example 5.7. Let $p: E \to B$ be a fibration and assume that B is a simply connected *compact* CW-space. Let C be the singular chain complex of E and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton. Then the cohomology Leray-Serre spectral sequence has

$$\mathcal{E}^1_{s,t} \cong H^{-s-t}C(-s,-s-1) \cong \prod_{s-\text{cells}} H^{-t}(F) \cong \hom\big(W_s, H^{-t}(F)\big)$$

(notation and proof as for proposition 4.10). The $\mathcal{E}^2_{**}$ term then becomes

$$\mathcal{E}^2_{s,t} \cong H^{-s}(B; H^{-t}(F)\,.$$

Here is some disturbing news: it is customary to switch the top and bottom indices as well as some of the signs, so that

$$\mathcal{E}^{s,t}_r := \mathcal{E}^r_{-s,-t}$$

which for the Leray-Serre cohomology spectral sequence means

$$\mathcal{E}^{s,t}_2 \cong H^s(B; H^t(F))\,.$$

This is nice because it means that the nonzero terms of the spectral sequence are all in the first quadrant ($s, t \geq 0$), but of course it also means that all the differentials go in the wrong direction. For example, in $\mathcal{E}^{**}_1$, differentials move one to the right and zero down, in $\mathcal{E}^{**}_2$ they move two to the right and one down, in $\mathcal{E}^{**}_r$ they move r to the right and $r-1$ down.
The spectral sequence converges to $H^*(E)$, of course, but to be more precise we would have to say that the $\mathcal{E}^{**}_\infty$ term is as follows:
$\mathcal{E}^{s,t}_\infty$ is a subquotient of $H^{s+t}(E)$, namely,

$$\frac{\text{kernel of restriction from } H^{s+t}(E) \text{ to } H^{s+t}(p^{-1}(B^{s-1}))}{\text{kernel of restriction from } H^{s+t}(E) \text{ to } H^{s+t}(p^{-1}(B^s))}\,.$$

At this point we can see the possibility of introducing products. For example, on the $\mathcal{E}_\infty^{**}$ term just described, we can use the cup product to get a bilinear map

$$\mathcal{E}_\infty^{s,t} \times \mathcal{E}_\infty^{m,n} \longrightarrow \mathcal{E}_\infty^{s+m,t+n}\,.$$

For this we only have to verify that the ordinary cup product $x \cup y$ of two elements in $H^*(E)$ restricts to 0 in $H^*(p^{-1}(B^{s+m}))$ provided x restricts to 0 in $H^*(p^{-1}(B^s))$ and y restricts to zero in $H^*(p^{-1}(B^m))$. (Verify this.) We can also very easily define products

$$\mathcal{E}_2^{s,t} \times \mathcal{E}_2^{m,n} \longrightarrow \mathcal{E}_2^{s+m,t+n}\,.$$

This amounts to specifying bilinear maps

$$H^s(B; H^t(F)) \times H^m(B; H^n(F)) \longrightarrow H^{s+m}(B; H^{t+n}(F)\,.$$

We take the "usual" cup product from $H^s(B; H^t(F)) \times H^m(B; H^n(F))$ to $H^{s+m}(B; H^t(F) \otimes H^n(F))$, and then apply the other "usual" cup product $H^t(F) \otimes H^n(F) \to H^{t+n}(F)$ to the coefficients.— The cup product on $\mathcal{E}_2^{**}$ is bilinear, associative, and commutative in the graded sense: $\alpha \cup \beta = (-1)^{|\alpha||\beta|}(\beta \cup \alpha)$ where $|\alpha| = s+t$ if $\alpha \in \mathcal{E}^2_{s,t}$. The cup product on $\mathcal{E}_\infty^{**}$ is also bilinear, associative, and commutative in the graded sense.

Summarizing, we have a cup product on the $\mathcal{E}_\infty^{**}$ term which is a piecemeal version of the honest cup product on the total space E of our fibration ; and we have another cup product on the $\mathcal{E}_2^{**}$ term. How are these two related ? The answer is easy. The differential d on $\mathcal{E}_2^{**}$ is compatible with the cup product, which means that $d(\alpha \cup \beta)$ equals $d(\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \cup d(\beta)$ for all α and β. Using this fact, you can define a cup product on $\mathcal{E}_3^{**}$ by choosing representatives and multiplying them as you would in $\mathcal{E}_2^{**}$. Again, the differential on $\mathcal{E}_3^{**}$ is compatible with the cup product on $\mathcal{E}_3^{**}$; you can use this fact to define a cup product on $\mathcal{E}_4^{**}$, and so on. You end up with a cup product in $\mathcal{E}_\infty^{**}$. But now $\mathcal{E}_\infty^{**}$ already has a cup product, as we saw. Now, as you might guess, agreement is supposed to take place...

Exercise 5.8. Write an essay about products in the cohomology Leray-Serre spectral sequence.

6. Homotopy Groups

Let X be a *pointed space*, i.e., a space with a distinguished point $* \in X$ which we will call the base point. For instance, in $\mathbb{S}^n$ the standard choice of base point is the north pole $(1, 0, 0, \ldots, 0)$. (For simplicity base points are generally denoted by $*$, even if they belong to different spaces.) A map between pointed spaces is a *pointed map* if it sends the base point to the base point. A homotopy (h_t) between pointed maps is a *pointed homotopy* if each h_t is a pointed map.

Definition 6.1. The n-th homotopy set $\pi_n(X, *)$ of a pointed space $(X, *)$ is the set of pointed homotopy classes of pointed maps

$$(\mathbb{S}^n, *) \longrightarrow (X, *)\,.$$

Clearly $\pi_0(X,*)$ is in bijection with the set of path components of X. There is a distinguished element in $\pi_0(X,*)$, that which is represented by the unique constant (and pointed) map from $\mathbb{S}^0$ to X. So the most elaborate structure we can put on $\pi_0(X,*)$ is the structure of a *pointed set.* On the other hand, you know already that $\pi_1(X,*)$ has a canonical group structure. It is not difficult to generalize this to all $n \geq 1$. To this end, let us think of $\mathbb{S}^n$ as something homeomorphic to the quotient space $I^n/\partial I^n$ (where $I = [0,1]$ and ∂I^n consists of all points $(x_1, x_2, \ldots, x_n)$ in the cube I^n for which at least one coordinate equals 0 or 1. Then we can also think of a pointed map from $\mathbb{S}^n$ to X as a map from I^n to X sending ∂I^n to the base point. If

$$f, g : I^n \longrightarrow X$$

are two such (continuous) maps, then we can define the *concatenation* $f \circledast g$ of f and g as follows. Let $A, B : I^n \to I^n$ be the maps given by

$$\begin{array}{rcl} (x_1, \ldots, x_{n-1}, x_n) & \mapsto & (x_1, \ldots, x_{n-1}, 2^{-1}x_n) \\ (x_1, \ldots, x_{n-1}, x_n) & \mapsto & (x_1, \ldots, x_{n-1}, 2^{-1} + 2^{-1}x_n) \end{array}$$

respectively. Then $\mathrm{im}(A) \cup \mathrm{im}(B) = I^n$. We define

$$(f \circledast g)(A(x)) = g(x), \qquad (f \circledast g)(B(x)) = f(x)$$

for $x = (x_1, \ldots, x_n)$ in I^n. This is well defined and continuous. Clearly the (pointed) homotopy class of $f \circledast g$ depends only on the (pointed) homotopy classes of f and g, and we can therefore define a multiplication on $\pi_n(X)$ by

$$[f][g] := [f \circledast g]$$

(square brackets for homotopy classes of pointed maps from $I^n/\partial I^n$ to X). This is seen to be associative. There is a neutral element given by the class of the (unique) constant pointed map, and each $[f]$ has an inverse given by

$$(x_1, x_2, \ldots, x_n) \mapsto f(x_1, x_2, \ldots, x_{n-1}, 1 - x_n).$$

Thus $\pi_n(X,*)$ is a group for $n \geq 1$.

Proposition 6.2. *The group $\pi_n(X,*)$ is commutative for $n \geq 2$.*

Proof. If $n > 2$ it is easy to construct embeddings $A_t, B_t : I^n \to I^n$ depending continuously on $t \in [0,1]$ such that the following are satisfied:

- $A_0 = A$ and $B_0 = B$
- $A_1 = B$ and $B_1 = A$
- if $A_t(x) = B_t(y)$, then x and y are boundary points of I_n.

Let $f \circledast_t g$ be defined by $A_t(x)) \mapsto g(x)$ and $B_t(x) \mapsto f(x)$ and $(f \circledast_t g)(y) =$ base point if $y \notin \mathrm{im}(A_t) \cup \mathrm{im}(B_t)$. Then the family $(f \circledast_t g)$ is a homotopy from $f \circledast_0 g = f \circledast g$ to $f \circledast_1 g = g \circledast f$. □

Occasionally one wants to know how $\pi_n(X,*)$ depends on the base point $*$. Suppose therefore that $*_0$ and $*_1$ are two points in X. Are the groups $\pi_n(X,*_0)$ and $\pi_n(X,*_1)$ isomorphic for all n ? The answer is yes if $*_0$ and $*_1$ belong to the same path component of X. In fact, let $\omega : I \to X$ be a path such that $\omega(0) = *_0$ and $\omega(1) = *_1$. We shall use this to define a homomorphism

$$\iota_\omega : \pi_n(X, *_0) \longrightarrow \pi_n(X, *_1)$$

for any $n \geq 0$. To do so, let's think of $\pi_n(X, *_0)$ as the set of homotopy classes of maps of pairs

$$(D^n, \mathbb{S}^{n-1}) \longrightarrow (X, *_0).$$

For such a map $f : D^n \to X$ (sending $\mathbb{S}^{n-1}$ to $*_0$), we let $\iota_\omega(f)$ be the map from D^n to X given by

$$z \mapsto \begin{cases} f(2z) & |z| < 1/2 \\ \omega(2|z| - 1) & 1/2 \leq |z| \leq 1 \end{cases}$$

where $|z|$ is the norm of $z \in D^n \subset \mathbb{R}^n$. This is a continuous map from D^n to X sending $\mathbb{S}^{n-1}$ to $*_1$ (the "other" basepoint). Passing to homotopy classes, we let

$$\iota_\omega([f]) = [\iota_\omega(f)]\,.$$

In the lemma which follows, we assume that λ is another path in X, from $*_1$ to another point $*_2$. Write $\omega\lambda$ for the concatenated path, from $*_0$ to $*_2$.

Lemma 6.3. *The homomorphism ι_ω depends only on the homotopy class (with endpoints $*_0$ and $*_1$ fixed) of ω. Furthermore, $\iota_{\omega\lambda} = \iota_\lambda \iota_\omega$. If $*_0 = *_1$ and ω is the constant path, then ι_ω is the identity homomorphism. Consequently ι_ω is an isomorphism for arbitrary ω (with arbitrary endpoints).* $\square$

Summarizing, we can say that $\pi_n(X, *_0)$ and $\pi_n(X, *_1)$ are isomorphic if $*_0$ and $*_1$ can be connected by a path ω in X, but the isomorphism may depend on the path. In particular, we could take $*_0 = *_1 (= *)$ and define a homomorphism from $\pi_1(X, *)$ to the automorphism group $\mathrm{Aut}(\pi_n(X, *))$ by

$$[\omega] \mapsto \iota_\omega\,.$$

(A homomorphism from one group to the automorphism group of another group is also called an *action*.) Of course, you know this action for $n = 1$: it is the conjugation action of $\pi_1(X, *)$ on itself. But for $n > 1$ it is something new and interesting.

Exercise 6.4.
- Choose any base point $*$ in $\mathbb{R}P^n$, where $n \geq 2$ is even. Show that the action of $\pi_1(\mathbb{R}P^n, *)$ on $\pi_n(\mathbb{R}P^n, *)$ is nontrivial.
- Let X be a path-connected topological group with neutral element $*$. Show that the action of $\pi_1(X, *)$ on $\pi_n(X, *)$ is trivial for all $n \geq 1$. In particular, this is true for $n = 1$, which means that $\pi_1(X, *)$ is commutative.

Exercise 6.5. The *Hopf invariant* of a map f from $\mathbb{S}^{2n-1}$ to $\mathbb{S}^n$, where $n > 1$, is defined as follows. Form the mapping cone, $\mathrm{cone}(f)$, as in exercise 1.17 Excision shows that

$$H^k(\mathrm{cone}(f)) \quad \cong \quad \begin{cases} \mathbb{Z} & \text{if } k = 0, n, 2n \\ 0 & \text{else} \end{cases}$$

(canonical isomorphisms can be chosen). Thus we have standard generators α, β for $H^n(\mathrm{cone}(f))$ and $H^{2n}(\mathrm{cone}(f))$, respectively. Then

$$\alpha \cup \alpha = \mu\beta$$

for some integer μ depending on f. This μ is the Hopf invariant of f.

- Show that the Hopf invariant is zero if n is odd.
- Show that the Hopf invariant is a homomorphism from $\pi_{2n-1}(\mathbb{S}^n)$ to $\mathbb{Z}$.
- Show that the Hopf maps $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$, $\mathbb{S}^{15} \to \mathbb{S}^8$ have Hopf invariant ± 1. (Compare exercise 1.17.)
- Show that there exists an element of Hopf invariant 2 in $\pi_{2n-1}(\mathbb{S}^n)$ for any even $n > 1$. (This is the difficult part of the exercise. Notice that you are asked to produce a CW-space X with only three cells, one in dimension 0 and one in dimension n and one in dimension $2n$, such that the cup square of the generator of $H^n(X) \cong \mathbb{Z}$ is $\pm$twice the generator of $H^{2n}(X) \cong \mathbb{Z}$. You should experiment with $\mathbb{S}^n \times \mathbb{S}^n$. This has a CW-structure with 4 cells.)

For each $n \geq 0$ we may regard π_n as a functor from the category of pointed spaces and pointed maps to the category of pointed sets (if $n = 0$) or to the category of groups ($n = 1$) or to the category of abelian groups ($n \geq 2$). For a pointed map $f : X \to Y$, the induced pointed map (homomorphism if $n \geq 1$)

$$f_* : \pi_n(X, *) \longrightarrow \pi_n(Y, *)$$

is given by composition with f. That is, $f_*([g]) = [fg]$. Then the following is of interest.

Proposition 6.6. *Let $f : X \to Y$ be a pointed map, and assume that f is an ordinary homotopy equivalence (not necessarily in the pointed sense). Then $f_* : \pi_n(X, *) \longrightarrow \pi_n(Y, *)$ is a bijection for all $n \geq 0$.*

Proof. Let $e : Y \to X$ be homotopy inverse to f, and let $\{h_t\}$ be a homotopy from ef to id_X. Let $*_1 = ef(*)$ and let ω be the path $t \mapsto h_t(*)$ from $*_1$ to $*$. We now have a diagram

$$\pi_n(X, *) \xrightarrow{f_*} \pi_n(Y, *) \xrightarrow{e_*} \pi_n(X, *_1) \longrightarrow \pi_n(X, *)$$

where the last arrow is the bijection ι_ω. Inspection shows that the composite map from $\pi_n(X, *)$ to itself is the identity. It follows that f_* is injective. By symmetry, e_* (the middle arrow in the diagram) is also injective. But the last arrow in the diagram is a bijection, so all arrows in the diagram must be bijections. In particular, f_* is a bijection. □

For a pair of spaces $A \subset X$ with base point $* \in A$ and any $n \geq 0$, there is a relative homotopy set $\pi_n(X, A, *)$. It is defined as the set of (pointed) homotopy classes of maps of pairs

$$(D^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)\,.$$

Often it is better to think of it as the set of homotopy classes of maps

$$I^n \longrightarrow X$$

taking all points of the form $(0, x_2, x_3, \dots, x_n)$ to A and taking the rest of ∂I^n to the base point. (Homotopies $\{h_t\}$ are only allowed if each h_t has these properties.) Then concatenation, defined using the same formula as before, leads to a group structure on $\pi_n(X, A, *)$ for $n \geq 2$:

$$[f][g] := [f \circledast g] \, .$$

(Why does this fail for $n = 1$?) For $n = 1$, we only have a pointed set $\pi_1(X, A, *)$, and $\pi_0(X, A, *) = \pi_0(X, *)$.

Just for the record, note that there is a restriction map

$$\pi_n(X, A, *) \longrightarrow \pi_{n-1}(A, *)$$

(restrict from D^n to S^n if you use the disk definition) and there is another forgetful map

$$\pi_n(X, *) \longrightarrow \pi_n(X, A, *) \, .$$

For the second of these, represent elements in $\pi_n(X, *)$ by maps from D^n to X taking $\mathbb{S}^{n-1}$ to the base point. Such a map also represents an element in $\pi_n(X, A, *)$. Both the restriction map and the forgetful map are homomorphisms when $n \geq 2$.

The relative homotopy groups are very useful in the study of homotopy properties of CW-spaces. Here is a technical notion.

Definition 6.7. A pair of spaces (X, A) has the *homotopy extension property* if the following holds. Given any map $f : X \to Y$, where Y is any other space, and given a homotopy $\{h_t : A \to Y\}$ such that h_0 is the restriction of f to A, there exists a homotopy $\{\bar{h}_t : X \to Y\}$ such that $\bar{h}_0 = f$ and each $\bar{h}_t$ restricts to h_t on A.

Proposition 6.8. *The following are equivalent:*

1. *(X,A) has the HEP (homotopy extension property) ;*
2. *$Z := X \times \{0\} \cup A \times I$ is a retract of $X \times I$, where $I = [0, 1]$;*
3. *Z is a strong deformation retract of $X \times I$ (details below).*

Proof. (1)⇒(2): Let f from $X \cong X \times \{0\}$ to Z be the inclusion, and define $h_t : A \to Z$ by $h_t(a) = (a, t) \in Z$. Use the HEP to construct a homotopy $\{\bar{h}_t : X \to Z\}$ extending $\{h_t\}$ and such that $\bar{h}_0 = f$. This homotopy is the desired retraction: in other words, the map

$$r : X \times I \longrightarrow Z \subset X \quad : \quad (x, t) \mapsto \bar{h}_t(x)$$

is a retraction (it equals the identity on Z).
(2)⇒(1): Let $r : X \times I \to Z$ be a retraction onto Z. For $f : X \to Y$ and a homotopy $\{h_t : A \to Y\}$ such that $h_0 = f_{|A}$, let $\bar{f} : Z \to Y$ be given by $\bar{f}(x, 0) = f(x)$ and $\bar{f}(a, t) = h_t(a)$. Then $\bar{f}$ is continuous and so is

$$\bar{f} r : X \times I \longrightarrow Y \, .$$

In fact, $\bar{f}r$ is the required homotopy extending $\{h_t\}$.
(2)⇒(3): Let $r : X \times I \to Z$ be the retraction onto Z. We can write

$$r(x,t) = (r_1(x,t), r_2(x,t)) \in X \times I\,.$$

For the strong deformation retraction we need, for every $(x,t) \in X \times I$, a path $\omega_{x,t}$ in $X \times I$ connecting (x,t) with $r(x,t)$. We want $\omega_{x,t}$ to depend continuously on (x,t), and we want it to be a constant path if $(x,t) \in Z$. Noting that $x = r_1(x,0)$ for all $x \in X$, we see that

$$\omega_{x,t}(s) := (r_1(x,st), (1-s)t + sr_2(x,t))$$

is a solution. □

Exercise 6.9. Show that A is closed in X if (X,A) has the HEP and X is Hausdorff.

Corollary 6.10. *Let $f : (X,A) \to (Y,B)$ be a map of pairs, homotopic (as a map of pairs) to a map with image contained in B. If (X,A) has the HEP, then f is homotopic rel A to a map whose image is contained in B.*

Proof. Choose a strong deformation retraction

$$\{\rho_t : X \times I \longrightarrow X \times I\}$$

where $t \in I$, each ρ_t is the identity on $Z \subset X \times I$, and ρ_0 is the identity whereas ρ_1 is a retraction onto Z. Using the assumption on f, let h from $X \times I$ to Y be a map such that $h(x,1) = f(x)$, $h(a,t) \in B$ for all $a \in A$ and all t, and $h(x,0) \in B$ for all x. Note that $h(Z) \subset B$. Define

$$\{k_t : (X,A) \longrightarrow (Y,B)\}$$

by $k_t(x) = h(\rho_t(x,1))$. Then $k_t(a) = h(a,1) = f(a)$, independent of t. This is what *rel* A means. Further,

$$\begin{aligned} k_0(x) &= h(\rho_0(x,1)) = h(x,1) = f(x)\;, \\ k_1(x) &= h(\rho_1(x,1)) \in h(Z) \subset B\;. \quad \square \end{aligned}$$

Example 6.11. The pair $(D^n, \mathbb{S}^{n-1})$ has the HEP. This follows easily from proposition 6.8. Applying corollary 6.10, we find: *A map f from $(D^n, \mathbb{S}^{n-1})$ to a pair (Y,B) represents the zero class in $\pi_n(Y,B,*)$ if and only if it is homotopic rel $\mathbb{S}^{n-1}$ to a map with image in B.* Of course, the base point in $B \subset Y$ is understood to be f(north pole).

Recall that the pushout $X \amalg_A Y$ of a diagram of spaces and maps

$$X \xleftarrow{\;i\;} A \xrightarrow{\;g\;} Y$$

is the identification space obtained from the disjoint union $X \amalg Y$ by making the identifications $i(a) \sim g(a)$ for all $a \in A$. There are canonical and obvious maps

$$X \longrightarrow X \amalg_A Y\,, \qquad Y \longrightarrow X \amalg_A Y\,.$$

If i is the inclusion of a closed subspace $A \subset X$, then

$$Y \longrightarrow X \amalg_A Y$$

is also an inclusion of a closed subspace. (More correctly, it is a homeomorphism onto its image, and the image is closed; but we will think of it as an inclusion map.)

