Prof Ranicki Notes by Adam Mole 14/10/08

Lecture 1. Problem:

$$\begin{array}{ccc} M^n & & M'^n \\ \downarrow f = \text{htpy eqvce} = f' & \downarrow & \mathcal{S}(X) = \{(M,f)\}/\sim \\ X & X & (M,f)\sim(M',f') \end{array}$$

No clear way of adding two members of $\mathcal{S}(X)$

1. The Q-groups of a chain complex

Let A be a ring with involution. (This allows us to have a good notion of quadratic forms).

involution is an operator $\bar{\ }: A \to A$, $a \mapsto \bar{a}$ st $\bar{1} = 1$, $\overline{a+b} = \bar{a}+\bar{b}$, $\overline{ab} = \bar{b}\bar{a}$, $\bar{\bar{a}} = a$

Eg. $A = \mathbb{Z}[\pi]$, for a group π, $\bar{g} := \pm g^{-1}$ ($g \in \pi$, $^{-1}$ is gp inverse)

or, for any commutative ring take $\bar{a} = a$

Defn $\mathrm{Proj}(A)$ = category of f.g. projective A-modules

(projective means $\exists N$ st $P \oplus N \cong A^k$ some k)

Involution on A gives a contravariant duality

$*: \mathrm{Proj}(A) \to \mathrm{Proj}(A)$ $\qquad A \times P^* \to P^*$

$P \mapsto P^* = \mathrm{Hom}_A(P, A)$ $\qquad (a, f) \mapsto (x \mapsto f(x)\bar{a})$

defining a left action on right A-module P^*

Since P is f.g. proj, the A-module morphism

$\mathrm{eval}: P \to P^{**}$, $x \mapsto (f \mapsto \overline{f(x)})$ is an isomorphism.

Pf

$ax \mapsto (f \mapsto \overline{f(ax)} = \overline{a f(x)} = \overline{f(x)}\,\bar{a} = a\,\overline{f(x)})$ so an A-module homom. ∎

More generally: $P \otimes_A Q \to P^{**} \otimes_A Q \cong \mathrm{Hom}_A(P^*, Q)$ slant isomorphism

$x \otimes y \mapsto (f \mapsto \overline{f(x)}.y)$

Check $\quad f(\bar{a}x)y = f(x)\bar{a}y = f(x)(ay)$ $\qquad$ use $P \times A \to P$

$\overline{f(\bar{a}x)}\,y = \bar{a}\,\overline{f(x)}\,y = \overline{f(x)}\,(ay)$ $\qquad (x,a) \mapsto \bar{a}x$

How to apply this to topology?

Consider the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x,x)$.

This has a $\pi_1(X)$-equivariant ($\swarrow$ commutes with $\pi_1(X)$ action) generalisation for the universal cover $\tilde{X}$ of X. Let $\pi = \pi_1(X)$; $\pi \times \tilde{X} \to \tilde{X}$, $(g, \tilde{x}) \mapsto g\tilde{x}$

Generalisation: $\tilde{\Delta}/_{\pi} : \underset{\substack{\parallel \\ X}}{\tilde{X}/_{\pi}} \longrightarrow \underset{\substack{\parallel \\ \tilde{X}\times\tilde{X}/_{(x,y)\sim(gx,gy)}}}{(\tilde{X}\times\tilde{X})/_{\pi}} = \tilde{X} \times_{\pi} \tilde{X}$ with $\tilde{X} \times \pi \to \tilde{X}$, $(\tilde{x}, g) \mapsto g^{-1}\tilde{x}$ right π-action

$$\begin{tikzcd} & \tilde{X} \times_{\pi} \tilde{X} \arrow[d] \\ X \arrow[ur, "\tilde{\Delta}/_{\pi}"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

$$\begin{aligned} H_n(X) \longrightarrow\ & H_n(\tilde{X} \times_{\pi} \tilde{X}) \\ &\parallel \\ & H_n(C(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} C(\tilde{X})) \\ \text{slant isom} \quad & \cong H_n(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\tilde{X})^*, C(\tilde{X}))) \end{aligned}$$

Assume X is a finite CW-cx, $C(\tilde{X})$ is the cellular, f.g. free $\mathbb{Z}[\pi]$-module chain complex, one $\mathbb{Z}[\pi]$ for every cell of X, not $\tilde{X}$

$C(\tilde{X})^* = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\tilde{X}), \mathbb{Z}[\pi])$ is the π-equivariant cochain complex

In general, $H^r(C(\tilde{X})^*) = \dfrac{\ker(d^* : C_r(\tilde{X})^* \to C_{r+1}(\tilde{X})^*)}{\mathrm{im}(d^* : C_{r-1}(\tilde{X})^* \to C_r(\tilde{X})^*)}$ is NOT $H^r(\tilde{X})$

Example $X = S^1$, $\tilde{X} = \mathbb{R}$ $\underset{-1}{|}\ \underset{0}{|}\ \underset{1}{|}\ \underset{2}{|} \longrightarrow \mathbb{R}$

$C_1(\tilde{X}) = \mathbb{Z}[z, z^{-1}] \xrightarrow{d = 1 - z} C_0(\tilde{X}) = \mathbb{Z}[z, z^{-1}]$

$C_0(\tilde{X})^* = \mathbb{Z}[z, z^{-1}] \xrightarrow{d^* = 1 - z^{-1}} C_1(\tilde{X})^* = \mathbb{Z}[z, z^{-1}]$ if $\iota(1) = 1$ then $\iota(z) \overset{?}{=} z^{-1}\iota(1) = z^{-1}$

$H^1(C(\mathbb{R})^*) = \mathrm{coker}(1 - z^{-1}) = \mathbb{Z} \neq 0 = H^1(\mathbb{R})$

From now on, $H^r(\tilde{X})$ means $H^r(C(\tilde{X})^*)$ (they are the same if π finite)

There is a relevance to cohomology with compact supports.

If C, D are chain complexes in $\mathrm{Proj}(A)$, the slant isom

$$C \otimes_A D \xrightarrow{\simeq} \mathrm{Hom}_A(C^*, D)$$

sends a homology class $\varphi \in H_n(C \otimes_A D)$ to a chain homotopy class $\hat{\varphi} \in H_n(\mathrm{Hom}_A(C^*, D))$ of A-module chain maps ~~with~~ $\hat{\varphi} : C^{n-*} \to D$ with

$$\begin{array}{ccl} (C^{n-*})_r & = & C^{n-r} = \mathrm{Hom}_A(C_{n-r}, A) \\ \downarrow d_{C^{n-*}} = \pm d_C^* & & \text{sign convention is } (-1)^r \\ (C^{n-*})_{r-1} & = & C^{n-r+1} \end{array}$$

So for any space X with $\tilde{X}$, $\pi_1(X) = \pi$,

$$(\tilde{\Delta}/\pi)_* : H_n(X) \longrightarrow H_n(\tilde{X} \times \tilde{X}) = H_0(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\tilde{X})^{n-*}, C(\tilde{X})))$$

$$x \longmapsto x \cap : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})$$

$$\cap : H_n(X) \otimes H^r(\tilde{X}) \longrightarrow H_{n-r}(\tilde{X})$$

<u>Big Example</u> If X is an n-dim'l mfd with fundamental class $[X] \in H_n(X)$, then $[X] \cap : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})_*$ is the Poincaré duality $\mathbb{Z}[\pi]$-module chain equivalence.

$$\mathbb{Z} = H^1(\mathbb{R}) \cong H_0(\mathbb{R}) = \mathbb{Z}$$

For any space X, $X \times X$ is a $\mathbb{Z}_2$-space with $T : X \times X \to X \times X$, $(x,y) \mapsto (y,x)$, $\mathbb{Z}_2 = \{1, T\}$

with fixed pts $(X \times X)^{\mathbb{Z}_2} = \{(x,x) \in X \times X\}$

If M_1, M_2 are subspaces then

$$\begin{array}{ccc} M_1 \cap M_2 & \longrightarrow & M_1 \times M_2 \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

And $M_1 \cap M_2 \cong (M_1 \times M_2) \cap (X \times X)^{\mathbb{Z}_2}$

$$\begin{array}{ccccc} C(X) & \xrightarrow{\Delta} & C(X \times X) & \xrightarrow{EZ} & C(X) \otimes C(X) \\ \downarrow & & T_0 \downarrow \quad T_1 \text{ chain htpy} & & \downarrow T = T_0 \\ C(X) & \xrightarrow{\Delta} & C(X \times X) & \xrightarrow{EZ} & C(X) \otimes C(X) \end{array}$$

$$T : C_p \otimes C_q \to C_q \otimes C_p$$
$$x \otimes y \mapsto (-1)^{pq} y \otimes x$$

For any X, $\exists$ natural chain htpy $T_1 : T_0(EZ) \simeq (EZ)T_0$

$\forall P, Q \in \mathrm{Proj}(A)$, $\mathrm{Hom}_A(P,Q) \cong \mathrm{Hom}_A(Q^*, P^*)$

$$f \mapsto f^* : \varphi \in Q^* \mapsto (p \mapsto \varphi(f(p)))$$

If $Q = P^*$, we get the duality isom $T : \mathrm{Hom}_A(P, P^*) \xrightarrow{\cong} \mathrm{Hom}_A(P, P^*)$

Notation, $T\varphi =: \varphi^*$ $\qquad \varphi \mapsto (\underbrace{p \mapsto (p' \mapsto \overline{\varphi(p')(p)})}_{\varphi^* = T\varphi \in \mathrm{Hom}_A(P, P^*)})$

Special case $P = A^k$, $\mathrm{Hom}_A(P, P^*) \xleftrightarrow{\cong}$ $k \times k$ matrices

$\alpha : P \to P^*$, $(x_1, -, x_k) \mapsto ((y_1, -, y_k) \mapsto \sum_i \sum_j y_i \alpha_{ij} \bar{x}_j)$

$\alpha^* = (\bar{\alpha}_{ji})$, conjugate transpose

$\mathrm{Hom}_A(P, P^*) \cong P^* \otimes_A P^*$, bilinear pairing

$(\varphi : P \to P^*) \mapsto \mathrm{adj}(\varphi) : P \times P \to A$, $(x,y) \mapsto \varphi(x)(y)$

$$\begin{tikzcd}
 & H_n(\tilde{X} \times_\pi \tilde{X}) \arrow[d, "T"] \arrow[drr, bend left, "T_1 \text{(chain equiv)}"] & \cong & H_0(\mathrm{Hom}_{\mathbb{Z}\pi}(C(\tilde{X})^*, C(\tilde{X}))) \arrow[d, "T = T_0"] \\
H_n(X) \arrow[ur, "\tilde{\Delta}/\pi"] \arrow[dr, "\tilde{\Delta}/\pi"'] & & & \\
 & H_n(\tilde{X} \times_\pi \tilde{X}) & \cong & H_0(\mathrm{Hom}_{\mathbb{Z}\pi}(C(\tilde{X})^*, C(\tilde{X})))
\end{tikzcd}$$

no problem commuting

T_0, T_1 beginning of Steenrod squares

For $x \in \ker d : C_n(X) \to C_{n-1}(X)$

"$T_1(x) : x \cap - \simeq T_0(x \cap -)$ $\quad C(\tilde{X})^{n-*} \to C(\tilde{X})$

so only symmetric up to chain homotopy"

Back to example. $X = S^1$

$$\begin{tikzcd}
C_0(\tilde{X})^* = A \arrow[r, shift left, "\varphi = 1"] \arrow[d, "d^* = 1 - z^{-1}"'] & C_1(\tilde{X}) = A \arrow[l, shift left, "\varphi^* = -z^{-1}"] \arrow[d, "d = 1 - z"] \\
C_1(\tilde{X})^* = A \arrow[r, shift left, "\varphi^* = 1"] \arrow[ur, bend left, "1"] & C_0(\tilde{X}) = A \arrow[l, shift left, "\varphi = -z"]
\end{tikzcd}$$

where $A = \mathbb{Z}[z, z^{-1}]$

and $1 : \varphi \simeq \varphi^* : C(\tilde{X})^{1-*} \to C(\tilde{X})$

Steenrod Squares: For any space X, $Sq^i : H^r(X) \to H^{r+i}(X)$ for r, i

$Sq^r(x) = x \cup x$, for $x \in H^r(X)$.

External product in cohomology $\quad H^p(X)\otimes H^q(Y) \longrightarrow H^{p+q}(X\times Y)$

(singular cohomology for arbitrary X, Y) $\qquad\qquad\qquad\qquad \overset{\shortparallel}{H^{p+q}}(C(X)\otimes C(Y))$

$$a\otimes b \longmapsto (a\times b : C(X)\otimes C(Y) \longrightarrow \mathbb{Z}\,.$$
$$x\otimes y \longmapsto a(x)b(y)),$$

$$H_p(X)\otimes H_q(Y) \longrightarrow H_{p+q}(X\times Y) = H_{p+q}(\overset{C(X\times Y)}{\overset{\shortparallel}{C(X)\otimes C(Y)}})$$
$$a\otimes b \longmapsto a\otimes b.$$

For internal product need $\Delta : X \longrightarrow X\times X$

Cup product
$$\begin{tikzcd}
\smile : H^p(X)\otimes H^q(X) \arrow[rr] \arrow[dr, "\times"'] & & H^{p+q}(X) \\
& H^{p+q}(X\times X) \arrow[ur, "\Delta^*"'] &
\end{tikzcd}$$

$$\frown : H^p(X)\otimes H_q(X) \longrightarrow H_{p+q}(X)$$

and $\qquad \langle a\smile b, c\rangle = \langle a, b\frown c\rangle \in \mathbb{Z}$

The symmetric Q-groups $Q^*(C)$ of ~~an~~ a left A-mod chain cx C, where A is a ring with involution;

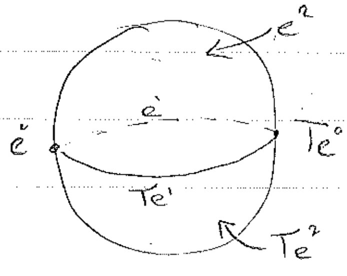
An elt $\varphi \in Q^n(C)$ is a homology class $\varphi_0 \in H_n(C\otimes C)$ with chain level reason for $\varphi_0 = T\varphi \in H_n(C\otimes_A C)$ where

$$T : C_p\otimes_A C_q \longrightarrow C_q\otimes_A C_p\ ;\ x\otimes y \longmapsto (-)^{pq}\, y\otimes x$$
$$\overset{\shortparallel}{C_p\otimes_{\mathbb{Z}} C_q} / \{ax\otimes y - x\otimes \bar{a}y \mid x\in C_p, y\in C_q, a\in A\}$$

The Borel construction for $\mathbb{Z}_2$; given a $\mathbb{Z}_2$-space X consider the product $S^\infty\times X$ where $S^\infty = \bigcup_{n=0}^{\infty} S^n$ with antipodal free $\mathbb{Z}_2$-action so $S^\infty\times X$ is a free $\mathbb{Z}_2$-space

$W = C(S^\infty)$ = cellular $\mathbb{Z}[\mathbb{Z}_2]$-module chain cx of $S^\infty = (e^0 \cup Te^0)\cup(e^1\cup Te^1)\cup\ldots$

$$\partial e^1 = e^0 - Te^0$$



$$W: \quad \cdots \xrightarrow{1-T} \underset{}{\overset{W_2}{\mathbb{Z}[\mathbb{Z}_2]}} \xrightarrow{1+T} \overset{W_1}{\mathbb{Z}[\mathbb{Z}_2]} \xrightarrow{1-T} \overset{W_0}{\mathbb{Z}[\mathbb{Z}_2]} \;(\longrightarrow \mathbb{Z} \longrightarrow 0)$$

canonical free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$ with trivial $\mathbb{Z}_2$-action. See Cartan + Eilenberg, 1952.

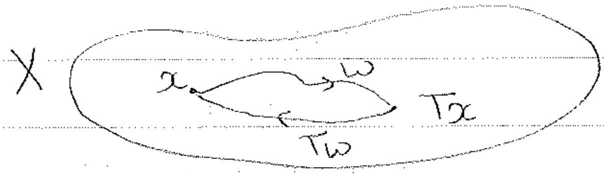
"Dual" Borel construction for $\mathbb{Z}_2$-space

$\mathrm{map}^{\mathbb{Z}_2}(S^\infty, X)$ homotopy $\mathbb{Z}_2$-fixed pt space

$\mathrm{map}^{\mathbb{Z}_2}(S^{1}, X) = \mathrm{map}\ [0,1] \xrightarrow{\omega} X$ st $\omega(1) = T\omega(0)$

Y^G = fixed pts of Y under

A htpy fixed pt $(\overset{x}{\cancel{X}}, \omega \in (X, X^{[0,1]}))$ is a pt $x \in X$ and a path $\omega : [0,1] \longrightarrow X$ with $\omega(0) = x$, $\omega(1) = Tx$



A $\mathbb{Z}_2$-equivariant map $\omega : S^\infty \longrightarrow X = \mathbb{Z}_2$-space (ie X is a $\mathbb{Z}_2$-space) is essentially the same as a pt $\omega_0 \in X$, a path $\omega_1 : [0,1] \longrightarrow X$ from $\omega_1(0) = \omega_0$ to $\omega_1(1) = T\omega_0$,

a htpy $\omega_2 : \omega_1 \simeq T\omega_1 : [0,1] \longrightarrow X$ rel $\{0,1\}$

a higher htpy $\omega_3 : \omega_2 \simeq T\omega_2, \ldots$ and so on.

Defn. $Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C))$

An elt $\varphi \in Q^n(C)$ is a $\mathbb{Z}[\mathbb{Z}_2]$-module chain ~~cx~~ homology class of $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps $\varphi : W \longrightarrow (C \otimes_A C)_{*+n} = S^n(C \otimes_A C)$

(Note: The suspension of chain cx C is SC with $(SC)_r = C_{r-1}$)

$$\begin{array}{ccccccccc}
W & \cdots \longrightarrow & W_2 & \xrightarrow{1+T} & W_1 & \xrightarrow{1-T} & W_0 & \longrightarrow & 0 \\
\varphi \downarrow & & W_r = \mathbb{Z}[\mathbb{Z}_2] & & \downarrow \varphi_1 & \searrow^{\varphi_1} & \downarrow \varphi_0 & & \downarrow \\
C \otimes C & & \longrightarrow & & (C \otimes_A C)_{n+1} & \xrightarrow{\varphi_0 - T\varphi_0} & (C \otimes_A C)_n & \longrightarrow & (C \otimes_A C)_{n-1}
\end{array}$$

$d_{C \otimes_A C} = d_C \otimes 1 \pm 1 \otimes d_C$

Note $d_{C\otimes_A C}(\varphi_0) = (d_C \otimes 1 \pm 1 \otimes d_C)\varphi_0 = 0 \in (C\otimes_A C)_{n-1}$

To compare with spaces, set $X = Y \times Y$, $C = C(Y)$ then

$$H_n(\mathrm{map}^{\mathbb{Z}_2}(S^\infty, Y\times Y)) \longrightarrow Q^n(C(Y))$$

An elt $\varphi \in Q^n(C)$ is represented by a collection $\{\varphi_s \in (C\otimes_A C)_{n+s} \mid s \geq 0\}$
s.t. $d_{C\otimes_A C}(\varphi_s) = \varphi_{s-1} + (-1)^s T\varphi_{s-1} \in (C\otimes_A C)_{n+s-1}$
and setting $\varphi_{-1} = 0$
φ_0 is symmetric up to higher chain homotopies.

In practice, C will be a f.g. proj A-module chain cx, for which the slant ~~map~~ products

$$C_p \otimes_A C_q \longrightarrow \mathrm{Hom}_A(C^p, C_q)$$
$$x \otimes y \longmapsto (f \mapsto \overline{f(x)}\, y)$$

are isomorphisms with

$$\begin{array}{ccc} C_p \otimes_A C_q & \xrightarrow{\cong} & \mathrm{Hom}_A(C^p, C_q) \\ \downarrow T & & \downarrow T \\ C_q \otimes_A C_p & \longrightarrow & \mathrm{Hom}_A(C^q, C_p) \end{array}$$

$T(\varphi : C^p \to C_q) = \varphi^* : C^q \longrightarrow C_p$ where
$\varphi^*(x)(y) = \overline{\varphi(y)(x)}$ $\quad x \in C^q, y \in C^p$
"φ^* is transpose complex conjugate of φ"

A cycle $\varphi_0 \in (C\otimes_A C)_n$ is a chain map $\varphi_0 : C^{n-*} \longrightarrow C$
where $(C^{n-*})_r = C^{n-r}$ and $d_{C^{n-*}} = \pm d_C^*$

A chain $\varphi_1 \in (C\otimes_A C)_{n+1}$ st. $d_{C\otimes_A C}(\varphi_1) = \varphi_0 - T\varphi_0$
is the same as an A-mod chain htpy $\varphi_1 : \varphi_0 \simeq T\varphi_0 : C^{n-*} \longrightarrow C$

$\varphi_2 \in (C \otimes_A C)_{n+2}$ is a higher chain homotopy $\varphi_2 : \varphi_1 \simeq T\varphi_1 : \varphi_0 \simeq T\varphi_0 : C^{n-*} \to C$

$$d_C\varphi_0 - \varphi_0 d_C^* = 0\ , \quad \varphi_0 - T\varphi_0 = d_C\varphi_1 + \varphi_1 d_C^*\ , \quad \varphi_1 + T\varphi_1 = d_C\varphi_2 + \varphi_2 d_C^*$$

$$\varphi \in Q^n(C) = H_n(\mathrm{hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C)))$$

Defn. An n-dim^l symmetric Poincaré cx over A (C, φ) is an n-dim f.g. free A-module chain cx $C : C_n \to C_{n-1} \to \ldots \to C_0$ up to chain equivalence together with an elt $\varphi \in Q^n(C)$ st $\varphi_0 \in H_n(C \otimes_A C) = H_n(\mathrm{Hom}_A(C^*, C))$ is a chain htpy class of A-mod chain equivalences $\varphi_0 : C^{n-*} \xrightarrow{\simeq} C$

Example. An oriented n-dim^l mfd M (or geometric Poincaré cx) determines an n-dim^l symmetric P. cx (C, φ) over $\mathbb{Z}[\pi_1(M)]$ with $C = C(\tilde{M})$ the cellular f.g. free $\mathbb{Z}[\pi_1(M)]$-module chain cx of the universal cover $\tilde{M}$.

$\varphi_0 := [M] \cap : C(\tilde{M})^{n-*} \xrightarrow{\simeq} C(\tilde{M})$ Poincaré duality with $\mathbb{Z}[\pi_1(M)]$-coefficients

Example (contd). If $\mathbb{Z}[\pi_1(M)] \to A$ is any morphism of rings with involution $A \otimes_A$

Example. A 0-dim^l SPC $(C : \ldots \to 0 \to C_0 \to 0 \to \ldots, \varphi)$ is a f.g. proj A-module C_0 together with $\varphi = \varphi_0 = \varphi_0^* \in C_0 \otimes_A C_0 = \mathrm{Hom}_A(C^*, C)$ $(C^0 = C_0^*,\ (C^0)^* = C_0)$, which is the same as a "sesquilinear symmetric" pairing

$$\varphi_0 : C^0 \times C^0 \longrightarrow A \quad \text{with} \quad \begin{aligned} \varphi_0(ax, by) &= b\,\varphi_0(x,y)\,\bar{a} \\ \varphi_0(y,x) &= \overline{\varphi_0(x,y)} \end{aligned}$$

and also $\varphi_0 = \varphi_0^* : C^0 \longrightarrow (C^0)^* = C_0$ is an A-mod isom
i.e. a "nonsingular symmetric form" φ_0 on C.
$Q^0(C)$ = all symmetric forms on C^0
Poincaré $\varphi \in Q^0(C)$ = non singular forms.

Eg. If $C_0 = A^k = A \otimes \dots \otimes A$ then an elt $\varphi \in Q^0(C)$ is a $k \times k$ matrix $(a_{ij})_{1 \le i,j \le k}$, $a_{ij} \in A$ with $a_{ij} = \overline{a_{ji}} \in A$.
(C, φ) Poincaré $\Leftrightarrow$ (a_{ij}) invertible

Example. Spse $n = 2i$ and $C : \dots \to 0 \to C_i \to 0 \to \dots$
concentrated in dim^n i.
An elt $\varphi \in Q^{2i}(C) = \ker(1 - (-1)^i T : \mathrm{Hom}_A(C^i, C_i) \longrightarrow \mathrm{Hom}_A(C^i, C_i))$
same as $d: W_i \to W_{i-1}$
is a $(-1)^i$-symmetric form on C^0.

Cf. Eg M^n, $n = 2i$. $H^i(M) \times H^i(M) \xrightarrow{\smile} H^{2i}(M) \cong \mathbb{Z}$
and $a \smile b = (-1)^i b \smile a$

The symmetric construction (Alexander-Whitney-Steenrod + Eilenberg-Zilbe).
For any space X and regular cover $\tilde{X}$ with gp π there is a natural transformation $\varphi_X : H_n(X) \longrightarrow Q^n(C(\tilde{X}))$

$$\tilde{X}/\pi = X \;\;\;\; * \longrightarrow \mathrm{map}^{\mathbb{Z}_2}(S^\infty, \tilde{X} \times_\pi \tilde{X})$$

$$\begin{array}{ccc} C(X) & \longrightarrow & C(\mathrm{map}^{\mathbb{Z}_2}(S^\infty, \tilde{X} \times_\pi \tilde{X})) \\ & {}_{\varphi_X}\searrow & \downarrow E^2 \\ & & \mathrm{Hom}(W, C(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} C(\tilde{X})) \end{array}$$

Natural means $f : X \to Y$ has $\begin{array}{ccc} H_n(X) & \xrightarrow{\varphi_X} & Q^n(C(\tilde{X})) \\ f_* \downarrow & & \downarrow \tilde{f} \otimes \tilde{f} \\ H_n(Y) & \xrightarrow{\varphi_Y} & Q^n(C(\tilde{Y})) \end{array}$ commutes

$$\begin{array}{ccc} X & \to & Y \\ \uparrow & & \uparrow \\ \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{Y} \end{array}$$

An A-module chain map $f: C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$f \otimes f : C \otimes_A C \longrightarrow D \otimes_A D \ ; \ x \otimes y \longmapsto f(x) \otimes f(y)$$

and hence $f^{\%} = f \otimes f : Q^n(C) \longrightarrow Q^n(D)$ might be f^{n} ?

Eg. If $\pi = \{1\}$, so $\tilde{X} = X$ then $\varphi_X : C(X) \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X) \otimes C(X))$

$$\begin{tikzcd} (\varphi_X)_0 = \ : C(X) \arrow[rr] \arrow[dr, "\Delta"'] & & C(X) \otimes C(X) \\ & C(X \times X) \arrow[ur, "\simeq"', "EZ"] & \end{tikzcd}$$

$[EZ_{-1} = 0]$

$(\varphi_X)_0 = (EZ_0)\Delta$

$(\varphi_X)_1 = (EZ_1)\Delta$

$$\begin{tikzcd} C(X \times X) \arrow[r, "EZ_0"', "\simeq"] \arrow[d, "geo = T"'] \arrow[dr, "EZ_1"] & C(X) \otimes C(X) \arrow[d, "T = alg"] \\ C(X \times X) \arrow[r, "EZ_0", "\simeq"'] & C(X) \otimes C(X) \end{tikzcd}$$

EZ_0 = chain map

$EZ_1 : T(EZ_0) \simeq (EZ_0)T$

$EZ_{s+1} : T(EZ_s) \simeq (EZ_s)T$ for $s \geq 0$

For $n = 2i$, $x \in H_{2i}(X)$, $y \in H^i(X) = H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X), S^i\mathbb{Z}))$

$$H_{2i}(X) \xrightarrow{\varphi_X} Q^{2i}(C(X)) \xrightarrow{y^{\%}} Q^{2i}(S^i\mathbb{Z})$$

$$x \longmapsto \langle y \cup y, x \rangle = \begin{cases} \mathbb{Z} & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd} \end{cases}$$

$$\begin{tikzcd} H_n(X) \arrow[r, "\varphi_X"] \arrow[rr, bend left, "(\varphi_X)_0"] \arrow[drr, "\Delta_0"'] & Q^n(C(X)) \arrow[r] & H_n(C(X) \otimes_{\mathbb{Z}} C(X)) \\ & & H_n(X \times X) \arrow[u, "\cong", "EZ_0"'] \end{tikzcd}$$

Ran 6

Andrew Ranicki, Lecture 3 — 22/10/08

Recall; $Q^n(C) = \cancel{\mathrm{Hom}} H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C))$
$=: H^n(\mathbb{Z}_2; C \otimes_A C)$ $\mathbb{Z}_2$-hypercohomology

Assume C is f.g. and proj, so $C \otimes_A C \cong \mathrm{Hom}_A(C^*, C)$

$\varphi \in Q^n(C)$ is an equivalence of collections
$\{\varphi_s : C^r = \mathrm{Hom}_A(C_r, A) \longrightarrow C_{n-r+s} \mid s \geq 0,\ \varphi_{-1} = 0\}$
st $d\varphi_s + \varphi_s d^* + \varphi_{s-1} + \varphi_{s-1}^* = 0$ (up to sign)

An A-mod chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-mod chain map
$f \otimes f : C \otimes_A C \longrightarrow D \otimes_A D$
$\mathrm{Hom}_A(C^*, C) \longrightarrow \mathrm{Hom}_A(D^*, D)$; $(\varphi : C^p \to C_q) \longrightarrow (f\varphi f^* : D^p \to D_q)$
$f^{\%} : Q^n(C) \longrightarrow Q^n(D)$; $\{\varphi_s\} \longmapsto \{f\varphi_s f^*\}$

<u>Thm 1</u> $(gf)^{\%} = g^{\%} f^{\%}$, $1^{\%} = 1$
If $f \simeq f' : C \to D$ (ie $f' - f = d_D h + h d_C$)
then $f^{\%} = f'^{\%}$
So if $f : C \to D$ is a chain equivlce then $f^{\%}$ is isom $Q^n(C) \to Q^n(D)$ $\forall n$.

Topological Background.
If $h : f \simeq f' : X \to Y$ is a htpy of spaces then
$\exists H_0 : f \times f \simeq f' \times f' : X \times X \longrightarrow Y \times Y$, $H_0(x_1, x_2, t) = \begin{cases} (f(x_1), h(x_2, 2t)) & 0 \leq t \leq \frac{1}{2} \\ (h(x_1, 2t-1), f'(x_2)) & \frac{1}{2} \leq t \leq 1 \end{cases}$

$H_0 T \neq H_0$
but $H_0 \simeq TH_0T$

Note that H_0 is not $\mathbb{Z}_2$-equivariant, ie $H_0(x_1, x_2, t) = (y_1, y_2)$ does <u>not</u> give $H_0(x_2, x_1, t) = (y_2, y_1)$

$$f \times f \xrightarrow[1 \times h]{\simeq} f \times f' \xrightarrow[h \times 1]{\simeq} f' \times f'$$

Similarly in algebra; if $h : f \simeq f' : C \to D$ chain htpy
$f \otimes h : f \otimes f \simeq f \otimes f'$ and $h \otimes f' : f \otimes f' \simeq f' \otimes f'$
$\mathbb{Z}$-mod chain htpy $f \otimes h + h \otimes f' : f \otimes f \simeq f' \otimes f'$ is not a $\mathbb{Z}_2$-mod chain htpy

OK, upto higher chain homotopy. $h\otimes h : (f\otimes h + h\otimes f')T - T(f\otimes h + h\otimes f') \overset{?}{\simeq} 0$

$$(C\otimes C)_* \to (D\otimes D)_{*+1}$$

In fact $(h\otimes f + f'\otimes h) - (h\otimes f' + f\otimes h) = (d\otimes 1 + 1\otimes d)h\otimes h + h\otimes h(d\otimes 1 + 1\otimes d)$

Eg. What is $Q^0(C)$ for $C: \ldots \to C_n \to C_{n-1} \to C_{n-2} \to \ldots$

$\varphi_0 : C^r \to C_{-r}$, $d\varphi_0 + \varphi_0 d^* = 0$

$$\begin{tikzcd}
C^{-1} \arrow[r, "\varphi_0"] \arrow[d] & C_1 \arrow[d] \\
C^0 \arrow[r, "\varphi_0"] \arrow[ur, "\varphi_1"] \arrow[d] & C_0 \arrow[d] \\
C^1 \arrow[r, "\varphi_0"] \arrow[ur, "\varphi_1"] \arrow[uur] & C_{-1}
\end{tikzcd}$$

$\varphi_0 - \varphi_0^* = d\varphi_1 + \varphi_1 d^*$, $\varphi_1 : \varphi_0 \simeq \varphi_0^*$

$\varphi_1 - \varphi_1^* = d\varphi_2 + \varphi_2 d^*$

$\varphi_s : C^r \to C_{-r+s}$

In general, $\varphi_{s+1} : \varphi_s \simeq \varphi_s^*$

If $s = 2r$ then $\varphi_s : C^r \to C_r$,

if $s = 2r+1$ then $\varphi_s : C^r \to C_{r+1}$

Homology Invariants of Elts $\varphi \in Q^n(C)$. $S^{n-r}A$ is $\to 0 \to 0 \to \overset{n-r}{A} \to 0 \to 0 \to$.

Defn. The Wu classes of $\varphi \in Q^n(C)$ are the functions

$$V_r(\varphi) : H^{n-r}(C) \to Q^n(S^{n-r}A) = \hat{H}^{n-2r}(\mathbb{Z}_2; A, (-)^{n-r})$$

$$(f : C_{n-r} \to A) \mapsto f^{\%}\varphi = \begin{cases} \{a \in A \mid \bar{a} = (-1)^r a\} & n = 2r \\ \{a\in A \mid \bar a = (-1)^{n-r}a\} / \{b + (-1)^{n-r}\bar{b} \mid b \in A\} & n > 2r \\ 0 & n < 2r \end{cases}$$

$$" \{f\varphi_{n-2r}f^*\}$$

$$A \xrightarrow{f^*} C^{n-r} \xrightarrow{\varphi_{n-2r}} C_{n-r} \xrightarrow{f} A$$

$\varphi_s : C^k \to C_{n-k+s}$, so $\varphi_{n-2r} : C^{n-r} \to C_{n-(n-r)+n-2r} = C_{n-r}$

Eg. $V_0(\varphi) : H^0(C) \to \hat{H}^0(\mathbb{Z}_2; A) = \{a \in A \mid \bar{a} = a\}$

$$f \mapsto f\varphi_0 f^*$$

Example. If M is a 1-conn 4-mfd, $\varphi_M : H_4(M) \to Q^4(C(M))$

$$[M] \mapsto \varphi_M([M])$$

M simply-conn so ~~$C(M)$~~ $H_*(M)$ is $\mathbb{Z} \to 0 \to H_2(M) \to 0 \to \mathbb{Z}$

Ran 7

22/10/08

Ex. ~~Note~~: $V_r(\varphi)(f+g) - V_r(\varphi)(f) - V_r(\varphi)(g) = \begin{cases} f\varphi_0 g^* + (-1)^r g\varphi_0 f^* & n=2r \\ 0 & n>2r \end{cases}$

$f,g\in H^{n-r}(C), a\in A$ $\quad V_r(\varphi)(af) = aV_r(\varphi)(f)\bar{a}$

Special case. $C = 0$-dim^l, ie $\ldots \to 0 \to 0 \to C_0 \to 0 \to \ldots$

$\varphi \in Q^0(C)$, $\varphi_0 = \varphi_0^* : C^0 \to C_0$, $\operatorname{adj}(\varphi_0) : C^0 \times C^0 \longrightarrow A$ symmetric form

$(f,g) \mapsto f\varphi_0 g^*$

$V_0(\varphi) : C^0 \longrightarrow H^0(\mathbb{Z}_2; A) = \{a\in A \mid a = \bar{a}\}$

$f \mapsto f^*\varphi_0 f = \operatorname{adj}(\varphi_0)(f,f) \qquad A \xrightarrow{f} C^0 \xrightarrow{\varphi_0} C_0 \xrightarrow{f^*} A$

So $V_0(\varphi)(f+g) = \operatorname{adj}(\varphi_0)(f+g, f+g)$

$= \operatorname{adj}(\varphi_0)(f,f) + \operatorname{adj}(\varphi_0)(g,g) + \operatorname{adj}(\varphi_0)(f,g) + \underbrace{\operatorname{adj}(\varphi_0)(g,f)}_{= (-1)^r \operatorname{adj}(\varphi_0)(f,g)}$

If (C^0, φ_0) is a nonsingular symmetric form over $\mathbb{Z}$

(ie $\varphi_0 : C^0 \to C_0$ isom) then $V_0(\varphi_0) : C^0 \to \mathbb{Z}$; $f \mapsto f^*\varphi_0 f$

is s.t. signature $(C^0, \varphi) \equiv V_0(\varphi)(f)$ mod 8

for any $f \in C^0$ with $\varphi_0(f)(g) = \varphi_0(g)(g)$ mod 3 (mod 2 probably)

Defn. $f\in C^0$ is characteristic if $\varphi_0(f)(g) \equiv \varphi_0(g)(g)$ mod 2 $\forall g\in C^0$

Connection between Wu class fn $V_r : Q^n(C) \longrightarrow \underset{\text{quadratic functions}}{QFun}(H^{n-r}(C); Q^n(S^{n-r}A))$

and the Steenrod squares $Sq^i : H^j(X;\mathbb{Z}_2) \longrightarrow H^{i+j}(X;\mathbb{Z}_2)$

$\downarrow$

For $\mathbb{Z}_2$-coeffs ;

$H_n(X) \xrightarrow[\text{symmetric construction}]{\varphi_X} Q^n(C(X)) \xrightarrow{V_r} \overbrace{QFun(H^{n-r}(X), Q^n(S^{n-r}A))}^{= \operatorname{Hom}(H^{n-r}(X),\mathbb{Z}_2)}$

$x \longmapsto (y \mapsto \langle Sq^r(y), x\rangle)$

and ~~$Sq^r(y)$~~ so $\langle Sq^r(y), x\rangle = v_r(\varphi_X(x))(y) \in \mathbb{Z}_2$

$H^{n-r}(X;\mathbb{Z}_2) \xrightarrow{Sq^r} H^n(X;\mathbb{Z}_2) \xrightarrow{\langle\ ,x\rangle} \mathbb{Z}_2$

$(\varphi_X)_s(x) : C^r(X;\mathbb{Z}_2) \longrightarrow C_{n-r+s}(X;\mathbb{Z}_2)$

a chain map

$\varphi_X(x)_0 = x \cap .. : C(X;\mathbb{Z}_2)^{n-*} \longrightarrow C(X;\mathbb{Z}_2)$

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, "y"] \arrow[rrr, bend right] & C^{n-r}(X;\mathbb{Z}_2) \arrow[r, "\varphi_X(x)_{n-2r}"] & C_{n-r}(X;\mathbb{Z}_2) \arrow[r, "y^*"] & \mathbb{Z}_2 \end{tikzcd}$$

$\langle Sq^i(y), x\rangle \in \mathbb{Z}_2$

$n = 2r, \quad \langle Sq^r(y), x\rangle = \langle y \cup y, x\rangle = \langle y, x \cap y\rangle \in \mathbb{Z}_2$

$$\begin{tikzcd} C(X)^0 \arrow[r, "\varphi_X(x)_0 = x\cap .."] \arrow[d] & C(X)_n \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ C(X)^n \arrow[uur, "\varphi(x)_n"] & C(X)_0 \end{tikzcd}$$

$\mathbb{Z} \xrightarrow{x} C(X)_n \xrightarrow{(\varphi_X)_s} (C(X)\otimes C(X))_{n+s}$

chain homotopy

$\varphi_X(x)_1 : \varphi_X(x)_0 \simeq T\varphi_X(x)_0$

$\varphi_X(x)_2 : \varphi_X(x)_1 \simeq T\varphi_X(x)_1$

higher chain homotopy

$\varphi_X(x)_n : C(X)^n \to C(X)_n, \quad \varphi_X(x)_n(y)(y) = \langle Sq^0(y), x\rangle = \langle y, x\rangle \in \mathbb{Z}_2$

Eg. For $r+i \le n$, $Sq^i : \underset{\mathbb{Z}_2}{H^r(\mathbb{R}P^n;\mathbb{Z}_2)} \to \underset{\mathbb{Z}_2}{H^{r+i}(\mathbb{R}P^n;\mathbb{Z}_2)}$ is id.

Def^n: The Thom space of an $\mathbb{R}^n$-bundle $\mathbb{R}^n \to E \xrightarrow{\eta} X$ (vec bundle) uses the sub-bundle $(D^n, S^{n-1}) \longrightarrow (E_{\le 1}, E_1) \longrightarrow X$

where $E_1 = \{$all vectors of length $1\}$

$E_{\le 1} = \{$ " " " " $\le 1\}$

Then the Thom space is $T(\eta) := E_{\le 1}/E_1$ ($\cong$ one pt compactification E^∞ if X cpct, Hausdorff)

The Thom class $U \in \tilde{H}^n(T(\eta))$ is s.t. for each $x \in X$, the inclusion

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, equal] \arrow[d] & \mathbb{R}^n \arrow[d] \\ \mathbb{R}^n \arrow[d, "\varepsilon"] & E \arrow[d, "\pi\eta"] \\ \{x\} \arrow[r, "x"] & X \end{tikzcd}$$

$U_\eta \in \tilde{H}^n(T(\eta)) \longrightarrow \tilde{H}^n(T(\varepsilon)) = \mathbb{Z} \ni 1$

$\downarrow z^*$, $\quad z : X \hookrightarrow T(\eta);\ x \mapsto$ zero vector over x

$e(\eta) \in H^n(X)$

Euler class

$T(\varepsilon) = D^n/S^{n-1}$

$= S^n$

Thm (Hopf) If $\eta = \tau_M$, $X = M^n$ then
$e(\tau_M) = \chi(M) \in H^n(M) = \mathbb{Z}$ $\chi(M)$ is Euler characteristic

Pf? $\tau_M = \nu_\Delta : M \hookrightarrow M\times M$, the normal bundle of diagonal

$$
\begin{array}{l}
[M] \in H_n(M) \xrightarrow{\Delta_* = (\varphi_n)_0} H_n(M\times M) \longrightarrow H_n(M\times M, M\times M\setminus\Delta_M) \\
\qquad\qquad\qquad = \sum H_r(M)\otimes H_{n-r}(M) = \cancel{\mathbb{Z}}\, H_n(T(\eta)) \\
\qquad\qquad\qquad = \sum \mathrm{Hom}(H_r(M), H_{n-r}(M)^*) \xrightarrow{\text{trace}} \cong H_0(M) = \mathbb{Z} \\
\qquad\qquad\qquad = \sum \mathrm{Hom}(H_r(M), H_r(M)) \ni 1 \\
\downarrow \\
H_n(M, M\setminus\{x\}) = H_n(S^n) \ni 1
\end{array}
$$

Note: $\chi(M) = e(\tau_M) = \sum(-1)^r \dim H_r(M)$

Thom isom $U\cup : H^*(X) \cong \tilde{H}^{*+n}(T(\eta))$
$U\cap : \tilde{H}_{*+n}(T(\eta)) \cong H_*(X)$

Lecture 4. 23/10/08

An elt $\varphi \in Q^n(C) = \cancel{\mathrm{Hom}_{\mathbb{Z}}} H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))) =: H_n(W^{\%}C)$
is an eqvce class of cycles $\varphi = \{\varphi_s : C^r \to C_{n-r+s} \mid s \geq 0\}$
$\in \ker(d : (W^{\%}C)_n \to (W^{\%}C)_{n-1})$

$d_{W\%C}\varphi = d_C\varphi_s + \varphi_s d_C^* + \varphi_{s-1} - \varphi_{s-1}^* = 0$, $\varphi_{-1} = 0$
with $\varphi \sim \varphi'$ if there exists a chain $\delta\varphi = \{\delta\varphi_s : C^r \to C_{n-r+s+1} \mid s\geq 0\}$
with $\varphi' - \varphi = d_{W\%C}(\delta\varphi)$
ie $\varphi_s' - \varphi_s = d_C(\delta\varphi_s) + \delta\varphi_s d_C^* + \delta\varphi_{s-1} - \delta\varphi_{s-1}^*$

In particular, $\varphi_0 : C^{n-*} \to C$ is a chain map (with $\varphi_1 : \varphi_0 \simeq \varphi_0^*$)
and $\delta\varphi_0 : \varphi_0 \simeq \varphi_0' : C^{n-*} \to C$ is a chain homotopy

$Q^n(C) \to H_n(\mathrm{Hom}_A(C^*, C))$ = abelian gp of chain htpy classes of A-mod chain maps $C^{n-*} \to C$
$\varphi \mapsto \varphi_0$

$$
\begin{tikzcd}
Q^n(C) \arrow[r] \arrow[rr, bend left] & H_n(\mathrm{Hom}_A(C^*, C)) \arrow[r, "1-T"] & H_n(\mathrm{Hom}_A(C^*, C))
\end{tikzcd}
$$

Composite is: $\varphi_0 \longmapsto \varphi_0 - T\varphi_0 = \varphi_0 - \varphi_0^* = 0$

Note: If $\frac{1}{2} \in A$ then every $\varphi \sim \varphi'$ with $\varphi_0' = \frac{1}{2}(\varphi_0 + \varphi_0^*)$, $\varphi_s' = 0 \ \forall s \geq$
and then $Q^n(C) = \ker\left(1-T : H_n(\mathrm{Hom}_A(C^*, C)) \longrightarrow H_n(\mathrm{Hom}_A(C^*, C))\right)$

For a space X with univ cover $\tilde{X}$, can use $\Delta : X = \tilde{X}/\pi_1 X \longrightarrow \tilde{X} \times_{\pi_1 X} \tilde{X}$
$x = [\tilde{x}] \longmapsto [\tilde{x}, \tilde{x}]$

To define cap products $H_n(X) \otimes H^r(\tilde{X}) \longrightarrow H_{n-r}(\tilde{X})$

$\xrightarrow{\Delta} H_n(\tilde{X} \times_{\pi_1 X} \tilde{X}) \otimes H^r(\tilde{X})$

where $H^r(\tilde{X}) = H^r(\mathrm{Hom}_{\mathbb{Z}[\pi_1 X]}(C(\tilde{X}), \mathbb{Z}[\pi_1 X])) = H^r(C(\tilde{X})^*)$
(ordinary cohomology of $\tilde{X}$ if $\pi_1 X$ is finite)

$$\left(C(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} C(\tilde{X})\right) \otimes_{\mathbb{Z}} C(\tilde{X})^* \longrightarrow C(\tilde{X})$$
$$(x \otimes y) \otimes z \longmapsto \overline{z(x)}\, y$$

Poincaré Duality Thm. If X is an oriented, conn n-mfd with univ cover $\tilde{X}$
then cap product with $[X] \in H_n(X)$, $[X] \cap : C(\tilde{X})^{n-*} \to C(\tilde{X})$ is a
$\mathbb{Z}[\pi_1 X]$-mod chain eqvce inducing $\mathbb{Z}[\pi_1 X]$-isom $H^r(\tilde{X}) \cong H_{n-r}(\tilde{X})$

$$H_n(X) \xrightarrow{\varphi_X} Q^n(C(\tilde{X})) \longrightarrow H_n(\mathrm{Hom}(C(\tilde{X})^*, C(\tilde{X})^*))$$
$$[X] \longmapsto \varphi_X[X]_0 = [X] \cap$$

Defn. An n-dim^l symm cx (C, φ) over A is an n-dim f.g. free
A-mod chain cx C with $\varphi \in Q^n(C)$.
The cx is Poincaré if $\varphi_0 : C^{n-*} \to C$ is a chain eqvce (iff $(\varphi_0)_* : H^*(C) \cong H_{n-*}(C$

Example If X is an n-mfd, $(C(\tilde{X}), \varphi = \varphi_X([X]))$ is an n-dim sym P cx over $\mathbb{Z}[\pi_1 X]$
If $f : X \to Y$ is a htpy eqvce of ~~spaces~~ mfds then $\tilde{f} : C(\tilde{X}) \to C(\tilde{Y})$ is a chain
eqvce with $f^{\%} \varphi_X[X] = \varphi_Y[Y]$

Ran 9

$\{\varphi_s\} \longmapsto \{f\varphi_s f^*\}$

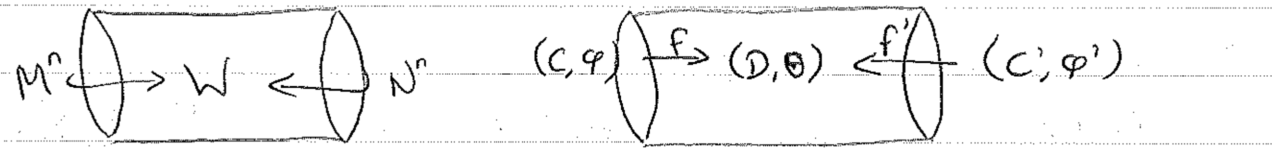
Any chain map $f: C \to D$ induces $f^{\%}: Q^n(C) \to Q^n(D)$

f chain eqvce $\Rightarrow$ $f^{\%}$ isom

A homotopy eqvce $f:(C,\varphi) \to (D,\theta)$ is a chain eqvce f st $f^{\%}(\varphi) = \theta$

An $f: X \to Y$ induces $\tilde{f}: (C(\tilde{X}), \varphi_X[X]) \to (C(\tilde{Y}), \varphi_Y[Y])$.

Cobordism of mfds — Cobordism of sym. P. cxs (C,φ) & (C',φ')



$\partial W = M \sqcup N$ — f, f' are chain maps

$[W] \cap : C(\tilde{W}, \tilde{M})^{n+1-*} \to C(\tilde{W}, \tilde{N})$ — $\varphi \in (W^{\%}C)_n$, $\varphi' \in (W^{\%}C')_n$

Poincaré - Lefschetz duality — $\theta \in (W^{\%}D)_{n+1}$

$f^{\%}(\varphi) - f'^{\%}(\varphi') = d\theta$

$\mathcal{C}(f)$ = alg mapping cone

$\mathcal{C}(f)_r = D_r \otimes C_{r-1}$, $d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-1)^r f \\ 0 & d_C \end{pmatrix}$

$$\begin{array}{ccccc} C^{n-*} & \xleftarrow{f^*} & D^{n-*} & \xrightarrow{f'^*} & C'^{n-*} \\ \varphi_0 \downarrow & & \downarrow\downarrow \theta_0 & & \downarrow \varphi_0' \\ C & \xrightarrow{f} & D & \xleftarrow{f'} & C' \end{array}$$

We want the chain map $\mathcal{C}(f)^{n+1-*} \xrightarrow{PL} \mathcal{C}(f')$ to a chain eqvce

$$\begin{array}{ccc} D^{n+1-r} \otimes C^{n-r} & \xrightarrow{\begin{pmatrix} \theta_0 & f\varphi_0 \\ \varphi_0' f'^* & 0 \end{pmatrix}} & D_r \otimes C'_{r-1} \\ \begin{pmatrix} d_D^* & 0 \\ f^* & d_C^* \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} d_D & f' \\ 0 & d_C \end{pmatrix} \\ D^{n+2-r} \otimes C^{n-r+1} & \longrightarrow & D_{r-1} \otimes C'_{r-2} \end{array}$$

$\theta_0 : f\varphi_0 f^* \simeq f'\varphi_0' f'^* : D^{n-*} \to D$ ie $f\varphi_0 f^* - f'\varphi_0' f'^* = d\theta_0 + \theta_0 d^*$

$PL = \begin{pmatrix} \theta_0 & f\varphi_0 \\ \varphi_0' f'^* & 0 \end{pmatrix}$; $\begin{pmatrix} d & f' \\ 0 & d \end{pmatrix} \begin{pmatrix} \theta_0 & f\varphi_0 \\ \varphi_0' f'^* & 0 \end{pmatrix} = \begin{pmatrix} \theta_0 & f\varphi_0 \\ \varphi_0' f'^* & 0 \end{pmatrix} \begin{pmatrix} d^* & 0 \\ f^* & d^* \end{pmatrix}$

Defn. $L^n(A)$ = ab gp of cobordism classes of n-dim sym P cxs (C,φ) over A with addition

$$(C,\varphi)+(C',\varphi') = (C\oplus C', \varphi\oplus\varphi')$$
$$(C,\varphi)+(C,-\varphi) = \cancel{(C\oplus C}\ 0 \in L^n(A)$$

Ex. $M \hookrightarrow \boxed{M\times I} \hookleftarrow M$ $\qquad (C,\varphi) \hookrightarrow \boxed{(C,0)} \hookleftarrow (C,-\varphi)$

Symmetric Signature map

$$\sigma^* : \Omega_n(X) = \{\text{bordism of } M^n \text{ with } f: M\to X\} \longrightarrow L^n(\mathbb{Z}[\pi_1 X])$$
$$(M,f) \longmapsto (C(\tilde{M}), \varphi_M[M])$$

Where do null-cobordant n-dim sym P cxs come from?

In general, as boundaries of $(n+1)$-dim sym cxs (D,θ)

$$\partial(D,\theta) = (C = \mathcal{C}(\theta_0 : D^{n+1-*} \to D)_{*+1}, \varphi)$$

where $\varphi_0 = \begin{pmatrix} \theta_1 & 1 \\ \pm 1 & 0 \end{pmatrix}$, $\varphi_s = \begin{pmatrix} \theta_{s+1} & 0 \\ 0 & 0 \end{pmatrix}$ $\forall s\geq 1$; $C_r = D_{r+1} \oplus D^{n+1-r}$

Note $d_D\theta_s + \theta_s d_D^* + \cancel{\oplus}\,\theta_{s-1} + \theta_{s-1}^* = 0$ $\qquad d_C = \begin{pmatrix} d_D & \theta_0 \\ 0 & d_D^* \end{pmatrix}$

$d_C\varphi_s + \varphi_s d_C^* + \varphi_{s-1} + \varphi_{s-1}^* = 0$

$(C,\varphi) \xrightarrow{f} (D,0)$ $\quad 0$ $\quad$, $f : \underbrace{D_{r+1}\oplus D^{n+1-r}}_{C_r} \longrightarrow D^{n+1-r}$

so $f\varphi_s f^* = 0$

A homotopy eqvce $f : (C,\varphi) \to (C',\varphi')$ determines a cobordism

$$C \xrightarrow{f} C' \xleftarrow{1} C'$$

Spse $C \xrightarrow{f} D \xleftarrow{f'} C'$ is an alg h-cobordism then

$f'^{-1}f : (C,\varphi) \to (C',\varphi')$ is a homotopy eqvce

Ran 10

~~What~~ Why is $L^0(A) \cong$ Witt gp of non-singular sym forms over A ? $\cdot$ isom : non-sing

A non-singular sym form over A $(P = \text{f.g. free } A\text{-mod}, \varphi = \varphi^* : P \to P^*)$

$\mathrm{adj}(\varphi) : P \times P \to A$

$\mathrm{adj}(\varphi)(x,y) = \overline{\mathrm{adj}(\varphi)(y,x)}$

Example. $(P,\varphi) = \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}\right)$ $= H(L,\lambda)$ hyperbolic forms, for any sym form (L^*, λ)

$\mathrm{adj}(\varphi) : L \oplus L^* \times L \oplus L^* \to A$

$((x,y),(x',y')) \mapsto y'(x) + \overline{y(x')} + \mathrm{adj}(\lambda)(y,y')$

Witt gp = ab gp of isom classes of non-sing sym forms over A

$(P,\varphi) \sim (P',\varphi')$ iff $(P,\varphi) \oplus H(L,\lambda) \cong (P',\varphi') \oplus H(L',\lambda')$ (isom)

for some $(L,\lambda), (L',\lambda')$

An elt $(C,\varphi) \in L^0(A)$ has C; $\cdots \to 0 \to C_0 \to 0 \to \cdots$

$\varphi \in Q^0(C) = \ker(1 - T : \mathrm{Hom}_A(C^0, C_0) \circlearrowleft)$ $\qquad C_0^* = (C^0)^*$

(C,φ) Poincaré $\Leftrightarrow \varphi_0 : C^0 \to C_0$ is an isom $\quad (\varphi_0 = \varphi_0^* : C^0 \to C_0)$

If $\left(C^0 = L \oplus L^*, \varphi_0 = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}\right)$ then

$C_0 \xrightarrow{f = (1,0)} D_0 \longleftarrow 0$

$\mathrm{Witt}(A) \xrightarrow{1} L^0(A)$

$(P,\varphi) \mapsto (C_0 = P^*, \varphi)$

Pf that if (C,φ) is null-cobordant then the form (C^0, φ_0) is isom to $\left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}\right)$

NULL COBORDANT $\quad C \xrightarrow{f} D \xleftarrow{f'} 0$

$$\begin{array}{ccc} & & D \\ & & \downarrow \\ C_0 & \xrightarrow{f} & D_0 \end{array} \qquad \begin{array}{ccc} D^0 & & \\ \downarrow f^* & & \\ C^0 & \xrightarrow{f\varphi_0} & D_0 \end{array}$$

$$D^0 \xrightarrow{f^*} C^0 \xrightarrow{\varphi_0} C_0 \xrightarrow{f} D_0$$

$= 0$

Then $\quad 0 \to D^0 \xrightarrow{f^*} C^0 \xrightarrow{f\varphi_0} D_0 \to 0 \quad$ is exact

Lemma. A non-singular sym form (P, φ) is isom to $\left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}\right)$ iff $\exists$ a direct summand $L \subset P$ with $L^\perp = L$ (say L is Lagrangian) where $L^\perp = \{x \in P \mid \varphi(x,y) = 0 \ \forall y \in L\}$

If A is a field, can define $L^{4k}(A) \xrightarrow{\text{an isom}} L^0(A)$; $(\overset{\text{f.g. free}}{C}, \varphi) \mapsto (\overset{\text{f.g. free}}{H_{2k}(C)}, \varphi_0)$

Eg. $L^0(\mathbb{R}) \cong$ Witt gp of $\mathbb{R} \xrightarrow[\cong]{\text{signature}} \mathbb{Z}$

$(P, \varphi) \mapsto$ signature $(P, \varphi) =$ no. of +ve eigenvals − no. of −ve evals of φ

$\Omega_{4k}(pt) =$ oriented cobordism of 4-mfds $\longrightarrow L^{4k}(\mathbb{Z}) \longrightarrow L^{4k}(\mathbb{R}) \cong \mathbb{Z} \ni \text{sign}(M)$

$M \longmapsto (C(M), \varphi) \mapsto (H_{2k}(M; \mathbb{R}), \varphi_0) =$ intersection for

Eg. $M = S^2 \times S^2 = \partial(D^3 \times S^2)$ $\quad \left(H_2(M;\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}, \ \varphi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

$$\downarrow f = (0,1)$$

$$H_2(D^3 \times S^2; \mathbb{R}) = \mathbb{R}$$

Thm. $L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (4) \ \ (n \equiv 0 \bmod 4) & \text{signature} \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \ (4) & \text{de Rham invariant} \\ 0 & \text{o/w} \end{cases}$

$L^n(\mathbb{F}_2) = \begin{cases} \mathbb{Z}_2 & n \equiv 0 \\ 0 & n \equiv 1 \\ \mathbb{Z}_2 & n \equiv 2 \\ 0 & n \equiv 3 \end{cases}$ mod 4

Quick Sketch

Signature : $L^{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$

$(C, \varphi) \mapsto (H_{2k}(C; \mathbb{R}), \varphi)$ signature

de Rham : $L^{4k+1} \ldots$

A morphism of rings with involution $f : A \to B$ induces covariantly

$f_* : L^n(A) \to L^n(B)$; $(C, \varphi) \mapsto B \otimes_A (C, \varphi)$

using $B \times B \times A \to B$; $(b, c, a) \mapsto b c f(a)$

Ran II

$$\sigma^* : \Omega_n(X \times S^1) \longrightarrow L^n(\mathbb{Z}\pi_1 X[\mathbb{Z}]) = L^n(\mathbb{Z}[\pi_1 X]) \oplus L^{n-1}(\mathbb{Z}[\pi_1 X])$$

$$(M^n \to X \times S^1) \longmapsto (C(\tilde{M}), \varphi)$$

$$\Omega_1(S^1) = \Omega_1(pt) \oplus \Omega_0(pt) \longrightarrow L^1(\mathbb{Z}[\mathbb{Z}]) = L^1\mathbb{Z} \oplus L^0\mathbb{Z}$$

$$S^1 \to S^1 \qquad = 0 \oplus \mathbb{Z} \qquad\qquad = L^0\mathbb{Z} = \mathbb{Z}$$

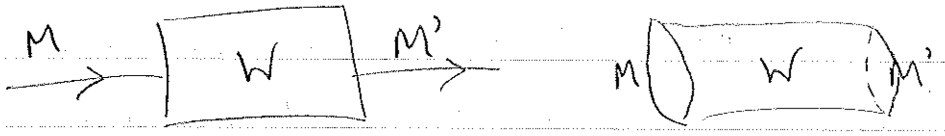
$$pt \mapsto pt$$

Lecture 5 28/10/08 28/10/08

Surgery Recollection

A surgery on an n-mfd M^n has input an embedding $S^r \times D^{n-r} \hookrightarrow M$ and output the n-mfd

$$M' = \underbrace{\overline{(M \setminus S^r \times D^{n-r})}}_{=: M_0} \cup (D^{r+1} \times S^{n-r-1}) \qquad \text{an } r\text{-surgery}$$

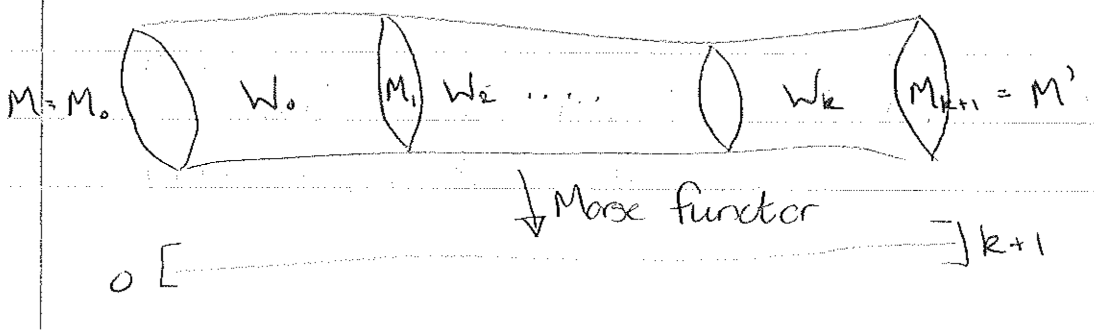


$(C(M), \varphi_M) \to \boxed{C(W)} \to (C(M'), \varphi_{M'})$

The trace of the surgery is the $(n+1)$-dim^l cobordism $(W; M, M')$ with $W = M \times I \cup_{S^r \times D^{n-r} \times \{1\} \subseteq M \times I} \underbrace{D^{r+1} \times D^{n-r}}_{\text{an } (r+1)\text{-handle}}$

The cobordism is elemtary, ie Morse functor $(W; M, M') \longrightarrow I$ with one critical point of index $r+1$.