Lemma 6.12. *If (X, A) has the HEP and $g : A \to Y$ is any map, then the pair $(X \amalg_A Y, Y)$ has the HEP.*

Proof. Let $f : X \amalg_A Y \to W$ be any map, and let $\{h_t : Y \to W\}$ be a homotopy such that h_0 agrees with $f_{|Y}$. Then the composition

$$X \longrightarrow X \amalg_A Y \xrightarrow{\ f\ } W$$

is a map $f^\flat : X \to W$, and the compositions

$$A \xrightarrow{\ g\ } Y \xrightarrow{\ h_t\ } W$$

make up a homotopy $\{h^\flat_t : A \to W\}$ such that $h^\flat_0$ agrees with $f^\flat$ on A. Since (X, A) has the HEP, we can find a homotopy $\{\bar{h}^\flat_t : X \to W\}$ extending $\{h^\flat_t\}$, and such that $\bar{h}^\flat_0 = f^\flat$. Finally we let

$$\begin{array}{lll} \bar{h}_t(x) & = \bar{h}^\flat_t(x) \in W & \text{for } x \in X \smallsetminus A \subset X \amalg_A Y \\ \bar{h}_t(y) & = h_t(y) \in W & \text{for } y \in Y \subset X \amalg_A Y\ . \end{array}$$

Then $\{\bar{h}_t\}$ is the required homotopy extending $\{h_t\}$. □

Corollary 6.13. *Any CW-pair (X, A) has the HEP. (In detail: X is a CW-space, and A is a closed union of cells in X. Then A is a CW-space in its own right.)*

Proof. It suffices to prove that the pair $(X^k \cup A, X^{k-1} \cup A)$ has the HEP for every $k \geq 0$ (where, as usual, $X^{-1} := \emptyset$). Reason for "suffices": Let $f : X \to Y$ be a map and let $\{h_t : A \to Y\}$ be a homotopy such that $h_0 = f_{|A}$. If $(X^k \cup A, X^{k-1} \cup A)$ has the HEP for all $k \geq 0$, then the homotopy can be extended step by step from $A = X^{-1} \cup A$ to $X^0 \cup A$, $X^1 \cup A$, $X^2 \cup A$, and finally to all of X. Make sure at each step that the homotopy starts with the appropriate restriction of f.
Now for the proof that $(X^k \cup A, X^{k-1} \cup A)$ has the HEP: We can write

$$X^k \cup A = K \amalg_{\partial K} L$$

where $L = X^{k-1} \cup A$, and K is a disjoint union of copies of D^k (and ∂K is the union of the boundary (k-1)-spheres). Apply example 6.11 and lemma 6.12. □

Proposition 6.14. *Let (X, A) be a CW-pair. Suppose that $\pi_n(X, A, *)$ is zero for all $n \geq 0$ (where $*$ is some base point in A). Then A is a strong deformation retract of X.*

Proof. Note that $\pi_n(X, A, *)$ is only a pointed set for $n = 0, 1$; we regard it as "zero" if it has only one element.
Note also that X is connected because $\pi_0(X, A, *) = \pi_0(X, *)$ is zero. Note furthermore that A is connected, because any point in A can be joined to the

base point by a path in X; this path can then be homotoped rel endpoints into A, because $\pi_1(X, A, *)$ is zero. (Here we use example 6.11.) This implies that $\pi_n(X, A, *_1) \cong \pi_n(X, A, *)$ for any "other" base point $*_1 \in A$ (proof: method of lemma 6.3). Hence $\pi_n(X, A, *_1)$ is zero for any $*_1 \in A$, and we conclude from example 6.11: (✠) *Any map of pairs* $(D^n, \mathbb{S}^{n-1}) \to (X, A)$, *where* $n \geq 0$, *is homotopic rel* $\mathbb{S}^{n-1}$ *to a map with image contained in* A. Now we construct a sequence of maps $f_i : X \to X$, where $i \geq -1$, with the following properties:

- f_{-1} is the identity ;
- $f_i(X^i \cup A) \subset A$;
- f_i is homotopic rel $X^i \cup A$ to f_{i+1}, for all $i \geq 0$.

Suppose that f_i has been constructed; then (✠) shows that the restriction of f_i to $X^{i+1} \cup A$ is homotopic rel $X^i \cup A$ to a map taking $X^{i+1} \cup A$ to A. Let $\{h_t\}$ be such a homotopy; using the HEP for the pair $(X, X^{i+1} \cup A)$, extend it to a homotopy $\{\bar{h}_t : X \to X\}$ such that $\bar{h}_0 = f_i$. Since $\bar{h}_t$ extends h_t, it is rel $X^i \cup A$. So the induction step is complete if we let $f_{i+1} = \bar{h}_1$. Of course, the (unlabelled) homotopies above from f_i to f_{i+1} are much more important than the maps f_i themselves. We can piece them together to get the required strong deformation retraction. In detail: Choose a homotopy rel $X^i \cup A$ from f_i to f_{i+1} ; reparametrize so that the parameter interval is

$$[1 - 2^{-i-1}, 1 - 2^{-i-2}].$$

Then the union of these homotopies is the required strong deformation retraction, with deformation parameter interval $[0, 1]$. Note that continuity problems do not seriously arise because of the "weakness" axiom which is part of the definition of a CW-space. □

Actually, the proof of proposition 6.14 also establishes the following:

Proposition 6.15. *Let* (X, A) *be a* CW*-pair such that* $\pi_k(X, A, *)$ *is zero for* $0 \leq k \leq n$ *(where* $* \in A$ *is some base point). Then there exists a homotopy* $\{h_t : X \to X\}$, *rel* A, *such that* $h_0 = \mathrm{id}_X$ *and* $h_1(X^n \cup A) \subset A$. □

We finish the section with the most elementary calculations one can do in homotopy theory.

Proposition 6.16. $\pi_k(\mathbb{S}^n, *)$ *is zero for* $0 \leq k < n$.

Proof. Represent an element in $\pi_k(\mathbb{S}^n, *)$ by a map

$$f : D^k \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$$

taking the boundary $\mathbb{S}^{k-1}$ to the base point. Use Stone-Weierstrass or other means to show that there exists a *smooth* map

$$f_1 : D^k \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$$

taking the boundary $\mathbb{S}^{k-1}$ to the base point, and such that the distance between $f(x)$ and $f_1(x)$ in $\mathbb{R}^{n+1}$ is less than some small number ϵ, for all $x \in D^k$. Then f and f_1 are homotopic rel $\mathbb{S}^{k-1}$ (why ?), and Sard's theorem tells us that f_1 is not onto (since it is smooth). Let $y \in \mathbb{S}^n$ be a point not in the image of f_1. Then $y \neq *$ and the class $[f_1] = [f]$ is in the image of the map induced by inclusion,

$$\pi_k(\mathbb{S}^n \smallsetminus \{y\}, *) \longrightarrow \pi_k(\mathbb{S}^n, *) \ .$$

But $\mathbb{S}^n \smallsetminus \{y\}$ is contractible, so π_k of it is zero. □

Proposition 6.17. *$\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$ for $n > 0$. The class* [id] *is a generator.*

Proof. Here it may be best to represent elements in $\pi_n(\mathbb{S}^n, *)$ by *smooth* maps $f : D^n \to D^n/\mathbb{S}^{n-1}$ taking $\mathbb{S}^{n-1}$ to the base point (which is the former boundary of the disk). We can then use euclidean coordinates to describe points in $D^n/\mathbb{S}^{n-1}$ other than the base point. By Sard's theorem we can find $p \in D^n$ arbitrarily close to the origin such that f is transverse to p. Performing a small homotopy if necessary, we may in fact assume that p is the origin. Then the inverse image $f^{-1}(p)$ is a finite subset $\{x_1, x_2, \dots, x_r\}$ of the interior of D^n. For each of the x_i, the differential of f at x_i is an invertible $n \times n$ matrix A_i. By the inverse function theorem, f is invertible in a suitable neighbourhood of each x_i. Performing another small homotopy if necessary, we can assume that f agrees with the linear map corresponding to A_i on a small open disk U_i with center x_i and radius ε independent of i, for $1 \le i \le r$. Using exercise 6.18 below, we can also arrange that A_i is an $n \times n$ identity matrix or a diagonal matrix with top left entry -1 and all other diagonal entries equal to 1. At this point it is easy to see that it is irrelevant what f looks like on the complement of the union of the U_i (as long as that complement is mapped to the complement of the origin). Therefore

$$[f] = \sum_{i=1}^{r} s_i[\mathrm{id}]\,, \quad \text{where } s_i = \begin{cases} 1 \text{ if } \det(A_i) > 0 \\ -1 \text{ if } \det(A_i) < 0\,. \end{cases}$$

(We should really use a $\prod$ symbol instead of a $\sum$ when $n = 1$.) We see that [id] does generate $\pi_n(\mathbb{S}^n, *)$. Now the degree of a map from $\mathbb{S}^n$ to itself is a homotopy invariant, and

$$\text{degree} : \pi_n(\mathbb{S}^n, *) \longrightarrow \mathbb{Z}$$

is a homomorphism. Since it maps the generator [id] to 1, it must be an isomorphism. □

Exercise 6.18. Show that the topological group $GL(n, \mathbb{R})$ has exactly two connected components for $n > 0$. (Two $n \times n$ matrices are in the same component if their determinant has the same sign.)

7. Long exact sequences of homotopy groups

For a pair of spaces (Y, A) (not necessarily CW) with base point $* \in A$ we have the following maps between homotopy sets, for all $n \geq 0$:

$$\begin{array}{rlc} i_* : \pi_n(A,*) \longrightarrow \pi_n(Y,*)\,, & & \text{(induced by inclusion)} \\ j_* : \pi_n(Y,*) = \pi_n(Y,\{*\},*) \longrightarrow \pi_n(Y,A,*)\,, & & \text{(ditto)} \\ \partial : \pi_{n+1}(Y,A,*) \longrightarrow \pi_n(A,*)\,, & & \text{(restriction to } \mathbb{S}^n\text{).} \end{array}$$

These are homomorphisms of groups for $n > 0$; for $n = 0$, they are still pointed maps between pointed sets.

Theorem 7.1. *The sequence of pointed sets and maps*

$$\cdots \longrightarrow \pi_{n+1}(Y,A,*) \xrightarrow{\partial} \pi_n(A,*) \xrightarrow{i_*} \pi_n(Y,*) \xrightarrow{j_*} \pi_n(Y,A,*) \xrightarrow{\partial} \cdots$$

is exact.

Note: Exactness means that the image of each map in the sequence is the inverse image of the base point under the next map. (In the sequel, we write ker for the inverse image of the base point.)

Proof. (1) $\ker(i_*) = \mathrm{im}(\partial) \subset \pi_n(A,*)$: Easy—think of D^{n+1} as a quotient of $\mathbb{S}^n \times I$.
(2) $\mathrm{im}(i_*) \subset \ker(j_*) \subset \pi_n(Y,*)$: obvious.
(3) $\mathrm{im}(i_*) \supset \ker(j_*) \subset \pi_n(Y,*)$: We think of $\pi_n(Y,*)$ as $\pi_n(Y,\{*\},*)$, so we represent an element in it by a map of pairs $f : (D^n, \mathbb{S}^n - 1) \to (Y, \{*\})$. Applying j_* means composing with the inclusion $(Y,\{*\}) \to (Y,A)$. From corollary 6.10 we know that if $j_*[f]$ is zero, then f is homotopy rel $\mathbb{S}^{n-1}$ to a map with image contained in A. That is all we need.
(4) $\mathrm{im}(j_*) \subset \ker(\partial) \subset \pi_n(Y,A,*)$: obvious.
(5) $\mathrm{im}(j_*) \supset \ker(\partial) \subset \pi_n(Y,A,*)$: Use the homotopy extension property for the pair $(D^n, \mathbb{S}^{n-1})$. □

Theorem 7.2. *(J.H.C. Whitehead) Let $f : X \to Y$ be a map betwen connected CW-spaces. If $f_* : \pi_n(X,*) \to \pi_n(Y, f(*))$ is an isomorphism for all $n > 0$ (for some base point $* \in X$) then f is a homotopy equivalence.*

Proof. We can assume that f is cellular, i.e., $f(X^k) \subset Y^k$ for all $k > 0$. (By the cellular approximation theorem, f is homotopic to a cellular map.) Let Z be the mapping cylinder of f ; this is the identification space

$$(X \times I) \amalg Y \big/ \sim$$

where $x \sim f(x)$ for $x \in X$. Since f is cellular, Z has a canonical CW-structure. The composition

$$X \cong X \times \{0\} \xrightarrow{\text{incl.}} Z \xrightarrow{\text{proj.}} Y$$

equals f, and since the mapping cylinder projection (second arrow) is a homotopy equivalence, we are reduced to proving: *The inclusion $X \hookrightarrow Z$ is a homotopy equivalence.* By assumption and by theorem 7.1, the relative homotopy groups $\pi_n(Z, X, *)$ vanish for all $n \geq 0$. By proposition 6.14, X is a strong deformation retract of Z. $\square$

Exercise 7.3. (This is a preparation for the next exercise, not directly related to theorem 7.2.) Let X and Y be pointed spaces ; let η_X, η_Y be the inclusions $x \mapsto (x, *)$, $y \mapsto (*, y)$ of X and Y into the product $X \times Y$. Show that the maps

$$\begin{array}{rcl} ([f],[g]) & \mapsto & [(f,g)] \\ ([f],[g]) & \mapsto & [\eta_X f] + [\eta_Y g] \end{array}$$

from $\pi_n(X, *) \times \pi_n(Y, *)$ to $\pi_n(X \times Y, (*, *))$ agree for $n > 0$. (Here $+$ denotes the addition in π_n ; of course this is not quite right if $n = 1$, and should be replaced by a multiplication dot.)

Exercise 7.4. (Stasheff) An *H-space* is a connected CW-space Y with base point $*$ and a map $\mu : Y \times Y \to Y$ for which the base point is a two-sided identity: $\mu(y, *) = \mu(*, y) = y$, for all $y \in Y$. Prove that any H-space has a *left inversion map* up to homotopy, which is a map $\rho : Y \to Y$ such that

$$y \mapsto \mu(\rho(y), y)$$

is homotopic to a constant map. (Hint: Experiment with self-maps of $Y \times Y$.)

Exercise 7.5. Show that $\mathbb{S}^{2k}$ for $k > 0$ does not admit an H-space structure. (Hint: Suppose it does—what does this mean for the cohomology ring $H^*(\mathbb{S}^{2k})$?) J.F.Adams showed (1959) that among spheres only $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ admit H-space structures.

Definition 7.6. A map $f : X \to Y$ is *n-connected* if, for any base point $* \in X$, the map $f_* : \pi_k(X, *) \to \pi_k(Y, f(*))$ is a bijection provided $k < n$, and a surjection provided $k = n$. (It is sufficient to check this for just one base point in each path component of X.)

Proposition 7.7. *Let $f : X \to Y$ be an n-connected map. Then for any CW-space B,*

$$f_* : [B, X] \longrightarrow [B, Y] \qquad \text{([,] for sets of homotopy classes)}$$

is a bijection if $\dim(B) < n$, *and a surjection if* $\dim(B) = n$.

Proof. Replacing Y by the mapping cylinder of f if necessary, we can assume that f is the inclusion of a subspace: $X \subset Y$. Then our assumption on f means that $\pi_k(Y, X, *)$ is zero for $0 \leq k \leq n$ and any $* \in X$. Given a CW-pair (L, K) and a map $g : (L, K) \to (Y, X)$, construct inductively maps $g_{-1}, g_0, g_1, g_2, \ldots, g_n$ from L to Y such that $g = g_{-1}$, $g_i(L^i) \subset X$, and $g_i \simeq g_{i+1}$ rel $L^i \cup K$. (For the induction step, use the vanishing of $\pi_i(Y, X, *)$ and corollary 6.10, example 6.11. Take $L = B$ and $K = \emptyset$ for surjectivity of f_* . Take $L = B \times I$ and $K = B \times \{0, 1\}$ for injectivity. $\square$

Theorem 7.8. *Let $p : E \to B$ be a pointed map and a fibration. Let $F = p^{-1}(*)$ be the fiber over the base point. Then*

$$p_* : \pi_n(E, F, *) \longrightarrow \pi_n(B, *)$$

is a bijection for $n > 0$.

Proof. Surjectivity: Writing $\mathbb{S}^n$ as a quotient of $\mathbb{S}^{n-1} \times I$, think of a pointed map g from $\mathbb{S}^n$ to B as a pointed homotopy $\{h_t\}$ from the constant pointed map $\mathbb{S}^{n-1} \to B$ to itself. Using the homotopy lifting property for p, we can find a homotopy $\{\bar{h}_t\, \mathbb{S}^{n-1} \to E\}$ such that $\bar{h}_0$ is the constant pointed map and $p\bar{h}_t = h_t$ for $t \in I$. Then $\bar{h}_1$ has image contained in $F = p^{-1}(*)$. We can think of $\{\bar{h}_t\}$ as a map $\bar{g}$ from the pair

$$\big((\mathbb{S}^{n-1}\times I/\mathbb{S}^{n-1}\times\{0\}), \mathbb{S}^{n-1}\times\{1\}\big) \cong (D^n, \mathbb{S}^{n-1})$$

to (E, F). It is not hard to see that $p_*[\bar{g}] = [g]$. Unfortunately, $\bar{g}$ need not be a pointed map. It does however take the base point in $\mathbb{S}^{n-1} \subset D^n$ to some point $*_1 \in F$ which is in the same path component as $* \in F$. (This is clear from the construction.) If ω is a path in F connecting $*_1$ to $*$, then $\iota_\omega[\bar{g}]$ belongs to $\pi_n(E, F, *)$ (compare lemma 6.3), and p_* of it is $[g]$.
Injectivity: Suppose first that $n > 1$ and that $e : (D^n, \mathbb{S}^{n-1}) \to (E, F)$ is a pointed map of pairs such that pe is nullhomotopic rel $\mathbb{S}^{n-1}$. Lifting such a nullhomotopy to E, we find that e is homotopic as a map of pairs to a map with image in F. By corollary 6.10, the map e is homotopic rel $\mathbb{S}^{n-1}$ to a map with image in F. Then clearly $[e] = 0$ in $\pi_n(E, F, *)$. When $n = 1$ we have to argue more carefully, because injectivity of p_* is not equivalent to triviality of the kernel of p_*. Here is a sketch: Let $e_\flat, e_\sharp$ be two pointed maps $(D^1, \mathbb{S}^0) \to (E, F)$. Suppose that $[pe_\flat] = [pe_\sharp]$ in $\pi_1(B, *)$. We want to show that $[e_\flat] = [e_\sharp]$ in $\pi_1(E, F)$. Now it is not hard to see that this is equivalent to: *The path in E obtained by running backwards through $e_\flat$ and then through $e_\sharp$ is homotopic (as a map of pairs $(D^1, \mathbb{S}^0) \to (E, F)$, without base points) to a map with image in F.* Such a homotopy can be obtained from the homotopy lifting property, using the hypothesis $[pe_\flat] = [pe_\sharp]$. □

Corollary 7.9. *(Notation of theorem 7.8.) There is a long exact sequence*

$$\cdots \longrightarrow \pi_{n+1}(B, *) \xrightarrow{\partial} \pi_n(F, *) \xrightarrow{i_*} \pi_n(E, *) \xrightarrow{p_*} \pi_n(B, *) \xrightarrow{\partial} \cdots$$

where i_* is induced by the inclusion of the fiber and p_* is induced by the bundle projection.

Exercise 7.10. Show that $\pi_n(\mathbb{S}^3, *) \cong \pi_n(\mathbb{S}^2)$ for $n > 2$. Show that $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Assuming $n \geq 0$, show that

$$\pi_n(\mathbb{S}^4, *) \cong \pi_n(\mathbb{S}^7, *) \oplus \pi_{n-1}(\mathbb{S}^3, *)\,, \qquad \pi_n(\mathbb{S}^8, *) \cong \pi_n(\mathbb{S}^{15}, *) \oplus \pi_{n-1}(\mathbb{S}^7, *)\ .$$

8. The Hurewicz theorems

Definition 8.1. A space X with base point is *n-connected* if the inclusion of the base point is an n-connected map. (Equivalently: $\pi_k(X, *)$ is zero for $0 \leq k \leq n$.)

Now let X be any pointed space. The *Hurewicz homomorphism*

$$z : \pi_k(X, *) \longrightarrow H_k(X)$$

(homology with integer coefficients) is defined by $z[f] := f_*[\alpha]$ where $[\alpha] \in H_k(\mathbb{S}^k) \cong \mathbb{Z}$ is the standard generator.

Theorem 8.2. *For a $(k-1)$-connected CW-space X (where $k > 1$) the Hurewicz homomorphism $z : \pi_k(X, *) \longrightarrow H_k(X)$ is an isomorphism. For a connected CW-space, the Hurewicz homomorphism from $\pi_1(X, *)$ to $H_1(X)$ is onto, and its kernel is the commutator subgroup.*

Proof. Note first that the theorem is true for $X = \mathbb{S}^k$; this follows easily from proposition 6.17. By a mild extension of proposition 6.17, it is also true for a wedge of spheres

$$\bigvee_\gamma \mathbb{S}^k$$

(indexed by the elements γ of some indexing set) ; use the same transversality argument, but pick one point distinct from the base point in each wedge summand. (Note: Here we use $k > 1$. What happens when $k = 1$?)
Another case we can handle is the case where X has exactly one 0-cell (the base point) and no cells of dimension i for $0 < i < k$. We will use the commutative diagram

$$\begin{array}{ccccc}
\pi_k(X^k, *) & \xrightarrow{(a)} & \pi_k(X^{k+1}, *) & \xrightarrow{(b)} & \pi_k(X, *) \\
{\scriptstyle (c)}\big\downarrow & & {\scriptstyle (d)}\big\downarrow & & {\scriptstyle (e)}\big\downarrow \\
H_k(X^k) & \xrightarrow{(f)} & H_k(X^{k+1}) & \xrightarrow{(g)} & H_k(X)
\end{array}$$

where all vertical arrows are Hurewicz homomorphisms and all horizontal arrows are induced by inclusion. By cellular approximation, (a) is onto and (b) is iso. Looking at cellular chain complexes, we find that (f) is onto and (g) is iso. Since X^k is a wedge of k-spheres, (c) is iso. It follows that (d) is onto, hence (e) is onto. Next, it is easy to see that the kernel of (a) contains all the classes homotopic to an attaching map for one of the $(k+1)$-cells. Hence (d) is injective. Therefore (e) is iso. (Again, we have assumed $k > 1$. What happens when $k = 1$?)
For the general case, we can use proposition 6.15 which tells us that there exists a homotopy $\{h_t : X \to X\}$ rel $*$ such that $h_0 = \mathrm{id}_X$ and $h_1(X^{k-1}) \subset \{*\}$. Then we can write h_1 as a composition of pointed maps

$$X \xrightarrow{\ q\ } X/X^{k-1} \xrightarrow{\ g\ } X$$

and it is still homotopic to the identity. Let $Y = X/X^{k-1}$. From the commutative diagram

$$\begin{array}{ccccc}
\pi_k(X, *) & \xrightarrow{q_*} & \pi_k(Y, *) & \xrightarrow{g_*} & \pi_k(X, *) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
H_k(X) & \xrightarrow{q_*} & H_k(Y) & \xrightarrow{g_*} & H_k(X)
\end{array}$$

where both compositions g_*q_* are identity maps and the middle vertical (Hurewicz) homomorphism is iso (if $k > 1$), we see that all vertical arrows must be iso. (What happens when $k = 1$?) □

There is a relative Hurewicz theorem (which contains the absolute version as a special case). For this we need the *relative* Hurewicz homomorphism. For a pair of spaces (Y, A) with base point $* \in A$ it takes the form

$$z : \pi_n(Y, A, *) \longrightarrow H_n(Y, A) \quad ; \quad z[f] = f_*[\beta] \ ,$$

where $[\beta]$ is the standard generator of $H_n(D^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$. Slightly more generally: It is convenient to define the relative homotopy sets $\pi_n(f)$ of a pointed map $f : X \to Y$ as $\pi_n(Z, X, *)$ where Z is the mapping cylinder of f and $X \cong X \times \{0\}$ is regarded as a subspace of Z. (Check that this is isomorphic to π_n of the pair (Y, X) if f happens to be an inclusion map.) Similarly, the homology groups $H_n(f)$ of $f : X \to Y$ are defined as $H_n(Z, X)$. One can also use notation like $H_n(X \to Y)$. Then we still have a Hurewicz homomorphism $z : \pi_n(f) \to H_n(f)$ for $n \geq 0$.