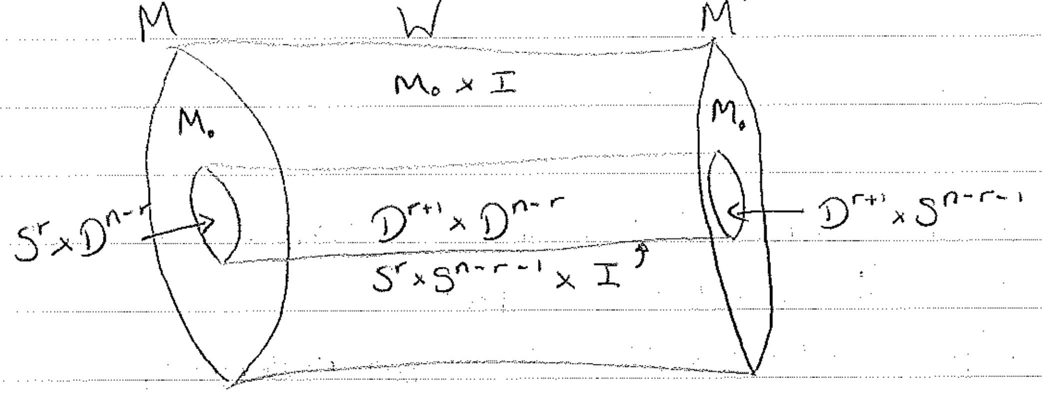
Thm. (Thom, Milnor, 1960) Every cobordism $(W; M, M')$ can be expressed (non-uniquely) as a union of elementary cobords $(W_i; M_i, M_{i+1})$ $(0 \le i \le k)$



st W_i is the trace of a surgery on $S^{r_i} \times D^{n-r_i} \subseteq M_i$ with $0 \le r_1 \le r_2 \le \dots \le r_k \le n+1$

The Effect of Surgery on Homotopy $\qquad W \simeq M \cup_{S^r} D^{r+1} \simeq M' \cup_{S^{n-r-1}} D^{n-r}$

~~Take $W \simeq M \cup_{S^r} D^{r+1}$~~. The trace W is obtained (upto homotopy) by attaching an $(r+1)$-cell. Note that $-W$ is the trace of surgery on $D^{r+1} \times S^{n-r-1} \subset M'$.



M' is obtained from M by first attaching an $(r+1)$-cell (to obtain W) and then detaching the complementary $(n-r)$-cell (from W)

SES of relative homotopy groups

$$\longrightarrow \pi_i(M) \longrightarrow \pi_i(W) \longrightarrow \pi_i(W, M) \longrightarrow \pi_{i-1}(M) \longrightarrow$$

$$\pi_i(W,M) = \begin{cases} 0 & \text{if } i \le r \\ \mathbb{Z}[\pi_1 M] & \text{if } i = r+1, \text{ if } M, W \text{ conn and } \pi_1 M \cong \pi_1 W \\ ? & \text{if } i \ge r+2 \end{cases}$$

$$\longrightarrow \pi_j(M') \longrightarrow \pi_j(W) \longrightarrow \pi_j(W, M') \longrightarrow \pi_{j-1}(M) \longrightarrow$$

$$\pi_j(W,M') = \begin{cases} 0 & \text{if } j \le n-r-2 \\ \mathbb{Z}[\pi_1 M'] & \text{if } j = n-r-1 \text{ if } M', W \text{ conn}, \pi_1 M' \cong \pi_1 W \\ ? & \text{if } j \ge n-r \end{cases}$$

We get $\pi_i(M) \cong \pi_i(W)$ for $i \le r-1$ and $\pi_r(M) = \pi_r(M) / \langle S^r \subset M \rangle$

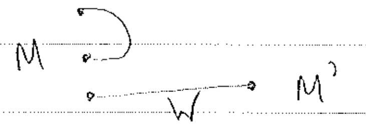
The effect of the surgery is to <u>kill</u> $(S^r \subset M) \in \pi_r(M)$

And $\pi_j(M') = \pi_j(W)$ if $j \le n-r-2$

$$\pi_{n-r}(W) = \frac{\pi_{n-r-1}(M')}{\langle S^{n-r-1} \subset M' \rangle}$$

In particular, if $2r \le n$, get $\pi_1(M) \cong \pi_1(W) \cong \pi_1(M')$, then $\pi_j(M') = \begin{cases} \pi_j(M) & \text{for } j<r \\ \pi_r(M)/\langle S^r \subset M \rangle & \text{for } j=r \end{cases}$

Example. A 0-surgery on 0-mfds.

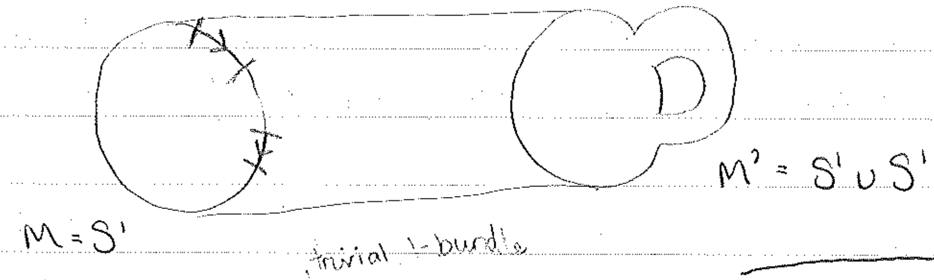


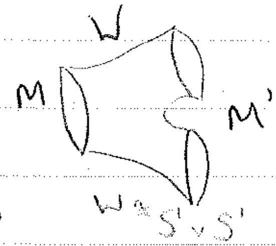
$S^0 \times D^0 \subset M^0$

Example. A 0-surgery on S^1.

Have 2 cases for $S^0 \times D^1 \subset S^1$

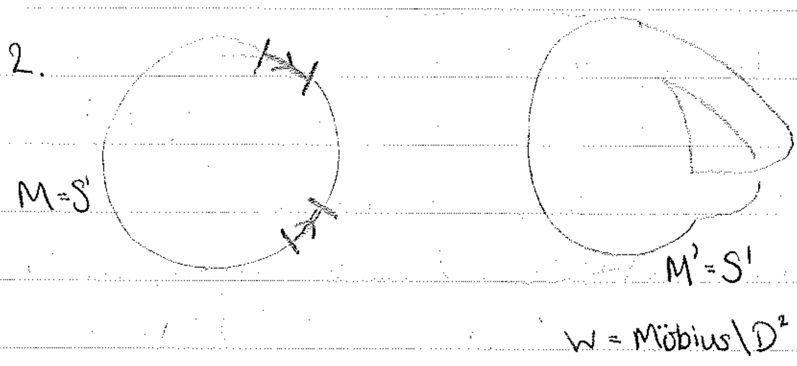
Case 1.

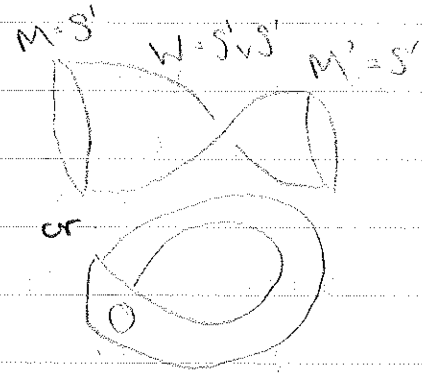




In this case, $\nu_{S^0 \subset S^1} \cong \underset{\text{trivial 1-bundle}}{\varepsilon}$, $\quad W = \overline{S^2 \setminus D^2 \cup D^2 \cup D^2}$

Case 2.





$Q^{n+1}(f,f')$ is defined to be to fit the LES. It is in fact mapping cone of other map in sequence. Sim for $Q^n(ff')$

~~$\to Q^{n}(f,f') \to Q^n(C)\oplus Q^n(C')$~~

Algebraic surgery

$$(C,\varphi) \xrightarrow{f} D \xleftarrow{f'} (C',\varphi')$$

$$\to Q^{n+1}(f,f') \longrightarrow Q^n(C) \oplus Q^n(C') \xrightarrow{(f^{\%}\ f'^{\%})} Q^n(D) \to Q^n(f,f') \to$$

$(\varphi, -\varphi')$

this is def^n of $Q^{n+1}(f,f')$

$$\to Q^{n+1}(f,f') \longrightarrow Q^n(C\oplus C') \xrightarrow{(ff')^{\%}} Q^n(D) \qquad (f,f') : C\oplus C' \to D$$

$$= Q^n(C)\oplus Q^n(C')\oplus H_n(C\otimes C')$$

Look at diagram

$$\begin{array}{ccccc} [M] \in H_n(M) & \xrightarrow{f_*} & H_n(W) & \xleftarrow{f'_*} & H_n(M') \ni [M'] \\ \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\ \Delta(M) = \varphi_M \in Q^n(C(M)) & \xrightarrow{f^{\%}} & Q^n(C(W)) & \xleftarrow{f'^{\%}} & Q^n(C(M')) \ni \varphi_{M'} \end{array}$$

$$f^{\%}\varphi_M = f'^{\%}\varphi_{M'}$$

Now spse given an $(n+1)$-dim^l sym Poincaré cobordism

$$((f,f') : C\oplus C' \longrightarrow D,\ (\delta\varphi,\ \varphi\oplus -\varphi')) \in Q^{n+1}(f,f')$$

$$(\delta\varphi, \varphi\oplus-\varphi') \in Q^n(C)\oplus Q^n(C') \mapsto 0 \in Q^n(D)$$

Thm. Can recover the chain homotopy type of (C',φ') from the $(n+1)$-dim^l symm (non-Poincaré) pair $(g : C \to \mathcal{C}(f'), (\theta,\varphi))$

$$(C,\varphi) \xrightarrow{f} (D,\delta\varphi) \xleftarrow{f'} (C',\varphi') \qquad \text{"}\ \theta = \delta\varphi/\varphi'\ \text{"}$$

Example. What about the symmetric Poincaré cobordism

$$((f\,f') : C(M)\oplus C(M') \longrightarrow C(W),\ (\delta\varphi_W,\ \varphi_M\oplus -\varphi_{M'}))$$

$$= ((ff') : C\oplus C' \to D,\ (\delta\varphi, \varphi\oplus-\varphi))$$

$C(W) \simeq \mathcal{C}(S^r\mathbb{Z} \xrightarrow{\varphi_0 g^*} C(M))$

$W \simeq M\cup D^{r+1} \simeq M' \cup$

associated to the trace $(W; M, M')$ of a surgery on $S^r\times D^{n-r} \subset M^n$

$$g : C(M) \longrightarrow \mathcal{C}(f') = C(W,M') \simeq S^{n-r}\mathbb{Z}([\pi,M])$$

$g^* \in H^{n-r}(M) \underset{\text{Poincaré duality}}{\cong} H_r(M)$ is the homology class of ~~$S^r\times D^{n-r}$~~ $S^r\times\{0\} \subseteq M$

$\varphi_0 g^*$

Note $C(W,M') = C(W/M')$

Note: $[M] \mapsto 0 \in H_n(W, M')$ $\quad \partial W = M \sqcup M'$

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$$\begin{array}{ccc} [M] \in H_n(M) & \xrightarrow{g} & H_n(W, M') \\ \Delta \downarrow & & \downarrow \Delta \\ \varphi_M \in Q^n(C(M)) & \xrightarrow{g^{\%}} & Q^n(C(W, M')) \end{array}$$

$$g^{\%}\varphi_M = g^{\%}\Delta(M) = \partial\theta \in (W^{\%}(S^{n-r}\mathbb{Z}))_n$$

where $\theta \in (W^{\%}(S^{n-r}\mathbb{Z}))_{n+1}$ depends on $S^r \times D^{n-r} \subseteq M$

Defn. Given an $(n+1)$-dim^l symmetric (in general non-Poincaré) pair $(g : C \to E, (\theta, \varphi))$

$$\to Q^{n+1}(g) \to Q^n(C) \xrightarrow{g^{\%}} Q^n(E) \to Q^n(g) \to$$
$$(\theta, \varphi) \mapsto \varphi \mapsto g^{\%}(\varphi) = \partial\theta \text{ on chain level}$$

define the effect of the algebraic surgery to be (C', φ') with $C'_r = C_r \oplus E_{r+1} \oplus E^{n-r+1}$ and

$$C'_r \longrightarrow C'_{r-1} \quad \text{by} \quad d_{C'} = \begin{pmatrix} d_C & 0 & \varphi_0 g^* \\ g & d_E & \theta_0 \\ 0 & 0 & d_E^* \end{pmatrix}$$

$\varphi_0 : C^{n-*} \to C$ chain map, $d\varphi_0 = \varphi_0 d^*$, $g\varphi_0 g^* = d_C\theta_0 + \theta_0 d_C^*$

ie $\theta_0 : g\varphi_0 g^* \simeq 0 : E^{n-*} \to E$

~~Sym. Poincaré cobordism~~

assumed Poincaré

$$(C, \varphi) \xrightarrow{h} \mathcal{C}(\varphi_0 g^* : E^{n-*} \to C) \xleftarrow{h'} (C', \varphi')$$
$$= F = C'/E_{*+1}$$

$g^{\%}\varphi = \partial\theta$

$F = \mathcal{C}(E_{*+1} \hookrightarrow C')$?

$$C' = \mathcal{C}\left(\begin{pmatrix} \varphi_0 g^* \\ \theta_0 \end{pmatrix} : E^{n+1-*} \longrightarrow \mathcal{C}(g : C \to E)_{*+1}\right)$$

Why is $\left(C \overset{\oplus}{*} C' \xrightarrow{(h\ h')} F, (0, \varphi \oplus -\varphi')\right)$ a sym Poincaré cobord?
Because $\mathcal{C}(h) \simeq E^{n+1-*} \simeq \mathcal{C}(h')^{n+1-*}$

Example (cont'd) $(C(M), \varphi_M)$, $g: \overset{C}{\overset{\shortparallel}{C(M)}} \longrightarrow E = \mathcal{C}(W, M') = S^{n-r}\mathbb{Z}$
so $E_i = \begin{cases} 0 & \text{for } i \neq n-r \\ \mathbb{Z} & \text{for } i = n-r \end{cases}$

Set $2r < n$ and $C_i' = C_i \oplus E_{i+1} \oplus E^{n-i+1}$
Get chain complex

$$C': \longrightarrow \underset{C_{r+2}}{\overset{\shortparallel}{C'_{r+2}}} \longrightarrow \underset{C_{r+1} \oplus \mathbb{Z}}{\overset{\shortparallel}{C'_{r+1}}} \xrightarrow{d\,\varphi_0 g^*} \underset{C_r}{\overset{\shortparallel}{C'_r}} \longrightarrow \underset{C_{r-1}}{\overset{\shortparallel}{C'_{r-1}}} \longrightarrow$$

attached an $(r+1)$-cell

and

$$\longrightarrow \underset{C_{n-r+1}}{\overset{\shortparallel}{C'_{n-r+1}}} \xrightarrow{\begin{pmatrix} d_C \\ g \end{pmatrix}} \underset{C_{n-r-1} \oplus \mathbb{Z}}{\overset{\shortparallel}{C'_{n-r-1}}} \xrightarrow{d_{C'} = d_C\, 0} \underset{C_{n-r-2}}{\overset{\shortparallel}{C'_{n-r-2}}} \longrightarrow$$

detached an $(n-r+1)$-cell

check indices

Get $H_r(C') = H_r(C)/(\varphi_0 g^*)$

Alg. surgery machine: Input: an n-dim^l sym P. cx (C, φ) and $(n+1)$-dim^l sym pair $(g: C \to E, (\theta, \varphi))$

Eg. for any geometric cobordism, take $C = C(M) \longrightarrow E = \cancel{C(W)}\ C(W, M')$

Output: an $(n+1)$-dim^l sym P. cobord $(C \oplus C' \to F, (0, \varphi \oplus -\varphi'))$

Eg $\quad C(M) \longrightarrow C(W) \longleftarrow C(M')$

$E|_C = C(W, M \cup M') \simeq C(W)^{n-*+1}$ Poincaré - Lefschetz duality

$C(M') \simeq C(M')^{n-*} \longleftarrow C(W)^{n-*} \simeq (E|_C)_{*+1} \longleftarrow C(W, M')^{n-*} \simeq E^{n-*}$

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Lecture 6 Surgery Theory 29/10/08

General principle: homotopy equivalent symmetric Poincaré cx s, pairs, cobordisms, etc are regarded as isomorphic. In particular, if the underlying chain cx s are chain contractible, regarded as isom to 0.

All possible because of the chain homotopy invariance of the Q-gps $Q^*(C)$.

$$(C, \varphi \in Q^n(C)) \quad C \simeq 0 \Rightarrow Q^n(C) = 0 \Rightarrow (C, \varphi) = 0$$

Thm. For any ring with involution A, there is a natural one-one correspondence of htpy eqvce classes

$$\left\{ \begin{array}{l} (n+1)\text{-dim}^l \text{ sym Poincaré} \\ \text{cobord } ((f\ f') : C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi')) \\ \quad C(M) \oplus C(M') \to C(W) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} (n+1)\text{-dim}^l \text{ sym pairs} \\ (g : C \to E, (\theta, \varphi)), (C, \varphi) \text{ Poincaré} \\ \quad C(M) \to C(W/M') \end{array} \right\}$$

Terminology. (C', φ') is the effect of an algebraic surgery on (C, φ) with input $(g : C \to E, (\theta, \varphi))$ "killing" $\operatorname{im}(E^{n-*} \xrightarrow{g^*} C^{n-*} \xrightarrow{\varphi_0} C)$

Proof

$$(C, \varphi) \xrightarrow{f} (D, \delta\varphi) \xleftarrow{f'} (C', \varphi')$$

Recall; $f^{\%}(\varphi) - f'^{\%}(\varphi) = d(\delta\varphi) \in (W^{\%} D)_n$

$$f \varphi_s f^* - f' \varphi'_s f'^* = d\,\delta\varphi_s + \delta\varphi_s d^* + \delta\varphi_{s-1} + \delta\varphi^*_{s-1}$$

Given $((f\ f') : C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi'))$ define $(g : C \to E, (\theta, \varphi))$ by

$$g = \begin{pmatrix} f \\ 0 \end{pmatrix} : C_r \to E_r = D_r \oplus C'_{r-1}, \quad d_E = \begin{pmatrix} d_D & f' \\ 0 & d_{C'} \end{pmatrix}$$

$$\theta_s = \begin{pmatrix} \delta\varphi_s & 0 \\ \varphi'_s f'^* & \varphi'^*_{s-1} \end{pmatrix} : E^{n-r+s+1} = D^{n-r+s+1} \oplus C'^{n-r+s} \to E_r = D_r \oplus C'_{r-1}$$

$$g \varphi_s g^* = d_E \theta_s + \theta_s d_E^* + \theta^*_{s-1} + \theta_{s-1}$$

Conversely, given $(g : C \to E, (\theta, \varphi))$ define

$$d_D = \begin{pmatrix} d_C & \varphi_0 g^* \\ 0 & d_E^* \end{pmatrix} : D_r = C_r \oplus E^{n-r+1} \longrightarrow D_{r-1} = C_{r-1} \oplus E^{n-r+2}$$

$$f = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \longrightarrow D_r = C_r \oplus E^{n-r+1}$$

$$d_{C'} = \begin{pmatrix} d_C & 0 & \varphi_0 g^* \\ g & d_E & \theta_0 \\ 0 & 0 & d_E^* \end{pmatrix} : C'_r = C_r \oplus \cancel{E^{n-r+1}_{r+1}} \oplus \cancel{E^{n-r+1}_{r+1}} \longrightarrow C'_{r-1} = C_{r-1} \oplus E_r \oplus E^{n-r+2}$$

$$f' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C'_r = C_r \oplus E_{r+1} \oplus E^{n-r+1} \longrightarrow D_r = C_r \oplus E^{n-r+1}$$

$$\varphi'_0 = \begin{pmatrix} \varphi_0 & 0 & 0 \\ g\varphi_1^* & \theta_1^* & 1 \\ 0 & 1 & 0 \end{pmatrix} : C'^{n-r} = C^{n-r} \oplus E^{n-r+1} \oplus E_{r+1} \longrightarrow C'_r = C_r \oplus E_{r+1} \oplus E^{n-r+1}$$

$$\varphi'_s = \begin{pmatrix} \varphi_s & 0 & 0 \\ g\varphi_1^* & \theta_{s+1}^* & 0 \\ 0 & 0 & 0 \end{pmatrix} : C'^{n-r+s} = C^{n-r+s} \oplus E^{n-r+1+s} \oplus E_{r-s+1} \longrightarrow C'_r = C_r \oplus E_{r+1} \oplus E^{n-r+1}$$

Can check that $f' \varphi'_s f'^* = f \varphi_s f^*$

so $f' \varphi'_s f'^* - f \varphi_s f^* = d\,\delta\varphi_s + \delta\varphi_s d^* + \delta\varphi_{s-1} + \delta\varphi_{s-1}^*$
$= 0$ with $\delta\varphi_s = 0$

To recover the stable homotopy type of M' from $(W/M', M)$, look at failure of Poincaré - Lefschetz

$$\mathcal{C}\left(C(W/M', M)^{n+1-*} \longrightarrow C(W/M') \right)_* \overset{?}{\simeq} C(M')_{*-1}$$

Note $C(W/M', M) \simeq C(W)$

Corollary For any ring with inv A $\exists$ nat^l 1-1 corresp of htpy eqvce classes

$$\left\{ \begin{array}{c} (n+1)\text{-dim}^l \text{ sym Poincaré pairs} \\ (f' : C' \to D, (\delta\varphi, \varphi')) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (n+1)\text{-dim}^l \text{ sym} \\ \text{cx } (E, \theta) \end{array} \right\}$$

$$C(M') \to C(W) \qquad\qquad C(W/M')$$

Proof See proof of thm, but set $(C, \varphi) = 0$.

Geometrically, forget about M so W is mfd with $\partial W = M'$.

have $\mathcal{C}(C(W/M')^{n+1-*} \xrightarrow{[W]\cap} C(W/M'))_{*} \simeq C(M')$

Terminology: (C', φ') is the boundary $\partial(E, \theta)$ with $C' = \mathcal{C}(\theta_0 : E^{n+1-*} \to E)_{*+1}$

Corollary. An n-dim^l symmetric Poincaré cx (C', φ') is the bdy of an $(n+1)$-dim^l symm Poincaré pair $(f' : C' \to D, (\delta\varphi, \varphi'))$ iff (C', φ') is homotopy eqvt to the bdy $\partial(E, \theta)$ of an $(n+1)$-dim^l sym cx (E, θ)

$(\Leftrightarrow (C', \varphi') = 0 \in L^n(A))$

Example. (of algebraic surgery)

If $(g : C \to E, (\theta, \varphi))$ has $E_i = 0 \ \forall i \neq n-r$

$2r < n$

$$\begin{array}{c} C'_n = C_n \\ \downarrow \\ \vdots \\ \downarrow \\ C'_{n-r+1} = C_{n-r+1} \\ \downarrow \\ C'_{n-r} = C_{n-r} \\ \downarrow \begin{pmatrix} d \\ g \end{pmatrix} \\ C'_{n-r-1} = C_{n-r-1} \oplus E_{n-r} \\ \downarrow (d \ 0) \\ C'_{n-r-2} = C_{n-r-2} \\ \downarrow \\ \vdots \\ \downarrow \\ C'_{r+2} = C_{r+2} \\ \downarrow \begin{pmatrix} d \\ 0 \end{pmatrix} \\ C'_{r+1} = C_{r+1} \oplus E^{n-r} \\ \downarrow (d \ \varphi_0 g^*) \\ C'_r = C_r \\ \vdots \end{array}$$

$n = 2r$, $E_i = 0 \ \forall i \neq r$

$$\begin{array}{ccc} C & & C' \\ \vdots & & \vdots \\ \downarrow d & & C_{r+2} \\ & & \downarrow \\ C_{r+1} & & C_{r+1} \oplus E^r \\ \downarrow d & & \downarrow (d \ 0) \\ C_r \xrightarrow{g} E_r & & C_r \\ \downarrow d & & \downarrow \begin{pmatrix} d \\ g \end{pmatrix} \\ C_{r-1} & & C_{r-1} \oplus E_{r-1} \\ \downarrow & & \downarrow \\ \vdots & & C_{r-2} \\ & & \downarrow \\ & & \vdots \end{array}$$

$n = 2k, \ n-r = k+1$

$$\begin{array}{ccc} C'_{k+1} = C_{k+1} & & \\ \downarrow \binom{d}{g} & & \\ C'_k = C_k \oplus E_{k+1} \oplus E^{k+1} & \text{from} & \begin{matrix} C^{k-1} \xrightarrow{\varphi_0} C_{k-1} \longrightarrow E_{n-r} = E_{k+1} \\ \downarrow \qquad\quad \downarrow \\ C^k \xrightarrow{\varphi_0} C_k \\ \downarrow \qquad\quad \downarrow \\ C^{k+1} \xrightarrow{\varphi_0} C_{k-1} \end{matrix} \\ \downarrow (d \ \ 0 \ \ \varphi_0 g^*) & & \\ C'_{k-1} = C_{k-1} & & \end{array} \qquad \varphi_0' = \begin{pmatrix} \varphi_0 & 0 & 0 \\ g\varphi_0^* & \theta_1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$W = M \times I \cup D^{r+1} \times D^{n-r}$, $M' = (M \setminus S^r \times D^{n-r}) \cup D^{r+1} \times S^{n-r-1}$

$W \simeq M \cup D^{r+1} \simeq M' \cup D^{n-r}$

If $(C, \varphi) = (C(M), \varphi_M)$ and $(W; M, M')$ is the trace of a surgery on $S^r \times D^{n-r} \subset M$ then $(C', \varphi') = (C(M'), \varphi_{M'})$ is obtained from, up to hpy eqvce, (C, φ) by an algebraic surgery on $(g : C(M) \longrightarrow C(W, M'), \{(\theta, \varphi)\})$ note: $C(W, M') \simeq S^{n-r}\mathbb{Z}$

determined by $S^r \subset M$ $\qquad$ determined by $S^r \times D^{n-r} \subset M$

$g \in H^{n-r}(M) \cong H_r(M)$ and $S^r \subset M \in \pi_r(M)$

Proof

The sym construction applied to the trace $(W; M, M')$ gives an $(n+1)$-dim^l sym Poincaré ~~ex~~ cobordism $((f f') : C \oplus C' \longrightarrow D, (\delta\varphi, \varphi \oplus -\varphi))$

$= (C(M) \oplus C(M') \longrightarrow C(W), \Delta([W], [M] \oplus [M']))$

Now apply the thm to obtain an $(n+1)$-dim^l sym pair $(g : C \longrightarrow E, (\theta, \varphi))$ st $E = \mathcal{C}(f') = C(W, M')$ and st $(C(M'), \varphi_{M'})$ is the effect of the algebraic surgery. $\square$

Example. Spse $M = S^r \times S^{n-r} = S^r \times D^{n-r} \cup S^r \times D^{n-r}$ and consider surgery on $S^r \times D^{n-r}$

$M' = D^{r+1} \times S^{n-r-1} \cup S^r \times D^{n-r} = \partial(D^{r+1} \times D^{n-r}) = S^n$

$C_n(M) = \mathbb{Z}$ $\qquad$ $C(M')$: $\mathbb{Z}$ $\qquad \mathbb{Z}$

$2r < n$. $\quad C^r(M) \longrightarrow C_{n-r}(M) = \mathbb{Z} \longrightarrow E_{n-r} = \mathbb{Z}$ $\qquad 0$

$\longrightarrow C_r(M) = \mathbb{Z}$ $\qquad \mathbb{Z} \simeq 0$

$\longrightarrow C_0(M) = \mathbb{Z}$ $\qquad \downarrow \mathbb{Z}$

0

$\mathbb{Z} \downarrow \mathbb{Z}$

0

$\mathbb{Z}$

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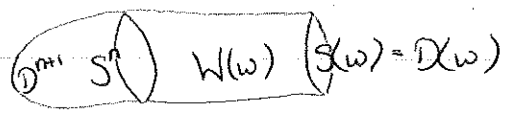
We can reverse the example; start with $S^n = \overset{D^{r+1}\times S^{n-r-1}}{D^{r+1}\times S^{n-r-1}} \cup S^r \times D^{n-r}$
to get $S^r \times S^{n-r}$.

Example. Given $\omega \in \pi_n(O(m-n)) = \pi_{n+1}(BO(m-n))$,
the effect of the n-surgery on

$$e_\omega : S^n \times D^{m-n} \hookrightarrow S^m = S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1}$$
$$(x,y) \longmapsto (x, \omega(x)(y))$$

is the $(m-n-1)$-sphere bundle

$$S^{m-n-1} \longrightarrow S(\omega) = D^{n+1}\times S^{m-n-1} \cup D^{n+1}\times S^{m-n-1} \longrightarrow S^{n+1} = D^{n+1} \cup_{S^n} D^{n+1}$$



$$(D^{m-n}, S^{m-n-1}) \longrightarrow (\overset{m-1}{D(\omega)}, \overset{m}{S(\omega)}) \to S^{n+1}$$

$(C(S(\omega)), \varphi_{S(\omega)})$ is obtained from $(C(S^m), \varphi_{S^m})$ by an alg. surgery on $(g : C(S^m) \longrightarrow S^{m-n}\mathbb{Z}, (\theta, \varphi_{S^m}))$

$$Q^m(C(S^m)) \xrightarrow[=0]{g^{\%}} Q^m(S^{m-n}\mathbb{Z})$$
$$\varphi \longmapsto 0$$

(assuming $0 < m < n$ so $g = 0 \in H^{m-n}(S^m) = 0$)

Have LES : $\longrightarrow \underset{\theta}{Q^{m+1}(S^{m-n}\mathbb{Z})} \longrightarrow Q^{m+1}(g) \longrightarrow Q^m(C(S^m)) \longrightarrow Q^m(S^{m-n}\mathbb{Z})$

$\pi_n(O(m-n))$

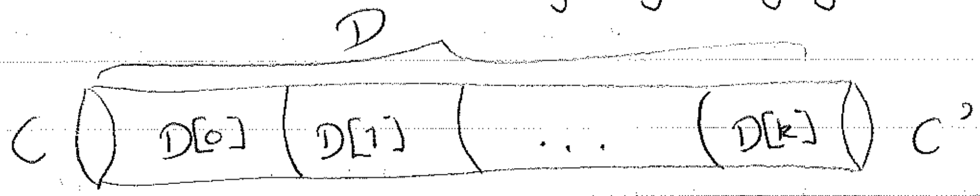
Question: what is θ?

Why isn't the symmetric signature map
$\sigma^* : \Omega_n(X) \longrightarrow L^n\mathbb{Z}[\pi_1 X]$ an isom? inj? surj? (OK for $\pi_1 X = \{1\}$)

Not surj $\because$ $\exists$ elts such as $\pi_1(X) = \mathbb{Z}_2$, $n = 4k$, $C_{2k} = \mathbb{Z}[\mathbb{Z}_2]$, $\varphi_0 = T$,
$C_r = 0$ for $r \neq 2k$, with $(C, \varphi) \notin \mathrm{im}(\sigma^*)$

If $(M, f: M \to X) \mapsto 0 \in L^n(\mathbb{Z}[\pi_1 X])$ the algebraic null-cobordism of $(C(\tilde{M}), \varphi_M)$ over $\mathbb{Z}[\pi_1 X]$ will not, in general, be realised geometrically. (eg failure of Hurewicz map to be an isom)

Algebraic analogue of the Thom - Milnor thm;
If $((f\,f'): C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi))$ is a sym Poincaré cobord and $\mathcal{C}(f') \simeq E$ = a f.g., finite, free chain cx then upto isom htpy eqvce each $D[i]$ = trace of an elementary alg surgery.



Lecture 7. 04/11/08

The mechanism for refining quadratic structures from symmetric structures.
"The algebra of sym. and quad. Poincaré cxs is the ideologically correct framework for the obstruction theory which decides if a geometric Poincaré cx is homotopy equivalent to a mfd. Especially the quadratic theory."

There are three types of Q-groups of a f.g. proj A-mod chain cxs over a ring with involution A.