Exercise 8.3. Let $f : X \to Y$ be a pointed map.

- Give another description of $\pi_n(f)$ in terms of pairs of maps g_1, g_2 as in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{g_1} & X \\ {\scriptstyle\subset}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ D^n & \xrightarrow{g_2} & Y \ . \end{array}$$

- Suppose in addition that f is a fibration. Let Φ be the fiber over the base point. Establish an isomorphism for $n > 1$ (bijection for $n = 1$)

$$\pi_n(f) \cong \pi_{n-1}(\Phi, *) \ .$$

  (Hint: there is an obvious map from right-hand side to left-hand side. Use theorem 7.1 and theorem 7.8.)

We will also need the Leray-Serre spectral sequence (with a "remark") in the proof of the relative Hurewicz theorem. The remark concerns the so-called *edge homomorphisms*. Let $f : X \to Y$ be a fibration (where Y is simply connected). Let Φ be the fiber over the base point, assuming that Y is pointed. Then we have another fibration $\Phi \to \{*\}$, which maps by inclusion to the first fibration $f : X \to Y$. In this case we can apply our naturality results from the section on naturality, especially corollary 5.2. Therefore we can describe the homomorphism $H_*(\Phi) \to H_*(X)$ induced by inclusion as follows:

$$H_t(\Phi) \cong \mathcal{E}^\infty_{0,t} \text{ of the spectral sequence made from } \Phi \to \{*\}$$

maps by naturality to

$$\mathcal{E}^\infty_{0,t} \text{ of the spectral sequence made from } f : X \to Y$$

which is a subgroup of $H_t(X)$. (Why is it a subgroup ?)

Similarly, we could compare the fibration $f : X \to Y$ with the identity fibration $Y \to Y$, and in this way interpret $f_* : H_*(X) \to H_*(Y)$ as an edge homomorphism. (Do it.)
Actually, we need a Leray-Serre spectral sequence for *pairs* of fibrations. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be fibrations over tha same base CW-space B, and let $g : E_1 \to E_2$ be a map over B. Assume that B is simply connected, with base point, and let $F_1 = p_1^{-1}(*)$, $F_2 = p_2^{-1}(*)$. Then there is a Leray-Serre spectral sequence with

$$\mathcal{E}^2_{s,t} = H_s(B; H_t(F_1 \to F_2))$$

converging to the (relative) homology $H_*(E_1 \to E_2)$. (The map from F_1 to F_2 to be used is of course the restriction of g.) One way to obtain this spectral sequence is to reduce to the case where g is an inclusion map (using mapping cylinders). Assuming this has been arranged, consider the singular chain complex $C(E_2)/C(E_1)$ of the pair (E_2, E_1). Much as in the absolute case, the skeleton filtration of B gives rise to a filtration of $C(E_1)/C(E_2)$, which then gives rise to a spectral sequence in the usual way.
Finally, it is possible to combine the two remarks, the one on edge homomorphisms and the one on pairs of fibrations. Rather than making it explicit, let's allow it to happen.

Theorem 8.4. *Let $f : X \to Y$ be a $(k-1)$-connected map between pointed spaces, where $k > 1$. Suppose also that X is simply connected. Then the Hurewicz homomorphism from $\pi_k(f)$ to $H_k(f)$ is an isomorphism.*

Proof. We can assume that f is a fibration. Let Φ be the fiber over the base point. Let $f_\flat : \Phi \to *$ be the restriction of f. Then we have a commutative diagram

$$\begin{array}{ccc} \pi_k(f_\flat) & \xrightarrow{z} & H_k(f_\flat) \\ \downarrow & & \downarrow \\ \pi_k(f) & \xrightarrow{z} & H_k(f) \end{array}$$

where the vertical arrows are induced by inclusion. We want to prove that the lower z is iso. From exercise 8.3, the left vertical arrow is iso, not just on π_k, but on all π_t for $t > 0$. It follows that we can apply the absolute Hurewicz theorem to the upper z ; note incidentally that

$$\pi_t(f_\flat) \cong \pi_{t-1}(\Phi, *) \quad , \quad H_t(f_\flat) \cong \tilde{H}_{t-1}(\Phi)$$

where the tilde means reduced homology. We see that the upper z is also iso, and moreover

$$(\clubsuit) \qquad H_t(f_\flat) = 0 \qquad \text{for } t < k\ .$$

(To make this argument work in the case $k = 2$, observe first that $\pi_1(\Phi, *)$ is abelian: the long exact homotopy sequence of the fibration f shows that it is a quotient of the abelian group $\pi_2(Y, *)$. Then the absolute Hurewicz applies.)

We can now complete the proof by showing that the right-hand vertical arrow in our square is iso. This is where a spectral sequence argument works wonders. The pair of fibrations that we apply it to consists of $f : X \to Y$ and the identity fibration $\mathrm{id} : Y \to Y$. Then f is a map over Y, from the total space of the first fibration to the total space of the second fibration, and by the above remarks we have a spectral sequence with

$$\mathcal{E}^2_{s,t} = H_s(Y; H_t(f_\flat))$$

converging to $H_*(f)$. By the above remarks again, the homomorphism from $H_k(f_\flat)$ to $H_k(f)$ induced by inclusion can be identified with the edge homomorphism

$$H_k(f_\flat) \cong H_0(Y; H_k(f_\flat)) \cong \mathcal{E}^2_{0,t} \quad \twoheadrightarrow \quad \mathcal{E}^\infty_{0,k} \subset H_k(f)\,.$$

But now this edge homomorphism is easily seen to be an isomorphism, due to the fact that all horizontal lines $\mathcal{E}^2_{*,t}$ are zero for $t < k$ (use (♣)) and all vertical lines $\mathcal{E}^2_{s,*}$ are zero for $s < 0$. □

Corollary 8.5. *(G. Whitehead) Let $f : X \to Y$ be a map between simply connected CW-spaces.*

1. *If $f_* : H_*(X) \to H_*(Y)$ is iso, then f is a homotopy equivalence.*
2. *If $f_* : H_*(X) \to H_*(Y)$ is iso for $* < k$ and onto for $* = k$, then f is n-connected (see definition 7.6).*

Proof. The assumption in (1) means that $H_*(f)$ is zero. Therefore $\pi_*(f)$ is zero—otherwise there exists a least integer k for which $\pi_k(f)$ is nontrivial, and this one will be isomorphic to $H_k(f)$, which is therefore also nontrivial. (Contradiction.)
The assumption in (2) means that $H_*(f)$ is zero for $* \le k$, and it follows as before that $\pi_*(f)$ is zero for $* \le k$. □

Exercise 8.6. (Freudenthal's theorem—this is a rather long exercise). For a CW-space Y with base point $*$ let ΣY be the reduced suspension of Y. (You can also use the unreduced suspension instead, which is the usual quotient space of $Y \times I$; the reduced suspension of Y is the unreduced suspension of Y quotiented out by the unreduced suspension of $\{*\}$. It is homotopy equivalent to the unreduced suspension of Y.) There is a very interesting map

$$\alpha : Y \to \Omega\Sigma Y$$

which sends $y \in Y$ to the loop $\omega_y : I \to \Sigma Y$ such that $\omega_y(t) = (y, t)$.

(a) Show that $\alpha : \mathbb{S}^n \to \Omega\Sigma\mathbb{S}^n$ induces an isomorphism on π_n.
(b) Conclude from (a) that $\alpha : \mathbb{S}^n \to \Omega\Sigma\mathbb{S}^n$ is $(2n-1)$-connected.

Notice also that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. For any map $f : \mathbb{S}^k \to \mathbb{S}^n$, we get another map Σf from $\Sigma\mathbb{S}^k$ to $\Sigma\mathbb{S}^n$ which simplifies to $\Sigma f : \mathbb{S}^{k+1} \to \mathbb{S}^{n+1}$. This defines a homomorphism

$$\pi_k(\mathbb{S}^n, *) \to \pi_{k+1}(\mathbb{S}^{n+1}, *) \quad ; \quad [f] \mapsto [\Sigma f]\ ,$$

the *suspension homomorphism*. Freudenthal's theorem states that it is an isomorphism for $k < 2n-1$ and onto for $k = 2n-1$. *Explain why Freudenthal's theorem is equivalent to (b).*
An interesting consequence is that the group $\pi_k(\mathbb{S}^n, *)$ depends only (up to isomorphism) on the difference $k-n$ as long as $k < 2n-1$. To express this fact, one writes π^s_{k-n}. For example, $\pi^s_0 \cong \mathbb{Z}$ because $\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$ for $n > 0$. In the next exercise, we shall see that $\pi^s_1 \cong \mathbb{Z}/2$, which means that $\pi_4(\mathbb{S}^3)$, $\pi_5(\mathbb{S}^4)$ and $\pi_6(\mathbb{S}^5)$ etc. are all isomorphic to $\mathbb{Z}/2$. Of course, you know already that these groups must be cyclic...

Exercise 8.7. (a) Calculate the cohomology ring $H^*(\Omega\mathbb{S}^n)$ where n is odd, $n \geq 3$. (Use the method of exercise 4.12, but this time use the *cohomology* version of the spectral sequence, with products.)
(b) Assuming $n > 2$ (even n permitted), show that $\Omega\mathbb{S}^n$ is homotopy equivalent to a CW-space with one cell in each dimension of the form $k(n-1)$, where $k \geq 0$. In particular, $\Omega\mathbb{S}^3$ is homotopy equivalent to a CW-space with one cell in each even dimension. The attaching map for the 4-cell corresponds to an element in $\pi_3(\mathbb{S}^2, *)$, which we know is isomorphic to $\mathbb{Z}$ via the Hopf invariant. What is the Hopf invariant of the attaching map ? Use your answer to (a).
(c) Use (b) and Freudenthal to show that the suspension homomorphism from $\pi_3(\mathbb{S}^2)$ to $\pi_4(\mathbb{S}^3)$ is onto, and that its kernel consists of all elements having even Hopf invariant. Conclude that $\pi^s_1 \cong \mathbb{Z}/2$.

9. Some stable homotopy theory

An excellent reference for the foundations of stable homotopy theory is Part III of the book by J.F.Adams, "Stable Homotopy and Generalised Homology", University of Chicago Press 1974.

Stable homotopy theory is the part of homotopy theory concerned with homotopy invariants which are not affected by suspension. For example, $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$ for a pointed CW-space X. Reduced homology groups are therefore stable invariants. For homotopy groups we also have a suspension homomorphism from $\pi_n(X, *)$ to $\pi_{n+1}(\Sigma X)$ (say $n > 0$) ; but it is not always an isomorphism. (Take $n = 3$ and $X = \mathbb{S}^2$.) Therefore homotopy groups are not stable invariants in general. But we can define stable homotopy groups $\pi^s_n(X)$ by

$$\pi^s_n(X) := \mathrm{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X, *) \ ;$$

these would clearly be stable invariants. In this sense, $\pi_4(\mathbb{S}^3, *)$ belongs to stable homotopy theory, but $\pi_3(\mathbb{S}^2, *)$ does not. Freudenthal's theorem tells us something about the frontier between the unstable and the stable world. There is a more general theorem in this direction (the homotopy excision theorem due to Blakers-Massey, below).
Informally, a *stable* map between pointed CW-spaces X and Y is a pointed map f from $\Sigma^n X$ to $\Sigma^n Y$, for some $n \geq 0$. Assuming that X and Y are

compact, we would call f a *stable homotopy equivalence* if, for some $k \geq 0$, the map $\Sigma^k f$ from $\Sigma^{n+k}X$ to $\Sigma^{n+k}Y$ is a homotopy equivalence. (It is not a good idea to use this definition when X and Y are noncompact—we shall see why later.)

Exercise 9.1. Give an example of two compact pointed CW-spaces which are stably homotopy equivalent without being homotopy equivalent. Can you find a pointed compact CW-space which is stably homotopy equivalent to a point without being contractible ?

It would clearly be good to have a category of "generalized" spaces (or generalized CW-spaces) in which stable maps are permitted as morphisms. One can be more radical and allow "generalized" CW-spaces in which the attaching maps for the cells are *stable* maps. This is roughly what we will do. The generalized pointed spaces are called *spectra* ; the generalized CW-spaces are called CW-spectra. For the following definition, note that the (reduced) suspension of a pointed CW-space X (where the base point is a 0-cell) has a canonical CW-structure, with one cell of dimension $n+1$ for each n-cell in X.

Definition 9.2.
(1) A *spectrum* is a sequence of pointed spaces X_n together with maps $\varepsilon_n : \Sigma X_n \to X_{n+1}$, one for each n.
(2) A spectrum as above is a *CW-spectrum* if each X_n is a CW-space (with base point equal to a 0-cell) and each ε_n is an isomorphism from ΣX_n to a CW-subspace of X_{n+1}.

We will usually work with CW-spectra. As a rule, the more general spectra in (1) are alright as "targets" (of maps whose source is a CW-spectrum), but hard to manage when they wnat to be "sources" of maps. At any rate, defining spectra and CW-spectra did not cost us much, but defining *maps* between spectra is more tricky. We will only define maps from a CW-spectrum to a spectrum.
First let $\mathbf{X} = \{X_n, \varepsilon_n\}$ be a CW-spectrum. For every cell $e \subset X_n$ distinct from the base point 0-cell, we let $\sigma e \subset X_{n+1}$ be the image of $e \times (0,1) \subset \Sigma X_n$ under ε_n. If e has dimension i, the σe has dimension $i+1$, and it is a cell in X_{n+1}.
A *CW-subspectrum* $\mathbf{X}'$ of $\mathbf{X}$ is a collection of CW-subspaces $X'_n \subset X_n$ such that $\varepsilon_n(\Sigma X'_n) \subset X'_{n+1}$. Such a subspectrum is *cofinal* if, for every n and every cell $e \subset X_n$ (distinct from the base point 0-cell), there exists $k \geq 0$ such that the cell $\sigma^k e \subset X_{n+k}$ belongs to X'_{n+k}. Note: The intersection of two cofinal subspectra is again cofinal.
Next, let $\mathbf{X} = \{X_n, \varepsilon_n\}$ be a CW-spectrum and let $\mathbf{Y} = \{Y_n, \rho_n\}$ be any spectrum. A *function* $\mathbf{f}$ from $\mathbf{X}$ to $\mathbf{Y}$ is a collection of pointed maps

$$f_n : X_n \to Y_n$$

making the diagrams

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\varepsilon_n} & X_{n+1} \\ \big\downarrow{\scriptstyle \Sigma f_n} & & \big\downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \xrightarrow{\rho_n} & Y_{n+1} \end{array}$$

commutative. By a *map* from $\mathbf{X}$ to $\mathbf{Y}$, we shall mean a function from some cofinal subspectrum of $\mathbf{X}$ to $\mathbf{Y}$. To be more precise, two such maps will be considered equal if they agree on the intersection of their domains (which is still a cofinal subspectrum of $\mathbf{X}$). To be perfectly precise, we would have to say: *A map from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is an equivalence class of pairs* $(\mathbf{X}', \mathbf{f})$*, where* $\mathbf{X}'$ *is cofinal in* $\mathbf{X}$ *and* $\mathbf{f}$ *is a function from* $\mathbf{X}'$ *to* $\mathbf{Y}$ *; two such pairs are equivalent...* I shall use boldface letters such as $\mathbf{f}$ for maps and/or functions indiscriminately. Little exercise: Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g} : \mathbf{Y} \to \mathbf{W}$ be maps between spectra, assuming that $\mathbf{X}$ and $\mathbf{Y}$ are CW-spectra ; how would you define $\mathbf{gf}$? As you can see, CW-spectra and the maps between them form a category.

Example 9.3. From any pointed space X we can make a spectrum $\{X_n, \varepsilon_n\}$ by taking $X_n = \Sigma^n X$ and

$$\varepsilon_n = \text{ standard identification } : \Sigma(\Sigma^n X) \cong \Sigma^{n+1} X \ .$$

This is the *suspension spectrum* of the pointed space X. Three-star exercise: Find a good label for it. (In the literature, you will often see $\Sigma^\infty X$; this is obviously loathsome.)
When $X = \mathbb{S}^k$, with the usual cell decomposition, I shall write $\mathbf{S}^k$ for the suspension spectrum. Here $k = 0$ is definitely allowed. Think of $\mathbf{S}^k$ as the *stable* version of $\mathbb{S}^k$. Another little exercise: What are the functions from $\mathbf{S}^1$ to $\mathbf{S}^0$? What are the maps from $\mathbf{S}^1$ to $\mathbf{S}^0$? It is quite easy to see that there is only one function from $\mathbf{S}^1$ to $\mathbf{S}^0$, and that one is uninteresting. But there are many interesting *maps*: For $k \geq 2$ let

$$p_k : \Sigma^k \mathbb{S}^1 \to \Sigma^k \mathbb{S}^0$$

be the $(k-2)$-fold suspension of the Hopf map from $\Sigma^2\mathbb{S}^1$ to $\Sigma^2\mathbb{S}^0$. Then we can think of the collection $\{p_k\}$ as a map from $\mathbf{S}^1$ to $\mathbf{S}^0$; it is defined on the cofinal subspectrum of $\mathbf{S}^1$ obtained by deleting all nontrivial cells in the zero-th term and the first term, and keeping all other cells.

Notation 9.4. Given two spaces with base point, say X and Y, let $X \wedge Y$ be the quotient $(X \times Y)/(X \vee Y)$. Examples: $\mathbb{S}^0 \wedge X \cong X$ for any pointed space X, and more generally $\mathbb{S}^n \wedge X \cong \Sigma^n X$.
Given a space W without base point, let W_+ be the space obtained from W by adding a disjoint base point: $W_+ = W \amalg \{*\}$.
Given a pointed space X and a spectrum $\mathbf{Y} = \{Y_n, \varepsilon_n\}$, we can make a new spectrum $X \wedge \mathbf{Y}$ from the spaces $X \wedge Y_n$ and the maps $\mathrm{id}_X \wedge \varepsilon_n$.
Given two spectra $\mathbf{X} = \{X_n, \varepsilon_n\}$ and $\mathbf{Y} = \{Y_n, \mu_n\}$, we can make another spectrum $\mathbf{X} \vee \mathbf{Y}$ from the spaces $X_n \vee Y_n$ and the maps $\varepsilon_n \vee \mu_n$; we can

make yet another spectrum $\mathbf{X} \times \mathbf{Y}$ from the spaces $X_n \times Y_n$ and the maps $\varepsilon_n \times \mu_n$. You can easily see that $\mathbf{X} \vee \mathbf{Y}$ is the categorical coproduct of $\mathbf{X}$ and $\mathbf{Y}$, and $\mathbf{X} \times \mathbf{Y}$ is the categorical product of $\mathbf{X}$ and $\mathbf{Y}$.

Given a CW-spectrum $\mathbf{X} = \{X_n, \varepsilon_n\}$ and a CW-subspectrum $\mathbf{A} \subset \mathbf{X}$, we would like to define the quotient $\mathbf{X}/\mathbf{A}$. Little exercise: Explain why it is not a good idea to use the spaces X_n/A_n. In fact, we should first replace $\mathbf{A}$ by its *saturation* $\mathbf{A}^\sharp$, the largest CW-subspectrum of $\mathbf{X}$ such that $\mathbf{A}$ is cofinal in $\mathbf{A}^\sharp$. Then we *define* $\mathbf{X}/\mathbf{A}$ as the spectrum made from the spaces $X_n/A^\sharp_n$.

Given a CW-spectrum $\mathbf{X}$ with CW-subspectrum $\mathbf{A}$ and a map $\mathbf{f} : \mathbf{A} \to \mathbf{Y}$ to another spectrum $\mathbf{Y}$, we can form the *pushout* (or amalgamation) $\mathbf{X} \amalg_{\mathbf{A}} \mathbf{Y}$. This is quite tricky ! Idea: Find cofinal subspectra $\mathbf{X}^\flat \subset \mathbf{X}$ and $\mathbf{A}^\flat \subset \mathbf{A}$ such that $\mathbf{A}^\flat$ is saturated in $\mathbf{X}^\flat$ and $\mathbf{f}$ is defined on $\mathbf{A}^\flat$ as a *function.* Then *define* $\mathbf{X} \amalg_{\mathbf{A}} \mathbf{Y}$ as $\mathbf{X}^\flat \amalg_{\mathbf{A}^\flat} \mathbf{Y}$ (and that is easy). Details left to you.

Two maps $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$ from a CW-spectrum to another spectrum are *homotopic* if there exists a map $\mathbf{h} : I_+ \wedge \mathbf{X} \to \mathbf{Y}$ such that $\mathbf{h}\iota_0 = \mathbf{f}$ and $\mathbf{h}\iota_1 = \mathbf{g}$. Here ι_0, ι_1 are the inclusions

$$\mathbf{X} \cong \{0\}_+ \wedge \mathbf{X} \subset I_+ \wedge \mathbf{X} \quad , \quad \mathbf{X} \cong \{1\}_+ \wedge \mathbf{X} \subset I_+ \wedge \mathbf{X} \ .$$

Homotopy is an equivalence relation, and the set of homotopy classes of maps from $\mathbf{X}$ to $\mathbf{Y}$ is denoted by $[\mathbf{X}, \mathbf{Y}]$.

Remark 9.5. Every CW-spectrum $\mathbf{X}$ has a skeleton filtration, as follows. For $k \in \mathbb{Z}$, let $\mathbf{X}^k$ be the subspectrum of $\mathbf{X}$ whose n-th term is the $(n+k)$-skeleton of the CW-space X_n. We call this the k-skeleton of $\mathbf{X}$. Then $\mathbf{X}^{k+1}$ is isomorphic as a CW-spectrum to the pushout (amalgamation) of a diagram

$$\mathbf{X}^k \xleftarrow{\ \mathbf{f}\ } \bigvee_\gamma \mathbf{S}^k \hookrightarrow \bigvee_\gamma \mathbf{D}^{k+1}$$

where $\mathbf{D}^{k+1}$ is the suspension spectrum of the disk D^{k+1} (with a base point in the boundary $\mathbb{S}^k$). This is not quite trivial—try it.