Symmetric: $Q^n(C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^*, C))) = H_n(W^{\%}C)$
$= \{\varphi_s : C^r \to C_{n-r+s} \mid d\varphi_s + \varphi_s d^* + \varphi_{s-1} + \varphi_{s-1}^* = 0,\ s \geq 0,\ \varphi_{-1} = 0\}$

Quadratic: $Q_n(C) = H_n(W_{\%}C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C))$
$= \{\psi_s : C^r \to C_{n-r-s} \mid d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* = 0,\ s \geq 1\}$

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Hyperquadratic: $\hat{Q}^n(C) = H_n(\hat{W}^{\%}C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, \mathrm{Hom}_A(C^*, C)))$

comes from vector bundles via Wu classes

$$= \{\chi_s : C^r \to C_{n-r+s} \mid d\chi_s + \chi_s d^* + \chi_{s-1} + \chi_{s-1}^* = 0,\ s \in \mathbb{Z}\}$$

Where $\hat{W}$ is given by (set $\Lambda := \mathbb{Z}[\mathbb{Z}_2] = \{a + bT \mid a, b \in \mathbb{Z},\ T^2 = 1\}$)

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & & \\
 & W^{-*-1} : & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & W^0 = \Lambda \arrow[r, "1-T"] \arrow[d] & W^1 = \Lambda \arrow[r, "1+T"] \arrow[d] & {} \\
H_*(\hat{W}) = 0 & \hat{W} : \arrow[from=u, "H"'] \arrow[d, "J"'] & \hat{W}_2 = \Lambda \arrow[r, "1+T"] \arrow[d] & \hat{W}_1 = \Lambda \arrow[r, "1-T"] \arrow[d] & \hat{W}_0 = \Lambda \arrow[r, "1+T"] \arrow[d] & \hat{W}_{-1} = \Lambda \arrow[r, "1-T"] \arrow[d] & \hat{W}_{-2} = \Lambda \arrow[r] \arrow[d] & {} \\
 & W : \arrow[d] & W_2 = \Lambda \arrow[r, "1+T"] & W_1 = \Lambda \arrow[r, "1-T"] & W_0 = \Lambda \arrow[r, "1+T"] & 0 \arrow[r] & 0 \arrow[r] & {} \\
\text{SES of ch cxs} & 0 & & & & &
\end{tikzcd}$$

Thm. The 3 types of Q-groups are related by a ~~lon~~ LES

$$\xrightarrow{J} \hat{Q}^{n+1}(C) \xrightarrow{H} Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \hat{Q}^n(C) \xrightarrow{H}$$

with

$1+T : Q_n(C) \to Q^n(C)\ ;\ \{\psi_s\} \longmapsto \left\{ (1+T)\psi_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0 \\ 0 & s > 0 \end{cases} \right\}$

$J : Q^n(C) \to \hat{Q}^n(C);\ \{\varphi_s\} \longmapsto \left\{ J\varphi_s = \begin{cases} \varphi_s & \text{if } s \geq 0 \\ 0 & s < 0 \end{cases} \right\}$

$H : \hat{Q}^{n+1}(C) \to Q_n(C);\ \{\chi_s\} \longmapsto \{ H\chi_s = \chi_{-s-1} \mid s \geq 0 \}$

Proof

This is the LES induced by the SES of $\mathbb{Z}$-mod ch cxs

$$0 \longrightarrow (W^{\%}C)_{*+1} \xrightarrow{J} \hat{W}^{\%}C \xrightarrow{H} (W_{\%}C)_{*-1} \longrightarrow 0$$

Example. $Q^{2n}(S^nA) = \{a \in A \mid \bar{a} = (-1)^n a\}$

$\hat{Q}^{2n+1}(S^nA) = \{a \in A \mid \bar{a} = (-1)^{n+1} a\} / \{x + (-1)^{n+1}\bar{x} \mid x \in A\}$

$Q_{2n}(S^nA) = A / \{b - (-1)^n \bar{b} \mid b \in A\}$

$\hat{Q}^{2n}(S^nA) = \{a \in A \mid \bar{a} = (-1)^n a\} / \{x + (-1)^n \bar{x} \mid x \in A\}$

$1+T : Q_{2n}(S^nA) \to Q^{2n}(S^nA)\ ;\ x \longmapsto x + (-1)^n \bar{x}$

Get sequence

$$0 \longrightarrow \hat{Q}^{2n+1}(S^nA) \longrightarrow Q_{2n}(S^nA) \xrightarrow{1+T} Q^{2n}(S^nA) \longrightarrow \hat{Q}^{2n}(S^nA) \longrightarrow 0$$

Example. $A = \mathbb{Z}$, $\bar{a} = a$

$$\hat{Q}^{2n+1}(S^n\mathbb{Z}) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases} \qquad Q_{2n}(S^n\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases} \xrightarrow{1+T = 2} Q^{2n}(S^n\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ 0 \end{cases} \qquad \hat{Q}^{2n} = \begin{cases} \mathbb{Z}_{} \\ \end{cases}$$

Example. (Steenrod, 1952).

$$\pi_{n+1}(BO, BO(n)) = H_{n+1}(BO, BO(n)) = Q_{2n}(S^n\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases}$$

generated by $\tau_{S^n} : S^n \to BO(n)$ with $\delta\tau_{S^n} : \tau_{S^n} \oplus \varepsilon^k \cong \varepsilon^{n+k}$ determined by the canonical framed embedding $S^n \times \{0\} \subset S^n \times D^k \subset S^n \times D^k \cup D^{n+1} \times S^{k-1} = S^{n+k}$

ε is trivial line bundle

This classifies n-plane bundles $\omega : S^n \to BO(n)$ with stable trivialisation $\delta\omega : \omega \oplus \varepsilon^k \cong \varepsilon^{n+k}$ (k large)

This computation also features in Smale (1959)'s classification of immersions $S^n \looparrowright S^{2n}$ upto regular homotopy $= \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases}$

Remark. If $\frac{1}{2} \in A$, then $\hat{Q}^*(C) = 0$; $1+T : Q_*(C) \xrightarrow{\cong} Q^*(C)$

Defn. An n-dim^l quadratic cx (C, ψ) over A is an n-dim^l f.g. proj A-mod chain cx C with $\psi \in Q_n(C)$.

The cx is Poincaré if $(1+T)\psi_0 : C^{n-*} \to C$ is a chain eqvce

ie if $(C, (1+T)\psi)$ is a sym. Poincaré cx.

Corresponding defn of quadratic pair $(f : C \to D, (\delta\psi, \psi))$, $(\delta\psi,\psi) \in Q_{n+1}(f)$

with $\to Q_{n+1}(D) \to Q_{n+1}(f) \to Q_n(C) \xrightarrow{f^{\%}} Q_n(D) \to$

Pictorially

$(C,\psi) \xrightarrow{f} (D,\delta\psi)$

$$d_C\psi_s + \psi_s d_C^* + \psi_{s+1} + \psi_{s+1}^* = 0, \quad s \geq 0$$
$$f\psi_s f^* = d_D\delta\psi_s + \delta\psi_s d_C^* + \delta\psi_{s+1} + \delta\psi_{s+1}^*$$

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Poincaré if $(f: C \to D, \cancel{(\delta\psi, \psi)}\ ((1+T)\delta\psi, (1+T)\psi))$ is Poincaré

$((1+T)\delta\psi_0, f(1+T)\psi_0) : \mathcal{C}(f)^{n+1-*} \xrightarrow{\simeq} D$
chain eqvce

$H^{n+1-*}(f) \cong H_*(D)$, quadratic Poincaré ~~Lesf~~ Lefschetz

Can now define the quadratic Poincaré cobordism group

$$L_n(A) = \frac{\{(C,\psi)\ n\text{-dim}^l \text{ quad } P.\}}{\{(C,\psi) \sim (C',\psi') \text{ if } \exists \text{ quad } P. \text{ pair } (ff') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi')\}}$$

Thm. These are isomorphic to the families Wall surgery obstruction groups. In particular, there are 4-periodicity isoms

$$L_n(A) \xrightarrow{\cong} L_{n+4}(A) ; (C,\psi) \mapsto (S^2 C, \bar{S}^2 \psi)$$

$$(\bar{S}^2\psi)_s = \psi_s ; (S^2 C)^{n+4-r} = C^{n+2-r} \longrightarrow (S^2 C)_{r+s} = C_{r+s-2}$$

The general theory of algebraic surgery works just as well (if not better) in the quadratic case.

i) There is a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{htpy eqvce classes of } (n+1)\text{-dim}^l \\ \text{quad } P. \text{ pairs } (f: C \to D, (\delta\psi, \psi)) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{htpy eqvce classes of} \\ (n+1)\text{-dim}^l \text{ quad cxs } (C,\psi) \end{array} \right\}$$

$$(f: C \to D, (\delta\psi, \psi)) \longmapsto (\mathcal{C}(f), \delta\psi/\psi)$$

ii) Given quad $P.$ cx (C,ψ) and a pair $(f: C \to D, (\delta\psi,\psi))$ (not nec $P.$) we can construct cobordism

$$\underset{\psi}{C} \xrightarrow{g} \underset{0}{E} \xleftarrow{g'} \underset{\psi'}{C'}$$

$$C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \qquad d_{C'} = \begin{pmatrix} d_C & 0 & (1+T)\psi_0 f^* \\ f & d_D & (1+T)\delta\psi_0 \\ 0 & 0 & d_D^* \end{pmatrix}$$

$\varphi_s : C^r \longrightarrow C_{n-r+s}$

$f\varphi_s f^* \neq 0$

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d] & C_n \arrow[d] \\
C^1 \arrow[r] \arrow[d] & C_{n-1} \arrow[d] \\
C^2 \arrow[r] \arrow[d] & C_{n-2} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
C^{n-2} \arrow[r] \arrow[d] & C_2 \arrow[d] \\
C^{n-1} \arrow[r] \arrow[d] & C_1 \arrow[d] \\
C^n \arrow[r, "\varphi_0"'] \arrow[ur, "\varphi_1"] \arrow[uur, "\varphi_2"] \arrow[uuuuuur, bend left, "\varphi_n"] & C_0
\end{tikzcd}$$

$d\varphi_s + \varphi_s d^* + \varphi_{s-1} + \varphi_{s-1}^* = 0$

$\psi_s : C^r \longrightarrow C_{n-r-s}$

$f\psi_s f^* = 0$

$$\begin{tikzcd}
C^0 \arrow[r, "\psi_0"] \arrow[d, "d^*"'] \arrow[dr, "\psi_1"] \arrow[ddr] & C_n \arrow[r, "f=1"] \arrow[d, "d_C"] & D_n = C_n \arrow[d] \\
C^1 \arrow[r] \arrow[d] \arrow[dr] & C_{n-1} \arrow[d] & 0 \arrow[d] \\
C^2 \arrow[r] \arrow[d] \arrow[ddddr, bend left, "\psi_n"] & C_{n-2} \arrow[d] & \vdots \\
\vdots \arrow[d] & \vdots \arrow[d] & \\
C^{n-2} \arrow[r] \arrow[d] & C_2 \arrow[d] & \\
C^{n-1} \arrow[r] \arrow[d] & C_1 \arrow[d] & \arrow[d] \\
C^n \arrow[r] & C_0 & 0
\end{tikzcd}$$

$C'_n = C_n$

$\downarrow d' = \binom{d}{1}$

$C'_{n-1} = C_{n-1} \oplus D_n = C_{n-1} \oplus C_n$

$\downarrow d' = (d \;\; 0)$

$C'_{n-2} = C_{n-2}$

$\vdots$

$C'_1 = C_1 \oplus C^n$

$\downarrow d' = (d \;\; (1+T)\psi_0)$

$C'_0 = C_0$

$d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* = 0$

Example $L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \equiv 0 \bmod 4 \\ 0 & n \equiv 1 \bmod 4 \\ \mathbb{Z}_2 & n \equiv 2 \bmod 4 \\ 0 & n \equiv 3 \bmod 4 \end{cases}$ $\quad L_{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}$

$(C, \psi) \longmapsto \frac{1}{8}$ signature of even sym form $\left(H^{2k}(C)/_{\text{torsion}}, (1+T)\psi_0 \right)$

$L_{4k+2}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}_2$; $(C, \psi) \longmapsto$ Arf invariant of the quadratic form $\left(H^{2k}(C; \mathbb{Z}_2), (1+T)\psi_0, \psi_0 \right)$

And ~~But~~ $L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \equiv 0 \ (4) \\ \mathbb{Z}_2 & 1 \ (4) \\ 0 & 2 \ (4) \\ 0 & 3 \ (4) \end{cases}$ $\quad L_{4k}(\mathbb{Z}) = \mathbb{Z} \xrightarrow{\times 8} L^{4k}(\mathbb{Z}) = \mathbb{Z}$

$L_{4k+2}(\mathbb{Z}) = \mathbb{Z}_2 \longrightarrow L^{4k+2}(\mathbb{Z}) = 0$

The mechanism for refining sym to quad structure

$\hat{Q}^{n+1}(C) \to Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \hat{Q}^n(C) \longrightarrow$

$\varphi \longmapsto J\varphi = 0 \Leftrightarrow \exists \psi$ st $(1+T)\psi = \varphi$

Defn. The suspension morphism

$$S : Q^n(C) \longrightarrow Q^{n+1}(SC) \; ; \; \{\varphi_s\}_{s\geq 0} \longmapsto \{\varphi_{s-1}\}_{s \geq 0} \quad \varphi_{-1} = 0$$

$\varphi_s : C^r \to C_{n-r+s}$, $(S\varphi)_s = \varphi_{s-1} : (SC)^r = C^{r-1} \to (SC)_{n+1-r+s} = C_{n+s-r}$

$Q^1(S\mathbb{Z}) \otimes Q^n(C) \longrightarrow Q^{n+1}(\underbrace{S\mathbb{Z} \otimes_{\mathbb{Z}} C}_{= SC})$

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$\{\psi_s\} \longmapsto \varphi = \left\{(1+T)\psi_s = \begin{cases}(1+T)\psi_0 & s=0\\ 0 & s>0\end{cases}\right\}$

$$\begin{tikzcd} Q_n(C) \arrow[r, "1+T"] \arrow[dr, bend right] & Q^n(C) \arrow[r, "J"] \arrow[d, "S"] & \hat{Q}^n(C) \\ & Q^{n+1}(SC) & \end{tikzcd}$$

$Q^{n+1}(SC) \ni (S\varphi)_s = \begin{cases}(1+T)\psi_0 & s=1\\ 0 & s\geq 2\end{cases} \overset{\text{if}}{=} 0 \in Q^{n+1}(SC)$

then null homologous, so is a bdy, of say θ

$(S\varphi)_s = d\theta_s + \theta_s d^* + \theta_{s-1} + \theta_{s-1}^*$ with $\theta_0 = \psi_0$, $\theta_s = 0$ $\forall s \geq 1$

Spse $\varphi \in \ker(S)$

$\varphi_{s-1} = d\theta_s + \theta_s d^* + \theta_{s-1} + \theta_{s-1}^*$, $s \geq 0$, $\theta_{-1} = 0$

$s=0$ $\quad 0 = d\theta_0 + \theta_0 d^*$

$s=1$ $\quad \varphi_0 = d\theta_1 + \theta_1 d^* + \theta_0 + \theta_0^* : C^{n-*} \longrightarrow C$

$\varphi_1 = d\theta_2 + \theta_2 d^* + \theta_1 + \theta_1^*$

Propn. $\varinjlim \left(Q^n(C) \xrightarrow{S} Q^{n+1}(SC) \xrightarrow{S} Q^{n+2}(S^2C) \to \dots\right) \cong \hat{Q}^n(C)$

For bounded C.

$$\begin{tikzcd} \varinjlim(\dots) & & \hat{Q}^n(C) \\ & Q^n(C) \arrow[ul] \arrow[ur, "J"'] & \end{tikzcd}$$

$$\begin{tikzcd} Q^n(C) \arrow[r] \arrow[d, "\subseteq"] & \hat{Q}^n(C) \arrow[d, "\cong"] \ni \{\chi_s\}_{s\in\mathbb{Z}} \\ Q^{n+1}(SC) \arrow[r] \arrow[d, "\subseteq"] & \hat{Q}^{n+1}(SC) \arrow[d, "\cong"] \ni \{\chi_{s-1}\}_{s\in\mathbb{Z}} \\ Q^{n+2}(S^2C) \arrow[r] & \hat{Q}^{n+2}(S^2C) \ni \{\chi_{s-2}\}_{s\in\mathbb{Z}} \end{tikzcd}$$

For bounded C, and any $\chi \in \hat{Q}^n(C)$ $\exists N > 0$ st $\chi_s = 0$ $\forall s < -N$ and $S^N\chi \in \operatorname{im}\left(Q^{n+N}(S^NC) \xrightarrow{J} \hat{Q}^{n+N}(S^NC)\right)$

$Q^n(C) \to Q^{n+k}(S^kC)$

Spse $(S^k\varphi)_s = d\chi_s + \chi_s d^* + \chi_{s-1} + \chi_{s-1}^*$, $s \geq 0$, ie $\varphi \mapsto S^k\varphi = 0$

Crucially $\quad 0 = (S^k\varphi)_{k-1} = d\chi_{k-1} + \chi_{k-1}d^* + \chi_{k-2} + \chi_{k-2}^*$

$\varphi_0 = (S^k\varphi)_k = d\chi_k + \chi_k d^* + \chi_{k-1} + \chi_{k-1}^*$

$\varphi_1 = (S^k\varphi)_{k-1} = d\chi_{k+1} + \chi_{k+1}d^* + \chi_k + \chi_k^*$

Set $\psi_0 = \chi_{k-1}$

$\psi_1 = \chi_k$

$\psi_2 = \chi_{k+1} \dots$

$d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* = 0$

$Q_n(C) \xrightarrow{1+T} Q^n(C)$

$\psi \longmapsto (1+T)\psi = \varphi$

Lecture 8. 05/11/08

Algebraic and Topological Suspension

$$S : Q^n(C) \longrightarrow Q^{n+1}(SC)\ ;\ \varphi = \{\varphi_s \mid s \geq 0\} \longmapsto S\varphi = \left\{ (S\varphi)_s = \begin{cases} \varphi_{s-1} & s \geq 1 \\ 0 & s = 0 \end{cases} \right\}$$

$$\Delta_X : C(X) \longrightarrow W^{\%} C(X) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(X) \otimes_{\mathbb{Z}} C(X))$$ symmetric construction

$$\Delta_X \cup \Delta_Y : C(X) \otimes C(Y) \xrightarrow{\Delta_X \otimes \Delta_Y} W^{\%} C(X) \otimes W^{\%} C(Y) \xrightarrow{U = \Delta_W^*} W^{\%}(C(X) \otimes C(Y))$$

$$\Delta_W : \underset{\overset{\shortparallel}{C(S^\infty)}}{W} \longrightarrow \underset{C(S^\infty)\otimes C(S^\infty)}{W \otimes W}\ ,\quad \Delta_W(1_s) = \sum_{r=0}^{s} 1_r \otimes T^r_{s-r}$$

Chain level Cartan formula is a natural chain homotopy

$$\begin{tikzcd} C(X \times Y) \arrow[rr, "\Delta_{X\times Y}"] \arrow[d, "EZ \simeq"'] \arrow[drr, bend left, "CEZ"] & & W^{\%} C(X \times Y) \arrow[d, "\simeq EZ^{\%}"] \\ C(X) \otimes C(Y) \arrow[rr, "\Delta_X \cup \Delta_Y"'] & & W^{\%}(C(X) \otimes C(Y)) \end{tikzcd}$$

$$\begin{tikzcd} H_n(X \times Y) \arrow[r, "\varphi_{X\times Y}"] \arrow[d, "EZ \cong"'] & Q^n(C(X \times Y)) \arrow[d, "\cong EZ^{\%}"] \\ H_n(C(X) \otimes C(Y)) \arrow[r, "\varphi_X \cup \varphi_Y"] & Q^n(C(X) \otimes C(Y)) \end{tikzcd}$$

commutes, but only upto chain htpy CEZ on chain level.

Apply this in the special case of X a pointed space, $Y = S^1$.

$C(X) = C(X, \{pt\})$, $\tilde{C}(X \wedge Y) \simeq \tilde{C}(X) \otimes \tilde{C}(Y)$ (reduced)

Top^{l} suspension $\Sigma X = X \wedge S^1$

ΣX is reduced suspension

$$\tilde{C}(\Sigma X) \simeq \tilde{C}(X) \otimes \tilde{C}(S^1) \simeq S\tilde{C}(X)$$

$$\because \tilde{C}(S^1) \simeq S\mathbb{Z}\ ,\quad \tilde{H}_*(X) = H_*(X, pt) = H_*(\tilde{C}(X))$$

Proposition. For any pointed space X, we get a commutative diagram

$$\begin{array}{ccccc} \tilde{H}_{n+1}(\Sigma X) & \xrightarrow{\tilde{\varphi}_{\Sigma X}} & Q^{n+1}(\tilde{C}(\Sigma X)) & \cong & \\ \cong & & & & Q^{n+1}(S\tilde{C}(X)) \\ \tilde{H}_n(X) & \xrightarrow{\tilde{\varphi}_X} & Q^n(\tilde{C}(X)) & \xrightarrow{S} & \end{array}$$

$$\begin{array}{ccc} H^r(\Sigma X) \otimes H^{n+1-r}(\Sigma X) & \xrightarrow{(\tilde{\varphi}_{\Sigma X})_0 = \cup} & H^{n+1}(\Sigma X) \\ \cong & & \cong \\ H^r(X) \otimes H^{n-r}(X) & \xrightarrow{0} & H^n(X) \end{array}$$

generalization of cup products vanishing in suspension

But on the chain level, there is only a chain homotopy

$$\begin{array}{ccc} \tilde{C}(\Sigma X) & \xrightarrow{\tilde{\varphi}_{\Sigma X}} & W^{\%}\hat{C}(\Sigma X) \\ \sigma^{\%} \cong & \searrow^{\psi_X} & \cong \sigma^{\%} \\ S\tilde{C}(X) & \xrightarrow{S\tilde{\varphi}_X} & W^{\%} S\tilde{C}(X) \end{array} \qquad \begin{array}{ccc} \tilde{H}^i(X;\mathbb{Z}_2) & \xrightarrow{S^r} & \tilde{H}^{i+r}(X;\mathbb{Z}_2) \\ \sigma \cong & \searrow^{\psi} & \cong \sigma \\ H^{i+1}(\Sigma X;\mathbb{Z}_2) & \xrightarrow{S^r} & H^{i+r+1}(X;\mathbb{Z}_2) \end{array}$$

Given a map $f: S^3 \to S^2$,
get $f = 0 : \tilde{C}(S^3) \to \tilde{C}(S^2)$
"ψ gives the Hopf invariant of f"

$$\begin{array}{ccc} H_2(S^2) & \xrightarrow{\varphi_{S^2}} & Q^2(\tilde{C}(S^2)) \\ 0 = f_* \ \cong & \searrow^{\psi} & \cong \ f_{\%} = 0 \\ H_2(S^1) & \xrightarrow{\varphi_{S^1}} & Q^2(\tilde{C}(S^1)) \end{array}$$

Thm. § A stable map $F : \Sigma^k X \to \Sigma^k Y$ (k large) induces a chain map $f : C(X) \to C(Y)$ and a quadratic construction
$\psi_F : \tilde{H}_n(X) \to Q_n(\hat{C}(Y))$
st $(1+T)\psi_F = f^{\%}\varphi_X - \varphi_Y f_*$

Remarks (a)
$$\begin{array}{ccc} H_n(X) & \xrightarrow{\varphi_X} & Q^n(C(X)) \\ f_* \downarrow & & \downarrow f^{\%} \\ H_n(Y) & \xrightarrow{\varphi_Y} & Q^n(C(Y)) \end{array}$$
will not commute for a general chain map $f : C(X) \to C(Y)$

(b) The diagram will commute ($f^{\%}\varphi_X = \varphi_Y f_*$) if f is induced by $F : X \to Y$

(c) If $\hat{C}(X) \to \hat{C}(Y)$ is induced by $F : \Sigma^k X \to \Sigma^k Y$

$$
\begin{tikzcd}
 & & Q^n(C(X)) \arrow[dr, bend left, "S^k"] \arrow[d, phantom, "\circlearrowleft"] & \\
\tilde{H}_n(X) \arrow[r, phantom, "\cong"] \arrow[urr, bend left, "\varphi_X"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_{n+k}(\Sigma^k X) \arrow[r, "\varphi_{\Sigma^k X}"] \arrow[d, "F_*"] & Q^{n+k}(C(\Sigma^k X)) \arrow[r, phantom, "\cong"] \arrow[d, "F^{\%}"] \arrow[dr, phantom, "\circlearrowleft"] & Q^{n+k}(S^k C(X)) \arrow[d, "f^{\%}"] \\
\tilde{H}_n(Y) \arrow[r, phantom, "\cong"] \arrow[drr, bend right, "\varphi_Y"'] & \tilde{H}_{n+k}(\Sigma^k Y) \arrow[r, "\varphi_{\Sigma^k Y}"] & Q^{n+k}(C(\Sigma^k Y)) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\circlearrowleft"] & Q^{n+k}(S^k C(Y)) \\
 & & Q^n(C(Y)) \arrow[ur, bend right, "S^k"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_n(X) \arrow[r, "\varphi_X"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\text{not comm}"] & Q^n(C(X)) \arrow[r, "S^k"] \arrow[d, "f^{\%}"] \arrow[dr, phantom, "\text{comm}"] & Q^{n+k}(S^k(C(X))) \arrow[d, "f^{\%}"] \\
H_n(Y) \arrow[r, "\varphi_Y"] & Q^n(C(Y)) \arrow[r, "S^k"] & Q^{n+k}(S^k C(Y))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H_n(X) \arrow[dl, "\psi_F"'] \arrow[d, "f^{\%}\varphi_X - \varphi_Y f_*"] & \\
Q_n(C(Y)) \arrow[r, "1+T"] & Q^n(C(Y)) \arrow[r] & \lim\limits_{k} Q^{n+k}(S^k \ldots)
\end{tikzcd}
$$

Example. For a map $F : \Sigma S^2 = S^3 \longrightarrow \Sigma S^1 = S^2$ we get

~~$\psi_F(S^2)$~~

$$\psi_F : H_2(S^2) \longrightarrow Q_2(C(S^1)) = Q_2(S\mathbb{Z}) = \mathbb{Z}_2$$

$$1 \longmapsto \text{mod } 2 \text{ Hopf invariant of } F$$

Can also be obtained by studying $\mathcal{C}(F) = S^2 \cup_F D^4$

and $\underset{\mathbb{Z}}{\underset{\shortparallel}{H^2(\mathcal{C}(F))}} \otimes \underset{\mathbb{Z}}{\underset{\shortparallel}{H^2(\mathcal{C}(F))}} \longrightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{H^4(\mathcal{C}(F))}}$

Properties of quadratic construction:

1.) If $F \simeq \Sigma^k F_0$ for some $F_0 : X \to Y$ then $\psi_F = 0$. So ψ_F is the primary desuspension obstruction

2.) If $F : \Sigma^k X \to \Sigma^k Y$, $G : \Sigma^k Y \to \Sigma^k Z$ induce $C(X) \xrightarrow{f} C(Y) \xrightarrow{g} C(Z)$ and $\psi_{GF} = g^{\%}\psi_F + \psi_G f_*$

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$Q_n(C(Y))$

3.) If $F \simeq F'$ then $\psi_F = \psi_{F'}$

4.) For $F : \Sigma X \to \Sigma Y$ we get $\psi_F : \hat{H}_n(X) \to \hat{H}_n(Y \wedge Y)$

$\psi \in \hat{H}_n(Y \wedge Y) \mapsto \left\{ \psi_s = \begin{cases} \psi & s=0 \\ 0 & s \geq 1 \end{cases} \right\} \in Q_n(C(Y))$

Eg. If $M^n \subset S^{n+1}$ is a codimension 1 framed submfd (eg Seifert surface)

so $M \times \mathbb{R} \subset S^{n+1}$

$F : S^{n+1} \longrightarrow S^{n+1} / S^{n+1} \setminus M \times \mathbb{R} = (M \times \mathbb{R})^{\infty} = \Sigma M^+$

$\hat{\psi}_F : H_n(S^n) \longrightarrow H_n(M \times M) \ni \hat{\psi}$ = Seifert fn

$\hat{\psi} + \hat{\psi}^* = [M]^n : C(M)^{n-*} \longrightarrow C(M)$

How vector bundles over spheres determine mfds with bdy and sym cxs

How do stably isom vec bundles determine quadratic cxs?

Consider $\pi_r(O(n-r)) \cong \pi_{r+1}(BO(n-r))$ = isom classes of $n-r$ plane bundles λ over S^{r+1}.

$(D^{n-r}, S^{n-r}) \longrightarrow (D(\omega)^{n+1}, S(\omega)) \longrightarrow S^{r+1}$

cpct $(n+1)$-dim^l mfd with bdy

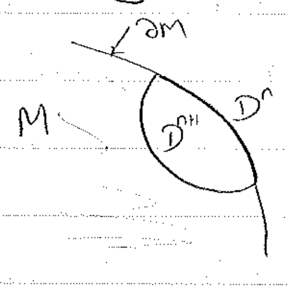
What is $\sigma^*(D(\omega), S(\omega))$ = $(n+1)$-dim^l sym Poincaré pair over $\mathbb{Z}$

Look at degree 1 map $(D(\omega), S(\omega)) \to (D^{n+1}, S^{n})$

If M^n is a closed mfd, $D^n \subset M \xrightarrow{\deg 1} M / M \setminus D = D^n / S^{n-1} = S^n$

For $(M, \partial M)$ use $(D^{n+1}, D^n) \subset (M, \partial M)$

$(M^{n+1}, \partial M) \xrightarrow{\deg 1} (M / M \setminus D^{n+1}, M / M \setminus D^n) = (D^{n+1}, S^n)$



deg 1 means $f_* : H_{n+1}(M, \partial M) \longrightarrow H_{n+1}(D^{n+1}, S^n)$, $1 \mapsto 1$

If $f:(C,\varphi) \longrightarrow (C',\varphi')$ is a chain map of sym Poincaré cxs
then $(C,\varphi) \simeq (C',\varphi') \oplus (\mathscr{C}(f')_{\cancel{**}}, \theta)$

$$\begin{tikzcd} C \arrow[r, "f"] & C' \\ C^{n-*} \arrow[u, "\varphi_0"] & C'^{n-*} \arrow[l, "f^*"] \arrow[u, "\varphi_0'"'] \end{tikzcd} \qquad f^{\%}\varphi = \varphi'$$

Eg. If $f:M \to M'$ is a deg 1 map of n-dim geo P. cx

$$\begin{tikzcd} H_r(M) \arrow[r, "f_*"] & H_r(M') \\ H^{n-r}(M) \arrow[u, "{[M]\cap}", "\cong"'] & H^{n-r}(M') \arrow[l, "f^*"] \arrow[u, "\cong", "{[M']\cap = f_*[M]\cap}"'] \end{tikzcd}$$

The Umbekor map is $f': C' \overset{\varphi_0'}{\simeq} C'^{n-*} \xrightarrow{f^*} C^{n-*} \overset{\varphi_0}{\simeq} C$
Have $f \circ f' \simeq 1 : C' \longrightarrow C$

$C' \xrightarrow{f'} C \xrightarrow{e} \mathscr{C}(f')$

$(C,\varphi) \simeq (C',\varphi') \oplus (\underbrace{\mathscr{C}(f')}_{\text{kernel, n-dim}^l \text{ sym Poincaré cx}}, e^{\%}\varphi)$

$f:M \to M'$ deg 1, $f': C(M') \simeq C(M')^{n-*} \longrightarrow C(M)^{n-*} \simeq C(M)$
Umbekor chain map, $f \circ f' \simeq 1 : C(M') \to C(M')$

$$\begin{array}{ccccc} K^{n-r}(M') & \longrightarrow & H^{n-r}(M^*) & \xleftarrow[f^*]{} & H^{n-r}(M') \\ \wr\wr & & \wr\wr & & \wr\wr \\ K_r(M) & \rightarrowtail & H_r(M) & \xrightarrow{f_*} & H_r(M') \end{array} \qquad K_r(M) := H_r(f')$$

The kernel of $(f,\partial f) : (D(\varepsilon^{n-r}), S(\varepsilon^{n-r})) \longrightarrow (D^{n+1}, S^n)$
$ \quad // \, (S^{r+1}\times D^{n-r}, S^{r+1}\times S^{n-r-1})$
is an n+1-dim^l sym P. pair $\sigma^*(f,\partial f)$ st
~~$\sigma^*(f,\partial f)$ st~~ $\sigma^*(f,\partial f) \oplus \sigma^*(D^{n+1}, S^n) \simeq \sigma^*(S^{r+1}\times D^{n-r}, S^{r+1}\times S^{n-r-1})$

~~In general,~~ For arbitrary $\omega \in \pi_r(SO(n-r))$ will now describe $\sigma^*(f,\partial f) =$
n+1-dim^l sym P. pair over $\mathbb{Z}$ st $\sigma^*(f,\partial f) \oplus \sigma^*(D^{n+1}, S^n) \simeq \sigma^*(D(\omega), S(\omega))$.

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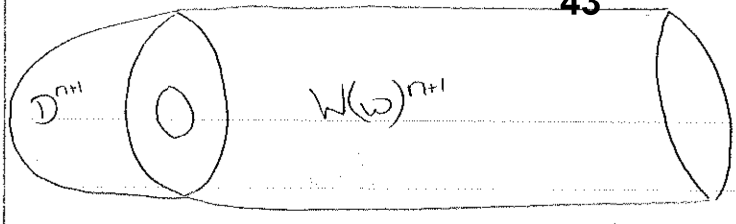

$S(\omega) = D^{r+1} \times S^{n-r-1} \underset{(x,y) \sim (x, \omega(y)(x))}{\cup} D^{r+1} \times S^{n-r}$

$e_\omega : S^r \times D^{n-r} \hookrightarrow S^r \times D^{n-r} \cup D^{r+1} \times S^{n-r-1}$

$(x, y) \longmapsto (x, \omega(x)(y))$ = effect of surgery on e_ω

$$\sigma^*(f, \partial f) \oplus \sigma^*(D^{n+1}, S^n) \simeq \sigma^*(D(\omega), S(\omega))$$

Apply the algebraic Thom construction

$$\underbrace{(f : C \to D, (\delta\varphi, \varphi))}_{\text{symmetric Poincaré pair}} \longmapsto \underbrace{(\mathcal{C}(f), \delta\varphi/\varphi)}_{\text{symmetric cx}} = (\overset{E}{\cancel{D}}, \theta)$$

$$\partial(E, \theta) \simeq (C, \varphi), \quad C \simeq \mathcal{C}(\theta_0 : E^{n+1-*} \to E)_{*+1} \longrightarrow D^{n+1-*}$$

Note: $C(S^n) = S^n\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(0,1)} C(D^{n+1}) = \mathbb{Z}$

$\partial(S^{n+1}\mathbb{Z}, 1) = \sigma'(S^n)$

Recall Thom space; $T(\omega) = D(\omega)/S(\omega) = S^{n-r} \cup D^{n+1}$

Note: $\mathcal{C}(C = C(S(\omega)) \xrightarrow{f} C(D(\omega))) \xrightarrow{\simeq} C(T(\omega)) = S^{n-r}\mathbb{Z} \oplus S^{n+1}\mathbb{Z}$

Defn. Steifel-Whitney class

$$\begin{array}{ccc} \tilde{H}^{n-r}(T(\omega)) & \xrightarrow{Sq^{r+1}} & H^{n+1}(T(\omega)) \\ \cong & & \cong \\ \mathbb{Z} = H^0(S^{r+1}) \ni 1 & \longmapsto \underbrace{w_{r+1}(\omega)}_{\text{Steifel-Whitney class}} & H^{r+1}(S^{r+1}) = \mathbb{Z} \end{array}$$

$$(C(S(\omega)) \to C(D(\omega)), (\overset{\Delta[D(\omega)]}{\overset{\parallel}{\delta\varphi}}, \varphi)) \rightsquigarrow (C(D(\omega))/C(S(\omega)), \delta\varphi/\varphi) = \overset{S^{n-r}\mathbb{Z} \oplus S^{n+1}\mathbb{Z}}{\overset{\parallel}{C(T(\omega))}}$$

$\delta\varphi/\varphi \in Q^{n+1}(C(T(\omega)))$

$$\begin{array}{ccccc} \mathbb{Z} = H_{n+1}(T(\omega)) & \xrightarrow{\varphi_{T(\omega)}} & Q^{n+1}(C(T(\omega))) & \longrightarrow & Q^{n+1}(S^{n-r}\mathbb{Z}) \; \cancel{C(T(\omega))} \\ 1 & & 1 & \longrightarrow & \overset{\in}{Sq^{r+1}} = w_{r+1}(\omega) \end{array}$$

cf Hopf

$$S^1 \to D^2\times S^1 \cup S^1\times D^2 = S^3 \to S^2$$

Eg $\omega = 1 \in \pi_1(\underset{S^1}{SO(2)}) = \mathbb{Z}$

$S^1\times D^2 \hookrightarrow S^1\times D^2 \cup D^2\times S^1 = S^3$

$(x,y) \mapsto (x, \omega(x)(y))$

$T(\omega) = D(\omega)/S(\omega) = \mathbb{CP}^2$

If $\omega\oplus\varepsilon^k \cong \omega'\oplus\varepsilon^k$ then $\omega_{r+1}(\omega) = \omega_{r+1}(\omega')$

$$\underset{T(\omega\oplus\varepsilon^k)}{\Sigma^k T(\omega)} \xrightarrow{\cong} \underset{T(\omega'\oplus\varepsilon^k)}{\Sigma^k T(\omega')}$$

$$H_{n+1}(T(\omega)) \xrightarrow{\psi_F} Q_{n+1}(\tilde{C}(T(\omega')))$$

Eg $\tau_{S^n}\oplus\varepsilon^k \cong \varepsilon^n\oplus\varepsilon^k$

$\pi_{n+1}(SO, SO(n)) \xrightarrow{\cong} Q_{2n}(S^n\mathbb{Z}) \ni \psi_F$

$(D^n, S^{n-1}) \to (D(\xi_n), S(\xi_n)) \to S^n$

$\qquad F \cong (D(\varepsilon^n), S(\varepsilon^n))$

Lecture 9. 11.11.08

Why is stable homotopy theory relevant to surgery theory?

Stable homotopy theory:

X = pointed space. Suspension of X is $\Sigma X = X\wedge S^1$

A stable map $F: X \rightsquigarrow Y$ is a map $F: \Sigma^k X \to \Sigma^k Y$ some $k\geq 0$ ($\rightsquigarrow$ means stable)

Two such stable maps $F: \Sigma^k X \to \Sigma^k Y$, $F': \Sigma^{k'}X \to \Sigma^{k'}Y$ are stably homotopic if for some $j\geq 0$ $\Sigma^{j+k'}F \simeq \Sigma^{j+k}F' : \Sigma^{j+k+k'}X \to \Sigma^{j+k+k'}Y$

A stable map $F: X\rightsquigarrow Y$ induces $F_*: \tilde{H}_*(X) \to \tilde{H}_*(Y)$ which is a stable htpy invt.

F is a stable htpy eqvce if $\exists G: Y\rightsquigarrow X$ st $FG \simeq_s 1_Y$, $GF\simeq_s 1_X$

Stable Whitehead Thm; for CW-cxs X, Y, F is a stable htpy eqvce iff $F_*: \tilde{H}_*(X) \to \tilde{H}_*(Y)$ is an isom.

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Motivating example. (Spanier, Whitehead, 1950s).
If $X \subset S^N$ is a finite subcx then the homotopy of X determines the ~~stablye~~ stable homotopy of $S^N \setminus X$ with
$$H_*(X) \cong H^{N-1-*}(S^N \setminus X), \quad H_*(S^N \setminus X) \cong H^{N-1-*}(X) \quad \text{for } * \neq 0, N-1$$

Def^n. The stable homotopy group X to Y is
$$\{X, Y\} := \varinjlim_k [\Sigma^k X, \Sigma^k Y]$$

Eg $[S^3, S^2] = \pi_3(S^2) = \mathbb{Z} \longrightarrow \{S^3, S^2\} = \pi_1^s = \mathbb{Z}_2$

$\tau_M \oplus \varepsilon^k \cong \varepsilon^{n+k}$

$[S^n, X] = \pi_n(X) \longrightarrow \pi_n^s(X) = \{S^n, X\}$ = bordism gp of framed mfds $M^n \to X$

"space" = finite CW-cx

Main Problem of Surgery Theory
Decide if a "space" X with n-dim^l Poincaré duality $H^*(X) \cong H_{n-*}(X)$ is homotopy eqvt to an n-dim^l mfd.

There always exists an embedding $X \subset S^N$ $(N \geq 2 \dim X)$
A regular nbhd $(W, \partial W)$ = codim 0 submfd $W \subset S^N$ with $X \subset W$ a htpy eqvce and

not in general a fibre bundle

(SNF)
Spivak normal fibration $\longrightarrow (D^{N-n}, S^{N-n-1}) \to (W, \partial W) \to X$ is a fibration
$\nu_X : X \to BG(N-n)$ (a homotopy analogue of $BO(N-n)$)

The Thom space $T(\nu_X) = W/\partial W$ is S-dual to $W^+ = X^+$

Def$^{\underline{n}}$. Two pted spaces X, Y are S-dual if for some $N \geq 0$ there exists a map $\alpha : S^N \to X \wedge Y$ with Hurewicz image $h(\alpha) \in \tilde{H}_N(X \wedge Y)$ and the cap product
$$\cap : \tilde{H}_N(X \wedge Y) \otimes \tilde{H}^r(X) \longrightarrow \tilde{H}_{N-r}(Y)$$
$$(x \otimes y) \otimes z \longmapsto z(x)\, y$$
st $h(\alpha) \cap - : \tilde{H}^r(X) \xrightarrow{\cong} \tilde{H}_{N-r}(Y)$

Example. If X is an n-dim^l Poincaré cx, $X \subset S^N$,

$$(D^{N-n}, S^{N-n-1}) \longrightarrow (W, \partial W) \xrightarrow{\nu_X} X \qquad \text{(SNF)}$$

then the composition ~~$\alpha: S^N \to S^N/\overline{S^N \setminus W} = W/\partial W = T(\nu_X)$~~

$$\alpha : S^N \xrightarrow{\rho_X} S^N/\overline{S^N\setminus W} = W/\partial W = T(\nu_X) \underset{x \mapsto (x,x)}{\xrightarrow{\Delta}} W\times W/(\partial W \times W) = W/\partial W \wedge W^+ \simeq T(\nu_X)\wedge X^+$$

is an S-duality map, since

$$\begin{tikzcd}
\tilde{H}^r(T(\nu_X)) \arrow[r, "\cong", "h(\alpha)\cap -"'] \arrow[d, "\cong"] & \tilde{H}_{N-r}(X^+) \\
H^r(W, \partial W) \arrow[r, "\cong", "\text{Poincaré Lefschetz}"'] & H_{N-r}(W) \arrow[u, "\cong"']
\end{tikzcd}$$

Thm. For $n \geq 5$, an n-dim^l Poincaré cx X is htpy eqvt to a differentiable n-mfd iff

(i) The SNF ν_X is realised by $\tilde{\nu}_X : X \longrightarrow BO(N-n)$ in which case

$$\begin{array}{ccc}
\rho_X : S^N & \longrightarrow & T(\nu_X) \simeq T(\tilde{\nu}_X) \\
\cup\!\!\mathsf{I} & & \cup\!\!\mathsf{I} \\
(\rho_X)^{-1}(X) = M^n & \xrightarrow{f} & X
\end{array}$$

ρ_X can be made differentially transverse at zero section $X \subset T(\tilde{\nu}_X)$ to obtain an n-dim^l mfd ~~M~~ $M^n = \rho_X^{-1}(X)$ with $M^n \subset S^N$

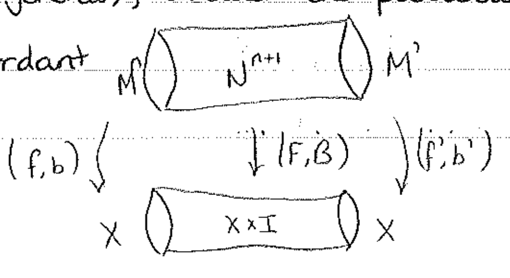
$$\begin{array}{ccccc}
\rho_M \in \cancel{\rho_M} \in \pi_N(T(\nu_{M\subset S^N})) & \xrightarrow{T(b)} & \pi_N(T(\nu_X)) \ni \rho_X & \ni \rho_X \\
\downarrow \qquad\quad \downarrow & & \downarrow & \downarrow \\
\tilde{H}_N(T(\nu_{M\subset S^N})) & & \tilde{H}_N(T(\nu_X)) & \\
[M] \in \ \cong H_n(M) & \xrightarrow{f_*} & H_n(X) & \ni [X]
\end{array}$$

(f,b) a degree 1 normal map $\begin{array}{ccc} \nu_X & \xrightarrow{b} & \tilde{\nu}_X \\ \downarrow & & \downarrow \\ M & \xrightarrow{B|_M = f} & X \end{array}$

In general, there are obstructions to (i)
the map $[X, BO(N-n)] \xrightarrow{J} [X, BG(N-n)]$ is not onto in general

$$X = S^n, \ \varinjlim_N [S^n, BO(N-n)] = \pi_n(BO) \longrightarrow \varinjlim_N [S^n, BG(N-n)] = \pi^S_{n-1} = \{S^{n-1}, S^n\}$$

(ii) Assuming (i) is possible, and choosing one particular possibility (infinite in general), consider the possibility of making the normal map $(f,b): M^n \to X$ normal bordant



(with same $\tilde{\nu}_X$ in target) to a htpy eqvce $f'? : M' \xrightarrow{\simeq} X$, by surgery on M which allows for extension of (f,b) to trace N.

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$\mathbb{Z}[\pi_1 X]$-module

Can all the kernels $K_*(\tilde M) = H_{*+1}(\tilde f)$ be killed by surgery?

This is possible iff the Wall surgery obstruction

$$\sigma_*(f,b) = (C,\psi) \in L_n(\mathbb{Z}[\pi_1 X]) \text{ is } 0$$

CAN USE STABLE $\pi_1(X)$-EQUIVARIANT HOMOTOPY THEORY + QUADRATIC CONSTRUCTION TO DEFINE ψ

Let (f,b) be a degree 1 normal map from an n-dim^l mfd to a geometric Poincaré cx X.

Assume $f_* : \pi_1(M) \xrightarrow{\cong} \pi_1(X)$

(If $n \geq 5$, this can be achieved by surgeries on $S^1 \times D^{n-1} \subset M$).

$K_1(M) = \ker(\pi_1(M) \xrightarrow{f_*} \pi_1(X)) = \mathrm{im}(\pi_2(f) \rightarrowtail \pi_2(X))$

Let $\tilde X$ be the universal cover of X, so $\tilde M = f^* \tilde X$ is the universal cover of M.

$$\begin{tikzcd} K_r(M) := \ker\big(\tilde f_* : H_r(\tilde M) \arrow[r] & H_r(\tilde X)\big) \\ H^{n-r}(\tilde M) \arrow[u, "{[M]\cap}", "\cong"'] & H^{n-r}(\tilde X) \arrow[l, "\tilde f^*"'] \arrow[u, "{[X]\cap}"', "\cong"] \end{tikzcd}$$

$\tilde f : C(\tilde M) \to C(\tilde X)$ is split by Umkehr chain map

$$f^{*!} : C(\tilde X) \simeq C(\tilde X)^{n-*} \xrightarrow{\tilde f^*} C(\tilde M)^{n-*} \simeq C(\tilde M)$$

We get $C(\tilde M) \underset{f'}{\overset{\tilde f}{\rightleftarrows}} C(\tilde X)$ with $\tilde f f' \simeq 1_{C(\tilde X)}$ ($\because f_*[M] = [X]$) f is deg 1

$$K_*(M) = \ker(\tilde f_*) = H_{*+1}(\tilde f) = H_*(f^{*!})$$

Lemma. If $f_* : \pi_1(M) \xrightarrow{\cong} \pi_1(X)$ then f is a ~~hom~~ htpy eqvce

$\Leftrightarrow \tilde f : C(\tilde M) \to C(\tilde X)$ is a chain eqvce

$\Leftrightarrow \tilde f^! : C(\tilde X) \to C(\tilde M)$ —— " ——

$\Leftrightarrow \mathscr{C}(f^!) \simeq 0$

Defn. $\mathscr{C}(f^!)$ = kernel $\mathbb{Z}[\pi_1 X]$-module chain cx.

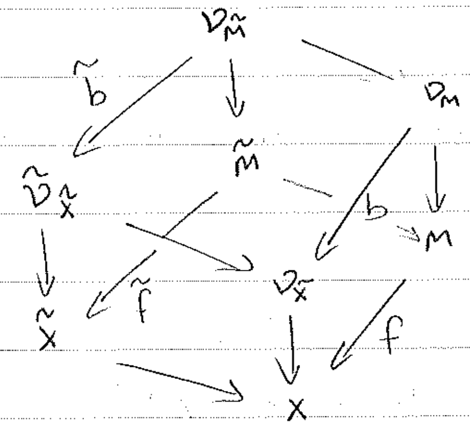
$\cancel{K_r} = H_r(C) \oplus H_r(\tilde{X})$
$= H_r(f^!) \oplus H_r(\tilde{X})$

$K_r(M)$
$\|$
$H_r(f^!) \rightleftarrows H_r(\tilde{M}) \underset{f'_*}{\overset{\tilde{f}_*}{\rightleftarrows}} H_r(\tilde{X})$, $H_r(\tilde{M}) = K_r(M) \oplus H_r(\tilde{X})$

If X, Y are pted spaces, then $[X,Y]_\pi$ = π-equivariant homotopy classes of π-maps $X \to Y$.

$T(\tilde{b}) \in \{T(\nu_{\tilde{M}}), T(\nu_{\tilde{X}})\}_{\pi_1(X)} = \varinjlim_k [\Sigma^k T(\nu_{\tilde{M}}), \Sigma^k T(\nu_{\tilde{X}})]_{\pi_1(X)}$
$\cong \{\tilde{X}^+, \tilde{M}^+\} \ni F$

$\pi_1(X)$-equivariant S-duality



is a stable $\pi_1(X)$ equivariant map
$F : \tilde{X}^+ \rightsquigarrow \tilde{M}^+$ inducing $f^! : C(\tilde{X}) \to C(\tilde{M})$.

$C(\tilde{X}) \xrightarrow{f^!} C(\tilde{M}) \xrightarrow{e} \mathcal{C}(f^!)$

$[X] \in H_n(X) \xrightarrow{\psi_F} Q_n(C(\tilde{M})) \xrightarrow{e_\%} Q_n(\mathcal{C}(f^!)) \ni \psi = e_\% \psi_F [X]$

ψ_F is the $\pi_1(X)$-equivariant quadratic construction

Let $\Lambda = \mathbb{Z}[\mathbb{Z}_2]$

W is : $\longrightarrow \underset{}{\overset{W_3}{\Lambda}} \xrightarrow{1-T} \overset{W_2}{\Lambda} \xrightarrow{1+T} \overset{W_1}{\Lambda} \xrightarrow{1-T} \overset{W_0}{\Lambda}$

$W[0,k-1]$ is : $0 \to W_{k-1} = \Lambda \longrightarrow \overset{W_{k-2}}{\Lambda} \longrightarrow \dots \longrightarrow \overset{W_1}{\Lambda} \xrightarrow{1-T} \overset{W_0}{\Lambda}$

So $W[0,k-1] = C(S^{k-1})$ with antipodal free $\mathbb{Z}_2$-action.

Statement. If $F : \Sigma^k X \to \Sigma^k Y$ is a k-stable map then there is a stable map

$$\psi_F^{[0,k-1]} : X \rightsquigarrow (S^{k-1})^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y)$$

inducing

$$\psi_F^{[0,k-1]} : H_*(X) \longrightarrow Q_*^{[0,k-1]}(C(Y)) = \hat{H}_*((S^{k-1})^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y))$$

st

(i) $(1+T)\psi_F^{[0,k-1]} = f^\% \varphi_X - \varphi_Y f_* : H_*(X) \longrightarrow Q^*(C(Y))$

There is a close relationship of $\psi_F^{[0,k-1]}$ with the sym construction φ on $\mathcal{C}(F) = \Sigma^k Y \cup_F C(\Sigma^k X)$

$$\varphi_{\mathcal{C}(F)} : H_*(F) \longrightarrow Q^*(\mathcal{C}(F))$$

$$\begin{tikzcd}
 & & Q^{n+1}(\mathcal{C}(f)) \arrow[d, "S^*"] \\
H_{n+k+1}(F) = H_{n+1}(f) \arrow[urr, dashed, "\varphi_{\mathcal{C}(F_0)}"] \arrow[rr, "\varphi_{\mathcal{C}(F)}"] \arrow[d, "\partial"] & & Q^{n+k+1}(S^k \mathcal{C}(f)) \arrow[d] \\
\hat{H}_{n+k+1}(\Sigma^{k+1} X) = \hat{H}_n(X) \arrow[r, "\psi_F^{[0,k-1]}"] & Q_n^{[0,k-1]}(C(Y)) \arrow[r, "g\%"] & Q_n^{[0,k-1]}(\mathcal{C}(f))
\end{tikzcd}$$

with $\mathcal{C}(F) = S^k\mathcal{C}(f)$.

$$\hat{C}(X) \xrightarrow{f} \hat{C}(Y) \xrightarrow{g} \mathcal{C}(f) \xrightarrow{h} S\hat{C}(X)$$

For $k=1$, $X = S^2$, $Y = S^1$, $n = 2$

$F : \Sigma X = S^3 \longrightarrow \Sigma Y = S^2$, $\mathcal{C}(F) = S^2 \cup_F D^4$

$$\psi_F^{[0,0]} : H_2(S^2) \longrightarrow \underset{\overset{\shortparallel}{S\mathbb{Z}}}{Q_2^{[0,0]}(\hat{C}(Y))} = \mathbb{Z}$$

$$\overbrace{H^2(\mathcal{C}(F))}^{\mathbb{Z}} \longrightarrow \overbrace{H^4(\mathcal{C}(F))}^{\mathbb{Z}} \; ; \; x \longmapsto x \cup x = \text{Hopf invariant}(F)$$

Lecture 10. 12/11/08

Immersions and Embeddings

Every elt $x \in \pi_n(M^m)$ is represented by a map $f : S^n \to M^m$

When can x be repr^d by an embedding $f : S^n \hookrightarrow M$ with trivial normal bundle, $S^n \times D^{m-n} \subset M$, st x can be "killed by surgery", st

$$M' = M \backslash S^n \times D^{m-n} \cup D^{n+1} \times S^{m-n-1}$$

with trace $(W = M \times I \cup D^{n+1} \times D^{m-n} ; M, M')$ has $\pi_n(W) = \pi_n(M)/\langle x \rangle$

Usually ok for $2n < m$, and certainly for a normal map $(f, b) : M \longrightarrow X$ with $x \in \ker(f_* : \pi_n(M) \to \pi_n(X))$, $m \geq 5$

For $m=2n$, we get self-intersection obstruction in

$$Q_{2n}\left(S^n\mathbb{Z}[\pi_1X]\right) = \mathbb{Z}[\pi_1X] \Big/ \left\{a-(-)^n\bar{a} \mid a\in\mathbb{Z}[\pi_1X]\right\}$$

Whitney founded the effective thm of diff^{ble} mfds embed $M^m \subset \mathbb{R}^{2m+1}$

Defn. An immersion of mfds $f: N^n \to M^m$ is a diff^{ble} map which is locally an embedding, or equivalently st the differential ~~$df: \tau_N \to \tau_M$~~ $d_xf: \tau_xN \to \tau_{f(x)}M$ is injective at each $x\in N$. write $f: N \looparrowright M$

f has a normal bundle ν_f st $\tau_N \oplus \nu_f \cong f^*\tau_M$

$\tau_N: N \to BO(n)$, $\tau_M: M \to BO(m)$, $\nu_f: N \to BO(m-n)$

Thm (i) For $2n\leq m$, every map $f: N^n \to M^m$ is homotopic to an immersion.

(ii) For $2n+1\leq m$, every map $f: N^n \to M^m$ is homotopic to an embedding.

(iii) For $n\geq 3$, $\pi_1(M)=\{1\}$, every map $f: N^n \to M^{2n}$ is homotopic to an embedding.

Defn. Two immersions $f,g: N^n \looparrowright M^m$ are regular homotopic if there a homotopy $h: f\simeq g$ st each h_t is an immersion.

Easy consequence, the normal bundles ν_f, ν_g are isomorphic.

$\nu_f \cong \nu_g : N \to BO(m-n)$

Smale, Hirsch. Assume $n\leq m-2$. Then the regular homotopy classes of immersions $f: N^n \looparrowright M^m$ are in one-to-one correspondence with the homotopy classes of diff^{ble} maps $f: N\to M$ st $df: \tau_N \to \tau_M$ is injective.

("Turning the sphere inside out" $e: S^2 \hookrightarrow \mathbb{R}^3$, $-e: S^2 \hookrightarrow \mathbb{R}^3$ are regularly homotopic)

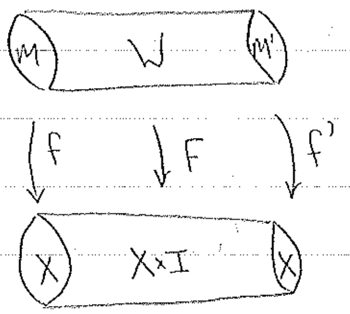
Given an immersion $f : N^n \looparrowright M^m$, when is it regular homotopic to an embedding f' (with trivial $\nu_{f'}$)

Thm. (Wall, 1969) If $(f,b): M^m \to X$ is an n-connected normal map, and $x \in K_n(M) = \ker(f_* : H_n(\tilde{M}) \to H_n(\tilde{X})) = \pi_{n+1}(f)$

(i) $2n \leq m$ and

then it is possible to realise x by an immersion $S^n \overset{\varphi}{\looparrowright} M$ with a null homotopy $\delta\varphi : f\varphi \simeq * : S^n \to X$ and trivial normal bundle $\nu_\varphi : S^n \to BO(m-n)$ uses Whitney (i) & Smale

(ii) For $2n+1 \leq m$, we can choose φ to be an embedding, allowing $x \in K_n(M)$ to be killed by surgery.



If $2n+1 < m$, we have

$K_n(M') = K_n(M)/(x)$ ← $\mathbb{Z}[\pi_1 X]$-submod gen by x

$K_r(M') = 0$ for $r < n$

Since $K_n(M)$ is a f.g. $\mathbb{Z}[\pi_1 X]$-mod (= first non-zero homology of a finite f.g. free $\mathbb{Z}[\pi_1 X]$-module chain cx)

Consequence (Wall). If $n \geq 5$. Every normal map $(f,b): M^m \to X$ with $m = 2n$ (quad fn) or $m = 2n+1$ (consists of quad fns) is normal bordant to an n-connected normal map

$(f', b') : M' \to X$ with $K_r(M') = 0$ for $r < n$.

Given an $f : N^n \looparrowright M^m$, consider

$f' : N^n \overset{f}{\looparrowright} M^m \hookrightarrow M \times \mathbb{R}^k$, $x \mapsto (x,0)$; $f' : N^n \looparrowright M \times \mathbb{R}^k$

for k so large that $2n+1 \leq m+k$

f' is an immersion with $\nu_{f'} = \nu_f \oplus f^* \nu_{M \subset M \times \mathbb{R}^k} = \nu_f \oplus \varepsilon^k$

Apply Whitney's embedding thm to f' to obtain a regular homotopic embedding $f'' : N^n \hookrightarrow M \times \mathbb{R}^k$ with

$$\nu_{f''} = \nu_{f'} = \nu_f \oplus \varepsilon^k$$

and then the Pontrjagin - Thom construction yields

$$F : (M \times \mathbb{R}^k)^\infty = \Sigma^k M^+ \longrightarrow M\times\mathbb{R}^k \big/ M\times\mathbb{R}^k \setminus E(\nu_{f''}) = T(\nu_{f''}) = \Sigma^k T(\nu_f)$$

Note: The stable homotopy class of F depends only on the regular homotopy class of ~~F~~ f.

If f is regular homotopic to an embedding $f_0 : N \hookrightarrow M$, we can take $k=0$ for f_0, and F is homotopic to $\Sigma^k F_0$ with $F_0 = PT$ of $f_0 : M^+ \to T(\nu_{f_0})$

Pontrjagin Thom $\qquad T(\nu_{f_0}) = T(\nu_f)$

In cases of interest to surgery $(2n=m)$, this is an if and only if provided $\pi_1(M)$ is taken into account and $n \geq 3$.

Defn. The double point set of a map $f : N \to M$ is

$$D_2(f) = \{(x,y) \mid x \neq y \in N,\ f(x) = f(y) \in M\} \big/ \mathbb{Z}_2\text{-action } (x,y)\sim(y,x)$$

Obvious: f is an embedding iff $D_2(f) = \phi$

Whitney: If $f : N^n \looparrowright M^m$ is an immersion without triple points (generally ok if $2n < 3m$: metastable) then $D_2(f)$ is an $(\overset{2n-m}{\cancel{m+2n}})$-mfd with an immersion

$$g : D_2(f)^{2n-m} \looparrowright M^m \ ; \ (x,y) \longmapsto f(x) = f(y)$$

s.t.

$$\begin{tikzcd}
\nu_g : D_2(f) \arrow[rr] \arrow[dr] & & BO(2n-2m) \\
& (\overset{N\times N}{\cancel{M\times M}})\big/\mathbb{Z}_2 \arrow[ur, "\nu_f \times \nu_f"'] &
\end{tikzcd}$$

$[x,y] \mapsto (x,y)$

The double cover $\overline{D}_2(f) = \{(x,y) \in N \times N \mid x \neq y,\ f(x) = f(y)\}$ of $D_2(f)$ is classified by

$$\begin{array}{ccc} \overline{D}_2(f) & \xrightarrow{\ \overline{c}\ } & S^{k-1} \\ \downarrow & & \downarrow \\ D_2(f) & \xrightarrow{\ c\ } & \mathbb{R}P^{k-1} \end{array} \qquad \text{for large } k \geq 1$$

$$\begin{array}{c} \mathbb{R}^{n-m} \\ \downarrow \\ E(\nu_f) \\ \downarrow \\ N \end{array}$$

$$\begin{array}{ccc} D_2(f) & \xrightarrow{\ \nu_g\ } & BO(2n-2m) \\ {\scriptstyle [x,y]}\downarrow \ \searrow{\scriptstyle e_2(c)} & & \uparrow{\scriptstyle e_2(\nu_f)} \\ (c(x,y),(x,y)) & S^{k-1} \times_{\mathbb{Z}_2} \left(\dfrac{M \times M}{N \times N}\right) & \end{array}$$

where

$$\begin{array}{c} \mathbb{R}^{2n-2m} \\ \downarrow \\ E(e_2(\nu_f)) = S^{k-1} \times_{\mathbb{Z}_2} (E(\nu_f) \times E(\nu_f)) \\ \downarrow \\ S^{k-1} \times_{\mathbb{Z}_2} (N \times N) \end{array}$$

Thm. (Ranicki 1980). If $F : \Sigma^k M^+ \longrightarrow \Sigma^k T(\nu_f)$ is the Umkehr map of $f : N^n \looparrowright M^m$ then

$$\begin{array}{ccc} H_m(M) & \xrightarrow{\ \psi_F\ } & Q_m^{[0,k]}(C(T(\nu_f))) \\ \downarrow{\scriptstyle g^!} & & Q_m^{[0,k-1]}(S^{m-n} C(N)) \\ H_{2n-m}(D_2(f)) & \xrightarrow{\ e_2(c)_*\ } & H_{2n-m}(S^{k-1} \times_{\mathbb{Z}_2} (N \times N)) \end{array}$$

If $D_2(f) = \phi$ then $\psi_F \equiv 0$ and conversely (if $m = 2n \geq 6$ & if $\pi_1 X$ is taken into account).

$m = 2n \geq 6$, $\exists F_0$ st $F \simeq \Sigma^k F_0 \Leftrightarrow \psi_F^{[0,k-1]} \equiv 0 \Leftrightarrow D_2(f) = \phi$ upto regular homotopy of f

Eg. If $m = 2n$, $\pi_1 M = \{1\}$, $Q_m^{[0,k-1]}(S^{m-n} C(M)) = \begin{cases} \mathbb{Z} & k = 1 \\ \mathbb{Z} & k \geq 2 \text{ even } n \\ \mathbb{Z}_2 & k \geq 2 \ n \text{ odd} \end{cases}$

Incorporating $\pi_1 M$...