Let $\mathbf{X}$ be a CW-spectrum and let $\mathbf{A} \subset \mathbf{X}$ be a CW-subspectrum. Then the pair $(\mathbf{X}, \mathbf{A})$ has the homotopy extension property. This follows from the corresponding statement for CW-pairs. However, be careful not to do an induction over skeletons, because the skeletons $\mathbf{X}^k$ may be nontrivial for arbitrary negative k. Sketch: Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{h} : I_+ \wedge \mathbf{A} \to \mathbf{Y}$ be given such that $\mathbf{h}\iota_0$ agrees with $\mathbf{f}$ on $\mathbf{A}$. Without loss of generality, $\mathbf{f}$ and $\mathbf{h}$ are *functions* defined on $\mathbf{X}$ and $I_+ \wedge \mathbf{A}$, respectively, and $\mathbf{A}$ is saturated in $\mathbf{X}$. (If $\mathbf{A}$ is not saturated, you must replace $\mathbf{X}$ by a smaller but cofinal subspectrum.) Now use the HEP for the CW-pair (X_0, A_0). Then you will find an extension of $\mathbf{h}$, with the required initial condition, to the subspectrum made from the space X_0 and its suspensions. Then apply the HEP for the CW-pair $(X_1, \Sigma X_0)$ to extend the homotopy further to the subspectrum made from X_1 and its supensions. And so on.

The same strategy will give you the proof of the *cellular approximation theorem*: Given a pair of CW-spectra $(\mathbf{X}, \mathbf{A})$ and a CW-spectrum $\mathbf{Y}$ and a map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ which is cellular when restricted to $\mathbf{A}$, there exists a cellular map $\mathbf{g} : \mathbf{X} \to \mathbf{Y}$ which is homotopic to $\mathbf{f}$ rel $\mathbf{A}$. Of course, a map between CW-spectra is cellular if it takes k-skeleton to k-skeleton for all $k \in \mathbb{Z}$.

Example 9.6. This was historically the first "genuine" example of a spectrum (around 1954) and it is the one which we shall study. Let $G_{p,q}$ be the space of p-dimensional linear subspaces of $\mathbb{R}^p \oplus \mathbb{R}^q$. This is compact. The tautological vector bundle over $G_{p,q}$ is the vector bundle

$$\gamma_{p,q} : E_{p,q} \longrightarrow G_{p,q}$$

such that $E_{p,q} = \{(V,x) \mid x \in V \in G_{p,q}\}$ and the bundle projection is given by $(V,x) \mapsto V$. (If you are not too familar with vector bundles, then you can at least admit that $\gamma_{p,q}$ is a fiber bundle and that each fiber has the structure of a real vector space of dimension p.) The space $E_{p,q}$ is locally compact ; let $\mathrm{Th}\, E_{p,q}$ be its one-point compactification. Use the point ∞ as base point.
Now let's vary q. This is not difficult: we get $G_{p,q} \subset G_{p,q+1}$ using the standard inclusion $\mathbb{R}^q \subset \mathbb{R}^{q+1}$; similarly $E_{p,q} \subset E_{p,q+1}$ and $\mathrm{Th}\, E_{p,q} \subset TE_{p,q+1}$. Then we take unions as $q \to \infty$, and denote the resulting spaces by G_p, E_p and $\mathrm{Th}\, E_p$. Caution: $\mathrm{Th}\, E_p$ is not the one-point compactification of $\mathrm{Th}\, E_p$ for $p > 0$ (it is not compact).
Next let's vary p. This is more difficult. Using the standard inclusion $\mathbb{R}^p \subset \mathbb{R}^{p+1}$, we can define maps

$$G_{p,q} \longrightarrow G_{p+1,q} \quad ; \quad V \mapsto V \oplus \langle b_{p+1} \rangle$$

where $b_{p+1} \in \mathbb{R}^{p+1}$ is the "last" standard basis vector in the direct summand $\mathbb{R}^{p+1}$. Think of these maps as inclusion maps. What does $\gamma_{p+1,q}$ look like when we restrict it to $G_{p,q}$? It looks like the Whitney sum (fiberwise direct sum) of $\gamma_{p,q}$ with a trivial line bundle (vector bundle with one-dimensional fibers). Therefore the one-point compactification of its total space looks like the reduced suspension of the one-point compactification of the total space of $\gamma_{p,q}$. Result: $\Sigma \mathrm{Th}\, E_{p,q} \subset \mathrm{Th}\, E_{p+1,q}$. All this is in some sense independent of q, so by the same reasoning:

$$\Sigma \mathrm{Th}\, E_p \subset \mathrm{Th}\, E_{p+1}.$$

So we have a spectrum made from the pointed spaces $\mathrm{Th}\, E_p$, for $p \geq 0$, and the inclusion maps $\Sigma \mathrm{Th}\, E_p \hookrightarrow TE_{p+1}$. This spectrum is denoted by $\mathbf{MO}$ (for various reasons difficult to explain. For example, the M in $\mathbf{MO}$ is in honor of Milnor, but Thom had a lot to do with that, too.)

Exercise 9.7. Let X be a compact CW-space, γ a vector bundle over X with total space E. Let $\mathrm{Th}\, E$ be the one-point compactification of E. How would you make $\mathrm{Th}\, E$ into a CW-space ? It can be done in such a way that $\mathrm{Th}\, E$ has exactly as many cells as X, plus the 0-cell at infinity which usually serves as base point. (Think of the special case where γ is a trivial vector bundle.)

The Grassmannians $G_{p,q}$ have canonical and very pretty CW-structures (see page 75 of *Characteristic Classes* by Milnor and Stasheff, Princeton University Press 1974). It follows that the spaces $\mathrm{Th}\, E_{p,q}$ above have canonical CW-structures. These are compatible with the various inclusions ; so **MO** is actually a CW-spectrum.

We have already defined spectra $\mathbf{S}^k$ for $k \geq 0$. An isomorphic definition which works for all $k \in \mathbb{Z}$ is as follows: Let $\mathbf{S}^k$ be the spectrum whose n-th term is $\mathbb{S}^{n+k}$ if $n + k \geq 0$, and $\{*\}$ if not. Now we can define the homotopy groups of an arbitrary spectrum by

$$\pi_k(\mathbf{X}) = [\mathbf{S}^k, \mathbf{X}] \qquad \text{for } k \in \mathbb{Z}\,.$$

Exercise 9.8. Show that $\pi_k(\mathbf{X})$ is isomorphic to the direct limit of the homotopy groups $\pi_{k+n}(X_n, *)$ as $n \to \infty$. Conclude that $\pi_k(\mathbf{X})$ is indeed an abelian group. Conclude also that $\pi_k(\mathbf{S}^n)$ is zero if $k < n$ and isomorphic to $\mathbb{Z}$ if $k = n$. Also, if $\mathbf{X}$ is a wedge of i copies of $\mathbf{S}^n$, then $\pi_n(\mathbf{X}) \cong \mathbb{Z}^i$.

Remark 9.9. If you don't know about direct limits alias colimits: The direct limit of a system of sets and maps

$$\cdots \to A_{-1} \to A_0 \to A_1 \to A_2 \to A_3 \to A_4 \to \dots$$

is the set of equivalence classes of pairs (n, a) where $a \in A_n$. Two such pairs (n, a), (m, b) are equivalent if there exists $k \geq m, n$ such that a and b have the same image in A_k under the appropriate maps in the system. If all the sets are groups and all the maps are homomorphisms, then the direct limit is also a group.

The homology group $H_k(\mathbf{X})$ of a spectrum $\mathbf{X} = \{X_n, \varepsilon_n\}$ can be defined as the direct limit of the (ordinary, reduced) homology groups $\tilde{H}_{k+n}(X_n)$ as $n \to \infty$. There is a Hurewicz homomorphism from $\pi_*(\mathbf{X})$ to $H_*(\mathbf{X})$; note (for later use) that it is iso if $\mathbf{X}$ is a wedge of copies of $\mathbf{S}^k$ and $* \leq k$.

If $\mathbf{X} = \{X_n, \varepsilon_n\}$ is a CW-spectrum, then we can also define the *cellular chain complex* $W(\mathbf{X})$ of $\mathbf{X}$. (Let's agree on notation: The suspension ΣC of a chain complex C is the same chain complex shifted one degree upwards.) The suspension $\Sigma W(X_n, \{*\})$ of the reduced cellular chain complex of X_n is canonically isomorphic to $W(\Sigma X_n, \{*\})$ which is contained in $W(X_{n+1}, \{*\})$. Therefore we can define $W(\mathbf{X})$ to be the direct limit of the system of chain complexes

$$\dots \hookrightarrow \Sigma W(X_{-1}) \hookrightarrow W(X_0) \hookrightarrow \Sigma^{-1} W(X_1) \hookrightarrow \Sigma^{-2} W(X_2) \hookrightarrow \dots\ .$$

Another definition of $W(\mathbf{X})$ which is also useful: Let $W_n(\mathbf{X})$ be the n-th homology of $(\mathbf{X}^n/\mathbf{X}^{n-1})$, where $\mathbf{X}^n$ is the n-skeleton (a CW-subspectrum) defined earlier. Define the boundary operator from $W_n(\mathbf{X})$ to $W_{n-1}(\mathbf{X})$ in the usual way.

The homology of $\mathbf{X}$ is, of course, isomorphic to the homology of the cellular chain complex $W(\mathbf{X})$. We define the *co*homology of $\mathbf{X}$ as the homology of $\hom(W(\mathbf{X}), \mathbb{Z})$.

A spectrum $\mathbf{X}$ is n-connected if $\pi_k(\mathbf{X}) = 0$ for all integers $k \le n$. For example, $\mathbf{MO}$ is (-1)-connected; $\mathbf{S}^k$ is $(k-1)$-connected, for all $k \in \mathbb{Z}$. A map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ between spectra is n-connected if $\mathbf{f}_* : \pi_k(X) \to \pi_k(\mathbf{Y})$ is iso for $k < n$ and onto for $k = n$. Equivalently, $\mathbf{f}$ is n-connected if $\pi_k(\mathbf{f})$ is zero for $k \le n$; the definition of $\pi_*(\mathbf{f})$ is left to you. For example, if $\mathbf{X}$ is CW-spectrum, then the inclusion $\mathbf{X}^n \hookrightarrow \mathbf{X}$ of the n-skeleton is n-connected.

Theorem 9.10. *For a $(k-1)$-connected CW-spectrum $\mathbf{X}$, the Hurewicz homomorphism $\pi_k(\mathbf{X}) \to H_k(\mathbf{X})$ is an isomorphism.*

Proof. Almost identical with that of theorem 8.2. Notice that k can be any integer, positive or negative. □

A relative Hurewicz theorem for spectra does exist, but we will see that it is a corollary of a more surprising theorem. To state and prove this, let's return to the setting of unstable homotopy theory for a while. The following theorem is a special case of the homotopy excision theorem (Blakers-Massey).

Theorem 9.11. *Let (Y, X) be an n-connected CW-pair with m-connected X, base point in X, and $m, n \ge 1$. Then $q_* : \pi_k(Y, X, *) \to \pi_k(Y/X)$ (induced by the quotient map q from Y to Y/X) is iso for $0 \le k \le m+n$ and onto for $k = m+n+1$.*

Proof. Write $Z = Y/X$. The first step is to replace $q : Y \to Z$ by a fibration $q^\sharp : Y^\sharp \to Z$ (where $Y^\sharp$ is homotopy equivalent to Y, and so on). Let Ψ be the fiber of $q^\sharp$ over the base point. We have the usual spectral sequence with $\mathcal{E}^2_{**}$ term

$$\mathcal{E}^2_{s,t} \cong H_s(Z; H_t(\Psi \to *)) \cong H_s(Z; \tilde{H}_{t-1}(\Psi))$$

converging to $H_*(q^\sharp) \cong H_*(q) \cong \tilde{H}_{*-1}(X)$. (See the proof of the relative Hurewicz theorem.) Since (Y, X) is n-connected, $\mathcal{E}^2_{s,t}$ is zero for $0 < s \le n$ (and for $s < 0$). Since X is m-connected, and

$$\tilde{H}_{*-1}(X) \cong H_*(q) \ ,$$

the map q is $(m+1)$-connected ; this means that Ψ is m-connected (from the long exact homotopy sequence of the fibration $q^\sharp$). Therefore $\mathcal{E}^2_{s,t}$ is zero for $t \le m+1$. So the $\mathcal{E}^2$ term looks like this:

$$\begin{array}{c|cccccc}
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
 & H_{m+3}\Psi & 0 & \dots & 0 & H_{n+1}(Z; H_{m+3}\Psi) & H_{n+2}(Z; H_{m+3}\Psi) & \dots \\
 & H_{m+2}\Psi & 0 & \dots & 0 & H_{n+1}(Z; H_{m+2}\Psi) & H_{n+2}(Z; H_{m+2}\Psi) & \dots \\
m+2 & H_{m+1}\Psi & 0 & \dots & 0 & H_{n+1}(Z; H_{m+1}\Psi) & H_{n+2}(Z; H_{m+1}\Psi) & \dots \\
 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\
 & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
\hline
 & 0 & & & & n+1 & &
\end{array}$$

The differentials on $\mathcal{E}^r_{**}$ still go r units to the left and $r-1$ units upwards. It follows easily that the edge homomorphisms of the spectral sequence, from $\mathcal{E}^2_{0,t}$ to $H_t(q^\sharp)$, are isomorphisms for $t \le m+n$ and onto for $t = m+n+1$. These edge homomorphisms are of course the homomorphisms

$$H_t(\Psi \to *) \xrightarrow{\ e_t\ } H_t\big(q^\sharp : Y^\sharp \to Z\big)$$

induced by the inclusion $\Psi \hookrightarrow Y^\sharp$.

At this point we can abandon the spectral sequence, and try to make sense of the result. Note that Ψ is the space of pairs (y, ω), where $y \in Y$ and ω is a path in Z such that $\omega(0) = q(y)$. Any $x \in X$ determines a pair (x, μ) in Ψ, where μ is the constant path with constant value $*$. So $X \subset \Psi$. The composition

$$H_t(X \to *) \xrightarrow{\ j_t\ } H_t(\Psi \to *) \xrightarrow{\ e_t\ } H_t\big(q^\sharp : Y^\sharp \to Z)\big) \cong \tilde{H}_{t-1}(X)$$

is an isomorphism for all t (see exercise cor-spectrquotient below). Since e_t is iso for $t \le m+n$ and onto for $t = m+n+1$, the same must be true for j_t, in other words: *the homomorphisms*

$$H_t(X) \longrightarrow H_t(\Psi)$$

induced by inclusion are isomorphisms for $t < m+n$ and onto for $t = m+n$. Therefore the inclusion $X \hookrightarrow \Psi$ is $(m+n)$-connected, by the absolute Hurewicz theorem. (Here we seem to be using the fact that Ψ is homotopy equivalent to a CW-space; this is Milnor's result. Also, one must check that Ψ is simply connected. To prove this, use the G. Whitehead theorem, corollary cor-GWhitehead, to see that q and $q^\sharp$ are 2-connected maps, and then use the long exact sequence of the fibration $q^\sharp$.)

Finally we use the Five lemma and the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_t(X,*) \arrow[r] \arrow[d] & \pi_t(Y,*) \arrow[r] \arrow[d] & \pi_t(Y,X,*) \arrow[r] \arrow[d] & \pi_{t-1}(X,*) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_t(\Psi,*) \arrow[r] & \pi_t(Y^\sharp,*) \arrow[r] & \pi_t(Y^\sharp,\Psi,*) \arrow[r] & \pi_{t-1}(\Psi,*) \arrow[r] & \cdots
\end{tikzcd}$$

(vertical arrows induced by inclusion) to conclude that the homomorphisms from $\pi_t(Y, X, *)$ to $\pi_t(Y^\sharp, \Psi, *)$ induced by inclusion are iso for $t \le m+n$ and onto for $t = m+n+1$. Note also that $q^\sharp$ induces an isomorphism $\pi_t(Y^\sharp, \Psi, *) \cong \pi_t(Z)$ by theorem 7.8, for all $t > 0$. Putting this together, we see that

$$q_* : \pi_t(Y, X, *) \longrightarrow \pi_t(Z, *)$$

is iso for $t \le m+n$ and onto for $t = m+n+1$. $\square$

Corollary 9.12. *For any pair of CW-spectra $(\mathbf{Y}, \mathbf{X})$ and any $n \in \mathbb{Z}$, the homomorphism $q_* : \pi_n(\mathbf{Y}, \mathbf{X}) \to \pi_n(\mathbf{Y}/\mathbf{X})$ induced by the quotient map q is an isomorphism.*

Proof. We write $(\mathbf{Y},\mathbf{X}) = \{(Y_n, X_n), \varepsilon_n\}$. *Surjectivity*: Any element in $\pi_n(\mathbf{Y}/\mathbf{X})$ can be represented by a pointed map

$$f : \mathbb{S}^{n+k} \longrightarrow Y_k/X_k$$

since $\pi_n(\mathbf{Y}/\mathbf{X})$ is the direct limit of the groups $\pi_{n+k}(Y_k/X_k, *)$ as $k \to \infty$. The same element is then also represented by

$$\mathbb{S}^{n+k+r} \cong \Sigma^r \mathbb{S}^{n+k} \xrightarrow{\ \Sigma^r f\ } \Sigma^r(Y_k/X_k) \cong \Sigma^r Y_k/\Sigma^r X_k \subset Y_{k+r}/X_{k+r}$$

where r can be as large as we please. In particular, we can take r so large that $2(r-1) > n+k+r$. Since $(\Sigma^r Y_k, \Sigma^r X_k)$ and $\Sigma^r X_k$ are both $(r-1)$-connected, the homotopy excision theorem 9.11 tells us that the class of $\Sigma^r f$ in $\pi_{n+k+r}(\Sigma^r Y_{k+r}/\Sigma^r X_{k+r}, *)$ comes from a class in $\pi_{n+k+r}(\Sigma^r Y_{k+r}, \Sigma^r X_{k+r}, *)$. Going backwards through the definitions, we find that this new class represents something in $\pi_n(\mathbf{Y},\mathbf{X})$ which maps to the class in $\pi_n(\mathbf{Y}/\mathbf{X})$ we started with.

Injectivity: Suppose that $[f],[g] \in \pi_n(\mathbf{Y},\mathbf{X})$ map to the same class in $\pi_n(\mathbf{Y}/\mathbf{X})$ under q_*. Arguing as above, we find that $[f]$ and $[g]$ can be represented by classes in $\pi_{n+k+r}(\Sigma^r Y_{k+r}, \Sigma^r X_{k+r})$, still with the same image in $\pi_{n+k+r}(\Sigma^r Y_{k+r}/\Sigma^r X_{k+r})$. Of course, k is large and r is very large. Applying theorem 9.11 once more, this time the injectivity part, we see that the two classes in $\pi_{n+k+r}(\Sigma^r Y_{k+r}, \Sigma^r X_{k+r})$ must be identical ; therefore $[f] = [g]$. $\square$

Exercise 9.13. Show that the inclusion $\mathbf{X} \vee \mathbf{Y} \hookrightarrow \mathbf{X} \times \mathbf{Y}$ (where $\mathbf{X}$ and $\mathbf{Y}$ are CW-spectra) is a homotopy equivalence. (Hint: Use corollary 9.12 to express $\pi_*(\mathbf{X} \vee \mathbf{Y})$ in terms of $\pi_*(\mathbf{X})$ and $\pi_*(\mathbf{Y})$.)

Exercise 9.14. Let T^2 be the 2-dimensional torus. Show that the suspension spectrum of T^2_+ is homotopy equivalent to $\mathbf{S}^0 \vee \mathbf{S}^1 \vee \mathbf{S}^1 \vee \mathbf{S}^2$.

10. Steenrod Operations

In this section, all homology groups and cohomology groups are taken with coefficients $\mathbb{Z}/2$, unless otherwise stated.

A *cohomology operation* θ is a natural transformation

$$\theta_Y : H^r(Y) \longrightarrow H^s(Y)$$

between contravariant functors from CW-spaces to abelian groups. Here the CW-space Y is regarded as a *variable*, whereas r and s are fixed.

In more detail: For each CW-space Y, we are given a map θ_Y from $H^r(Y)$ to $H^s(Y)$; and for each continuous $f : X \to Y$, the diagram

$$\begin{array}{ccc} H^r(Y) & \xrightarrow{f^*} & H^r(X) \\ {\scriptstyle\theta_Y}\downarrow & & {\scriptstyle\theta_X}\downarrow \\ H^s(Y) & \xrightarrow{f^*} & H^s(X) \end{array}$$

commutes. *Example*: Choose your favorite $r > 0$, and choose another integer $p > 0$. For each Y, define θ_Y from $H^r(Y)$ to $H^{pr}(Y)$ by $\theta_Y(z) := z^p$ (the p-th cup product power).
A cohomology operation θ, as above, has a unique extension to CW-pairs. ("Extension" is meaningful because every space Y can be regarded as a pair $(Y, \emptyset)$.) Uniqueness of the extension follows from the commutativity of the diagram

$$\begin{array}{ccccc} H^r(Y/A) & \longleftarrow & H^r(Y/A,*) & \longrightarrow & H^r(Y,A) \\ {\scriptstyle \theta_{Y/A}}\big\downarrow & & {\scriptstyle \theta_{Y/A,*}}\big\downarrow & & {\scriptstyle \theta_{Y,A}}\big\downarrow \\ H^s(Y/A) & \longleftarrow & H^r(Y/A,*) & \longrightarrow & H^s(Y,A) \end{array}$$

in which the horizontal arrows are induced by certain maps of pairs. (The left-hand horizontal arrows are injective, the right-hand ones are iso.) The same diagram can be used to define the extension.
A *stable* cohomology operation λ of degree s is a natural transformation

$$\lambda_Y : H^*(Y) \longrightarrow H^{*+s}(Y)$$

(Y variable, $*$ variable, but s fixed) with the following additional property: for every CW-pair (Y, A), the square

$$\begin{array}{ccc} H^*(A) & \xrightarrow{\delta} & H^{*+1}(Y,A) \\ {\scriptstyle \lambda_A}\big\downarrow & & {\scriptstyle \lambda_{Y,A}}\big\downarrow \\ H^{*+s}(A) & \xrightarrow{\delta} & H^{*+s+1}(Y,A) \end{array}$$

commutes. (Note that we have used the automatic extension from spaces to pairs.)

Exercise 10.1. Show that any stable cohomology operation is *additive*, in other words $\lambda_Y(w+z) = \lambda_Y(w) + \lambda_Y(z)$ whenever these expressions are meaningful. (Hint: Show first that a stable cohomology operation commutes with the suspension isomorphisms. Then search for unusual definitions of the addition in $H^*(\Sigma Y)$.) Compare this with the example above, where $\theta_Y(z) = z^p$. For which values of p is θ additive ?

Examples of stable cohomology operations are not so easy to find. Steenrod found many. The following theorem is mostly due to him except for the Cartan formula, which is presumably due to H. Cartan.

Theorem 10.2. *There exist stable cohomology operations*

$$Sq^i : H^*(Y) \longrightarrow H^{*+i}(Y) \qquad (i = 0, 1, 2, 3, \dots)$$

(variable Y) with the following properties.

- *Sq^0 is the identity.*
- *$Sq^i(z) = z \cup z$ for any cohomology class z of degree i, and $Sq^i(z) = 0$ if z has degree $< i$.*

- (Cartan formula) $Sq^k(v \cup w) = \sum_{i+j=k} Sq^i(v) \cup Sq^j(w)$ *for arbitrary cohomology classes* v *and* w *in* $H^*(Y)$ *(any* Y*).*

These operations are called *Steenrod operations* or *Steenrod squares*. Eventually we shall see that the properties in theorem 10.2 characterize them, but right now existence seems more important.
There is a very pretty geometric idea behind theorem 10.2. Let $\mathbb{S}^\infty$ be the set of vectors of unit length in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$. We make this into a CW-space in such a way that the n-skeleton is exactly $\mathbb{S}^n = \mathbb{S}^\infty \cap \mathbb{R}^{n+1}$. Note that $\mathbb{S}^\infty$ has two cells in each dimension ≥ 0. For a CW-space Y, we then form

$$(\mathbb{S}^\infty \times Y \times Y)/\sim$$

where the equivalence relation identifies points which correspond to one another under the involution $(z, y_1, y_2) \mapsto (-z, y_2, y_1)$. Since the involution is cellular and has no fixed points, the space $(\mathbb{S}^\infty \times Y \times Y)/\sim$ inherits a CW-structure from the product $\mathbb{S}^\infty \times Y \times Y$. It is easy to describe the cellular chain complex of $(\mathbb{S}^\infty \times Y \times Y)/\sim$ in terms of the cellular chain complex of X, and this will give us a way to construct interesting classes in the cohomology of $(\mathbb{S}^\infty \times Y \times Y)/\sim$. Then we note that $(\mathbb{S}^\infty \times Y \times Y)/\sim$ contains $\mathbb{R}P^\infty \times \Delta Y$ where ΔY is the diagonal in $Y \times Y$. By restriction, the interesting cohomology classes in $(\mathbb{S}^\infty \times Y \times Y)/\sim$ will give us classes in

$$H^*(\mathbb{R}P^\infty \times \Delta Y) \quad \cong \quad \bigoplus_i H^{*-i}(Y)\,.$$

(The isomorphism uses the Künneth theorem ; note that the coefficient group is still $\mathbb{Z}/2$.)

We start with some chain complex algebra, however. Let $G = \{1, T\}$ be the cyclic group of order 2. (We must distinguish between the group G, which acts on spaces like $\mathbb{S}^\infty \times Y \times Y$, and the field $\mathbb{Z}/2$ which we need in order to define cohomology rings.) Let $R = \mathbb{Z}/2[G]$ be the group algebra. So elements in R are formal linear combinations $\sum_{g \in G} a_g g$, with $a_g \in \mathbb{Z}/2$ for all $g \in G$, and multiplication in R is defined in such a way that it satisfies the distributive law and extends the multiplication in G. In particular, we think of G as a subset of R (a subgroup of the group of units of R, to be more precise). Every vector space over $\mathbb{Z}/2$ with an action of G (by linear maps) is automatically an R-module: extend the action from G to R by distributivity.
Example: G acts on the CW-space $\mathbb{S}^\infty$ by the antipodal action, and the action is by cellular maps. So the cellular chain complex of $\mathbb{S}^\infty$, with $\mathbb{Z}/2$ coefficients, is a chain complex on which G acts ; so it is a chain complex of R-modules. Call it W. Since there are two cells in each dimension, and G permutes these, we find that $W_n \cong R$ as an R-module. (Choose the isomorphism in such a way that $1 \in R$ corresponds to the “northern” n-cell in $\mathbb{S}^\infty$, the one containing points whose $(n+1)$-st coordinate is positive.)

With these identifications, W has the following appearance. (Note that $+$ and $-$ is the same thing.)

$$W: \qquad R \xleftarrow{1+T} R \xleftarrow{1+T} R \xleftarrow{1+T} R \xleftarrow{1+T} R \xleftarrow{1+T} R \xleftarrow{1+T} \dots$$

Each of the boundary maps is multiplication by $(1+T)$, and of course W starts in degree 0. Write $\alpha : W_0 \to \mathbb{Z}/2$ for the augmentation.

Exercise 10.3. Show that, up to chain homotopy, there is only one R-module chain map e from W to $W \otimes W$ inducing an isomorphism on H_0. (Here you should equip $W \otimes W$ with the "diagonal" G-action, and extend to R as usual). Give an explicit description of e. What does this tell you about the cup product in $H^*(\mathbb{R}P^\infty)$? (Note that the cellular chain complex of $\mathbb{R}P^\infty$, with coefficients $\mathbb{Z}/2$, is isomorphic to $W \otimes_R \mathbb{Z}/2$.) Your answer should be: Everything.

More example: Let C be the cellular chain complex with $\mathbb{Z}/2$ coefficients of some CW-space Y. Then $C \otimes C$ is isomorphic to the cellular chain complex of $Y \times Y$ (not by the Künneth theorem, but because you can see it). G acts on $Y \times Y$ by permuting factors, and the corresponding action of G on $C \otimes C$ also permutes the (tensor) factors. It follows easily that the cellular chain complex of $(\mathbb{S}^\infty \times Y \times Y)/\sim$ is isomorphic to

$$\big(W \otimes C \otimes C\big) \otimes_R \mathbb{Z}/2$$

where G, or R, acts diagonally on $W \otimes C \otimes C$. *Notation*: We will regard this cellular chain complex as a quotient of $W \otimes C \otimes C$ (which it is), so we write elements as sums of terms of the form $w \otimes c_1 \otimes c_2$.
Now suppose that $v : C \to \mathbb{Z}/2$ is a cocycle of dimension k, with class $[v] \in H^k(C)$. Then we define a cocycle

$$v^\diamondsuit : \big(W \otimes C \otimes C\big) \otimes_R \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$$

of dimension $2k$ by $v^\diamondsuit(w \otimes c_1 \otimes c_2) := \alpha(w) \cdot v(c_1) \cdot v(c_2)$. This is in fact a cocycle (check !), and its class $[v^\diamondsuit]$ depends only on the class $[v]$ (ditto). We may write

$$[v] \in H^k(Y) \quad , \quad [v^\diamondsuit] \in H^{2k}\big((\mathbb{S}^\infty \otimes Y \otimes Y)/\sim\big)\,.$$

Finally we apply the "enhanced" diagonal map

$$\blacktriangle : \mathbb{R}P^\infty \times Y \longrightarrow (\mathbb{S}^\infty \otimes Y \otimes Y)/\sim$$

which gives us (together with the Künneth theorem)

$$\blacktriangle^*[v^\diamondsuit] \in H^{2k}(\mathbb{R}P^\infty \times Y) \cong \bigoplus_{0 \le j \le 2k} H^j(Y)\,.$$

The component of $\blacktriangle^*[v^\diamondsuit]$ in $H^{k+i}(Y)$ is what we call $Sq^i[v]$. For $i > k$, we *define* $Sq^i[v] = 0$. Notice that we have inadvertently defined $Sq^i[v]$ for some negative values of i, namely, $-k \le i < 0$. It will turn out that $Sq^i[v] = 0$ for $i < 0$.

Now we have to check that the properties in theorem 10.2 are satisfied. We start with naturality: Given a map of CW-spaces $f : X \to Y$, and integers i and $k \geq 0$, we want to know whether

$$\begin{array}{ccc} H^k(Y) & \xrightarrow{f^*} & H^k(X) \\ {\scriptstyle Sq^i}\big\downarrow & & {\scriptstyle Sq^i}\big\downarrow \\ H^{k+i}(Y) & \xrightarrow{f^*} & H^{k+i}(X) \end{array}$$

commutes. It does of course because f is homotopic to a cellular map.
Next, we can use the naturality of Sq^i to show that Sq^i is zero for $i < 0$. In fact, let $\theta : H^k(X) \to H^s(X)$ be any cohomology operation (variable X) where $s < k$. Let $q : X \to X/X^{q-1}$ be the quotient map. Then the upper horizontal arrow in the commutative diagram

$$\begin{array}{ccc} H^k(X/X^{k-1}) & \xrightarrow{q^*} & H^k(X) \\ {\scriptstyle \theta}\big\downarrow & & {\scriptstyle \theta}\big\downarrow \\ H^s(X/X^{k-1}) & \xrightarrow{q^*} & H^s(X) \end{array}$$

is onto, and the lower left corner is zero. Conclusion: $\theta = 0$.
For the second property in theorem 10.2, suppose that $[v] \in H^i(Y)$. Then $Sq^i[v]$ (same i) is the image of $[v^\diamondsuit]$ under the map in cohomology induced by

$$Y \xrightarrow{\subset} \mathbb{R}P^\infty \times Y \xrightarrow{\blacktriangle} (\mathbb{S}^\infty \times Y \times Y)/\sim \ .$$

But now we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\subset} & \mathbb{R}P^\infty \times Y \\ {\scriptstyle \Delta}\big\downarrow & & {\scriptstyle \blacktriangle}\big\downarrow \\ Y \times Y & \xrightarrow{\subset} & (\mathbb{S}^\infty \times Y \times Y)/\sim \ . \end{array}$$

The lower horizontal arrow sends (y_1, y_2) to the equivalence class of $(*, y_1, y_2)$, where $* = (1, 0, 0, \dots) \in \mathbb{S}^\infty$. By inspection, the induced map in cohomology sends $[v^\diamondsuit]$ to the external product $v \times v \in H^{2k}(Y \times Y)$, which goes to $v \cup v$ under Δ^*.
To prove the Cartan relations, the third item in theorem 10.2, let us find out what the construction $[v] \mapsto [v^\diamondsuit]$ does to external products in cohomology. (The point is that external products in cohomology are defined at the chain complex level, whereas internal cup products are not.) Let $e : W \to W \otimes W$ be an R-module chain map as in exercise 10.3.
Let B and C be chain complexes of vector spaces over $\mathbb{Z}/2$. Let $u : B \to \mathbb{Z}/2$ and $v : C \to \mathbb{Z}/2$ be cocycles of dimensions j and k, respectively. (Later, but not now, we will assume that B and C are the cellular chain complexes of CW-spaces X and Y, respectively.) *Notation*: $W \int C$ means

$$(W \otimes C \otimes C) \otimes_R \mathbb{Z}/2 \ ,$$

and elements in this chain complex will be written as sums of terms

$$w \otimes c_1 \otimes c_2 \ .$$

We can use e to produce a chain map

$$(\maltese) \qquad W \int (B \otimes C) \longrightarrow (W \int B) \otimes (W \int C)$$

(provisionally send $w \otimes (b_1 \otimes c_1) \otimes (b_2 \otimes c_2)$ to $e(w) \otimes (b_1 \otimes b_2) \otimes (c_1 \otimes c_2)$, then note that $e(w)$ is a sum of terms $w' \otimes w''$, then push the w'' terms past the terms b_1 and b_2 to have it all in the right order). It is easy to see that the cocycle $u^\diamondsuit \times v^\diamondsuit$ (external product) of dimension $2j+2k$ goes to $(u \times v)^\diamondsuit$ under ($\maltese$).

The geometric meaning of this is as follows. Let $\eta : \mathbb{R}P^\infty \to \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ be a cellular approximation to the diagonal. Lifting this to universal covers gives a cellular map $\tilde{\eta} : \mathbb{S}^\infty \to \mathbb{S}^\infty \times \mathbb{S}^\infty$, and passage to cellular chain complexes gives a chain map $W \to W \otimes W$. This is now an R-module chain map (why ?), so we may call it e. Further, suppose that B and C are the cellular chain complexes of CW-spaces X and Y. We can use $\tilde{\eta}$ to produce a cellular approximation of the map

$$\begin{array}{rccc} \nabla : \dfrac{\mathbb{S}^\infty \times (X \times Y) \times (X \times Y)}{\sim} & \longrightarrow & \dfrac{\mathbb{S}^\infty \times X \times X}{\sim} \times \dfrac{\mathbb{S}^\infty \times Y \times Y}{\sim} \\ (z, x_1, y_1, x_2, y_2) & \mapsto & (z, x_1, x_2, z, y_1, y_2) \, . \end{array}$$

The map of cellular chain complexes induced by this cellular approximation of ∇ will then simply be ($\maltese$). So now we know that

$$\nabla^*([u^\diamondsuit] \times [v^\diamondsuit]) = [(u \times v)^\diamondsuit] \, .$$

Then we can use the commutative diagram

$$\begin{array}{ccc} \mathbb{R}P^\infty \times (X \times Y) & \xrightarrow{\ \zeta\ } & (\mathbb{R}P^\infty \times X) \times (\mathbb{R}P^\infty \times Y) \\ {\scriptstyle\blacktriangle}\big\downarrow & & {\scriptstyle\blacktriangle \times \blacktriangle}\big\downarrow \\ \dfrac{\mathbb{S}^\infty \times (X \times Y) \times (X \times Y)}{\sim} & \xrightarrow{\ \nabla\ } & \dfrac{\mathbb{S}^\infty \times X \times X}{\sim} \times \dfrac{\mathbb{S}^\infty \times Y \times Y}{\sim} \end{array}$$

(where $\zeta(p, x, y) = (p, x, p, y)$) to conclude that

$$\begin{array}{rcl} \blacktriangle^*[(u \times v)^\diamondsuit] & = & \blacktriangle^* \nabla^*([u^\diamondsuit] \times [v^\diamondsuit]) \\ = \ \zeta^*(\blacktriangle \times \blacktriangle)^*([u^\diamondsuit] \times [v^\diamondsuit]) & = & \zeta^*\big(\blacktriangle^*[u^\diamondsuit] \times \blacktriangle^*[v^\diamondsuit]\big). \end{array}$$

This is an equation in $H^{2j+2k}(\mathbb{R}P^\infty \times (X \times Y)$, which we think of as a direct sum of cohomology groups of $X \times Y$. Comparing coefficients, we see that

$$Sq^i([u] \times [v]) = \sum_{s+t=i} (Sq^s[u]) \times (Sq^t[v]) \, .$$

This is the external product version of the Cartan formula, and you can easily deduce the internal version, item 3 in theorem 10.2, from it. (Actually, the external version is equivalent to the internal version—why ?)

Last not least, we have to prove that Sq^0 is the identity. This is harder to prove than the other properties. Note first that if we want to define or caculate $Sq^i[w]$ for some $[w] \in H^k(Y)$, some Y, then it is enough to regard or define $[w^\diamondsuit]$ as a class in $H^{2k}\big((\mathbb{S}^{i+k} \times Y \times Y)/\sim\big)$, and to pull it back using an enhanced diagonal of the form

$$\blacktriangle : \mathbb{R}P^{i+k} \times Y \longrightarrow (\mathbb{S}^{i+k} \times Y \times Y)/\sim\ .$$

To simplify further, suppose that $k > 0$, and that Y is pointed. Then $[w]$ comes from $H^k(Y, *)$ and it follows easily that $[w^\diamondsuit]$ comes from H^{2k} of a quotient space of $(\mathbb{S}^{i+k} \times Y \times Y)/\sim$, namely, $(\mathbb{S}^{i+k} \times (Y \wedge Y))/\sim$. Saving symbols, we may write

$$\blacktriangle : \mathbb{R}P^{i+k} \times Y \longrightarrow (\mathbb{S}^{i+k} \times (Y \wedge Y))/\sim\ ,$$

$$[w^\diamondsuit] \in H^{2k}\big((\mathbb{S}^{i+k} \times (Y \wedge Y))/\sim\big).$$

We apply these observations in a very special case: $Y = \mathbb{S}^1$, and $[w] \in H^1(\mathbb{S}^1)$ is the nonzero class. We want to prove that $Sq^0[v]$ is still nonzero. So we examine

$$\text{(!)} \qquad \blacktriangle : \mathbb{R}P^1 \times \mathbb{S}^1 \longrightarrow (\mathbb{S}^1 \times (\mathbb{S}^1 \wedge \mathbb{S}^1))/\sim$$

and ask what this does to $[w^\diamondsuit]$. Now the domain in (!) is a 2-dimensional torus, and the codomain has the same (co)-homology as a product $\mathbb{S}^1 \times \mathbb{S}^2$ (coefficients $\mathbb{Z}/2$), and $[w^\diamondsuit]$ is the only nonzero class in H^2 of the codomain. We still have to check that (!) is nonzero on H^2, or on H_2. Changing CW-structures, we can say that (!) is the inclusion of a CW-subspace, and then we have a long exact homology sequence involving the homology of the supspace, homology of the ambient space, and homology of the quotient. Showing that (!) is nonzero on H_2 is therefore equivalent to showing that H_3 of the quotient has dimension < 2. But the quotient is homeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^2)/(\mathbb{S}^1 \times *)$. We conclude that Sq^0 acts like the identity on $H^1(\mathbb{S}^1)$. Now the general case follows easily: The Cartan formula (external version) tells us that Sq^0 acts like the identity on $H^n((\mathbb{S}^1)^n)$. Naturality of Sq^0 and the fact that we have a map from $(\mathbb{S}^1)^n$ to $\mathbb{S}^n$ inducing an isomorphism on H^n then implies that Sq^0 acts like the identity on $H^n(\mathbb{S}^n)$. If X is a wedge of n-spheres, then we have an injection

$$H^n(X) \longrightarrow \prod_\alpha H^n(\mathbb{S}^n)$$

induced by the inclusions of the various wedge summands, and again naturality shows that Sq^0 acts like the identity on $H^n(X)$. If X is an arbitrary CW-space, then we have homomorphisms

$$H^n(X) \longleftarrow H^n(X/X^{n-1}) \longrightarrow H^n(X^n/X^{n-1})$$

induced by quotient map and inclusion, respectively. The first of these is surjective, the second is injective. Of course, X^n/X^{n-1} is a wedge of spheres. Again, naturality implies that Sq^0 is the identity on $H^n(X)$. ☐

Exercise 10.4. Explain why *stability* in theorem 10.2 follows from the properties 1 and 2 of the Steenrod operations. (Hint: Show first that the Steenrod operations commute with the suspension isomorphism. For a pair (Y, A), note that $\tilde{H}^*(\text{cone}) \cong H^*(Y, A)$, where "cone" is the mapping cone of the inclusion map $A \to Y$. Then note that the quotient of the cone by Y is homeomorphic to the unreduced suspension of A.)

Exercise 10.5. (1) Let X be a CW-space, and let $\alpha : E \to X$ be a vector bundle over X, of fiber dimension n. When X is compact, we define the Thom space $\mathrm{Th}\, E$ to be the one-point compactification $E \cup \infty$ of E, and we use ∞ as the base point. Then $\mathrm{Th}\, E$ is a pointed CW-space; apart from the 0-cell $\{\infty\}$ it has one cell of dimension $n + k$ for every k-cell in X. In more detail, if e is a k-cell in X, then $\alpha^{-1}(e) \subset E \subset \mathrm{Th}\, E$ is the corresponding $n + k$-cell. *Show that the reduced cellular chain complex of* $\mathrm{Th}\, E$ *is isomorphic to the cellular chain complex of* X*, up to a degree shift by* n *units.* Here you have to use coefficients in $\mathbb{Z}/2$, otherwise it's not quite true. Even if you do use coefficients in $\mathbb{Z}/2$, it is a little tricky. You will need a practical definition of the degree of a map from $\mathbb{S}^k$ to itself. Try transversality.

The resulting isomorphism $H^k(X) \cong \tilde{H}^{n+k}(\mathrm{Th}\, E)$ is the *Thom isomorphism.* The image of the class $1 \in H^0(X)$ under the Thom isomorphism is the Thom class $U_\alpha \in H^n(\mathrm{Th}\, E)$. All this works equally well when X is non-compact, but you must not define $\mathrm{Th}\, E$ as the one-point compactification. Instead, note as before that $\mathrm{Th}\, E$ has a nice cell decomposition, and give it the CW-topology.

(2) Let $\beta : E' \to Y$ be another vector bundle, of fiber dimension m. Then $\alpha \times \beta : E \times E' \to X \times Y$ is a vector bundle over $X \times Y$ of fiber dimension $m + n$, and its Thom space is homeomorphic to $\mathrm{Th}\, E \wedge \mathrm{Th}\, E'$. Using this identification, show that

$$U_{\alpha\times\beta} = U_\alpha \times U_\eta$$

(the Thom class of the external product $\alpha \times \beta$ is the external product of the Thom classes).

(3) In general, the Thom isomorphism fails to commute with the action of the Steenrod operations (incidentally, also with cup products). This makes the following definition reasonable and interesting: For $k > 0$, the k-th Stiefel Whitney class of the vector bundle $\alpha : E \to X$ is the class

$$w_k(\alpha) \in H^k(X)$$

which corresponds to $Sq^k(U_\alpha)$ under the Thom isomorphism. Show that $w_0(\alpha) = 1$, and $w_k(\alpha) = 0$ if k is greater than the fiber dimension of α.

(4) Show that for an external product $\alpha \times \beta : E \times E' \to X \times Y$ of vector bundles, the following holds:

$$w_k(\alpha \times \beta) = \sum_{i+j=k} w_i(\alpha) \times w_j(\beta)\,.$$

If $X = Y$, then we can also define the Whitney sum $\alpha \oplus \beta$, which is a vector bundle on X, and then

$$w_k(\alpha \cup \beta) = \sum_{i+j=k} w_i(\alpha) \cup w_j(\beta)\,.$$

(5) Show that the Stiefel-Whitney classes are *characteristic classes* in the following sense: Let $\alpha : E \to X$ be a vector bundle as usual, and let $f : V \to X$ be a map. Then $w_k(f^*\alpha) = f^*(w_k(\alpha)$, where $f^*\alpha$ is the induced vector bundle over V.

(6) Show that w_1 of the nontrivial one-dimensional vector bundle over $\mathbb{S}^1$ is nonzero. (Hint: the Thom space is homeomorphic to $\mathbb{R}P^2$.)

(7) Suppose that α and β are vector bundles over X such that the Whitney sum $\alpha \oplus \beta$ is a trivial vector bundle. Explain how the Stiefel-Whitney classes of α determine those of β.

(8) Let τ be the tangent bundle of $\mathbb{R}P^n$, and let η be the canonical or tautological line bundle. (Line bundle always means: vector bundle of fiber dimension 1. We have a tautological line bundle over $\mathbb{R}P^n$ because $\mathbb{R}P^n$ is also the Grassmannian of 1-dimensional linear susbspaces of $\mathbb{R}^{n+1}$.) Finally let ε be the trivial line bundle over $\mathbb{R}P^n$, with total space $\mathbb{R} \times \mathbb{R}P^n$. Show that

$$\tau \oplus \varepsilon \cong (n+1)\eta$$

where $(n+1)\eta$ means a Whitney sum of $n+1$ copies of η.

(9) Let $f : \mathbb{R}P^n \to \mathbb{R}^{n+k}$ be a smooth immersion, iė; every point $x \in \mathbb{R}P^n$ has a neighbourhood W such that $f_{|W}$ is a smooth embedding. (For example, there is the "figure 8" immersion of $\mathbb{R}P^1$ in $\mathbb{R}^2$.) The existence of such an f implies the existence of a vector bundle β over $\mathbb{R}P^n$, of fiber dimension k, such that $\tau \oplus \beta$ is a trivial vector bundle. Why ?

(10) Show that $\mathbb{R}P^4$ cannot be immersed in $\mathbb{R}^6$, and more generally that $\mathbb{R}P^k$ cannot be immersed in $\mathbb{R}^{2k-2}$ if k is a power of 2. (Use (9), (7) and the last part of (3).)