Let $f : N^n \looparrowright M^m$ then $N \hookrightarrow M \times \mathbb{R}^k$ via $N \looparrowright M \hookrightarrow M \times \mathbb{R}^k$

Then $F : (M \times \mathbb{R}^k)^\infty = \Sigma^k M^+ \longrightarrow \Sigma^k T(\nu_f)$

$$C(M) \overset{PD}{\simeq} C(M)^{m-*} \xrightarrow{\ f^*\ } C(N)^{m-*} \overset{PD}{\simeq} C(N)_{*+n-m}$$

$$\xrightarrow{\ F = f^!\ } \tilde{C}(T(\nu_f)) \qquad \simeq \text{ Thom chain eqvce}$$

Assume $\pi_1(N) = \{1\}$, $\pi_1(M)$ arbitrary

$$\begin{tikzcd} \pi_1(M)\times N = \tilde{N} \arrow[r, hook, "\tilde{f}"] \arrow[d] \arrow[rr, bend left, "\pi_1 M\text{-equivariant}"] & \tilde{M} \arrow[r, hook] \arrow[d] & \tilde{M}\times\mathbb{R}^k \arrow[d] \\ N \arrow[r, hook, "f"] & M \arrow[r, hook] & M\times\mathbb{R}^k \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} \pi_1(M)\times N = \tilde{N} \arrow[r, hook, "\tilde{f}'"] \arrow[d] & \tilde{M}\times\mathbb{R}^k \arrow[d] \\ N \arrow[r, hook, "f'"] & M\times\mathbb{R}^k \end{tikzcd}$$

$\tilde{F} : (\tilde{M}\times\mathbb{R}^k)^\infty = \Sigma^k \tilde{M}^+ \longrightarrow \Sigma^k T(\nu_{\tilde{f}}) = \bigvee_{\pi_1 M} \Sigma^k T(\nu_f)$

$\pi_1(M)$-equivariant stable Umkehr.

f.g. free $\mathbb{Z}[\pi_1 M]$-mod chain cx

$\Lambda = \mathbb{Z}[\pi_1 M]$, $\bar{g} := g^{-1}$ for $g\in\pi_1 M$

Thm. $$\begin{tikzcd} H_m(M) \arrow[r, "\psi_{\tilde{F}}^{[0,k-1]}"] \arrow[d] & Q_m^{[0,k-1]}(\dot{C}(T(\nu_{\tilde{f}}))) \\ & = Q_{2m}^{[0,k-1]}(S^{m-n}C(\tilde{N})) \\ H_{2n-m}(D_2(f)) \arrow[r] & = H_{2m-n}(S^{k-1}\times_{\mathbb{Z}_2}(\tilde{N}\times_{\pi_1 M}\tilde{N})) = \mathbb{Z}[\pi_1 M]/\{a-(-)^n\bar{a} \mid a\in\mathbb{Z}[\pi_1 M]\} \end{tikzcd}$$

$k\geq 2$

$[M] \in H_m(M) \mapsto \mu(f) = \sum_{x\in D_2(f)} \mu(x)$

For $N = S^n \overset{f}{\looparrowright} M^{2n=m}$, we get Wall's μ fn.

(Wall, 1970) f is regular homotopic to an embedding iff $\mu(f) = 0$

Back to an n-dim normal map $(f,b) : M^m \longrightarrow X$

Define Umkehr chain map $f^! : C(X) \simeq C(X)^{m-*} \xrightarrow{f^*} C(M)^{m-*} \simeq C(M)$

Claim. $f^!$ is induced by an Umkehr stable map $F : \Sigma^k X^+ \longrightarrow \Sigma^k M^+$ which depends on b.

$\Psi_F : H_m(X) \longrightarrow Q_m(C(M))$

$$\begin{tikzcd} S^n \arrow[r, hook, "g"] \arrow[d] & M^m \arrow[d, "f"] \\ D^{n+1} \arrow[r, "\delta g"] & X \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^\infty T(\nu_g) \arrow[d] & \Sigma^\infty M^+ \arrow[l, "G"'] \\ * & \Sigma^\infty X^+ \arrow[l] \arrow[u, "F"] \end{tikzcd}$$

$GF \simeq * : \Sigma^\infty X^+ \to \Sigma^\infty T(\nu_g)$

$\Psi_{GF} = 0 = g^! \Psi_F + \Psi_g g^! f^!$

Ran28

$[x] \longmapsto \psi_f[x]$

$$\begin{array}{ccccc} H_m(X) & \xrightarrow{\psi_F} & Q_m(C(M)) & \to & Q_m(\mathscr{C}(f^!)) \\ \cong \downarrow f^! & \circlearrowleft & \downarrow g^! & & \\ H_m(M) & \xrightarrow{\psi_g} & Q_m(\dot{C}(T(\nu_g))) & & \end{array}$$

global ($Q_m(C(M))$), local ($Q_m(\dot{C}(T(\nu_g)))$)

$g^! : C(M) \to C(T(\nu_g)) = S^{m-n}\mathbb{Z}$

$g^!$ represents $H^{m-n}(M) \cong H_n(M)$

$$\begin{array}{ccc} K_n(M) & & \\ \downarrow & & \\ H_n(M) & \ni & \varphi \\ \downarrow f_* & & \downarrow \\ H_n(X) & \ni & 0 \end{array}$$

Lecture 11

17.11.08

S-duality is used to associate ~~a~~ to a vec bundle $\eta : X \to BO(k)$ (or even just a $(k-1)$-spherical fibration $\eta : X \to BG(k)$) a "chain bdle" over $\mathbb{Z}[\pi_1(X)]$: a $\mathbb{Z}[\pi_1(X)]$-module ch cx $C = C(\tilde{X})$, $\tilde{X}$ = univ cover of X (assume X is a finite CW-cx) and a hyperquadratic class $\gamma(\eta) \in \hat{Q}^0(C(\tilde{X})^{-*})$.

In fact, $\gamma(\eta) = \gamma(\eta \oplus \varepsilon)$, so chain bdle will only depend on the stable isom classes of η, not k.

$$\hat{\sigma}^*(X, \eta) = (C(X), \gamma(\eta) \in \hat{Q}^0(C(\tilde{X})^{-*}))$$
$$= \text{a } 0\text{-dim}^l \text{ hyperquadratic cx}$$

For any morphism of rings with involution $\mathbb{Z}[\pi_1 X] \to A$ also have a chain bdle

$$A \otimes_{\mathbb{Z}[\pi_1 X]} (C(\tilde{X}), \gamma(\eta)) = (A \otimes_{\mathbb{Z}[\pi_1 X]} C(\tilde{X}), \mathrm{id} \otimes \gamma(\eta)) \text{ over } A$$

Eg for $\mathbb{Z}[\pi_1 X] \xrightarrow{\text{augment}} \mathbb{Z}_2$ get $\mathbb{Z}_2 \otimes (C(\tilde{X}), \gamma(\eta)) = (C(X, \mathbb{Z}_2), \mathrm{id} \otimes \gamma(\eta))$, over $\mathbb{Z}_2$

$\sum n_g g \longmapsto \sum n_g$

Will prove that this is determined by the Wu class $v_r(\eta) \in H^r(X; \mathbb{Z}_2)$

In fact, $\gamma(\eta) \in \hat{Q}^0(C(\tilde{X})^{-*})$ is largely determined by $v_r(\eta)$ also.

How is $\gamma(\eta)$ defined?

Need S-dual of Y of the Thom space $T(\eta)$ with any S-duality map $\alpha : S^N \longrightarrow T(\eta) \wedge Y$ (actually $S\pi_1(X)$-dual in general)

$$\text{Thom class } U_\eta \in \dot{H}^k(T(\eta)) \cong H_{n-k}(Y) \longrightarrow Q^{N-k}(\dot{C}(Y))$$
$$= Q^{N-k}(\dot{C}(T(\eta))^{N-*})$$
$$= Q^{N-k}(\dot{C}(\tilde{X})^{N-k-*})$$
$$\downarrow$$
$$\hat{Q}^0(C(\tilde{X})^{-*})$$
$$\gamma(\eta) \in$$

S-duality

$(\dot{C}(Y) \simeq \dot{C}(T(\eta))^*$

$\dot{C}(T(\eta)) \simeq S^k C(X)$)

Thom isom

Example If X^n is a mfd (or Poincaré cx) with $\nu_X \subset S^N$ $X \to BO(k)$ the normal bundle. Take $Y = X^+$, get $N-k=n$, and

$$\cancel{U_{\nu_X}} \in \dot{H}^k(T(\nu_X)) \overset{\text{S-duality}}{\cong} H_n(X) \xrightarrow{\varphi_X} Q^n(C(X)) \underset{\text{Poincaré duality}}{\simeq} Q^n(C(X)^{n-*})$$

$U_{\nu_X} \longmapsto [X]$ only for P. cx

defining property of SNF

$$\downarrow J \qquad\qquad \downarrow$$

$$\gamma(\nu_X) \in \hat{Q}^n(C(X)) \simeq \hat{Q}^0(C(X)^{-*})$$

$\dot{\sigma}^*(X) = (C(X), \varphi_X[X])$, an n-dim'l sym P. cx

unstable, if suspend we lose P. duality

Get $J\sigma^*(X) = \hat{\sigma}^*(X, \nu_X)$, chain cx version of Wu & Thom

$$\begin{tikzcd} & \dot{C}(T(\nu_X)) \arrow[d, "S\text{-duality}"] \\ S^k C(X) \arrow[ur, "\text{Thom isom}", "\cong"'] & S^k C(X)^{n-*} \arrow[l, "{[X]^\frown}", "\sim"'] \end{tikzcd}$$

Poincaré duality

Claim/Example

ν_X = trivial

$\Rightarrow \gamma(\nu_X) = 0$

$\Rightarrow \varphi_X(U_{\nu_X}) \in \ker J$

$\Rightarrow \varphi_X(U_{\nu_X}) \in \mathrm{Im}(1+T)$

So comes from quadratic form in $Q_n(C(X))$

$$S^N \xrightarrow{\rho} T(\nu_X) \xrightarrow{\Delta} X^+ \wedge T(\nu_X)$$

Pontrjagin Thom

diagonal

this gives S-duality map

Formula of Wu and Thom

$$H^{n-r}(X;\mathbb{Z}_2) \xrightarrow{Sq^r} H^n(X;\mathbb{Z}_2)$$

$Sq^r = v_r(\nu_X)\cup -$ where $v_r(\nu_X) \in H^r(X;\mathbb{Z}_2)$ is the Wu class

The original defn of the Wu classes used the "dual" to the Steenrod squares, $\chi(Sq)^r$, instead of the Steenrod squares used to define the Stiefel-Whitney classes of $\eta : X \to BO(k)$.

$$\begin{array}{ccccccc} H^0(X;\mathbb{Z}_2) \cong & \dot{H}^k(T(\eta);\mathbb{Z}_2) & \xrightarrow{Sq^r} & \dot{H}^{k+r}(T(\eta);\mathbb{Z}_2) & \cong & H^r(X;\mathbb{Z}_2) \\ 1 & U_\eta & & & & \text{Stiefel-Whitney } W_r(\eta) \end{array}$$

Dual Steenrod Squares

For any space Y with S-dual $S^N \to Y \wedge Z$

$$\begin{array}{ccc} H^k(Z;\mathbb{Z}_2) & \xrightarrow{Sq^r} & H^{k+r}(Z;\mathbb{Z}_2) \\ \text{S-duality } \wr\wr & & \wr\wr \text{ S-duality} \\ H_{N-k}(Y;\mathbb{Z}_2) & & H_{N-k-r}(Y;\mathbb{Z}_2) \\ \text{univ coeff thm } \wr\wr & & \wr\wr \text{ univ coeff thm} \\ H^{N-k}(Y;\mathbb{Z}_2)^* & \xrightarrow{(\chi(Sq^r))^*} & H^{N-k-r}(Y;\mathbb{Z}_2)^* \end{array}$$

where

$$\chi(Sq)^r : H^j(Y;\mathbb{Z}_2) \longrightarrow H^{j+r}(Y;\mathbb{Z}_2)$$

remember $Sq^r : H^j(Y;\mathbb{Z}_2) \longrightarrow H^{j+r}(Y;\mathbb{Z}_2)$

Connection $\sum_{i+j=k} Sq^i\,\chi(Sq)^j = \begin{cases} 1 & k=0 \\ 0 & \text{o/w} \end{cases}$

$\chi(Sq)^0 = \text{id}$, $\chi(Sq^1) = Sq^1$, $Sq^2 + Sq^1\chi(Sq^1) + \chi(Sq)^2 = 0$, etc

$$\begin{array}{ccccc} \dot{H}^k(T(\eta);\mathbb{Z}_2) & \longrightarrow & \dot{H}^{k+r}(T(\eta);\mathbb{Z}_2) \cong H^r(X;\mathbb{Z}_2) & & \\ U_\eta & \xrightarrow{Sq^r} & W_r(\eta) & & \text{Thom's defn of Stiefel-Whitney class} \\ U_\eta & \xrightarrow{\chi(Sq)^r} & V_r(\eta) & & \text{Thom's defn of Wu class} \end{array}$$

(Steenrod squares tell you cup product structure)

So
$$\boxed{\begin{array}{l} V_r(\eta) = \sum_{p+q=r} \chi(Sq)^r(W_q(\eta)) \\ W_r(\eta) = \sum_{p+q=r} Sq^r(V_r(\eta)) \end{array}}$$

Standard 1950s application of " $J\sigma^*(X) = \hat{\sigma}^*(\nu_X)$ " :

If X is an n-dim^l Poincaré duality space with SNF $\nu_X : X \to BG(k)$

then $H^{n-r}(X) \xrightarrow{Sq^r} H^n(X) = \mathbb{Z}_2 \quad Sq^r = V_r(\nu_X) \cup -$

$Sq^r(x) = V_r(\nu_X) \cup x \in H^n(X;\mathbb{Z}_2)$ for $x \in H^{n-r}(X;\mathbb{Z}_2)$ Top dim^n only

Hyperquadratic Wu classes of $\hat{\sigma}^*(\eta)$ = usual Wu classes ~~of~~ $V_*(\eta) \in H^*(X;\mathbb{Z}_2)$

$n=2r \quad H^r(X) \xrightarrow{Sq^r} H^{2r}(X) = \mathbb{Z}_2$

$\qquad x \longmapsto x \cup x \cong \langle x \cup x, [X] \rangle = \langle x \cup V_r(\nu_X), [X] \rangle$

<u>Example</u> $\dim X = 4$, $V_2(\nu_X) \in H^2(X;\mathbb{Z}_2)$. Take $X = \mathbb{C}P^2 = S^2 \cup_\eta D^4$

where $\eta : S^3 \to S^2$ is Hopf map

$w_2(\nu_{\mathbb{C}P^2}) = v_2(\nu_{\mathbb{C}P^2}) = 1_2 \in H^2(\mathbb{C}P^2;\mathbb{Z}_2) = \mathbb{Z}_2$

$1_2 \cup 1_2 = 1_4 \in H^4(\mathbb{C}P^2;\mathbb{Z}_2) = \mathbb{Z}_2$

<u>Def^n</u> An S-duality between ptd spaces X, Y is a map $\alpha : S^N \to X \wedge Y$

st the slant products with $\alpha[S^N] \in \dot{H}_N(X \wedge Y)$ are isoms

$$\alpha[S^N] \backslash : \dot{H}^*(X) \xrightarrow{\cong} \dot{H}_{N-*}(Y)$$

(We will always assume X, Y are finite CW-cxs.)

<u>Recall</u> $\backslash : \dot{H}_N(X \wedge Y) \otimes \dot{H}^r(X) \longrightarrow \dot{H}_{N-r}(Y)$

$$(x \otimes y) \otimes z \longmapsto z(x)y$$

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<u>Theorem</u> (1) Every X admits an S-dual Y

(2) If $\alpha : S^N \longrightarrow X \wedge Y$ is an S-dual, then so is

$$\Sigma^k \alpha : \Sigma^k S^N = S^{N+k} \longrightarrow X \wedge \Sigma^k Y$$

(3) If $\alpha : S^N \longrightarrow X \wedge Y$ is an S-dual, then

Recall $\{W;X\}$ = stable homotopies $\{W;X\} := \lim_{\substack{\to \\ k}} [\Sigma^k W, \Sigma^k X]$

$$\{W;X\} \cong \{W \wedge Y; S^N\}$$

$$(W \wedge S^N \xrightarrow{1 \wedge \alpha} W \wedge X \wedge Y = X \wedge W \wedge Y \xrightarrow{1 \wedge g} X \wedge S^N) \longleftarrow\!\shortmid (W \wedge Y \xrightarrow{g} S^N)$$

$$\{X;W\} \cong \{S^N; Y \wedge W\}$$

$$(X \xrightarrow{f} W) \longmapsto (S^N \xrightarrow{\alpha} X \wedge Y \xrightarrow{f \wedge 1} W \wedge Y)$$

are isomomorphisms for any W.

(0) If $\alpha : S^N \longrightarrow X \wedge Y$ is an S-dual, then so is

$$T\alpha : S^N \longrightarrow Y \wedge X$$

$\left(\alpha[S^N] \backslash : C(X)^{N-*} \to C(X) \text{ a chain eqvce}, \ T\alpha[S^N] \backslash - = (\alpha[S^N] \backslash -)^* \right)$

<u>Proof</u>

(0) is kind of obvious, (2) is obvious

(3) First do special case $W = S^m$

$\{S^m; X\} \xleftarrow{\cong} \{S^m \wedge Y; S^N\}$ $\qquad H^{N-m}(Y) \cong H_m(X)$ using stable

$\pi_m^s(X) \longleftarrow \pi_s^{N-m}(Y)$ $\qquad \Rightarrow \pi_s^{N-m}(Y) \cong \pi_m^s(X)$ Hurewicz

and induction on dimension.

(1) assuming (3). Detach cells

Given S-dual $\alpha : S^N \longrightarrow X \wedge Y$, let $X' = X \cup_f D^m$ for $f : S^{m-1} \to X$

So $f \in \pi_{m-1}^s(X) \cong \pi_s^{N-m+1}(Y) \ni f'$, so $f' : \Sigma^k Y \longrightarrow S^{N+k-m+1}$

Define $Y' = S^{N+k-m+1} \cup_{f'} \mathscr{C}(\Sigma^k Y)$ $\qquad H^*(Y') = H_{N'-*}(X')$

$$\alpha' : S^{N+k-m+2 = N'} \longrightarrow X' \wedge Y'$$

So have nice exact seq of chain cxs, mapping cone of single cell $\to X$

$$0 \longrightarrow C(X) \longrightarrow C(X') \longrightarrow S^n \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow S^{N'} \mathbb{Z} \longrightarrow C(Y') \longrightarrow S^{k+1} C(Y) \longrightarrow 0$$

"$\square$"

$W \overset{\text{htpy eq'vce}}{\simeq} X$

Let $X \subset S^N$ be a finite subcx, with closed, regular nbhd $(W, \partial W)$ W a codim 0 submfld of S

Claim $W/\partial W$ is an S-dual of $X^+ = X \amalg \{pt\}$

$$\alpha : S^N \longrightarrow S^N/_{S^N \setminus W} \simeq W/_{\partial W} \xrightarrow{\Delta} \frac{W \times W}{W \times \partial W} = W^+ \wedge W/_{\partial W} \simeq X^+ \wedge W/_{\partial W}$$

$$\begin{array}{ccc} \alpha[S^N])\backslash - : \dot{H}^*(X^+) = H^*(X) & \longrightarrow & H_{N-*}(W, \partial W) \\ \cong & \nearrow \cong & [W]\cap - \text{ Poincaré - Lefschetz duality} \\ H^*(W) & & \end{array}$$

Lecture 12. 20.11.08

What is the point of chain bundles?

i) To explain, on the chain level, the relationship between the SNF of a geo P. cx X and the algebraic topology of X itself.

In particular, if $(f,b): M \longrightarrow X$ is a normal map determined by a TOP reduction $\tilde{\nu}_X : X \longrightarrow BTOP$ then the quadratic structure $\psi_F[X] \in Q_n(\mathcal{C}(f^!))$ representing the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1 X])$ is the chain level expression of the matching ~~ub~~ up by b of the chain bundles of ν_M and $\tilde{\nu}_X$

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \tilde{\nu}_X \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

$\rho \in \pi_{n+k}(T(\nu_X)) \xrightarrow{\text{Hurewicz}} H_{n+k}(T(\nu_X)) = H_n(X) \ni [X]$

$[X] \cap - : H^{n-*}(X) \xrightarrow{\cong} H_*(X)$

$$\begin{array}{ccc} S^{n+k} & \longrightarrow & T(\nu_X) \\ \cup & & \cup \\ \rho^{-1}(X) = M & \xrightarrow{(f,b)} & X \end{array}$$

$$\begin{array}{c} BTOP \\ \downarrow \\ X \longrightarrow BG \end{array}$$

TOP (or O) transversality!

$$\varinjlim_k \pi_{n+k}(T(\tilde{\nu}_X)) = \Omega_n^{TOP}(X, \tilde{\nu}_X)$$

$\rho \longmapsto$ bordism class of (f,b)

Cannot decide from transversality construction of (f,b) if $f: M \to X$ is a homotopy equivalence. (In fact, for $n \geq 5$, it is necessary and sufficient that $\sigma_*(f,b) = (C(f^!), e_\% U_F[X])$ be 0.)

$$\begin{tikzcd}
G/TOP \arrow[r] & BTOP \arrow[r] & BG \arrow[r] & B(G/TOP) \\
 & & X \arrow[u] \arrow[ur, dashed, "\text{topological K-theory obstruction}"'] &
\end{tikzcd}$$

ii) Can use a "local" version of the chain cx theory and in particular the chain bundle theory, st the question of whether X is htpy eqvt to a TOP mfd is entirely decided on the chain level.

The <u>total surgery obstruction</u> is

$s(X) \in \mathcal{S}_n(X)$ = abelian gp st for $n \geq 5$

$X \simeq$ TOP mfd $\Leftrightarrow s(X) = 0$

$$\to H_n(X;\mathbb{L}) \to L_n(\mathbb{Z}[\pi_1 X]) \to \mathcal{S}_n(X) \to H_{n-1}(X;\mathbb{L}) \to$$

<u>Chain bdles over a ring with involution, A</u>

<u>Defn.</u> A <u>chain bundle over A</u>, (C,γ), is an A-module chain cx with a cycle $\gamma \in (\hat{W}^{\%} C^{-*})_0$ (corresp to an elt $[\gamma] \in \hat{Q}^0(C^{-*})$) as represented by γ_s,

$\gamma_s : C_{-r} \to C^{r-s} \quad (s \in \mathbb{Z})$

st $d^*\gamma_s + \gamma_s d + \gamma_{s-1} + \gamma_{s-1}^* = 0$

An <u>equivalence</u> $\chi : \gamma \simeq \gamma'$ of chain bdles has $\gamma_s - \gamma'_s = d^*\chi_s + \chi_s d + \chi_{s-1} + \chi_s^*$

where $\chi_s : C_{-r} \to C^{r-s-1}$

Example. Spse $C_r = 0 \ \forall r \neq 0$, $\gamma = \{\gamma_0 : C_0 \to C^0\}$ st $\gamma_0 + \gamma_0^* = 0$

Example. $2\gamma \simeq 0$ with $\chi_s = \gamma_{s+1}$, for symmetric $2\gamma_0 = \gamma_0 + \gamma_0^* \simeq 0$

$$C_0 = A \oplus A \Rightarrow \gamma_0 = \begin{pmatrix} a & b \\ a\bar{b} & c \end{pmatrix} \overset{?}{=} \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \bar{b} & 0 \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

$C_0 = \mathbb{Z}$ then get seq of 1's and 0's.

Note: Always have $\hat{Q}^n(C \oplus D) = \hat{Q}^n(C) \oplus \hat{Q}^n(D)$ is additive.

$$\begin{array}{ccccccc}
Q_n(C\oplus D) & = & Q_n(C) & \oplus & Q_n(D) & \oplus & H_n(C \otimes_A D) \\
\downarrow 1+T & & \downarrow 1+T & & \downarrow & & \wr\wr \\
Q^n(C\oplus D) & = & Q^n(C) & \oplus & Q^n(D) & \oplus & H_n(C \otimes_A D) \\
\downarrow & & \downarrow & & \downarrow & & \\
\hat{Q}^n(C\oplus D) & = & \hat{Q}^n(C) & \oplus & \hat{Q}^n(D) & &
\end{array}$$

$\hat{Q}^0(A \oplus A) = \hat{Q}^0(A) \oplus \hat{Q}^0(A)$, and $\hat{Q}^0(A) = \{a \in A \mid a = \bar{a}\} / \{b + \bar{b} \mid b \in A\}$ all that matters are Wu classes

Thm. The hyperquadratic Wu classes

$$\hat{v}_r : \hat{Q}^n(C) \longrightarrow \mathrm{Hom}_A(H^{n-r}(C), \hat{Q}^{(-1)^r}(A))$$
$$\gamma \longmapsto (f \longmapsto f\gamma_{n-2r} f^*)$$

are $\mathbb{Z}$-module morphisms, which are the cpts of an isom

$$\hat{v}_* : \hat{Q}^{*}(C) \longrightarrow \prod_{r \in \mathbb{Z}} H_{*-r}(C; \hat{Q}^{(-1)^r}(A))$$ both are GHTs

Proof (idea)

(certainly true for $A = \mathbb{Z}[\pi]$), proved by looking at functional properties of

$$C \longrightarrow \hat{Q}^*(C)$$
$$C \longrightarrow \prod \cancel{\mathrm{Hom}_A(H^{n-r}(C), \hat{Q}}\ H_{n-r}(C; \hat{Q}^{(-1)^r}(A))$$

Natural transformation of GHTs that are isom on a point

SES of chain cxs Ran 32

$0 \to C \to D \to E \to 0$ induces a LES

$$\to \hat{Q}^n(C) \to \hat{Q}^n(D) \to \hat{Q}^n(E) \to \hat{Q}^{n-1}(C) \to$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\to \hat{\Pi} H_{n-r}(C; \ ) \to \hat{\Pi} H_{n-r}(D; \ ) \to \hat{\Pi} H_{n-r}(E; \ ) \to \hat{\Pi} H_{n-1-r}(C; \hat{Q}^{(-1)^r}(A)) \to$$

__Example__ $C: C_1 \xrightarrow{d} C_0$, $[\gamma] \in \hat{Q}^0(C^{-*})$

$$\begin{tikzcd} C_1 \arrow[d, "d"'] \arrow[dr, "\tilde{\gamma}_1"] \arrow[ddr, "\gamma_2"'] & \\ C_0 \arrow[r, "\gamma_0"] \arrow[dr, "\gamma_1"'] & C^0 \arrow[d, "d^*"] \\ & C^1 \end{tikzcd}$$

Have	Get 4 eqn s
$\gamma_0 : C_0 \to C^0$	$\gamma_0 - \gamma_0^* = 0$
$\gamma_{-1} : C_0 \to C^1$	$\gamma_{-1} - \tilde{\gamma}_{-1}^* = d^*\gamma_0$
$\tilde{\gamma}_{-1} : C_1 \to C^0$	$\tilde{\gamma}_{-1} - \gamma_{-1}^* = \gamma_0 d$
$\gamma_{-2} : C_1 \to C^1$	$d^*\tilde{\gamma}_{-1} + \gamma_{-1} d = \gamma_{-2} + \gamma_{-2}^*$

$$\hat{Q}^0(C^{-*}) \longrightarrow H^0(C; \hat{Q}^0(A)) \oplus H^1(C; \hat{Q}^1(A))$$

cohomology $\gamma \longmapsto (\gamma_0, \gamma_{-2})$

Defining the classes γ_0, γ_{-2} to be represented by: $\to x \in H_0(C)$ $x \mapsto \gamma_0(x)(x)$ $\quad y \mapsto \gamma_{-2}(y)(y)$ $y \in H_1(C)$

A chain map $f: C \to D$ induces a LES

$$\to Q^n(C) \xrightarrow{f^{\%}} Q^n(D) \to Q^n(f) \to$$
$$\to Q_n(C) \xrightarrow{f_{\%}} Q_n(D) \to Q_n(f) \to$$
$$\to \hat{Q}^n(C) \xrightarrow{\hat{f}^{\%}} \hat{Q}^n(D) \to \hat{Q}^n(f) \to$$
$$\hat{Q}^n(f) \cong \hat{Q}^n(\mathcal{C}(f))$$

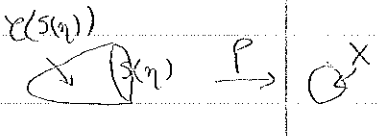
?

$$\begin{tikzcd} & & Q^n(f) \arrow[r] \arrow[dr] & Q^{n-1}(C) \\ & Q^n(D) \arrow[ur] & & Q^n(\mathcal{C}(f)) \arrow[u] \\ Q^n(C) \arrow[ur] & & & \end{tikzcd}$$

There are two chain bdle constructions:

i) Start with a sphere bdle $S^{k-1} \longrightarrow S(\eta) \xrightarrow{f} X$ and apply the hyperquadratic construction.

$T(\eta) = X \cup_\rho \mathcal{C}(S(\eta))$ = mapping cone of ρ



$$H^k(T(\eta)) = H_{N-k}(Y) \longrightarrow \hat{Q}^{N-k}(C(Y))$$

$$U_\eta \in \quad \xrightarrow{t\,\Theta_{T(\eta)}} \quad \hat{Q}^{N-k}(C(T(\eta))^{N-*}) = \hat{Q}^{N-k}(C(X)^{N-k-*})$$

Thom class

$$\longrightarrow \gamma(\eta) \in \hat{Q}^0(C(X)^{-*})$$

The Wu classes are the same $\tilde{v}_*(\gamma(\eta)) = v_*(\eta) \in H^*(X;\mathbb{Z}_2)$

ii) An n-dim^l symm Poincaré cx (C,φ) has a "Spivak normal" chain bdle $(C, \gamma_\varphi \in \hat{Q}^0(C^{-*}))$, defined by:

$$Q^n(C) \xrightarrow{J} \hat{Q}^n(C) \underset{\varphi_0 \text{ chain eqvce}}{\cong} \hat{Q}^n(C^{n-*}) \cong \hat{Q}^0(C^{-*})$$

$$\varphi \longmapsto \gamma_\varphi$$

More precisely, if $\varphi \in (W^{\%}C)_n$ is a cycle of a sym P. cx (C,φ) then there is a "natural" eqvce class of pairs (γ,χ) with $\gamma \in (\hat{W}^{\%}C^{-*})_0 = (\hat{W}^{\%}C^{n-*})$ a chain bdle.

"Recall" $\quad S : Q^n(C) \longrightarrow Q^{n+1}(SC)$

$$\{\varphi_s\} \longmapsto \{(S\varphi)_s := \begin{cases} \varphi_{s-1} & s \geq 1 \\ 0 & s = 0 \end{cases}\}$$

This is not an isom. However

$$S : \hat{Q}^n(C) \longrightarrow \hat{Q}^{n+1}(SC)$$

$$\{\chi_s\} \longmapsto \{(S\chi)_s := \chi_{s-1}\}$$

is an isom. We can put the suspension of symmetric forms into an LES

$$\longrightarrow H_n(C \otimes_A C) \longrightarrow Q^n(C) \xrightarrow{S} Q^{n+1}(SC) \longrightarrow H_{n-1}(C \otimes_A C) \longrightarrow$$

$$\Phi \longmapsto \Phi_0$$

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Given $\varphi \in Q^n(C)$, we have $\varphi_0 : C^{n-*} \longrightarrow C_*$

We also have a map $J : Q^n(C) \longrightarrow \hat{Q}^n(C)$ ~~$\hat{W}^{\%}(C) \ldots$~~

So we get an induced map

Purely notational to distinguish from $\varphi_0^{\%} : W^{\%}C^{n-*} \to W^{\%}C$ — $(\hat{\varphi}_0)^{\%} : (\hat{W}^{\%}C^{n-*} \longrightarrow \hat{W}^{\%}C_*)$

which gives $\hat{Q}^n(C) \cong \hat{Q}^n(C^{n-*})$

Thus

Note $S(C^{-*}) = C^{k+1-*}$

$$Q^n(C) \xrightarrow{J} \hat{Q}^n(C) \overset{(\hat{\varphi}_0)^{\%}}{\cong} \hat{Q}^n(C^{n-*}) \overset{S^n}{\cong} \hat{Q}^0(C^{-*})$$

$$\varphi \longmapsto J\varphi \longmapsto \gamma$$

In general, $\varphi \in (W^{\%}C)_n$, $\gamma \in (\hat{W}^{\%}C^{-*})_0$ satisfy

$$(\hat{\varphi}_0)^{\%} : S^n\gamma \longmapsto J\varphi$$

$$(\hat{W}^{\%}C^{-*})_0 \xrightarrow{S^n} (\hat{W}^{\%}C^{n-*})_n \xrightarrow{(\hat{\varphi}_0)^{\%}} (\hat{W}^{\%}C)_n \xleftarrow{J} (W^{\%}C)_n$$

$$\gamma \longmapsto S^n\gamma \longmapsto J\varphi \longleftarrow \varphi$$

Poincaré-Spivak-Wu-Thom: $J\varphi - (\hat{\varphi}_0)^{\%}(S^n\gamma) = d\chi$

for some $\chi \in (\hat{W}^{\%}C)_{n+1}$

So in particular, $d\varphi = 0 \iff d\gamma = 0$

Looking at Wu classes:

If $(C, \varphi) = \sigma^*(X)$, then the Wu classes are characterised by

$$\begin{array}{ccc} H^{n-r}(C) & \xrightarrow{v_r(\varphi)} & Q^{(-1)^r}(A) \\ \cong\downarrow \varphi_0 & & \downarrow J \\ H_r(C) & \xrightarrow{\hat{v}_r(\gamma)} & \hat{Q}^{(-1)^r}(A) \end{array}$$

$Sq^r = v_r(X) \cup - : H^{n-r}(X; \mathbb{Z}_2) \longrightarrow H^n(X; \mathbb{Z}_2)$

Moreover, these are the Wu classes of the SNF $\nu_X : X \longrightarrow BG$ (?)

$$v_r(X) = v_r(\varphi) = v_r(\nu_X) = \sum_{p+q=r} \chi(Sq)^p w_q(\nu_X)$$

<u>Defn</u>. An <u>n-dim^l normal space</u> (X, ν_X, ρ_X) is a space X together with spherical fibration $S^{k-1} \longrightarrow S(\nu_X) \longrightarrow X$ and map $\rho_X : S^{n+k} \longrightarrow T(\nu_X)$

Example An n-dim^l P. cx X is a normal space with ν_X the SNF

Defn. An n-dim^l normal chain cx $(C, \varphi, \gamma, \chi)$ is a chain cx C together with

$$\varphi \in (W^{\%} C)_n \quad , \quad \gamma \in (\hat{W}^{\%} C^{-*})_0 \quad , \quad \chi \in (\hat{W}^{\%} C)_{n+1}$$

st

$$d\varphi = 0 \; , \; d\gamma = 0 \quad \text{and} \quad J\varphi - (\hat{\varphi}_0)^{\%}(S^n \gamma) = d\chi$$

Preview for next week:

For any chain bundle (C, γ), we can define a long exact sequence

$$\longrightarrow Q_n(C,\gamma) \longrightarrow Q^n(C) \xrightarrow{J_\gamma} \hat{Q}^n(C) \longrightarrow Q_{n-1}(C,\gamma) \longrightarrow$$

$$(\varphi, \chi) \in \qquad \varphi' \longmapsto J\varphi' - (\hat{\varphi}'_0)^{\%}(S^n \gamma)$$

Put in non-linear term to restore linearity:

$$(\tilde{\varphi}, \tilde{\chi}) + (\varphi', \chi') := (\tilde{\varphi} + \varphi', \; \tilde{\chi} + \chi' + \tilde{\varphi}_0 \gamma \varphi_0'^{*})$$

If $(C, \gamma) = (C(X), \gamma(\nu))$ for some spherical fibration $S^{k-1} \longrightarrow S(\nu) \longrightarrow X$ then we get

$$[X] \in \; \cong H_n(X)$$

$$\begin{tikzcd}
\longrightarrow \Pi'_{n+k}(T(\nu)) \arrow[r] \arrow[d] & \pi_{n+k}(T(\nu)) \arrow[r, "\text{Hurewicz}"] \arrow[d] \arrow[r, phantom, "\text{commutes}", shift right=20] & H_{n+k}(T(\nu)) \arrow[r] \arrow[d, "\varphi_X"] & \Pi'_{n+k-1}(T(\nu)) \longrightarrow \arrow[d] \\
\longrightarrow \hat{Q}^{n+1}(C) \arrow[r] & Q_n(C(X), \gamma(\nu_X)) \arrow[r] & Q^n(C(X)) \arrow[r, "J_{\gamma(\nu_X)}"] & \hat{Q}^n(C) \longrightarrow
\end{tikzcd}$$

with ρ_C over $\pi_{n+k}(T(\nu))$, and

$$(\varphi_X[X], \chi(\nu_X)) \longmapsto \varphi_X[X]$$

where $\Pi'_{n+k}(T(\nu))$ is some group, an early example of quadratic forms.

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Ranicki. Lecture 13

Normal Spaces

Defn (Quinn 1972) An n-diml normal space (X, ν_X, ρ_X) is a space X (usually a finite CW-cx) together with a spherical fibration* $\nu_X : X \longrightarrow BG(k)$ (= classifying space for fibrations $S^{k-1} \longrightarrow S(\nu_X) \xrightarrow{p_{\nu_X}} X$) and a map

$$\rho_X : S^{n+k} \longrightarrow T(\nu_X) = \text{mapping cone of } p_{\nu_X} = X \cup_{p_{\nu_X}} \mathcal{C} S(\nu_X)$$

"spherical Thom class" = Thom space of ν_X.

The fundamental class of X is the image of ρ_X under the composite

$$\rho_X \in \pi_{n+k}(T(\nu_X)) \xrightarrow{\text{Hurewicz}} \tilde{H}_{n+k}(T(\nu_X)) \xrightarrow{\text{Thom isom}} H_n(X) \ni [X]$$

The chain map $[X] \cap - : C(X)^{n-*} \longrightarrow C(X)$ is the "would be" duality chain map.

Example. An n-diml geometric Poincaré cx X carries the structure (ν_X, ρ_X) of an n-diml normal space with ν_X the Spivak Normal Fibration, using $X \subset S^{n+k}$ with regular nbhd $(W, \partial W)$

$$S^{k-1} \longrightarrow \partial W \hookrightarrow W \simeq X \qquad \partial W = S(\nu_X)$$

$$\rho_X : S^{n+k} \longrightarrow S^{n+k}/S^{n+k} \backslash W \simeq W/\partial W = T(\nu_X)$$

In this case, the "would be" duality is an actual duality.

Defn. An (n+1)-diml normal pair

$(X, \partial X \subset X, \nu_X : X \longrightarrow BG(k), \rho_{\partial X} : S^{n+k} \longrightarrow T(\nu_X|_{\partial X}))$ st

$$\begin{tikzcd} S^{n+k+1} \arrow[r, "\rho_X"] \arrow[dr, "\Sigma \rho_{\partial X}"'] & T(\nu_X)/T(\nu_X|_{\partial X}) \arrow[d] \\ & \Sigma T(\nu_X|_{\partial X}) \end{tikzcd}$$

commutes

Similarly for cobordism of normal spaces $(X; \partial_0 X, \partial_1 X)$
Fundamental class $[X] \in H_{n+1}(X, \partial X)$ image of $\rho_X \in \pi_{n+k+1}\left(T(\nu_X)/T(\nu_X|_{\partial X})\right)$

Example. An $(n+1)$-dim^l geometric Poincaré $\left\{\begin{array}{l}\text{complex}\\ \text{pair}\\ \text{cobordism}\end{array}\right.$ is an $(n+1)$-dim^l normal $\left\{\begin{array}{l}\text{complex}\\ \text{pair}\\ \text{cobordism}\end{array}\right.$ with duality.

Thm. Given a space X, and a spherical fibration $\nu : X \to BG(k)$,
let $\Omega^N_n(X, \nu)$ be the bordism group of n-dim^l normal spaces (Y, ν_Y, ρ_Y),
with a normal map $(f, b) : (Y, \nu_Y) \longrightarrow (X, \nu)$
(ie $f : Y \to X$ and $b : \nu_Y \longrightarrow \nu$ is a stable map of spherical fibrations)
then

(i) The normal space Pontrjagin-Thom map $\Omega^N_n(X, \nu) \longrightarrow \varinjlim_k \pi_{n+k}(T(\nu))$ $\quad \pi_{n+k+1}(T(\nu \oplus \epsilon)) \cong \pi_{n+k+1}(\Sigma T(\nu))$
is an isom. $\quad (Y, f, b) \longmapsto T(b)_* \rho_Y$
It has inverse $\varinjlim_k \pi_{n+k}(T(\nu)) \longrightarrow \Omega^N_n(X, \nu)$
$$(\rho : S^{n+k} \to T(\nu)) \longmapsto (X, \nu, \rho)$$

(ii) For $n \geq 5$ ~~and~~ an exact sequence
$$\longrightarrow L_n(\mathbb{Z}[\pi_1 X]) \longrightarrow \Omega^P_n(X, \nu) \longrightarrow \Omega^N_n(X, \nu) \xrightarrow{\sigma_*} L_{n-1}(\mathbb{Z}[\pi_1 X]) \to$$

where $\Omega^P_n(X, \nu)$ is bordism of normal maps $(f, b) : Y \to X$ with
Y an n-dim^l Poincaré cx.
$$\sigma_* : \Omega^N_n(X, \nu) \longrightarrow L_{n-1}(\mathbb{Z}[\pi_1 X])$$
an $(n-1)$-dim^l quadratic Poincaré cx
$$(Y, \nu_Y, \rho_Y) \longmapsto (\mathcal{C}([Y] \cap - : \tilde{C}(Y)^{n-*} \to \tilde{C}(Y))_{*+1}, \psi)$$

Example. Let $(f, b) : M \to X$ be a normal map in the sense of Browder-Wall. So M is an ~~A-dim~~ n-mfd, X is an n-Poincaré cx, $f : M \to X$ is a degree one map, $b : \nu_M \longrightarrow \tilde{\nu}_X$ stable bundle map

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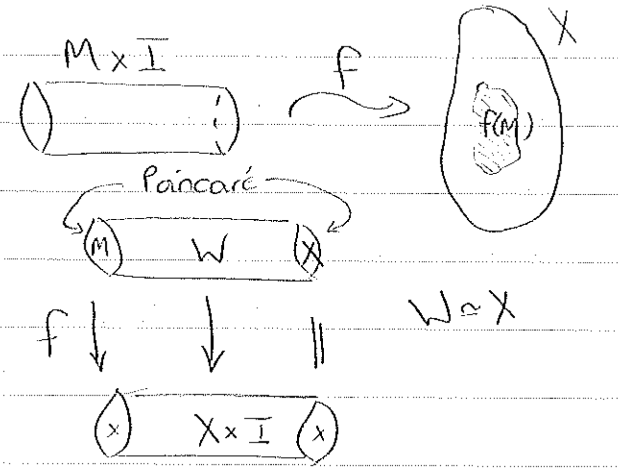
Let W = mapping cylinder of f,

$W = M \times I \cup_{(m,1)=f(m)} X$



Then $(W; M\times\{0\}, X)$ is an $(n+1)$-dim^{l} normal space bordism over X with "would be" duality chain map

$$\mathscr{C}(\underbrace{C(W,X)}_{\simeq 0}{}^{n+1-*} \longrightarrow C(\overset{X}{\overset{\wr\wr}{W}},M))_{*+1}$$

$$\simeq \mathscr{C}(f^! : C(X) \longrightarrow C(M)) \qquad \mathscr{C}(f)^{n-*} \simeq \mathscr{C}(f)_{*+1} \simeq \mathscr{C}(f^!)$$

The obstruction to this (normal, Poincaré) bordism being rel-∂ bordant to a Poincaré cobordism is $\sigma_*(f,b)$ = Wall surgery obstruction $= (\mathscr{C}(f^!), \psi) \in L_n(\mathbb{Z}[\pi_1(X)])$

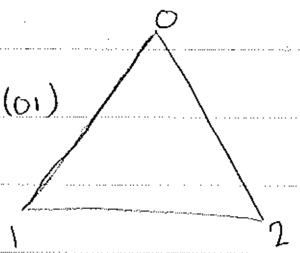
Where do normal ~~maps~~ spaces come from?

From any n-dim^{l} Poincaré cx X. For convenience, assume that X is a finite simplicial cx, with barycentric subdivision X'. At each simplex σ in X, define the dual cell $D(\sigma, X)$ to be

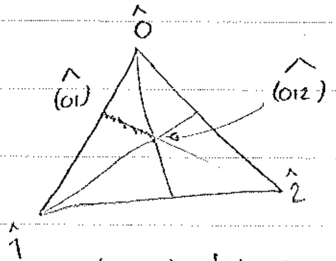
$$D(\sigma, X) = \{\hat{\sigma}_0\hat{\sigma}_1\ldots\hat{\sigma}_k \mid \sigma \leq \sigma_0 \leq \ldots \leq \sigma_k \text{ in } X\}$$

where $\hat{\sigma}$ is the barycentre of σ.

Eg $X = \Delta^2$ $\quad \sigma = (01)$



X'



$01 \leq 01 \leq 012$

?

$\partial D(\sigma, X) = \{\widehat{(012)}, \widehat{(01)}\}$

$D(\sigma, X) = \{\hat{01}, \hat{012}, (\hat{01}\hat{012})\}$

So $D(\sigma, X)$ = contractible space, the cone on $\hat{\sigma}$

$= \hat{\sigma} * \partial D(\sigma, X)$

Claim. Each $(D(\sigma,X), \partial D(\sigma,X))$ is an $(n-|\sigma|)$-dim^l normal pair with would be duality $[D(\sigma,X)] \cap - : C(D(\sigma,X))^{n-|\sigma|-*} \longrightarrow C(D(\sigma,X), \partial D(\sigma,X))$ is a chain eqvce $\Leftrightarrow$ $H_*(\partial D(\sigma,X)) \cong H_*(S^{n-|\sigma|-1})$ $\forall \sigma$ in X

$\Leftrightarrow$ X is an homology n-mfd

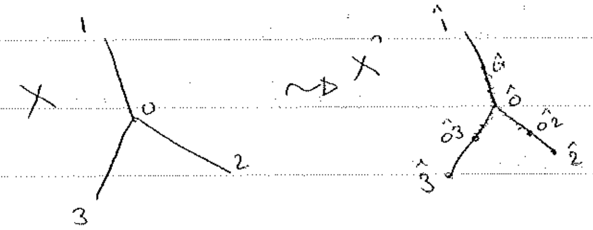
$\Leftrightarrow$ $\forall x \in X$, $H_*(X, X\backslash\{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) = \begin{cases} \mathbb{Z} & \text{if } *=n \\ 0 & \text{o/w} \end{cases}$

Note that for any simplicial complex X; $H_*(D(\sigma,X), \partial D(\sigma,X)) \overset{\text{suspension isom}}{\cong} H_{*+|\sigma|}(X, X\backslash\{\hat{\sigma}\})$

And $D(\sigma,X) = \text{star}_{X'}(\hat{\sigma})$ = all sxs τ for which $\exists$ sx ρ with $\hat{\sigma} \in \rho$, $\tau \leq \rho$

$\partial D(\sigma,X) = \text{Link}_{X'}(\hat{\sigma}) = \{\tau \in \text{star}_{X'}(\hat{\sigma}) \mid v \notin \tau\}$

Eg



$\sigma = \{0\}$, $D(\sigma,X) = \{\hat{0}, \hat{0}\hat{1}, \hat{0}\hat{2}, \hat{0}\hat{3}, \hat{0}\hat{0}\hat{1}, \hat{0}\hat{0}\hat{2}, \hat{0}\hat{0}\hat{3}\}$

$\partial D(\sigma,X) = \{\hat{0}\hat{1}, \hat{0}\hat{2}, \hat{0}\hat{3}\}$

~~Thus an n-dim^l geo P. cx X determines a collection $\{(C(D(\sigma,X), \partial D(\sigma,X))_{*+1}, \psi) \mid \sigma$ in $X\}$ of $(n-|\sigma|-1)$-dim^l quad. P. cxs over $\mathbb{Z}$ normal pairs and hence a collection $\{(C(D(\sigma,X)), \partial D(\sigma,X))_{*+1}, \psi$~~

Thus an n-dim^l geo P. cx X determines a collection $\{(D(\sigma,X), \partial D(\sigma,X)) \mid \sigma \in X\}$ of $(n-|\sigma|)$-dim^l normal pairs, and hence a collection

$\{C(\sigma) := \mathcal{C}(C(D(\sigma,X))^{n-|\sigma|-*} \longrightarrow C(D(\sigma,X), \partial D(\sigma,X)))_{*+1} \mid \sigma \in X\}$ of

$(n-|\sigma|-1)$-dim^l quad. P. cxs over $\mathbb{Z}$.

MAIN THM OF THE COURSE

For $n \geq 5$, X is htpy eqvt to an n-dim^l top^l mfd iff the collection $\{(C(\sigma), \psi(\sigma)) \mid \sigma \in X\}$ is cobordant to ~~a zero~~ 0 by a collection of quad. P. cobordisms $\{(C(\sigma) \to \delta C(\sigma), (\delta\psi(\sigma), \psi(\sigma))) \mid \sigma \in X\}$ with assembly $A_\sigma(\delta C(\sigma)) \simeq 0$ over $\mathbb{Z}[\pi_1 X]$ ie, $A(C) \simeq 0$

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"Assembly" of a "collection" $\{B(\sigma) \mid \sigma \in X\}$ of $\mathbb{Z}$-module chain cxs and chain maps $\{B(\sigma) \longrightarrow B(\tau) \mid \sigma \geq \tau \text{ in } X\}$ is the $\mathbb{Z}[\pi_1(X)]$-module chain ~~homom~~ cx

also a simplicial cx

$A(B) := B(\tilde{X}) = \sum_{\tilde{\sigma} \in \tilde{X}} B(p(\tilde{\sigma}))$ where $p : \tilde{X} \longrightarrow X$ is univ cover

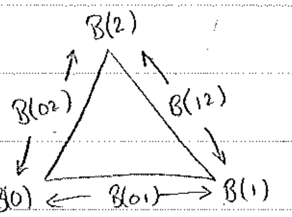
$= \sum_{\sigma \in X} \mathbb{Z}[\pi_1 X] \otimes_{\mathbb{Z}} B(\sigma)$

For $C = \left\{ C(\sigma) = \mathcal{C}\left([D(\sigma, X) \cap -] : C(D(\sigma, X))^{n-|\sigma|-*} \longrightarrow C(D(\sigma, X), \partial D(\sigma, X))\right) \mid \sigma \in X \right\}$

~~$\mid \sigma \in X$ and $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{\hat{\sigma}\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & * \neq n \end{cases}$~~

$A(C) = \mathcal{C}\left([X] \cap - : C(\tilde{X})^{n-*} \xrightarrow{\simeq} C(\tilde{X})\right) \simeq 0$

Eg $X = \partial \Delta^2$



A $\mathbb{Z}[t, t^{-1}]$-module chain cx

$B(01) \to B(1) \leftarrow B(12) \to B(2) \leftarrow B(02) \to B(0) \leftarrow tB(01) \to tB(1) \leftarrow tB(1\ldots$

Lecture 14 26. 11. 08

How will the main thm of the course be proved?

Recall: Main Thm. For $n \geq 5$, an n-dim^l geo P. cx is htpy eqvt to an n-dim^l top^l mfd iff the system of ~~geo~~ quadratic P. cxs over $\mathbb{Z}$ measuring the failures of the links of simplices in X (assumed to be a simp cx) to be homology spheres are $\mathbb{Z}[\pi_1 X]$-contractibly quadratic null-cobordant.

We use the traditional Browder - Novikov - Sullivan - Wall 2-stage (top^l K-theory and algebraic L-theory) obstruction theory.

$X \simeq$ top^l mfd $\Leftrightarrow$ $\exists$ top reduction $\tilde{\nu}_X$ of SNF ν_X st the corresp normal map $(f, b) : M \longrightarrow X$ has

$\sigma_*(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1 X])$

Exotic spheres (Milnor, 1956) used the failure of the Hirzebuch signature thm for mfds with bdy to distinguish diff^{ble} structures on particular htpy 7-spheres Σ^7 (= 7 dim^l diff^{ble} mfds of the type $S(\omega)$ for $\omega \in \pi_4(BSO(4)) = \pi_3(SO(4))$).

$$(D^4, S^3) \longrightarrow (\underbrace{D(\omega)^8}_{\text{3-conn 8-mfd with } \partial}, S(\omega)) \longrightarrow S^4$$

$S(\omega) \overset{\text{htpy eqvce}}{\simeq} S^7$ and in fact homeom but not diffeom

(Milnor, 1958) "On the Whitehead homomorphism J"

$$J : \pi_{r-1}(SO(n)) \longrightarrow \pi_{n+r-1}(S^n) \overset{PT}{=} \{\overbrace{\text{framed } (r-1)\text{-mfds}}^{M^{r-1}\times D^n} \subseteq S^{n+r-1}\}$$

$$(\omega : S^{r-1} \to SO(n)) \longmapsto (S^{r-1}\times D^n \hookrightarrow \overbrace{S^{r-1}\times D^n \cup D^r \times S^{n-1}}^{S^{r-1}\times D^n \cup D^r\times S^{n-1}})$$

$$(x, y) \longmapsto (x, \omega(x)(y))$$

Thm 1. (Milnor, 1958). Let ξ be the $\mathbb{R}^n$-bundle over S^r corresp to an elt $\lambda \in \pi_{r-1}(SO(n)) = \pi_r(BSO(n))$. ie $\lambda \in \ker(J)$

If $J(\lambda) = 0 \in \pi_{n+r-1}(S^n)$ then

$\exists$ oriented mfd $M^r \subseteq S^{n+r}$ st for some degree 1 map $g : M^r \to S^r$

$\exists$ a bundle map $b : \nu_{M\subseteq S^{n+r}} \longrightarrow \xi$

(first example of a normal map)

The modern take on this thm:

If X is an r-dim^l Poincaré cx (in thm, $X = S^r$) and there is given a vector bundle reduction $\hat\nu_X$ of ν_X (in thm, $\nu_{S^r\subseteq S^{n+r}} \underset{\text{can be chosen canonically}}{\cong} \varepsilon^n$)

~~where ν_X is the normal bundle of any embedding $X \subseteq S^{n+r}$, r large, (in thm, $\tilde\nu_{S^r\subseteq S^{n+r}} = \xi$).~~

where ν_X is the Spivak normal fibration (= trivial for $X = S^r$ since $\nu_{S^r\subseteq S^{n+r}} \cong \varepsilon^n$, $\hat\nu_X = \xi$ = n-plane vec' bdle over S^r with $J(\xi) = 0 \in \pi_{n+r-1}(S^r)$)

$$\begin{tikzcd}
S^{n+r} \arrow[r, "\binom{0}{1}"] & T(\nu_{S^r\subseteq S^{n+r}}) \cong T(\xi) \\
 & = T(\varepsilon^n) = S^n \vee S^{n+r} \\
M^r \arrow[uu, "b"] \arrow[r, "g = \text{restriction}"] & S^r \arrow[u]
\end{tikzcd}$$

fibre htpy eqvce

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(Kervaire, Milnor, 1961) Classification of exotic spheres in dim$^n \geq 5$ using surgery methods, homotopy theory, and quadratic form theory: in fact they computed $L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & n \equiv 0\,(4) \text{ signature} \\ \mathbb{Z}_2 & n \equiv 2\,(4) \text{ Arf invt} \\ 0 & n \equiv 1,3\,(4) \end{cases}$

Defn. $\theta_n :=$ group of h-cobordisms classes of $\Sigma^n \overset{\text{htpy eqvce}}{\simeq} S^n$ htpy eqvce ⇒ homeom

First example of the surgery exact sequence

$$\longrightarrow L_{n+1}(\mathbb{Z}) \longrightarrow \underset{\mathcal{S}^{Diff}(S^n)}{\overset{\parallel}{\theta_n}} \longrightarrow \pi_n(G/O) \longrightarrow L_n(\mathbb{Z}) \longrightarrow$$

$$\pi_n(G/O) \downarrow \pi_n(BO) = \pi_{n-1}(O) \xrightarrow{J} \pi_n(BG) = \pi^S_{n-1}$$

(Browder, 1962). Used K-M methods to prove that for $n \geq 5$ a simply-connected n-dim P. cx X is htpy eqvt to a diff^{ble} r-mfd $\exists \mathbb{R}^n$-bdl. ξ over X with a degree one map $S^{n+r} \longrightarrow T(\xi)$

$$L_r(\mathbb{Z}) = \begin{cases} \text{OK for } n \text{ odd.} & \text{converse of Hirzebuch th}^m \\ \text{Signature}(X) = \langle \mathcal{L}(\xi), [X] \rangle & \text{for } r = 4s \\ \text{Arf invt constructed} \in \mathbb{Z}_2 & \text{for } r = 4s+2 \end{cases}$$

Simplicial Sets

We know that BO classifies vector bundles. Similarly BPL - PL bundles, BTOP - TOP bundles, BG - spherical fibrations.

Defn. A simplicial set K is a sequence of sets $K^{(0)}, K^{(1)}, K^{(2)}, \ldots$ with fns $\partial_i : K^{(n)} \longrightarrow K^{(n-1)}$ for $0 \leq i \leq n$

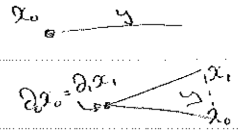
satisfying $\partial_i \partial_j = \partial_{j-1} \partial_i$ ~~for~~ for $i < j$

(and also degeneracies $S_i : K^{(n)} \longrightarrow K^{(n+1)}$)

Δ-sets := simp sets without degeneracies and each $K^{(n)}$ is a pted set. $\phi^{(n)} \in K^{(n)}$

Main Example. For any space X, let $K =$ "singular cx Δ-set",
ie $K^{(i)} = \mathrm{map}(\Delta^i, X) =$ singular simplices $\Delta^i \longrightarrow X$
$\partial_i : \Delta^{n-1} \longrightarrow \Delta$ inclusion of i^{th} face, $\partial_i^* = \partial_i : K^{(n)} \longrightarrow K^{(n-1)}$

Rather complicated defn of $\pi_*(K)$ which is much easier for Δ-sets with Kan extension condition. for every collection $x_0, x_1, -, x_n \in K^{(n)}$ with the necessary compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for $i<j$
$\exists y \in K^{(n+1)}$ with $\partial_i y = x_i$ for $i = 0, -, n$ call this a Kan-Δ-set.



$\underline{n=0}$ $\forall x_0 \in K^{(0)}$ $\exists y \in K^{(1)}$ st $\partial_0 y = x_0$
$\underline{n=1}$ $\forall x_0, x_1 \in K^{(1)}$ $\exists$ st $\partial_0 x_0 = \partial_1 x_1$

Then can define $\pi_n(K) =$ set of $x \in K^{(n)}$ with $\partial_i x = \phi$
with $x \sim x'$ if $\exists y \in K^{(n+1)}$ st $\partial_0 y = x$, $\partial_1 y = x'$, $\partial_i y = \phi$ $\forall i \geq 2$

Addition in ~~$\mathbb{R}$~~ $\pi_n(K)$: Given $x, x' \in K^{(n)}$ with $\partial_i x = \partial_i x' = \phi$
$\exists y \in K^{(n+1)}$ st $\partial_0 y = x$, $\partial_1 y = x'$, $\partial_2 y = - = \partial_n y = \phi$
(Abelian for $n \geq 2$)

Two examples: 1) If $(X,*)$ is a pted space, $K = X^\Delta$ is a Kan-Δ-set.
$K^{(n)} = \mathrm{map}(\Delta^n, X)$, with $\phi^{(n)} : \Delta^n \longrightarrow X$, $d \longmapsto *$ (base pt)
Then $\pi_n(K) = \pi_n(X, *)$

2) $\pi_n(\Omega) = \Omega_n$. Take any bordism category:
oriented mfds, framed unoriented cxs (alg or geo. P. cxs)
and define a collection of Kan-Δ-sets Ω_k for $k \geq 0$ by
$\Omega_k^{(n)} = (n+k)$-dim^l object M with bdy $\partial M = \partial_0 M \cup \ldots \cup \partial_n M$
where $\partial_i M =$ codim $0 \subseteq \partial M$ and st $\partial_0 M \cap \ldots \cap \partial_n M = \phi$

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Eg in the geometric cases as modelled by

$$(M, \partial M) \xrightarrow{f} (\Delta^n, \partial\Delta^n)$$

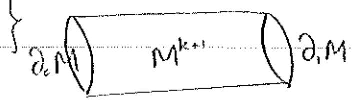
$\partial_i M \cap \partial_j M$ = codim 1 $\subseteq \partial M$

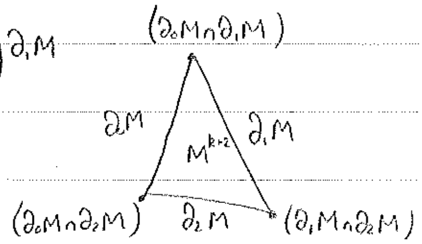
$\partial_{i_0} M \cap \ldots \cap \partial_{i_r} M$ = codim r $\subseteq \partial M$

In geometric cases;

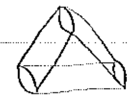
$\Omega_k^{(0)} = \{$closed k-dim^l objects $M^k \mid \partial M = \phi\}$

$\Omega_k^{(1)} = \{(k+1)$-dim^l cobordisms$\}$





$\Omega_k^{(2)} = \{(k+2)$-dim^l tribordism$\}$



Get $\ldots \Omega_k^{(2)} \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; \Omega_k^{(1)} \rightrightarrows \Omega_k^{(0)}$

Then

$$\pi_n(\Omega_k) = \Omega_{n+k} = \text{cobordism group of } (n+k)\text{-dim}^l \text{ closed objects in } \Omega$$

? (S_0) $\pi_n(\Omega_0) = \pi_{n-1}(\Omega_1)$

$\Omega_k \simeq$ loop space of Ω_{k-1} (spectrum)

Defn. A generalised homology theory on a category of spaces X (eg all spaces, or simplicial cxs) is a functor

$$h_* : \text{Spaces} \longrightarrow \mathbb{Z}\text{-graded abelian gps}$$

st 1. a htpy eqvce $f : X \to Y$ induces isom $f_* : h_*(X) \xrightarrow{\cong} h_*(Y)$

2. there is a Mayer-Vietoris sequence

Ω = loop space

A spectrum (of spaces or Kan-Δ-sets) $\{Y_0, Y_1, Y_2, \ldots\}$ with $Y_k = \Omega Y_{k-1}$ determines a GHT

$$h_n(X) := \varinjlim_k \pi_{n+k}(X \wedge Y_?)$$

Lecture 15. 02.12.08

What is left to do?

1. to define for any simplicial cx X the algebraic surgery exact seq

(i) $\longrightarrow \underset{\text{GHT}}{H_n(X;\mathbb{L}_\bullet)} \xrightarrow{A} \underset{\text{alg quad bordism}}{L_n(\mathbb{Z}[\pi_1 X])} \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X;\mathbb{L}) \longrightarrow$

with $\mathbb{L}$ the Ω-spectrum of Kan Δ-sets realising $\pi_*(\mathbb{L}) = L_*(\mathbb{Z})$

and A the assembly map

and ~~$\mathcal{S}$~~ $\mathcal{S}_n(X)$ (measuring failure of A to be an isom) a certain algebraic bordism group.