(11) Let $\wp = \wp(w_1, w_2, w_3, \dots)$ be a monomial in finitely many of the symbols w_1, w_2, etc; of total degree n. (For example, the total degree of $w_1^3 w_2^5 w_4 w_6^2$ is $3 \cdot 1 + 5 \cdot 2 + 1 \cdot 4 + 2 \cdot 6$.) If γ is a vector bundle over X, we can substitute $w_i(\gamma)$ for w_i in the monomial $\wp$, and interpret the product as a cup product, so that $\wp(w_1(\gamma), w_2(\gamma), \dots)$ becomes an element in $H^n(X)$. If $X = M$ happens to be a closed connected smooth n-manifold, and $\gamma = \tau$ is the tangent bundle, then $H^n(M) \cong \mathbb{Z}/2$, so the element which we have constructed from $\wp$ is a "number" (in $\mathbb{Z}/2$). It depends only on M, since τ is determined by M. If M is not connected (but still closed and smooth) we can sum over all components, or better: we can evaluate $\wp(w_1(\gamma), w_2(\gamma), \dots)$ on the fundamental class (scalar product of a homology class with a cohomology class). This type of number (in $\mathbb{Z}/2$) associated with M is called a *characteristic number*. Show that characteristic numbers

are bordism invariants. In other words: If M^n as above, with tangent bundle τ, is the boundary of a compact smooth $(n+1)$-manifold with boundary, and if $\wp$ is a polynomial as above of total degree n, then $\wp(w_1(\tau), w_2(\tau), \dots)$ evaluated on the fundamental class of M gives 0. (This has a lot to do with Stokes theorem, although you should not use Stokes theorem to prove it.)

(12) Using (11), show that $\mathbb{R}P^2$, $\mathbb{R}P^4$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$ are not bordant to zero, and that $\mathbb{R}P^4$ is not bordant to $\mathbb{R}P^2 \times \mathbb{R}P^2$. (We say that two closed smooth manifolds of the same dimension are *bordant* if their disjoint union is the boundary of a compact smooth manifold, of one dimension higher.)

11. $H^*(\mathbf{MO})$ and $H_*(\mathbf{MO})$

We continue to use $\mathbb{Z}/2$ coefficients for homology and cohomology groups.

Recall (from example 9.6) that G_p is the Grassmannian of p-dimensional linear subspaces of $\mathbb{R}^p \oplus \mathbb{R}^\infty$, that $\mathrm{Th}\, E_p$ is the Thom space of the tautological vector bundle of fiber dimension p over G_p, and that we made the spectrum $\mathbf{MO}$ from the spaces $\mathrm{Th}\, E_p$ and certain inclusion maps $\Sigma \mathrm{Th}\, E_p \hookrightarrow \mathrm{Th}\, E_{p+1}$. Calculating the homology or cohomology of $\mathbf{MO}$ is much the same then as calculating the homology or cohomology of the spaces $\mathrm{Th}\, E_p$. By the Thom isomorphism, this is again the same as calculating the homology or cohomology of G_p. So this is where we start.

Before we really start, remember that what we are after is $\pi_*(\mathbf{MO})$. In fact, $\pi_n(\mathbf{MO})$ is isomorphic to the bordism group of n-dimensional closed smooth manifolds. This is the content of the Thom-Pontryagin construction, expounded in a talk. As a first step towards calculating $\pi_*(\mathbf{MO})$, we might try to calculate $H_*(\mathbf{MO})$. We may hope, quite unreasonably, that the Hurewicz homomorphism from $\pi_*(\mathbf{MO})$ to $H_*(\mathbf{MO})$ is injective.

For the calculation of $H^*(G_p)$, we proceed by induction on p. So we should start by understanding the relationship between G_p and G_{p+1}. Of course, we have the embedding $G_p \hookrightarrow G_{p+1}$. We can convert it into a fibration, and ask what the fibers are ; if we can see what they are, then the spectral sequence machinery can be applied. There is a standard procedure for converting maps into fibrations, but if we use it here we might not be able to see what the fibers are ; so we have to be more inventive.

Let $G_p^\sharp$ be the space of all pairs (V, z) where V is a $(p+1)$-dimensional linear subspace of $\mathbb{R}^{p+1} \oplus \mathbb{R}^\infty$ and $z \in V$ is a unit vector. Forgetting z gives a map

$$\phi : G_p^\sharp \longrightarrow G_{p+1}$$

which is seen to be a fiber bundle with fibers homeomorphic to $\mathbb{S}^p$. Forgetting V gives a map ψ from $G_p^\sharp$ to the unit sphere in $\mathbb{R}^{p+1} \oplus \mathbb{R}^\infty$; this is also a fibration, this time with contractible base space. The fiber over the unit vector b_{p+1} (standard basis vector in the direct summand $\mathbb{R}^{p+1}$) is exactly the image of the embedding $G_p \to G_{p+1}$. So we have factorized the embedding as

$$G_p \xrightarrow{\ e\ } G_p^\sharp \xrightarrow{\ \phi\ } G_{p+1}$$

where e is a homotopy equivalence (the inclusion of a fiber of ψ) and ϕ is a fibration with fibers homeomorphic to $\mathbb{S}^p$.

Exercise 11.1. (1) Let $q : D \to Y$ be a fibration with simply connected base space Y, and assume the fibers are homotopy equivalent to $\mathbb{S}^n$ for some $n > 0$. Show that there exist long exact sequences

$$\cdots \to H_n(D) \xrightarrow{q_*} H_n(Y) \to H_{n-k-1}(Y) \to H_{n-1}(D) \xrightarrow{q_*} H_{n-1}(Y) \to \cdots$$

$$\cdots \to H^{n-1}(Y) \xrightarrow{q^*} H^{n-1}(D) \to H^{n-k-1}(Y) \to H^n(Y) \xrightarrow{q^*} H^n(D) \to \cdots .$$

These are called the *Gysin sequences*, and they are obviously very similar to the *Wang sequence*, exercise 4.14. Instead of the spectral sequence method, which is the obvious choice, you could also use a Thom space argument. Namely, the exact sequences suggest that the mapping cone of q has the same (co-)homology as Y except for a dimension shift. Can you indicate why ? Incidentally, all this is true if (co-)homology with coefficients $\mathbb{Z}$ is used.

(2) Explain why the assumption on $\pi_1(Y)$ is superfluous if we use coefficients $\mathbb{Z}/2$ (which we do). In general, the asssumption "simply connected base space" which we tend to make when using Leray-Serre spectral sequences is too strong—how would you weaken it ?

Let $\mathbb{Z}/2[w_1, w_2, \ldots, w_p]$ be the ring of polynomials in the variables w_1, w_2, $\ldots$, w_p , with coefficients in $\mathbb{Z}/2$. Define the degree of w_k to be k; this makes the polynomial ring into a graded ring. Then we have a homomorphism of graded rings

$$\Theta_p : \mathbb{Z}/2[w_1, w_2, \ldots, w_p] \longrightarrow H^*(G_p) \quad ; \quad w_k \mapsto w_k(\gamma_p)$$

i.e., we send the variable w_k to the k-th Stiefel-Whitney class of the tautological bundle γ_p . (Strictly speaking, γ_p was not defined in example 9.6. It is the vector bundle over G_p whose fiber over $V \in G_p$ is the vector space V.)

Theorem 11.2. *For all p, the homomorphism Θ_p is an isomorphism.*

Proof. We first prove injectivity. To do so, we define a map

$$f : (G_1)^p \to G_p$$

(where $(G_1)^p = G_1 \times G_1 \times \cdots \times G_1$) as follows. A point in $(G_1)^p$ is a collection $(L_1, L_2, \ldots L_p)$ of lines (through 0) in $\mathbb{R}^1 \oplus \mathbb{R}^\infty$. Rearranging coordinates, we can regard the direct sum of these lines L_i as a p-dimensional linear subspace of

$$\mathbb{R}^p \times (\mathbb{R}^\infty)^p.$$

Choosing a vector space isomorphism from $(\mathbb{R}^\infty)^p$ to $\mathbb{R}^\infty$, we can the regard the direct sum of the L_i as a p-dimensional linear subspace of $\mathbb{R}^p \oplus \mathbb{R}^\infty$, or as a point $f(L_1, \ldots, L_p)$ in G_p.

The important thing is that the vector bundle $f^*(\gamma_p)$ on $(G_1)^p$ is isomorphic to what we should call the tautological bundle over $(G_1)^p$, namely,

$$\gamma_1 \times \gamma_1 \times \cdots \times \gamma_1 \ .$$

Note also that G_1 is really infinite projective space $\mathbb{R}P^\infty$, and γ_1 is the usual tautological line bundle over $\mathbb{R}P^\infty$. We know that the Stiefel-Whitney classes of γ_1 are given by $w_0 = 1$ (as always), w_1 is the nonzero element which we may call v, all other w_i equal to 0. (Why ?) We know from the Künneth formula that $H^*((G_1)^p)$ is a polynomial ring in generators $v_1, v_2, \dots v_p$, all in degree 1. We know from the formula for Stiefel-Whitney classes of external products that the k-th Stiefel-Whitney classes of $\gamma_1\times\cdots\times\gamma_1$ is $\sigma_k(v_1, \dots, v_p)$, the k-th elementary polynomial in the variables $v_1, \dots v_p$. The composite homomorphism

$$\mathbb{Z}/2[w_1, w_2, \dots, w_p] \xrightarrow{\Theta_p} H^*(G_p) \xrightarrow{f^*} H^*((G_1)^p)$$

is injective, since it sends w_k to $\sigma_k(v_1, \dots v_k)$. This proves injectivity of Θ_p. We use induction on p to complete the proof. The square

$$\begin{tikzcd}
\mathbb{Z}/2[w_1, w_2, \dots, w_p] \arrow[r, "\Theta_p"] & H^*(G_p) \\
\mathbb{Z}/2[w_1, w_2, \dots, w_p, w_{p+1}] \arrow[u] \arrow[r, "\Theta_{p+1}"] & H^*(G_{p+1}) \arrow[u]
\end{tikzcd}$$

commutes (left vertical arrow: send w_{p+1} to zero, right vertical arrow: induced by embedding). Suppose that Θ_p is iso. Then the right-hand vertical arrow in the square is onto (because the left-hand vertical arrow is onto), and then we get an upper bound on the size of $H^*(G_{p+1})$ from the Gysin sequence of the fibration

$$\phi : G_p^\sharp \to G_{p+1}.$$

(Details left to you.) As a consequence, it is enough to know that Θ_{p+1} is *injective*, which we do know. $\square$

Let G_∞ be the union of the G_p (not the disjoint union ; use the embeddings $G_p \hookrightarrow G_{p+1}$). It follows from theorem 11.2 that

$$H^*(G_\infty) \cong \mathbb{Z}/2[w_1, w_2, w_3 \dots] \, .$$

(Sketch: The universal coefficient theorem tells you that $H_*(G_p)$ is the dual of $H^*(G_p)$; then a direct limit argument tells you what $H_*(G_\infty)$ is ; then another application of the universal coefficient theorem gives you $H^*(G_\infty)$.) Here we can no longer define the isomorphism by sending w_k to the k-th Stiefel-Whitney class of some vector bundle over G_∞, unless we are willing to allow certain types of infinite dimensional vector bundles (it would not be absurd). Let us just say that the isomorphism sends w_k to the class in $H^*(G_\infty)$ which maps to $w_k(\gamma_p) \in H^*(G_p)$ under the homomorphism induced by inclusion, for all p.

The homology $H_*(G_\infty)$, which is the dual of $H^*(G_\infty)$, seems less interesting than the cohomology since it has no cup product. But it has another product which will turn out to be rather more important. To define this, we think of $H_*(G_\infty)$ as the direct limit of the graded groups $H_*(G_p)$. Given classes v, w in $H_*(G_\infty$, we can therefore assume that $v \in H_m(G_p)$ and $w \in H_n(G_q)$ (large p and q). Then we define the product

$$v \cdot w \in H_{m+n}(G_{p+q}) \subset H_{m+n}(G_\infty)$$

as $\mu(v \times w)$, where μ is the following map:

$$\mu : G_p \times G_q \longrightarrow G_{p+q} \quad ; \quad (V_1, V_2) \mapsto V_1 \oplus V_2 \ .$$

In more detail: V_1 is a p-dimensional linear subspace of $\mathbb{R}^p \oplus \mathbb{R}^\infty$, and V_2 is a q-dimensional linear subspace of $\mathbb{R}^q \oplus \mathbb{R}^\infty$, and therefore $V_1 \oplus V_2$ is a $(p+q)$-dimensional linear subspace of $\mathbb{R}^{p+q} \oplus \mathbb{R}^\infty \oplus \mathbb{R}^\infty$. Now choose a linear isomorphism of $\mathbb{R}^\infty \times \mathbb{R}^\infty$ with $\mathbb{R}^\infty$, as in the proof of theorem 11.2.
The product on $H_*(G_\infty)$ is called *Pontryagin product*. It is commutative and associative. (Why ?) Something similar can be defined on the homology of any H-space. We did not show that G_∞ is an H-space, but in defining μ above (for arbitrarily large p and q) we came close to it.
Now let's see what $H_*(G_\infty)$ looks like as a ring (with the Pontryagin product). We have the embedding $G_1 \hookrightarrow G_\infty$, and G_1 is still the same as $\mathbb{R}P^\infty$. Denote by a_k the nonzero element in $H_k(G_1)$, assuming $k > 0$. Use the same letter for its image in $H_k(G_\infty)$.

Theorem 11.3. *$H_*(G_\infty)$ is a polynomial ring generated by the a_i :*

$$H_*(G_\infty) \cong \mathbb{Z}/2[a_1, a_2, a_3, \dots] \ .$$

Proof. This is surprisingly easy. Earlier (in the proof of theorem 11.2) we defined a map

$$f : (G_1)^p \longrightarrow G_p$$

(take p "large") which turned out to induce an injection in cohomology and therefore a surjection in homology. Compose this with the inclusion $G_p \hookrightarrow G_\infty$; the composition

$$\bar{f} : (G_1)^p \longrightarrow G_\infty$$

will still induce a surjection in H_* for $* \le p$. By the Künneth theorem, every element in $H_*((G_1)^p)$ is a sum of terms of the form

$$z_1 \times z_2 \times z_3 \cdots \times z_p$$

(external homology product) with $z_i \in H_*(G_1)$. By definition of the Pontryagin product, the image of such an element under $\bar{f}_*$ takes the form

$$\prod_{1 \le i \le p} (\bar{f}\alpha_i)_*(z_i)$$

where $\alpha_i : G_1 \to (G_1)^p$ is the inclusion of the i-th axis. But $\bar{f}\alpha_i : G_1 \to G_\infty$ is clearly homotopic to the standard embedding, so $(\bar{f}\alpha_i)_*(z_i)$ must be equal to some a_k in $H_*(G_\infty)$, or to 0. We see that the image of $\bar{f}_*$ is contained in the subring of $H_*(G_\infty)$ generated by the elements a_k. But we also know that it is surjective in dimensions $\le p$. Moreover, p was arbitrary, so we must conclude: $H_*(G_\infty)$ is generated (as a ring) by the elements a_k for $k > 0$. The question is now whether it is freely generated by the a_k. If not, then for some $n > 0$, the dimension of $H_n(G_\infty)$ must be $<$ the dimension of the degree n part of the polynomial ring $\mathbb{Z}/2[a_1, a_2, \dots]$. Fortunately we know from theorem 11.2 that this is not the case: the dimensions are equal. □

Let's interpret all this. Here are a few challenging hypotheses:

- By dint of exercise 10.5 part (1), we should expect "Thom isomorphisms":
$$H_*(G_\infty) \cong H_*(\mathbf{MO}) \quad , \quad H^*(G_\infty) \cong H^*(\mathbf{MO})\,.$$
- There is no reasonable cup product in the cohomology of an arbitrary spectrum. Therefore, although $H^*(G_\infty)$ appears to be isomorphic to $H^*(\mathbf{MO})$ as a graded abelian group, we may not be able to make any use of the cup product in $H^*(G_\infty)$.
- The Pontryagin product in $H_*(G_\infty)$ may well be relevant to our problem (which is to determine $\pi_*(\mathbf{MO})$). Following Thom, we make the identification $\mathfrak{N}_* \cong \pi_*(\mathbf{MO})$, where $\mathfrak{N}_k$ is the abelian group of bordism classes of closed smooth k-manifolds. Then $\mathfrak{N}_*$ is a graded abelian group, but in fact it is a graded ring: Given $[M_1] \in \mathfrak{N}_s$ and $[M_2] \in \mathfrak{N}_t$, their (well defined) product in $\mathfrak{N}_{s+t}$ is the bordism class of $M_1 \times M_2$. Is it possible that the Hurewicz homomorphism
$$\mathfrak{N}_* \cong \pi_*(\mathbf{MO}) \longrightarrow H_*(\mathbf{MO}) \cong H_*(G_\infty)$$
is a ring homomorphism (Pontryagin product on $H_*(G_\infty)$) ?

Item (1): By general nonsense or by the calculation we did earlier in this section, the obvious homomorphism
$$\mathrm{colim}_p H_*(G_p) \longrightarrow H_*(G_\infty)$$
(remember remark 9.9) is an isomorphism. By definition,
$$H_*(\mathbf{MO}) = \mathrm{colim}_p H_{*+p}(\mathrm{Th}\, E_p)$$
where $\mathrm{Th}\, E_p$ is the p-th space in the spectrum $\mathbf{MO}$, namely, the Thom space of the vector bundle γ_p on G_p. Then by exercise 10.5 part (1), there is a Thom isomorphism from $H_*(G_p)$ to $H_{*+p}(\mathrm{Th}\, E_p)$, for every p. These Thom isomorphisms are compatible (for variable p), so they do lead to an isomorphism $H_*(G_\infty) \cong H_*(\mathbf{MO})$.
Item (2): If you think there is a reasonable cup product in the cohomology of an arbitrary spectrum, define it.
Item (3): We defined the Pontryagin product in $H_*(G_\infty)$ using certain maps
$$\mu : G_p \times G_q \longrightarrow G_{p+q} \subset G_\infty\,.$$

Without changing too much, we can replace the infinite Grassmannians by the appropriate Thom spaces (and products by smash products) to get

$$\widehat{\mu} : \mathrm{Th}\, E_p \wedge \mathrm{Th}\, E_q \longrightarrow \mathrm{Th}\, E_{p+q}\,.$$

In detail: Recall that $\mathrm{Th}\, E_p = E_p \cup \{\infty\}$, and E_p consists of all pairs (U, y) such that $y \in U \in G_p$. For (U, y) in E_p and (V, z) in E_q, let

$$\widehat{\mu}((U, y), (V, z)) := (\mu(U, V), y \oplus z)$$

(and $\mu(U, V)$ is identified with $U \oplus V$ as a vector space, so that $y \oplus z$ is in it). We could use $\widehat{\mu}$ instead of μ to define products in $H_*(\mathbf{MO})$: Given classes $x \in H_m(\mathbf{MO})$ and $y \in H_n(\mathbf{MO})$, we may suppose they come from $H_{n+p}(\mathrm{Th}\, E_p)$ and $H_{m+q}(\mathrm{Th}\, E_q)$, respectively, for sufficiently large p and q. Then their external product can be regarded as a class in $H_{m+n+p+q}(\mathrm{Th}\, E_{p+q})$, which in turn determines a class in $H_{m+n}(\mathbf{MO})$. The following diagram is commutative:

$$\begin{array}{ccc} H_m(G_p) \times H_n(G_q) & \longrightarrow & H_{m+n}(G_{p+q}) \\ \big\downarrow & & \big\downarrow \\ \tilde{H}_{m+p}(\mathrm{Th}\, E_p) \times \tilde{H}_{n+q}(\mathrm{Th}\, E_q) & \longrightarrow & H_{m+n+p+q}(\mathrm{Th}\, E_{p+q}) \end{array}$$

(the vertical arrows are Thom isomorphisms, the horizontal ones are given by external product and applying μ or $\widehat{\mu}$ as appropriate). As a consequence, it does not really matter whether we use μ or $\widehat{\mu}$ to define the multiplication in $H_*(\mathbf{MO})$. But for geometric purposes and interpretations, $\widehat{\mu}$ is better. For example, we can also use $\widehat{\mu}$ (but not μ) to define a product on $\pi_*(\mathbf{MO})$ (please try to forget that we already have one): Suppose $[f] \in \pi_m(\mathbf{MO})$ and $[g] \in \pi_n(\mathbf{MO})$ are represented by pointed maps

$$f : \mathbb{S}^{m+p} \longrightarrow \mathrm{Th}\, E_p \quad , \quad g : \mathbb{S}^{n+q} \longrightarrow \mathrm{Th}\, E_q$$

for some large p and q. Then the composite

$$\mathbb{S}^{m+n+p+q} \cong \mathbb{S}^{m+p} \wedge \mathbb{S}^{n+q} \xrightarrow{\ f\wedge g\ } \mathrm{Th}\, E_p \wedge \mathrm{Th}\, E_q \xrightarrow{\ \widehat{\mu}\ } \mathrm{Th}\, E_{p+q}$$

represents an element in $\pi_{m+n+p+q}(\mathrm{Th}\, E_{p+q})$, or if we prefer in $\pi_{m+n}(\mathbf{MO})$. This is the product $[f] \cdot [g]$. It makes $\pi_*(\mathbf{MO})$ into a graded ring. Furthermore, the Hurewicz homomorphism from $\pi_*(\mathbf{MO})$ to $H_*(\mathbf{MO})$ becomes a ring homomorphism if we use this product on $\pi_*(\mathbf{MO})$ and the Pontryagin product on $H_*(\mathbf{MO})$.

Now please do remember that we already had a product on $\pi_*(\mathbf{MO})$. To be more precise, we had a product on $\mathfrak{N}_*$, defined geometrically by taking products of representing manifolds. To complete this discussion, we need to check that it agrees (under the isomorphism $\mathfrak{N}_* \cong \pi_*(\mathbf{MO})$) with the Pontryagin-type product just defined on $\pi_*(\mathbf{MO})$. To this end, recall how the isomorphism is defined: Given $[f] \in \pi_m(\mathbf{MO})$, we choose a representative $f : \mathbb{S}^{m+p} \to \mathrm{Th}\, E_p$ which is smooth on the complement of $f^{-1}(\infty)$ and transverse to the zero section of E_p. (This makes sense because the image of f, being compact, will be contained in $\mathrm{Th}\, E_{p,q} \subset \mathrm{Th}\, E_p$ for some large q, and $\mathrm{Th}\, E_{p,q}$ without the base point is certainly a smooth manifold. See example 9.6 for notation.) Then the inverse image of the zero section under f is a closed smooth manifold. Its bordism class is what we map $[f]$ to. Having recalled all this, we only need to check the following: *If $f : \mathbb{S}^{m+p} \to \mathrm{Th}\, E_p$ and $g : \mathbb{S}^{n+q} \to \mathrm{Th}\, E_q$ are smooth (where possible) and transverse to the zero sections, then so is the composition*

$$\mathbb{S}^{m+n+p+q} \cong \mathbb{S}^{m+p} \wedge \mathbb{S}^{n+q} \xrightarrow{f \wedge g} \mathrm{Th}\, E_p \wedge \mathrm{Th}\, E_q \xrightarrow{\widehat{\mu}} \mathrm{Th}\, E_{p+q}\,.$$

This is (almost) obvious. Moreover, $(\widehat{\mu}(f \wedge g))^{-1}$(zero section) is clearly the product of f^{-1}(zero section) and g^{-1}(zero section).

Summary 11.4. *The Hurewicz homomorphism*

$$\mathfrak{N}_* \cong \pi_*(\mathbf{MO}) \longrightarrow H_*(\mathbf{MO})$$

is a ring homomorphism (no ambiguity) and the ring $H_*(\mathbf{MO})$ is isomorphic to the ring $H_*(G_\infty)$ (described in theorem 11.3) by a Thom isomorphism. The image under this isomorphism of $a_k \in H_k(G_\infty)$ will be denoted by α_k, so that $H_*(\mathbf{MO})$ is a polynomial algebra over $\mathbb{Z}/2$ with polynomial generators $\alpha_1, \alpha_2, \alpha_3, \dots$.

Finally we activate the Steenrod operations. Two remarks are in order.

First remark: Suppose that X is a space such that $H_n(X)$ is finite dimensional for all n. Then, by the universal coefficient theorem, $H^n(X)$ is the dual space of $H_n(X)$ and $H_n(X)$ is the dual of $H^n(X)$. We may define

$$\overline{S}q^i : H_{n+i}(X) \longrightarrow H_n(X)$$

as the dual (better transpose) of $Sq^i : H^n(X) \to H^{n+i}(X)$, so that

$$\langle x, \overline{S}q^i(y)\rangle = \langle Sq^i(x), y\rangle$$

for $x \in H^n(X)$ and $y \in H_{n+i}(X)$. Note that $\overline{S}q^i$ lowers degrees by i. We have a Cartan formula for external products in *homology*, which looks just like the one in cohomology:

$$\overline{S}q^k(v \times w) = \sum_{i+j=k} \overline{S}q^i(v) \times \overline{S}q^j(w)\,.$$

Second remark: Let $\mathbf{X} = \{X_k, \varepsilon_k\}$ be a spectrum such that $H_n(X_k)$ is finite dimensional for all k and n. Then we can define $\overline{S}q^i : H_{n+i}(\mathbf{X}) \to H_n(\mathbf{X})$ in the most obvious way, by using representatives: If $v \in H_{n+i}(\mathbf{X})$ is represented by some element in $H_{n+i+k}(X_k)$, then apply $\overline{S}q^i$, which gives something in $H_{n+k}(X_k)$, which represents something in $H_n(\mathbf{X})$. The stability property of the Steenrod operations shows that this is well defined. (As a matter of fact, the technical assumption that $H_n(X_k)$ be finite dimensional, $\forall n, k$, is superfluous. There are more systematic ways to define $\overline{S}q^i$. See for instance Adams' Chicago Lecture Notes.)

So now let's find out how the operations $\overline{S}q^i$ act on $H_*(\mathbf{MO})$. You may wonder why this is necessary. It is necessary because it will help us to determine the image of the Hurewicz homomorphism from $\pi_*(\mathbf{MO})$ to $H_*(\mathbf{MO})$. Namely, suppose that $y \in H_n(\mathbf{MO})$ belongs to the image of the Hurewicz homomorphism. *Then $\overline{S}q^i(y) = 0$ for all $i > 0$.* Proof: By assumption there exists a map of spectra $\mathbf{f}$ from $\mathbf{S}^n \to \mathbf{MO}$ such that $\mathbf{f}_*(x) = y$, where $x \in H_n(\mathbf{S}^n)$ is the nonzero element. Then $\overline{S}q^i(y) = \overline{S}q^i(\mathbf{f}_*(x)) = \mathbf{f}_*(\overline{S}q^i(x))$, which is zero because $Sq^i(x) \in H_{n-i}(\mathbf{S}^n) = 0$. (No special properties of $\mathbf{MO}$ were used in this argument—it works just as well for any other spectrum.)

Lemma 11.5. *For $v, w \in H_*(\mathbf{MO})$ and $k \geq 0$:*

$$\overline{S}q^k(v \cdot w) = \sum_{i+j=k} \overline{S}q^i(v) \cdot \overline{S}q^j(w)$$

where $\cdot$ is the Pontryagin product.

Proof. Represent v and w by some $v' \in H_{m+p}(\mathrm{Th}\, E_p)$ and $w' \in H_{n+q}(\mathrm{Th}\, E_q)$. Then $v \cdot w$ is represented by $\hat{\mu}_*(v' \times w')$, which belongs to $H_{m+n+p+q}(\mathrm{Th}\, E_{p+q})$. Then $\overline{S}q^k(v \times w)$ is represented by $\overline{S}q^k(\hat{\mu}_*(v' \times w'))$ which equals

$$\hat{\mu}_*(\overline{S}q^k(v' \times w')).$$

Now use the Cartan formula for external products in homology to write $\overline{S}q^i(v' \times w')$ as a sum $\sum \overline{S}q^i(v') \times \overline{S}q^j(w')$. □

The lemma, together with summary 11.4, means that we only have to calculate $\overline{S}q^i(\alpha_k)$ for all $k > 0$. We have the following commutative diagram:

$$\begin{tikzcd}
H_k(G_1) \arrow[r] \arrow[d, "\text{Thom}"'] & H_k(G_\infty) \cong \mathrm{colim}_p H_k(G_p) \arrow[d, "\text{Thom}"'] \\
\tilde{H}_{k+1}(\mathrm{Th}\, E_1) \arrow[r] & H_k(\mathbf{MO}) = \mathrm{colim}_p \tilde{H}_{k+p}(\mathrm{Th}\, E_p)
\end{tikzcd}$$

where the vertical arrows are Thom isomorphisms. By construction, the class α_k comes from $H_k(G_1)$, and we normally go from there to $H_k(\mathbf{MO})$ via $H_k(G_\infty)$, but now we choose the other path via $\tilde{H}_{k+1}(\mathrm{Th}\, E_1)$. The point is that the (co-)homology of $\mathrm{Th}\, E_1$, including action of the Steenrod operations, is easy to understand. Moreover the homomorphisms in the direct system

$$\tilde{H}_{k+1}(\mathrm{Th}\, E_1) \to \tilde{H}_{k+2}(\mathrm{Th}\, E_2) \to \tilde{H}_{k+3}(\mathrm{Th}\, E_3) \to \tilde{H}_{k+4}(\mathrm{Th}\, E_4) \to \dots$$

(which "approaches" $H_k(\mathbf{MO})$) are all induced by genuine maps, e.g., from $\Sigma \mathrm{Th}\, E_p$ to $\mathrm{Th}\, E_{p+1}$, so they commute with the Steenrod operations. In short, we have to understand how $\overline{S}q^i$ acts on $\tilde{H}_*(\mathrm{Th}\, E_1)$. Recall that $\mathrm{Th}\, E_1$ is the Thom space of the tautological line bundle on $G_1 \cong \mathbb{R}P^\infty$. Surprisingly, this Thom space is again homeomorphic to $\mathbb{R}P^\infty$ (more suggestive, to $\mathbb{R}P^{\infty+1}$). More generally:

Lemma 11.6. *The Thom space of the tautological line bundle on $\mathbb{R}P^n$ (where $n = \infty$ is allowed) is homeomorphic to $\mathbb{R}P^{n+1}$.*

Proof. The Thom space in question consists of the point ∞ and all pairs (L, z) where L is a line through 0 in $\mathbb{R}^{n+1}$, and $z \in L$. Map ∞ to the line through 0 in $\mathbb{R}^{n+2}$ spanned by the last standard basis vector b_{n+2}. Map a pair (L, z) to the line through 0 in $\mathbb{R}^{n+2}$ spanned by $v + \langle v, z\rangle b_{n+2}$, where v can be any nonzero vector in L. □

So now we have to know how the Steenrod operations act on $H_*(\mathbb{R}P^\infty)$ or on $H^*(\mathbb{R}P^\infty)$. You are expected to know that $H^*(\mathbb{R}P^\infty)$ is a (graded) polynomial ring on one generator $e \in H^1(\mathbb{R}P^\infty)$:

$$H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}/2[e]\,.$$

Then $Sq^0(e) = e$ by property 1 in theorem 10.2, and $Sq^1(e) = e^2 \neq 0$ by property 2 in the same theorem, and finally

$$Sq^i(e^{k+1}) = \begin{cases} 0 & \text{if } i > k+1, \\ \binom{k+1}{i} e^{k+1+i} & \text{if } i \leq k+1 \end{cases}$$

by those same properties. Dualizing this result, and noting that the dual of e^{k+1} in $\tilde{H}_{k+1}(\mathbb{R}P^\infty) \cong \tilde{H}_{k+1}(\mathrm{Th}\, E_1)$ maps to what we called α_k in $H_k(\mathbf{MO}) \cong \mathrm{colim}_p \tilde{H}_{k+p}(\mathrm{Th}\, E_p)$, we get:

Proposition 11.7. *In* $H_*(\mathbf{MO})$,

$$\overline{S}q^i(\alpha_{k+i}) = \begin{cases} \binom{k+1}{i} \alpha_k & i \leq k+1 \\ 0 & i > k+1\,. \end{cases}$$

(where $\alpha_0 = 1$ *by definition).*

Together with lemma 11.5 and summary 11.4, this gives us a complete description of $H_*(\mathbf{MO})$ with the action of the operations $\overline{S}q^i$.

12. The action of the Steenrod algebra on $H^*(\mathbf{MO})$

Suppose that $\mathbf{H}\mathbb{Z}/2$ is a CW-spectrum such that $\pi_0(\mathbf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2$ and $\pi_k(\mathbf{H}\mathbb{Z}/2) = 0$ for all $k \neq 0$. (Existence of such an $\mathbf{H}\mathbb{Z}/2$ will be proved later.) By definition, $\mathbf{H}\mathbb{Z}/2$ is (-1)-connected ; therefore the Hurewicz homomorphism from $\pi_0(\mathbf{H}\mathbb{Z}/2)$ to $H_0(\mathbf{H}\mathbb{Z}/2)$ is an isomorphism, and then the universal coefficient theorem shows that $H^0(\mathbf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2$. (The universal coefficient theorem is a theorem about chain complexes, not about spaces. You can apply it to the cellular chain complex of a CW-space, or to the cellular chain complex of a CW-spectrum.) We now define a natural transformation

$$\rho : [\mathbf{X}, \mathbf{H}\mathbb{Z}/2] \longrightarrow H^0(\mathbf{X})$$

(where $\mathbf{X}$ is a "variable" CW-spectrum) by the formula

$$\rho([\mathbf{f}]) := \mathbf{f}^*(u) \quad ; \quad 0 \neq u \in H^0(\mathbf{H}\mathbb{Z}/2)\,.$$

Lemma 12.1. *ρ is an isomorphism (for all spectra $\mathbf{X}$).*

Proof. Fix $\mathbf{X}$, and let W be the cellular chain complex of $\mathbf{X}$ (with coefficients $\mathbb{Z}/2$). Let $\mathbf{X}^n$ be the n-skeleton, and remember that this can be nontrivial for negative n. Think of $H^0(\mathbf{X}) \cong H^0(W)$ as the set of homotopy classes of chain maps $W \to \mathbb{Z}/2$ (here $\mathbb{Z}/2$ means a chain complex concentrated in dimension 0). If $\mathbf{f} : \mathbf{X} \to \mathbf{H}\mathbb{Z}/2$ is a cellular map, then we have an "induced" chain map $f : W \to \mathbb{Z}/2$ which is defined as follows (on W_0):

$$W_0 = H_0(\mathbf{X}^0/X^{-1}) \to H_0(\mathbf{X}/\mathbf{X}^{-1}) \xrightarrow{\ \mathbf{f}_*\ } H_0(\mathbf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2\,.$$

Terminology: $\mathbf{f}$ is a *realization* of f.
Now let's prove two things: firstly, any $f : W \to \mathbb{Z}/2$ has a realization. Secondly, if $\mathbf{A} \subset \mathbf{X}$ is a CW-subspectrum with corresponding chain subcomplex $V \subset W$, then any realization $\mathbf{A} \to \mathbf{H}\mathbb{Z}/2$ of $f_{|V}$ can be extended to a realization of f.
None of this is difficult. We start by constructing a cellular map

$$\mathbf{X}^0/\mathbf{X}^{-1} \to \mathbf{H}\mathbb{Z}/2$$

such that the induced map

$$W_0 = H_0(\mathbf{X}_0/\mathbf{X}_1) \to H_0(\mathbf{H}\mathbb{Z}/2) \cong \mathbb{Z}/2$$

agrees with f. Since $\mathbf{X}^0/X^{-1}$ is a wedge of spheres, this is easy. We can of course regard the map constructed so far as a map $\mathbf{f}^0$ from $\mathbf{X}^0$ to $\mathbf{H}\mathbb{Z}/2$. The fact that f is a chain map means that, for any attaching map $\mathbf{g} : \mathbf{S}^0 \to \mathbf{X}^0$ of a 1-cell of $\mathbf{X}$, the induced map

$$H_0(\mathbf{S}^0) \xrightarrow{\ g_*\ } H_0(\mathbf{X}) \xrightarrow{\ \mathbf{f}^0_*\ } H_0(\mathbf{H}\mathbb{Z}/2)$$

is zero—which means that $\mathbf{f}^0\mathbf{g}$ represents the zero class in $\pi_0(\mathbf{H}\mathbb{Z}/2)$, which means that $\mathbf{f}^0\mathbf{g}$ is nullhomotopic. Choosing explicit nullhomotopies means choosing an extension of $\mathbf{f}^0$ to a cellular map $\mathbf{f}^1$ from $\mathbf{X}^1$ to $\mathbf{H}\mathbb{Z}/2$. Next, we extend this to $\mathbf{X}^2$, using the fact that $\pi_1(\mathbf{H}\mathbb{Z}/2) = 0$; and so on, until we have a map defined on all of $\mathbf{X}$. This proves the first claim: existence of a realization. The second part (existence of "relative" realizations) is very similar and left to you.
The existence claim means that $\rho : [\mathbf{X}, \mathbf{H}\mathbb{Z}/2] \to H^0(\mathbf{X})$ is *onto*. The relative existence claim proves injectivity, as follows: Suppose that $\mathbf{f} : \mathbf{X} \to \mathbf{H}\mathbb{Z}/2$ and $\mathbf{f}' : \mathbf{X} \to \mathbf{H}\mathbb{Z}/2$ are such that $\rho(\mathbf{f}) = \rho(\mathbf{f}')$. We must show that $\mathbf{f} \simeq \mathbf{f}'$. We can assume that $\mathbf{f}$ and $\mathbf{f}'$ are cellular, and then we know at least that the corresponding chain maps $f, f' : W \to \mathbb{Z}/2$ are homotopic. Now let $\bar{W}$ be the cellular chain complex of $\mathbf{X} \wedge I_+$, where I is the unit interval with the usual CW-structure. Note that $\bar{W}$ contains a copy of $W \oplus W$, as the cellular chain complex of $\mathbf{X} \wedge (\partial I)_+$. Since f and f' are homotopic chain maps, their direct sum $f \oplus f' : W \oplus W \to \mathbb{Z}/2$ extends to a chain map $\bar{f} : \bar{W} \to \mathbb{Z}/2$. By the relative existence claim, there exists a realization of $\bar{f}$ which extends the map given by $\mathbf{f}$ and $\mathbf{f}'$ on $\mathbf{X} \wedge (\partial I)_+$. This shows that $\mathbf{f}$ and $\mathbf{f}'$ are homotopic. ☐

Here is a construction of $\mathbf{H}\mathbb{Z}/2$. We construct inductively pointed spaces K_n for $n > 0$ such that $\pi_n(K_n) \cong \mathbb{Z}/2$ and $\pi_i(K_n) = 0$ for $i \neq n$. For K_1 we take the space $\mathbb{R}P^\infty$ (with some base point). Suppose that K_n has already been constructed ; we try to make K_{n+1} by starting with the space $Y(0) := \Sigma K_n$ and improving where necessary. Clearly $Y(0)$ has the correct homotopy groups in degrees $\leq n+1$, by the absolute Hurewicz theorem. If $\pi_{n+2}(Y(0))$ is nonzero, choose cellular maps $f_j : \mathbb{S}^{n+2} \to \mathbf{Y}(0)$ whose homotopy classes generate all of $\pi_{n+2}(Y(0))$. Let $Y(1)$ be the space obtained from $Y(0)$ by attaching $(n+3)$-cells with attaching maps f_j. Then the inclusion $Y(0) \subset Y(1)$ is $(n+2)$-connected, so the induced map on π_{n+2} must be onto ; but it is also zero by construction, so $\pi_{n+2}(Y(1)) = 0$, and $\pi_i(Y(1))$ is still "correct" for $i < n+2$. Next, if $\pi_{n+3}(Y(1)) \neq 0$, choose cellular maps from $\mathbb{S}^{n+3}$ to $Y(1)$ whose classes generate $\pi_{n+3}(Y(1))$. Form $Y(2)$ by attaching $(n+4)$-cells to $Y(1)$ using these attaching maps. Then $\pi_{n+3}(Y(2)) = 0$; and so on. Finally let K_{n+1} be the union of the $Y(i)$.
By construction, K_{n+1} contains ΣK_n for all $n > 0$. If we call the inclusion ε_n, then we have a spectrum $\{K_n, \varepsilon_n : \Sigma K_n \hookrightarrow K_{n+1}\}$ which we may call $\mathbf{H}\mathbb{Z}/2$ because it has the required properties.

Recall that, in the category of spectra, we have suspension isomorphisms for homotopy groups (easy consequence of corollary 9.12). Therefore

$$\pi_i(\mathbb{S}^k \wedge \mathbf{H}\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = k \\ 0 & i \neq k \end{cases}$$

Essentially repeating the proof of lemma 12.1, we get:

Lemma 12.2. *There are natural isomorphisms*

$$[\mathbf{X}, \mathbb{S}^n \wedge \mathbf{H}\mathbb{Z}/2] \cong H^n(\mathbf{X})$$

where $\mathbf{X}$ *can be any CW-spectrum.*

This has two surprising corollaries. Firstly, we can interpret elements in the cohomology $H^n(\mathbf{H}\mathbb{Z}/2)$ as *stable cohomology operations* of degree n. Secondly, we can make $H^*(\mathbf{H}\mathbb{Z}/2)$ into a graded ring so that the multiplication corresponds to composition of stable cohomology operations. Let's start with the ring structure: Given $x \in H^m(\mathbf{H}\mathbb{Z}/2)$ and $y \in H^n(\mathbf{H}\mathbb{Z}/2)$. We can assume $m, n \geq 0$ because $\mathbf{H}\mathbb{Z}/2$ is (-1)-connected. Now

$$H^m(\mathbf{H}\mathbb{Z}/2) \cong [\mathbf{H}\mathbb{Z}/2, \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2] \quad , \quad H^n(\mathbf{H}\mathbb{Z}/2) \cong [\mathbf{H}\mathbb{Z}/2, \mathbb{S}^n \wedge \mathbf{H}\mathbb{Z}/2]$$

by lemma 12.2 (take $\mathbf{X} = \mathbf{H}\mathbb{Z}/2$). Let

$$\bar{x} : \mathbf{H}\mathbb{Z}/2 \to \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2 \ , \qquad \bar{y} : \mathbf{H}\mathbb{Z}/2 \to \mathbb{S}^n \wedge \mathbf{H}\mathbb{Z}/2$$

be the maps corresponding to x and y (better: homotopy classes of maps). Then the composition

$$\mathbf{H}\mathbb{Z}/2 \xrightarrow{\bar{x}} \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2 \xrightarrow{\mathrm{id} \wedge \bar{y}} \mathbb{S}^m \wedge \mathbb{S}^n \wedge \mathbf{H}\mathbb{Z}/2$$

is an element in $[\mathbf{H}\mathbb{Z}/2, \mathbb{S}^{m+n}\mathbf{H}\mathbb{Z}/2] \cong H^{m+n}(\mathbf{H}\mathbb{Z}/2)$. We call it $y \cdot x$ (the order is important). This multiplication is bilinear (over $\mathbb{Z}/2$), so it makes $H^*(\mathbf{H}\mathbb{Z}/2)$ into a graded (associative) algebra.
Given $x \in H^m(\mathbf{H}\mathbb{Z}/2)$, let $\bar{x}$ be the corresponding map

$$\mathbf{H}\mathbb{Z}/2 \to \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2\,.$$

Let θ_x be the stable cohomology operation defined like this:

$$H^i(\mathbf{X}) \cong [\mathbf{X}, \mathbb{S}^i \wedge \mathbf{H}\mathbb{Z}/2] \longrightarrow [\mathbf{X}, \mathbb{S}^i \wedge \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2] \cong H^{i+n}(\mathbf{X})$$

where the middle arrow is composition with

$$\mathrm{id} \wedge \bar{x} : \mathbb{S}^i \wedge \mathbf{H}\mathbb{Z}/2 \longrightarrow \mathbb{S}^i \wedge \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2]\,.$$

This is in fact easily seen to be stable. It works for spectra $\mathbf{X}$. If you want it to operate on the cohomology of a space Y, use the suspension spectrum of Y_+ (which has the same cohomology as Y.)

Definition 12.3. The algebra $H^*(\mathbf{H}\mathbb{Z}/2)$ is called the *Steenrod algebra*, abbreviated $\mathcal{A}^*$.

In the following, we assume that $\mathbf{X}$ is a (-1)-connected spectrum, that $H^m(\mathbf{X})$ is finite dimensonal for all m, and that there is no odd torsion in $\pi_*(\mathbf{X})$. (These assumptions are of the "not really necessary" type, but in any case the spectrum $\mathbf{MO}$ qualifies. Remember that every nonzero element in $\pi_*(\mathbf{MO}) \cong \mathfrak{N}_*$ has order two.) Note that $H^*(\mathbf{X})$ is a graded module over $\mathcal{A}^*$: Given

$$x \in H^m(\mathbf{X}) \cong [\mathbf{X}, \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2] \quad , \quad a \in \mathcal{A}^n \cong [\mathbf{H}\mathbb{Z}/2, \mathbb{S}^n \wedge \mathbf{H}\mathbb{Z}/2]\,,$$

define $ax \in H^{m+n}(\mathbf{X}) \cong [\mathbf{X}, \mathbb{S}^{m+n} \wedge \mathbf{H}\mathbb{Z}/2]$ as the composition

$$\mathbf{X} \xrightarrow{\ x\ } \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2 \xrightarrow{\mathrm{id}\wedge a} \mathbb{S}^n \wedge \mathbb{S}^m \wedge \mathbf{H}\mathbb{Z}/2\,.$$

Theorem 12.4. *Suppose that $H^*(\mathbf{X})$ is free as an $\mathcal{A}^*$-module (explanation follows). Then the Hurewicz homomorphism*

$$\pi_*(\mathbf{X}) \longrightarrow H_*(\mathbf{X}) \cong \mathrm{hom}(H^*(\mathbf{X}), \mathbb{Z}/2)$$

is injective, and its image consists of all $f : H^*(\mathbf{X}) \to \mathbb{Z}/2$ such that $f(ax) = 0$ for arbitrary $x \in H^{*-i}(\mathbf{X})$ and $a \in \mathcal{A}^i$, where $i > 0$.

Proof. Say that $H^*(\mathbf{X})$ is *free* if there exist elements $x_1, x_2, \dots$ in $H^*(\mathbf{X})$ such that every element of $H^*(\mathbf{X})$ can be written in a unique way as a linear combination of the x_i with coefficients in $\mathcal{A}^*$. Suppose now that this is the case, and that the degree of x_i is m_i. Each x_i is or corresponds to a map from $\mathbf{X}$ to $\mathbb{S}^{m_i} \wedge \mathbf{H}\mathbb{Z}/2$. Then we have a map of spectra

$$\prod_i x_i : \mathbf{X} \longrightarrow \prod_i \mathbb{S}^{m_i} \wedge \mathbf{H}\mathbb{Z}/2\,. \tag{!}$$

We also have the inclusion map of the wedge into the product,

$$\bigvee_i \mathbb{S}^{m_i} \wedge \mathbf{H}\mathbb{Z}/2 \hookrightarrow \prod_i \mathbb{S}^{m_i} \wedge \mathbf{H}\mathbb{Z}/2\,. \tag{!!}$$

By exercise 9.13, (!!) is a homotopy equivalence. (Actually, it is a little more difficult because there can be infinitely many wedge summands. However, for every $N > 0$, only finitely many of the positive integers m_i will be less than N, and using this you can still verify that (!!) is iso on homotopy groups.) Given that (!!) is a homotopy equivalence, we know what the cohomology of the target spectrum in (!) is: it is isomorphic as an $\mathcal{A}^*$-module to

$$\bigoplus_i \Sigma^{m_i}\mathcal{A}^*$$

where Σ means a shift in grading as usual. To put it differently: it is free, with one basis element u_i of degree m_i for each i. The map (!!) induces a homomorphism in cohomology which sends u_i to x_i. Since it is a homomorphism of $\mathcal{A}^*$-modules, and since the x_i form a basis, it must be an isomorphism. Equivalently, the map in *homology* induced by (!) is an isomorphism. Of course, this is still homology with $\mathbb{Z}/2$ coefficients.

Now let's prove that (!) induces an isomorphism in homology with integer coefficients. Suppose not; let $p \geq 0$ be the least integer such that $H_p((!)\,;\mathbb{Z})$, the p-th homology of the map (!), is nonzero. By the Hurewicz theorem, this will be isomorphic to $\pi_p((!))$, and by our assumption on $\mathbf{X}$ (and a certain long exact homotopy sequence) it will be an abelian group without odd torsion. Then the universal coefficient theorem tells us that $H_p((!)\,;\mathbb{Z}/2)$ is still nonzero—contradicting what we know already.

So now we know, from the J.H.C. Whitehead theorem for spectra, that (!) is a homotopy equivalence. Since (!!) is also a homotopy equivalence, we have reduced the proof to the case where $\mathbf{X}$ is a wedge sum of spectra of the form $\mathbb{S}^m_i \wedge \mathbf{H}\mathbb{Z}/2$. It is easy to reduce further to the case where the wedge sum has only one summand, and this case is obvious. $\square$

This is where the general nonsense ends ; now we want to apply theorem 12.4 to **MO**. At this stage we need some explicit information about the Steenrod algebra $\mathcal{A}^*$. Here I have to quote from other sources. The cohomology (coefficients $\mathbb{Z}/2$ again) of the spaces K_m defined earlier in this section can be calculated by induction on m. In more detail, $H^*(K_1) = H^*(\mathbb{R}P^\infty)$ and $H^*(K_{m+1})$ can be calculated from $H^*(K_m)$ using the Leray-Serre spectral sequence and the "Borel theorem". The result is as follows:

Theorem 12.5. *(Quotation:) For all $m > 0$, the ring $H^*(K_m)$ is a polynomial algebra on generators*

$$Sq^{i_1}Sq^{i_2}\dots Sq^{i_s}(u_m)$$

where u_m is the unique nonzero element in $H^m(K_m)$, and

(a) $i_1 \geq 2i_2,\ i_2 \geq 2i_3,\ \dots,\ i_{s-1} \geq 2i_s,\ i_s \geq 1$;
(b) $(i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{s-1} - 2i_s) + i_s < m$.

Terminology: $(i_1, i_2, \dots, i_s)$ is an *admissible sequence* if it satisfies property (a). In that case its *excess* is the left-hand side of (b). The excess is also equal to $i_1 - i_2 - \cdots - i_s$.

It is an easy consequence of theorem 12.5 that the inclusion $\Sigma K_m \hookrightarrow K_{m+1}$ induces an isomorphism in cohomology in dimensions $< 2m+1$ (since it sends u_{m+1} to the suspension of u_m). In particular,

$$H^i(\mathbf{H}\mathbb{Z}/2) \cong H^{m+i}(K_m) \qquad \text{for large } m$$

which means (1) that we know $H^*(\mathbf{H}\mathbb{Z}/2)$, and (2) that the Steenrod operations act on $H^*(\mathbf{H}\mathbb{Z}/2)$ (cf. the two remarks after summary 11.4).

Corollary 12.6. *$H^*(\mathbf{H}\mathbb{Z}/2)$ is a graded vector space with a graded basis consisting of all elements of the form*

$$Sq^{i_1}Sq^{i_2}\dots Sq^{i_s}(u)$$

where $u \in H^0(\mathbf{H}\mathbb{Z}/2)$ is the unique nonzero element and $(i_1, i_2, \dots, i_s)$ is admissible. □

Note that this does not tell us anything about the (composition) product in $H^*(\mathbf{H}\mathbb{Z}/2)$. (The composition product is not closely related to the cup product mentioned in theorem 12.5.)
Earlier in this section we saw that an element in $H^m(\mathbf{H}\mathbb{Z}/2)$ determines a stable cohomology operation of degree m. It is utterly plausible that the cohomology operation determined by $Sq^{i_1}Sq^{i_2}\dots Sq^{i_s}(u)$ is $Sq^{i_1}Sq^{i_2}\dots Sq^{i_s}$. Nevertheless, it requires proof. This is left to you as an exercise. Granting this, we know in principle what $H^*(\mathbf{MO})$ looks like as a module over $\mathcal{A}^*$, because we know how the Steenrod operations Sq^i act on it (summary 11.4, lemma lem-Cartaninhomol, proposition 11.7, dualized). Now we have to make it more explicit, in order to show that $H^*(\mathbf{MO})$ is free over $\mathcal{A}^*$. It is convenient to work with $H_*(\mathbf{MO})$ and $H^*(\mathbf{MO})$ simultaneously. Write

$$H_*(\mathbf{MO}) \cong \mathbb{Z}/2[\alpha_1, \alpha_2, \alpha_3, \dots] \cong \mathfrak{B}_* \otimes \mathfrak{C}_*$$

where $\mathfrak{B}_*$ is the polynomial algebra generated by those α_k for which $k+1$ is a power of 2, and $\mathfrak{C}_k$ is the polynomial algebra generated by the remaining α_k. Both of these can be regarded as subrings of $H_*(\mathbf{MO})$, and the isomorphism of $\mathfrak{B}_* \otimes \mathfrak{C}_*$ is then given by replacing tensor products $x \otimes y$ by Pontryagin products $x \cdot y$. Dualizing (which means taking $\hom(—, \mathbb{Z}/2)$) and writing $\mathfrak{B}^*$, $\mathfrak{C}^*$ for the duals of $\mathfrak{B}_*$ and $\mathfrak{C}_*$, respectively, we have

$$H^*(\mathbf{MO}) \cong \mathfrak{B}^* \otimes \mathfrak{C}^* \,.$$

Here $\mathfrak{B}^* \cong \mathfrak{B}^* \otimes \bar{1}$ and $\mathfrak{C}^* \cong \bar{1} \otimes \mathfrak{C}^*$ should be considered as graded subgroups of $H^*(\mathbf{MO})$ (and $\bar{1}$ is the nonzero element in $\mathfrak{C}^0$ or $\mathfrak{B}^0$, as appropriate.)

Lemma 12.7.
- *$\mathfrak{B}^* \subset H^*(\mathbf{MO})$ is an $\mathcal{A}^*$-submodule, and as such it is free on one generator (the nonzero element in degree 0).*
- *$\mathfrak{C}_* \subset H_*(\mathbf{MO})$ is closed under the operations $\bar{S}q^i$ for all i.*

Proof. The first thing to note is that

$$\binom{k+1}{i} \equiv 0 \mod 2 \quad \text{if } k+1 = 2^p \text{ and } i \neq 0,\ i \neq k+1\,.$$

Together with proposition 11.7 this shows that $\mathfrak{C}_*$ is stable under $\bar{S}q^i$ for all i. The same argument shows that the ideal $\mathcal{I}$ (not subring) in $H_*(\mathbf{MO})$ generated by the α_k, where $k+1$ is not a power of 2, is closed under the operations $\bar{S}q^i$. Noting that the dual of $H_*(\mathbf{MO})/\mathcal{I}$ is exactly $\mathfrak{B}^* \subset H^*(\mathbf{MO})/\mathcal{I}$, we conclude that $\mathfrak{B}^*$ is closed under the operations Sq^i for all i. Since these operations generate $\mathcal{A}^*$ as an algebra, $\mathfrak{B}^*$ is an $\mathcal{A}^*$-submodule. It remains to be shown that $\mathfrak{B}^*$ is free on one generator. To start somewhere, let us show that $\dim \mathfrak{B}^n = \dim \mathcal{A}^n$; this will make the rest of the proof easier. Remember that $\mathcal{A}^*$ has a graded vector space basis consisting of the admissible monomials in the Sq^i. On the other hand, $\mathfrak{B}^*$ is dual to $H_*(\mathbf{MO})$, which is a graded ring isomorphic to

$$\mathbb{Z}/2[\alpha_1, \alpha_3, \alpha_7, \alpha_{15}, \dots], \qquad \mathrm{degree}(\alpha_k) = k\,.$$

This has a vector space basis consisting of the monomials in the generators. We now set up a bijection between these bases by

$$\begin{array}{ccc} Sq^{i_1}Sq^{i_2}\dots Sq^{i_s} & \mapsto & \alpha_1^{n_1}\alpha_3^{n_2}\alpha_7^{n_3}\alpha_{15}^{n_4}\dots \\ (n_1, n_2, n_3, \dots n_s) & = & (i_1 - 2i_2, i_2 - 2i_3, \dots, i_s)\,. \end{array}$$

You can easily see that it is a bijection, and that it preserves degree:

$$i_1 + i_2 + \cdots + i_s = 1n_1 + 3n_2 + 7n_3 + \cdots + 2^{s-1}n_s\,.$$

So the dimensions are correct. *Notation*: In the situation above write

$$\begin{array}{rl} (i_1, i_2, i_3, \dots, i_s) = I, & (n_1, n_2, n_3, \dots n_s) = N \\ Sq^{i_1}Sq^{i_2}\dots Sq^{i_s} = Sq^I, & \alpha_1^{n_1}\alpha_3^{n_2}\alpha_7^{n_3}\alpha_{15}^{n_4}\cdots = \alpha^N \\ N = & \varepsilon(I). \end{array}$$

To proceed further, let $0 \neq \theta \in \mathcal{A}^*$ and write

$$\theta = Sq^I + \text{ other admissible monomials in the } Sq^i$$

where $I = (i_1, i_2, \dots, i_s)$ has been selected to be minimal in the lexicographic ordering. That is, among all admissible monomials which appear with nonzero coefficient in the expansion of θ there is none having a smaller i_1 ; among those having the same i_1, there is none having a smaller i_2 ; and so on. Let $\bar{1}$ be the nonzero element in $\mathfrak{B}^0$, as usual. We shall verify that

$$\langle \theta(\bar{1}), \alpha^N \rangle = \langle Sq^I(\bar{1}), \alpha^N \rangle = 1 \qquad (N = \varepsilon(I)).$$

(Here $\langle\ ,\ \rangle$ is the scalar product relating $\mathfrak{B}^*$ to its dual, and α^N lives in the dual.) This will complete the proof, since it shows that $\theta \mapsto \theta(\bar{1})$ is an injection and therefore a bijection (by the dimension count).

Note that

$$\langle \theta(\bar{1}), \alpha^N \rangle = \langle \bar{1}, \bar{\theta}(\alpha^N) \rangle = \bar{\theta}(\alpha^N) \in H_0(\mathbf{MO}) \cong \mathbb{Z}/2$$

where $\bar{\theta}$ is the transpose (better: adjoint) of θ, operating on the *homology* $H_*(\mathbf{MO})$, and lowering degrees. Then

$$\begin{array}{rcl} \bar{\theta} & = & \bar{S}q^I + \text{ adjoints of other admissible monomials in the } Sq^i \\ \bar{S}q^I & = & \bar{S}q^{i_s}\dots\bar{S}q^{i_2}\bar{S}q^{i_1}\,. \end{array}$$

(Note that the order of composition is reversed ; transposition has this property.) Recall from proposition 11.7 and the Pascal triangle modulo 2 that if $k+1$ is a power of 2, then

$$(*)\qquad \bar{S}q^j(\alpha_{k+j}) = \begin{cases} \alpha_k & j=0,\ j=k+1 \\ 0 & 0\neq j\neq k+1 \end{cases}$$

where α_0 should be read as 1. Note also that $I=(i_1,i_2,\dots,i_s)$ can be recovered from $N=\varepsilon(I)=(n_1,n_2,\dots,n_s)$ by

$$(**)\ \ i_1 = n_1+2n_2+4n_3+\cdots+2^{s-1}n_s, \quad i_2 = n_2+2n_3+4n_4+\cdots+2^{s-2}n_s\ ,$$

and so on. Using this and the Cartan formula, you will see that

$$\bar{S}q^{i_1}\alpha^N = \alpha^{N'} \qquad\qquad N'=(n_2,n_3,n_4,\dots,n_s)$$

and then by induction, noting that $N'=\varepsilon(I')$ where $I'=(i_2,\dots,i_s)$:

$$\bar{S}q^I(\alpha^N)=1\,.$$

(If you need help: write $\alpha^N=\alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_n}$, $\quad n=n_1+n_2+\cdots+n_s$. Then use the Cartan formula as in lemma 11.5, and you will find that $\bar{S}q^{i_1}\alpha^N$ is a sum of many terms of the form

$$\bar{S}q^{j_1}\alpha_{k_1}\bar{S}q^{j_2}\alpha_{k_2}\dots\bar{S}q^{j_n}\alpha_{k_n}, \qquad j_1+j_2+\cdots+j_n=i_1\,.$$

But (*) and (**) imply that the sum has only one nonzero term—the term which you get by taking $j_i=(k_i+1)/2$.)
The same reasoning shows that $\bar{S}q^i(\alpha^N)=0$ if $i>i_1$. By the minimality of I, and by induction, this shows that $\bar{S}q^J(\alpha^N)=0$ if Sq^J is any other admissible monomial which appears with nonzero coefficient in the expansion of θ. □

Now we can show that $H^*(\mathbf{MO})$ is a free $\mathcal{A}^*$-module. Recall:

$$H^*(\mathbf{MO})\cong\mathfrak{B}^*\otimes\mathfrak{C}^*\,.$$

Theorem 12.8. *Any (graded) vector space basis of*

$$\mathfrak{C}^*\cong\bar{1}\otimes\mathfrak{C}^*\subset\mathfrak{B}^*\otimes\mathfrak{C}^*\cong H^*(\mathbf{MO})$$

is an $\mathcal{A}^*$-basis of $H^*(\mathbf{MO})$.

Proof. Let $\mathfrak{C}^{\le n}$ be the n-skeleton of the graded vector space $\mathfrak{C}^*$. We shall prove, by induction on n, that the composition

$$\mathcal{A}^*\otimes\mathfrak{C}^{\le n}\xrightarrow{\ \mu\ }\mathfrak{B}^*\otimes\mathfrak{C}^*\xrightarrow{\text{projection}}\mathfrak{B}^*\otimes\mathfrak{C}^{\le n}$$

is an isomorphism of graded vector spaces. Here

$$\mu(\theta\otimes c):=\theta(\bar{1}\otimes c)\,.$$

Suppose that this has been shown for a particular n. Then $\mu(\mathcal{A}^*\otimes\mathfrak{C}^{\le n})$ has trivial intersection with $\bar{1}\otimes\mathfrak{C}^{n+1}$, because the latter is contained in the kernel of the projection to $\mathfrak{B}^*\otimes\mathfrak{C}^{\le n}$. Choose a vector space basis $\{c_1,c_2,\dots,c_r\}$ for $\mathfrak{C}^{n+1}$. Let $\mathfrak{D}_*$ be the graded sub-vector space of $H_*(\mathbf{MO})$ dual to $H^*(\mathbf{MO})/\mu(\mathcal{A}^*\otimes\mathfrak{C}^{\le n})$ (a graded quotient vector space of $H^*(\mathbf{MO})$).

Then $\mathfrak{D}_*$ is closed under all operations $\bar{\theta}$ for $\theta \in \mathcal{A}^*$. Since $\bar{1} \otimes c_1, \ldots, \bar{1} \otimes c_r$ are linearly independent in the quotient $H^*(\mathbf{MO})/\mu(\mathcal{A}^* \otimes \mathfrak{C}^{\leq n})$, we can find $d_1, \ldots, d_r$ in $\mathfrak{D}_{n+1}$ such that $\langle \bar{1} \otimes c_i, d_j \rangle = \delta_{ij}$ (Kronecker delta).
Suppose now that the composition

$$\mathcal{A}^* \otimes \mathfrak{C}^{\leq n+1} \xrightarrow{\mu} \mathfrak{B}^* \otimes \mathfrak{C}^* \xrightarrow{\text{projection}} \mathfrak{B}^* \otimes \mathfrak{C}^{\leq n+1}$$

is *not* an isomorphism. Then it is not injective (dimension counting) and we conclude that there exist $\theta_1, \ldots, \theta_r$ in $\mathcal{A}^k$ (some $k \geq 0$), not all zero, such that

$$(*) \qquad \sum_i \theta_i(\bar{1} \otimes c_i) \in \mu(\mathcal{A}^* \otimes \mathfrak{C}^{\leq n}).$$

Without loss of generality, $\theta_1 \neq 0$. Choose $b \in \mathfrak{B}_k$ such that $\bar{\theta}_1(b) = 1$ (we know from lemma 12.7 that it can be done). Then

$$\begin{array}{rcccl} & \langle \sum_i \theta_i(\bar{1} \otimes c_i), b \cdot d_1 \rangle & = & \sum_i \langle \bar{1} \otimes c_i, \bar{\theta}_i(b \cdot d_1) \rangle & \\ \overset{!}{=} & \sum \langle \bar{1} \otimes c_i, \bar{\theta}_i(b) \cdot d_1 \rangle & = & \sum_i \theta_i(b) \langle \bar{1} \otimes c_i, d_1 \rangle & = \quad 1. \end{array}$$

(For the equation labelled !, write $\bar{\theta}$ as a composition of operations $\bar{S}q^i$, where $i > 0$, use the formula of proposition 11.7, and note that $\bar{S}q^i(d_1) = 0$ since $\mathfrak{D}_*$ is zero for $* \leq n$.) But if we use (*) to write $\sum_i \theta_i(\bar{1} \otimes c_1)$ as a sum of elements of the form $\theta'(\bar{1} \otimes c')$ with $c' \in \mathfrak{C}^{\leq n}$, and repeat the calculation, we find

$$\langle \sum_i \theta_i(\bar{1} \otimes c_i), b \cdot d_1 \rangle = \cdots = 0. \quad \square$$

Corollary 12.9. *The composition*

$$\bar{1} \otimes \mathfrak{C}^* \hookrightarrow H^*(\mathbf{MO}) \xrightarrow{\text{projection}} \mathbb{Z}/2 \otimes_{\mathcal{A}^*} H^*(\mathbf{MO})$$

is an isomorphism. $\square$

(If the tensor product over $\mathcal{A}^*$ makes you feel uncomfortable, a simpler description of the right-hand term is as follows: it is what you get by introducing relations $\theta(x) = 0$ for any $x \in H^*(\mathbf{MO})$ and any $\theta \in \mathcal{A}^n$, any $n > 0$, but not $n = 0$. If you want to understand the tensor product notation: Think of $\mathbb{Z}/2$ as a graded vector space concentrated in dimension 0, and make it into a graded module over $\mathcal{A}^*$ in the only possible way. If you think of $H^*(\mathbf{MO})$ as a left module over $\mathcal{A}^*$, then you should think of $\mathbb{Z}/2$ as a right module over $\mathcal{A}^*$.)

Corollary 12.10. *The composition*

$$\pi_*(\mathbf{MO}) \hookrightarrow H_*(\mathbf{MO}) \xrightarrow{\text{projection}} \mathfrak{C}_*$$

is an isomorphism.

Proof. Modulo theorem 12.4, this is simply the dualized version of corollary 12.8. Note that $\mathfrak{C}_*$ is a polynomial ring over $\mathbb{Z}/2$ with one generator α_k in degree k for each k not of the form $2^p - 1$. The projection map from

$$H_*(\mathbf{MO}) \cong \mathbb{Z}/2[\alpha_1, \alpha_2, \alpha_3, \ldots]$$

is a ring homomorphism (but it is not "compatible" in any sense with the action of the operations $\bar{S}q^i$). It takes α_k to 0 if $k = 2^p - 1$, and to α_k if k does not have this form. The Hurewicz homomorphism is also a ring homomorphism by summary 11.4, and moreover the ring structure on $\pi_*(\mathbf{MO})$ is "equal" to the geometrically defined ring structure on $\mathfrak{N}_* \cong \pi_*(\mathbf{MO})$. Summarizing, the isomorphism in corollary 12.10 is a *ring* isomorphism. $\square$

Theorem 12.11. *The bordism ring $\mathfrak{N}_*$ is isomorphic to a graded polynomial ring $\mathbb{Z}/2[\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8, \dots]$, with one polynomial generator α_k in degree k for each k not of the form $2^p - 1$.*

For example: $\dim(\mathfrak{N}_0) = 1$, $\dim(\mathfrak{N}_1) = 0$, $\dim(\mathfrak{N}_2) = 1$, $\dim(\mathfrak{N}_3) = 0$, $\dim(\mathfrak{N}_4) = 2$, $\dim(\mathfrak{N}_5) = 1$, $\dim(\mathfrak{N}_6) = 3$, $\dim(\mathfrak{N}_7) = 1$, $\dim(\mathfrak{N}_8) = 5$, $\dim(\mathfrak{N}_9) = 3$, $\dim(\mathfrak{N}_{10}) = 8$, and so on.