$L_n(\mathbb{Z}[\pi_1 X])$ = cobords of n-dim^l quad P. cxs over $\mathbb{Z}[\pi_1 X]$ with involution

$H_n(X;\mathbb{L})$ = — " — $(\mathbb{Z},X)$ using $(\mathbb{Z},X)$-mods with ch duality fg free $\mathbb{Z}$-mods $M = \sum_{x\in X} M_x$

$\mathcal{S}_n(X;\mathbb{L})$ = — " — $(n-1)$ — " — $(\mathbb{Z},X)$ st $A(C,\psi)=0$

ii) Relate (i) to $G/_{TOP}$ = homotopy fibre of $BTOP \longrightarrow BG$

($BTOP$ = TOP version of BO, BG = homotopy theory version of BG (spherical fibrations))

$\mathcal{S}$ And $G/_{TOP} \simeq$ ~~$\mathbb{L}$~~ $\mathbb{L}$ $\quad\therefore\ \pi_*(G/_{TOP}) \cong L_*(\mathbb{Z})$

iii) For an n-dim^l P. cx X, use the algebraic normal cxs (C,φ,γ,χ) to define the total surgery obstruction

$$S(X) = \partial(C,\varphi,\gamma,\chi) \in \mathcal{S}_n(X)$$

with $C = C(X^b)$ = a $(\mathbb{Z},X)$-mod chain cx (X^b = barycentric subdiv^n of X)

measuring the failures of the links $\text{link}_X(x) \simeq \partial D(x,X) = \partial(\text{Dual cell})$ to be $\simeq S^{n-|x|-1}$

ie failure of X to be a homology mfd.

Then for $n \geq 5$, X is htpy eqvt to an n-dim^l TOP mfd $\Longleftrightarrow S(X)=0$

iv) If X is an n-dim^l top^l mfd ~~set~~ let $\mathcal{S}^{TOP}(X)$ be the structure set of pairs (M = n-dim^l top^l mfd, $f: M \xrightarrow[\text{eqvt}]{\text{htpy}} X$)

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with $(M,f) \sim (M',f')$ if $\exists$ homeom $h: M \to M'$ st $f = f' \circ h$

Want to establish an isom of exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1 X]) & \longrightarrow & \mathcal{S}^{TOP}(X) & \longrightarrow & \underset{\text{normal maps } (f,b): M \to X}{[X, G/TOP]} \; H^0(X;\mathbb{L}) & \xrightarrow{\theta} & L_n(\mathbb{Z}[\pi_1 X]) & \to \\
& \| & & \simeq \downarrow s & & \downarrow [X]_{\mathbb{L}} \cap & & \| & \\
\longrightarrow & L_{n+1}(\mathbb{Z}[\pi_1 X]) & \longrightarrow & \mathcal{S}_{n+1}(X) & \longrightarrow & H_n(X;\mathbb{L}_\cdot) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1 X]) & \to
\end{array}$$

$s(f) = s\,\partial(\text{mapping } \cancel{\text{cone}} \text{ cylinder of } f: M \to X)$

$\quad = s\partial((n+1)\text{-dim}^l \text{ P. cobordism from } M \text{ to } X)$

Thus $s(f) = 0 \in \mathcal{S}_{n+1}(X) \iff (X,1) = (M,f) \in \mathcal{S}^{TOP}(X)$

$\iff$ the htpy eqvce $f: M \to X$ is htpic to a homeom

Can give direct defn of $s(f) = \{(C(\mathcal{C}(f^{-1}(D(x,X)) \to D(x,X))), \psi) \mid x \in X\} \in \mathcal{S}_{n+1}(X)$

chain cx of failures of "pt inverses" $f^{-1}(\hat{x}) \subset M$ to be contractible

Recall that $f: M \to X$ is a htpy eqvce of mfds here.

<u>Δ-sets</u> (= simplicial sets without degeneracies)

$K = \{K^{(n)} \mid n \geq 0\}$ with $\partial_i : K^{(n)} \to K^{(n-1)}$ $\quad 0 \leq i \leq n$

pointed sets, base pt is $\phi \in K^{(n)}$ $\quad$ st $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$

<u>Defn</u> The <u>realisation</u> of K, $|K|$, is the actual top^l space

$$|K| = \Big(\coprod_{n \geq 0} \Delta^n \times K^{(n)}\Big)/\sim \qquad (a, \partial_i b) \sim (\partial_i a, b)$$

with $\partial_i : \Delta^n \to \Delta^{n+1}$ the inclusion into i^{th} face

(htpy theory of Kan Δ-sets) = (htpy theory of top^l spaces)

$$K \longleftrightarrow |K|$$

$$(\Omega K)^{(n)} = \{x \in K^{(n+1)} \mid \partial_{n+1} x \neq 0,\ \partial_0 \partial_1 \ldots \partial_n x = \phi \in K^{(0)}\}$$

Eg $K^{(0)} = \{\phi\}$, $K^{(1)} = \{x, \phi\}$, $\partial_0 x = \partial_1 x = \phi$

So $|K| = \Delta^1 \times \{x\} / (0,x) \sim (1,x) \quad = S^1 \qquad (\Delta^1 = [0,1])$

$\cancel{\pi_n(K) =}$

$\pi_n(\Omega K) = \pi_{n+1}(K) = \{x \in K^{(n+1)} \mid \partial_i x = \phi\} / \sim$

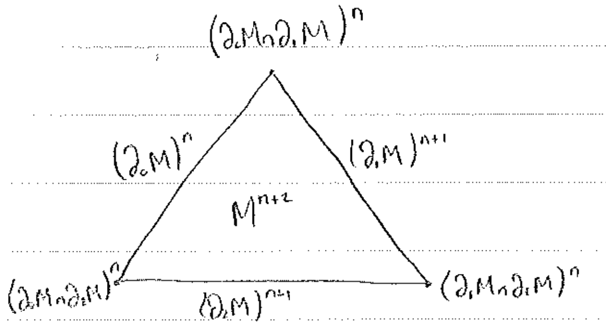
An $\underline{\Omega}$-spectrum $\mathbb{F} = \{\mathbb{F}_n \mid n \in \mathbb{Z}\}$ is a seq of Kan Δ-sets with $\Omega \mathbb{F}_n \simeq \mathbb{F}_{n+1}$

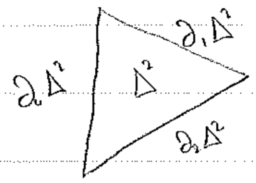
so $\pi_n(\mathbb{F}) = \pi_{n+k}(\mathbb{F}_k)$ for any $k, n \in \mathbb{Z}$

thus $\pi_n(\mathbb{F}_0) = \pi_n(\mathbb{F}_0) = \pi_0(\mathbb{F}_{-n})$

<u>Main Example</u>; Geometric bordism

$\underline{\Omega} = \{ \underline{\Omega}_n$ = Δ-sets with k-simplices, $(n+k)$-dim' mfds $(M, \partial M)$ with a map $f : (M^{n+k}, \partial M) \to (\Delta^k, \partial \Delta^k)$ st $f^{-1}(\partial_i \Delta) = \partial_i M \subset M$ codim 1 submfd $\}$





For $k = 0$, have closed n-mfds & ϕ = empty n-mfd

$k = 1$, cobordism of n-mfds

$\pi_n(\underline{\Omega})$ = cobordism gp of n-dim' mfds.

A GHT is a covariant functor h_* : {finite simp cxs} $\to$ $\mathbb{Z}$-graded $\mathbb{Z}$-modules satisfying

1. htpy invariance
2. MV-sequence

(Also h^* GCT and have $h_*(X) = h^{n-*}(S^n \setminus X)$, Alexander duality)

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<u>Theorem</u> (G. W. Whitehead, 1962) There is a 1-1 correspondence

GHTs $\longleftrightarrow$ Ω-spectra of Kan Δ-sets

$\pi_n(X \wedge \mathbb{F}) := \varinjlim_k \pi_{n+k}(X \wedge \mathbb{F}_{-k})$

$h_n(X) = \pi_n(X \wedge \mathbb{F}) \longleftarrow \mathbb{F} = \{\mathbb{F}_n \simeq \Omega \mathbb{F}_{n-1} \mid n \in \mathbb{Z}\}$

$h^n(X) = [X, \mathbb{F}_n]$ = htpy classes of maps $X \longrightarrow \mathbb{F}_n$

<u>Example</u>. Ordinary homology and cohomology with $\mathbb{F}_n = \underset{\text{Eilenberg Maclane space}}{K(\mathbb{Z}, -n)}$

$h_n(X) = H_n(X)$, $h^n(X) = H^n(X)$

<u>Example</u>. Bordisms: $h_n(X)$ = bordism gps of $f: M \to X$ $= \Omega_n(X)$

for MV, $\partial: \Omega_n(X_1 \cup X_2) \longrightarrow \Omega_{n-1}(X_1 \cap X_2)$

$(f: M \to X_1 \cup X_2) \longmapsto (g: N^{n-1} \longrightarrow X_1 \cap X_2)$

$g = f|_{N^{n-1}}$, $N^{n-1} := f^{-1}(X_1 \cap X_2)$

$L_n(\mathbb{Z}[\pi_1(X_1 \cup X_2)]) \longrightarrow L_{n-1}(\mathbb{Z}[\pi_1(X_1 \cap X_2)])$?

$H_n(X_1 \cup X_2; \mathbb{L}) \longrightarrow H_{n-1}(X_1 \cap X_2; \mathbb{L})$ ✓

Roughly speaking, an elt $c \in h_n(X) = \overset{\text{dst?}}{\pi_n}(X \wedge \mathbb{F})$ is a collection $\{c(x) \in \mathbb{F}_{n-m}^{(m-|x|)} \mid x \in X\}$ with $\partial_i c(x) = c\,(\delta_i x)$ compatability condition

where $\delta_i: (\partial \Delta^{m+1})^{m-k} \longrightarrow (\partial \Delta^{m+1})^{(m-k+1)}$

$\sigma \longmapsto \sigma \cup j_i$

$i: \{0, 1, \ldots, m+1\} \longmapsto \{j_0, \ldots, j_k\}$

<u>~~Example. The bordism~~</u>

<u>Motivation</u>. A simplicial map $f: M^n \to \underset{\text{barycentric subdiv}^n}{X'}$ can be reconstructed from $(n-|x|)$-mfds $(M(x), \partial(M(x)))$ for $x \in X$

with $f(x): M(x) \longrightarrow D(x, X)$ = dual cell

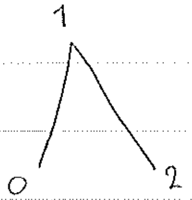
$\partial M(x) = \bigcup_{y > x} M(y)$ compatability condition

We do this by setting $M = \bigcup_{x \in X} M(x)$ = assembly of M

for $A : h_n(X) \longrightarrow h_n(pt)$, $\{C(x) \mid x \in X\} \longmapsto$ "$\bigcup_{x\in X} C(x)$"

where gluing defined using Kan extension condition.

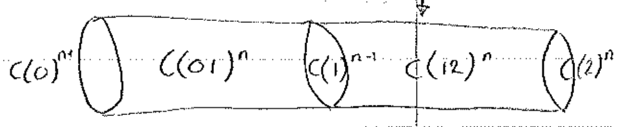
<u>Example</u>.

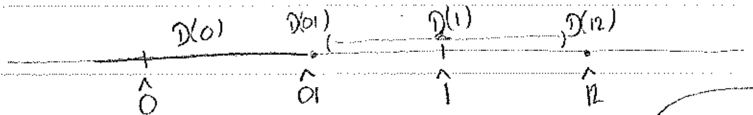


$X = \{0, 1, 2, 01, 12\} \subseteq \partial \Delta^2$

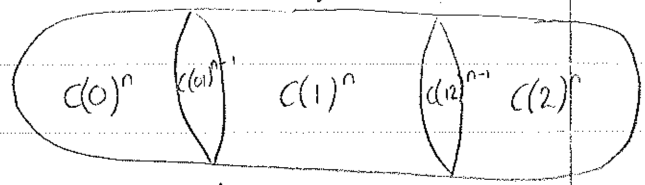
$C \in H_n(X;\mathbb{F})$

?, Not this one!





This is right picture!



closed and subdivided $= C(0) \cup C(1) \cup C(2)$

$H_n(X;\mathbb{F}) \xrightarrow[\text{Assemby } A]{A} H_n(pt;\mathbb{F})$
$= \pi_n(\mathbb{F})$

$H_n(X;\mathbb{L}_\cdot) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1 X])$

$A_{\{1\}}$ $\qquad\qquad\qquad \downarrow$

$H_n(pt;\mathbb{L}_\cdot) \quad = \quad L_n(\mathbb{Z})$

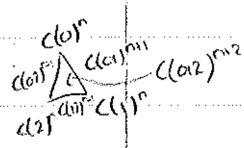
<u>Defn</u>. $\mathbb{L}_\cdot(A)$ = quadratic L-theory Kan Ω-spectrum for any ring with inv^n A

$= \{\mathbb{L}_n(A) \simeq \Omega \mathbb{L}_{n-1}(A) \mid n \in \mathbb{Z}\}$

$\mathbb{L}_n(A)^{(0)} = \{$ n-dim^l quad P. cxs (C,ψ) over $A\}$ $\quad (C,\psi) \xrightarrow{f} (D,\delta\psi) \xleftarrow{f'} (C',\psi')$

$\mathbb{L}_n(A)^{(1)} = \{$cobordisms of n-dim^l quad P. cx over $A\}$

$\mathbb{L}_n(A)^{(2)} = \{$tribordisms —— " —— $\}$



$\pi_n(\mathbb{L}_\cdot(A)) = L_*(A) = \pi_0(\mathbb{L}_n(A))$

= cobordism gp of n-dim^l quad P cxs over A

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Lecture 16 03.12.08

Description of $A: H_*(X;\mathbb{L}_\cdot) \longrightarrow L_n(\mathbb{Z}[\pi_1 X])$ and the algebraic surgery exact sequence using the additive category $\mathbb{A}(\mathbb{Z},X)$ with chain duality.

[Could perhaps use the simplicial ring $\mathbb{Z}[\Omega X]$ instead of $\mathbb{A}(\mathbb{Z},X)$

$$A: \Omega X \longrightarrow H(\Omega X) = \mathbb{Z}[\pi_1 X]$$
]

Recall; $H_*(X;\mathbb{L}_\cdot) = L_*(\mathbb{A}(\mathbb{Z},X))$

= cobord gps of quad P. cxs in $\mathbb{A}(\mathbb{Z},X)$

which are essentially the cycles $\{(C(x),\psi(x)) \mid x\in X\}$

with $(C(x),\psi(x))$ essentially quadratic Poincaré pairs over $\mathbb{Z}$

with compatibility $\partial(C(x),\psi(x)) = \bigcup_{y>x}(C(y),\psi(y))$

<u>Defn</u> of $\mathbb{A}(\mathbb{Z},X)$ (an additive category): for X a simplicial cx

The objects are f.g., free, ab gps A with a decomposition $A = \sum_{x\in X} A(x)$ (a $(\mathbb{Z},X)$-module)

The morphisms $f: A\longrightarrow B$ are $\mathbb{Z}$-module homoms $A\longrightarrow B$

st $f(A(x)) \subseteq \sum_{y\geq x} B(y)$

ie the matrix $\{f(x,y): A(x)\longrightarrow B(y)\}_{x,y}$ is upper triangular

$(f(x,y)=0$ for $x>y)$

Thus f is an isom iff $f(x,x): A(x)\longrightarrow B(x)$ is an isom $\forall x\in X$

(general property about inverting upper triangular matrices)

<u>Example</u> The simp. chain cx $C(X')$ is a $(\mathbb{Z},X)$-chain cx

with $C(X')(x)_r = C(D(x,X),\partial D(x,X))_r = \mathbb{Z}[\hat\sigma_0\hat\sigma_1\ldots\hat\sigma_r]$

where $x = \sigma_0 < \sigma_1 < \ldots < \sigma_r \in X$

$$\partial(\hat\sigma_0 \ldots \hat\sigma_r) = \sum_{q=0}^{r}(-1)^q(\hat\sigma_0\ldots\hat\sigma_{q-1}\hat\sigma_{q+1}\ldots\hat\sigma_r)$$

$$\in \begin{cases} C(X')(x)_{r-1} & \text{if } q\neq 0,\ x=\sigma_0 \\ C(X')(\sigma_1)_{r-1} & \text{if } q=0,\ x<\sigma_1 \end{cases}$$

Eg cont'd

More generally, if $f: Y \longrightarrow X'$ is any simp map then the simp chain cx $C(Y)$ is also a $(\mathbb{Z}, X)$-module chain cx with $C(Y)(x) = C(f^{-1}D(x,X), f^{-1}\partial D(x,X))$

Example $C(X)^{n-*}$ is a $(\mathbb{Z}, X)$-module chain cx with

$$(C(X)^{n-*})_r = C(X)^{n-r} = \mathbb{Z}[x^* \mid x \in X^{(n-r)}]$$

$$d_{C(X)^{n-*}} = (d_{C(X)})^* : C(X)^{n-*} \longrightarrow C(X)^{n+1-*}$$

$$d_{C(X)^{n-*}}(x^*) = \sum_{\substack{w \in X^{(0)};\, x<y \\ y=(v_0 \ldots v_{n-r} w)}} y^* \quad \text{where } x = (v_0 \ldots v_{n-r})$$

So

$$d_{C(X)^{n-*}} : C(X)^{n-r}(x) = \mathbb{Z}[x^*] \longrightarrow \sum_{\substack{y \geq x \\ |y|=|x|+1}} C(X)^{n+1-r}(y) = \mathbb{Z}[y^* \mid x < y]$$

Claim. X is an n-dim' homology mfd ~~iff~~ with fundamental cycle $[X] \in C_n(X)$ iff $-\cap[X] : C(X)^{n-*} \longrightarrow C(X')$ is a $(\mathbb{Z}, X)$-~~chain~~ module chain eqvce

$$C(X')(x) = C(D(x,X), \partial D(x,X))$$

Note: $C(X')^{n-*}(x) = \mathbb{Z}[x^*] \xrightarrow{\sim} C(D(x,X), \partial D(x,X))$

iff $H_*(D(x,X), \partial D(x,X)) = \begin{cases} \mathbb{Z} & \text{if } * = n-|x| \\ 0 & \text{if } * \neq n-|x| \end{cases}$

Defn. A $(\mathbb{Z}, X)$-module chain map $f : C \to D$ is a $(\mathbb{Z}, X)$-mod ch eqvce iff $f(x,x) : C(x) \to D(x)$ is a $\mathbb{Z}$-mod ch eqvce $\forall x \in X$

Defn. Given $(\mathbb{Z}, X)$-mods A, B define a new $(\mathbb{Z}, X)$-mod by

$$A \otimes_{(\mathbb{Z},X)} B := \sum_{\substack{x,y \in X \\ x \cap y \neq \phi}} A(x) \otimes_{\mathbb{Z}} B(y)$$

and here $(A \otimes_{(\mathbb{Z},X)} B)(z) = \sum_{\substack{x,y \in X \\ x \cap y = z}} A(x) \otimes_{\mathbb{Z}} B(y)$

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$$\begin{array}{ccc} K\times_X L & \longrightarrow & |K| \\ \downarrow \text{pullback} & & \downarrow f \\ \text{Think } |L| & \xrightarrow{g} & |X'| \end{array}$$ (see example below)

<u>Example</u>. If $f: K \to X'$, $g: L \to X'$ are simp maps then
$C(K\times_X L) \simeq_{(\mathbb{Z},X)} C(K) \otimes_{(\mathbb{Z},X)} C(L)$

<u>Defn</u>. A <u>chain duality</u> $(T;e)$ on an additive category $\mathbb{A}$ is a contravariant functor $T: \mathbb{A} \to \mathbb{B}(\mathbb{A})$; $A \mapsto T(A)$
where $\mathbb{B}(\mathbb{A})$ = additive cat of bounded ch cxs in $\mathbb{A}$
together with a natural transformation
$$e(A): T(T(A)) \longrightarrow A \quad = \text{0-dim}^l \text{ ch cx in } \mathbb{A}$$
st

(i) $e(T(A)) \circ T(e(A)) \overset{\text{chain homotopic}}{\simeq} 1$

$$T(A) \xrightarrow{T(e(A))} T(T(T(A))) \xrightarrow{e(T(A))} T(A)$$

(ii) $e(A): T^2(A) \to A$ is a chain eqvce

<u>Example</u>. If each $T(A)$ is a 0-dim^l ch cx, ie an object of $\mathbb{A}$, then this is an involution on $\mathbb{A}$.

1. Need to define the chain duality on the $(\mathbb{Z},X)$-mod cat $\mathbb{A}(\mathbb{Z},X)$
2. Define the assembly map
$$A: \mathbb{A}(\mathbb{Z},X) \longrightarrow \mathbb{A}(\mathbb{Z}[\pi_1 X]) = \{\text{f.g. proj } \mathbb{Z}[\pi_1 X]\text{-mods}\}$$
and check that ch duality $\longmapsto$ involution $P \mapsto \mathrm{Hom}(P^*, \mathbb{Z}[\pi_1 X])$ (a 0-dim^l duality)

This will induce
$$A: L_*(\mathbb{A}(\mathbb{Z},X)) = H_*(X;\mathbb{L}.) \longrightarrow L_*(\mathbb{A}(\mathbb{Z}[\pi_1 X])) = L_*(\mathbb{Z}[\pi_1 X])$$

$$A: \mathbb{A}(\mathbb{Z},X) \longrightarrow \mathbb{A}(\mathbb{Z}[\pi_1 X]) \qquad p: \tilde{X} \to X$$
$$\sum_{x\in X} M(x) \longmapsto \sum_{\tilde{x}\in\tilde{X}} M(p(\tilde{x}))$$, a $\mathbb{Z}[\pi_1 X]$-module, where $\tilde{X}$ is univ cover

Given $f : M \longrightarrow N$ a $(\mathbb{Z}, X)$-module ~~chain map~~ morphism induces

$$A(f) : \textstyle\sum_{\tilde{x} \in \tilde{X}} M(\rho(\tilde{x})) \longrightarrow \sum_{\tilde{y} \in \tilde{X}} N(\rho(\tilde{y})) \quad \text{a } \mathbb{Z}[\pi, X]\text{-mod morphism}$$

is given by $\{ f(\rho(\tilde{x}), \rho(\tilde{y})) \mid \tilde{x} \leq \tilde{y} \in \tilde{X} \}$

$$\begin{tikzcd} \mathbb{A}(\mathbb{Z}, X) \arrow[rr, "A"] \arrow[dr, "\rho^!"'] & & \mathbb{A}(\mathbb{Z}[\pi, X]) \\ & \mathbb{A}(\mathbb{Z}, \tilde{X})^{\pi, X} \arrow[ur, "q_!"'] & \end{tikzcd} \qquad \begin{tikzcd} \tilde{X} \arrow[r, "q"] \arrow[d, "\rho"] & \{pt\} \\ X & \end{tikzcd}$$

Assembly $A = q_! \rho^!$

<u>Defn</u>. The chain duality $T : \mathbb{A}(\mathbb{Z}, X) \longrightarrow \mathbb{B}(\mathbb{A}(\mathbb{Z}, X))$ is defined by

$$T(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{(\mathbb{Z}, X)}(C(X)^{-*}, A), \mathbb{Z}) = \text{a ch cx in } \mathbb{A}(\mathbb{Z}, X)$$

<u>Eg</u> For $X = pt$, $T(A) = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$

~~In general~~, ie $T(A)_r(x) = \begin{cases} \sum_{y \geq x} \mathrm{Hom}_{\mathbb{Z}}(A(y), \mathbb{Z}) & \text{if } r = -|x| \\ 0 & \text{if } r \neq -|x| \end{cases}$

<u>Example</u>. $X = \Delta^1 = \{0, 1, 01\}$. $\quad f = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix}$

A $(\mathbb{Z}, \Delta^1)$-module map $A = A(0) \oplus A(1) \oplus A(01) \longrightarrow B(0) \oplus B(1) \oplus B(01)$

The dual of A is the chain cx TA with

$TA_0(0) = A(0)^* \oplus A(01)^* \qquad A(0)^* = \mathrm{Hom}_{\mathbb{Z}}(A(0), \mathbb{Z})$ etc

$TA_0(1) = A(1)^* \oplus A(01)^*$

$TA_{\cancel{0}\,-1}(01) = A(01)^*$

$$d = (0\ 1\ 0\ 1) : \underbrace{(A(0)^* \oplus A(01)^*) \oplus (A(1)^* \oplus A(01)^*)}_{TA_{\cancel{*}\,0}} \longrightarrow \underbrace{A(01)^*}_{TA_{-1}}$$

$$C(\Delta^1)^{-*} : \mathbb{Z} \xrightarrow{\binom{1}{1}} \mathbb{Z} \oplus \mathbb{Z}$$

and $H_*(TA) = H_*(A^{-*})$

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Example. $TC(X)^{-*} \simeq C(X')$
$TC(X') \simeq C(X^*)^{-*}$

Key Property. T is adjoint, in sense of $\mathrm{Hom}_{(\mathbb{Z},X)}(A,B) \simeq_{(\mathbb{Z},X)} TA \otimes_{(\mathbb{Z},X)} B$ and $\mathrm{Hom}_{(\mathbb{Z},X)}(TA, B) \simeq_{(\mathbb{Z},X)} A \otimes_{(\mathbb{Z},X)} B$

Recall that for simp maps

$$\begin{array}{ccc} M \times_X N & \longrightarrow & M \\ \downarrow & & \downarrow f \\ N & \xrightarrow{g} & X' \end{array} \qquad \begin{aligned} C(M \times_X N) &\simeq C(M) \otimes_{(\mathbb{Z},X)} C(N) \\ &\simeq \mathrm{Hom}_{(\mathbb{Z},X)}(TC(M), C(N)) \end{aligned}$$

Theorem (Poincaré Duality)

$$C(X) = C(X' \times_X X') \xrightarrow[\text{Alexander-Whitney diagonal ch cx}]{\Delta} C(X \times_{\{pt\}} X) \stackrel{\text{Eilenberg-Zilber}}{\simeq} C(X) \otimes_{\mathbb{Z}} C(X)$$
$$\simeq \mathrm{Hom}_{(\mathbb{Z},X)}(TC(X')^{-*}, C(X')) \qquad = \mathrm{Hom}_{\mathbb{Z}}(C(X)^{-*}, C(X))$$

The homology class $[X] \in H_n(X) = H_n(\mathrm{Hom}_{(\mathbb{Z},X)}(TC(X'), C(X')))$
$= H_n(\mathrm{Hom}_{(\mathbb{Z},X)}(C(X)^{n-*}, C(X')))$
is a $(\mathbb{Z},X)$-module chain homotopy class $[X] : C(X)^{n-*} \longrightarrow C(X')$

"Proof" (?) $\Delta : X = X \times_X X \longrightarrow X \times_{\{pt\}} X = X \times X$

$$\begin{array}{ccc} [X] \in H_n(X) & \longrightarrow & H_n(X \times X) \ni \Delta[X] \\ \parallel & & \parallel \\ H_0(\mathrm{Hom}_{(\mathbb{Z},X)}(C(X)^{n-*}, C(X'))) & \longrightarrow & H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X)^{n-*}, C(X))) \end{array}$$

$$X = X \times_X X \xrightarrow{\tilde{\Delta}} \tilde{X} \times_{\pi_1 X} \tilde{X}$$
$$x \longmapsto (\tilde{x}, \tilde{x})$$

$$\begin{array}{ccc} H_n(\mathrm{Hom}_{(\mathbb{Z},X)}(C(X)^{n-*}, C(X'))) & \longrightarrow & H_n(\mathrm{Hom}_{\mathbb{Z}[\pi_1 X]}(C(\tilde{X})^{n-*}, C(\tilde{X}))) \\ \parallel & & \parallel \\ H_n(X) & \xrightarrow{\tilde{\Delta}_*} & H_n(C(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} C(\tilde{X})) \end{array}$$

if X is a mfd, $[X] \longmapsto$ Poincaré Duality

Define $L_n(\mathbb{A})$ for any additive cat $\mathbb{A}$ with chain duality as the cobordism gp of n-dim^l quad P. cxs
(C, ψ) where $\psi \in Q_n(C) = H_n(W \otimes_\Delta (C \otimes_A C))$
st $(1+T)\psi_0 : TC^{n-*} \longrightarrow C$ is a chain eqvce $\quad C \otimes C \simeq \mathrm{Hom}(TC^{*}, C)$

Claim $L_n(\mathbb{A}(\mathbb{Z}, X)) \cong H_n(X; \mathbb{L}_\cdot)$
"Pf"
An n-dim^l q. P. cx in $\mathbb{A}(\mathbb{Z}, X)$ is exactly the same as a cycle in
$\mathbb{H}(X; \mathbb{L}_\cdot) = [\Sigma^m, \Sigma^m \setminus \bar{X}, \mathbb{L}_n(\mathbb{Z})]$
where $\bar{X} = \partial\Delta^{m+1} \setminus X$, $\Sigma^m = \partial\Delta^{m+1}$, $\bar{X}$ = supplement of X

Lecture 17

09.12.08

The construction of the total surgery obstruction ~~$s(X) \in \mathcal{S}_n(X) =$~~
$$s(X) \in \mathcal{S}_n(X) = \pi_{n+1}(\underset{\uparrow\ \text{assembly map}}{A} ; \overset{\text{LOCAL}}{\mathbb{H}(X; \mathbb{L}_\cdot)}) \longrightarrow \mathbb{L}_\cdot(\overset{\text{GLOBAL}}{\mathbb{Z}[\pi_1 X]})$$
$\mathbb{L}_\cdot$ homology spectrum $\quad$ quadratic L-spectrum

of an n-dim^l geo P. cx X using alg & geo normal cxes.

We need alg normal cxs to measure the failure upto alg cobordism of the links of X (assumed to be the polyhedron of a finite simplicial cx) to be spheres.

Recall that X is an n-dim^l homology mfd iff each $\underset{\text{bdy of dual cell } D(x) \text{ in } X'}{\partial D(x)} \subset X'$ is a homology $(n-|x|-1)$-sphere

$s(X) = 0 \iff X$ is htpy eqvt to an n-dim^l top^l mfd. $\quad (n \geq 5)$

Ran 44

(Alg) Normal Cxs and Quadratic Structures

An n-dim^{l} normal cx $(C, \varphi, \gamma, \chi)$ (in an additive category $\mathbb{A}$ with chain duality $T : \mathbb{A} \longrightarrow \mathbb{B}(\mathbb{A})$, & in particular $\mathbb{A} = \{$f.g., proj A-mods$\}$ for A a ring with involution) is an n-dim^{l} chain cx C with ① an n-dim^{l} symmetric structure $\varphi \in (W^{\%} C)_n = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)_n = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{A}}(TC, C))_n$
st

$$\varphi_0 : (TC)^{n-r} \longrightarrow C_r, \quad \varphi_s : (TC)^{n-r+s} \longrightarrow C_r \quad (\text{for } s \geq 0)$$

and $d\varphi_s + \varphi_s d^* + \varphi_{s-1} + \varphi_{s-1}^* = 0 \qquad (\varphi_{-1} := 0)$

In particular, φ_0 is a chain map, it is a chain eqvce iff (C, φ) is Poincaré.

② a chain bundle $\gamma \in (\hat{W}^{\%} TC)^0$ so

$$\gamma_s : C_r \longrightarrow TC^{-r-s} \quad (s \in \mathbb{Z})$$

st $d^*\gamma_s + \gamma_s d + \gamma_{s-1} + \gamma_{s-1}^* = 0$

③ and a clutching chain $\chi \in (\hat{W}^{\%} C)_{n+1}$

st $d\chi = J\varphi - (\varphi_0)^{\%}(S^n \gamma)$ Wu-Thom identity

ie $\varphi_s - \varphi_0 \gamma_{s-n} \varphi_0^* = d\chi_s + \chi_s d^* + \chi_{s-1} + \chi_{s-1}^* \qquad (\varphi_s = 0 \ \forall s \leq -1)$

Example An n-dim^{l} geo normal cx $(X, \nu_X : X \to BG(k), \rho_X : S^{n+k} \to T(\nu_X))$
(classifying space for S^{k-1}-fibrations over X; Thom space)

determines an n-dim^{l} alg normal cx (i) over $\mathbb{Z}[\pi_1 X]$ global
(ii) over $(\mathbb{Z}, X)$ local

(i) Here $\mathbb{A} = \{$f.g., ~~proj~~ free $\mathbb{Z}[\pi_1 X]$-mods with involution$\}$
$C = C(\tilde{X})$, where $\tilde{X}$ is the universal cover of X.
φ_X is symmetric construction, $\varphi = \varphi_X([X])$.
So $\varphi_0 \doteq -\cap[X] : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})$

S-dual $\qquad$ sym constr^n

$$H^k(T(\nu_X)) \cong H_{N-k}(T(\nu_X)^*) \xrightarrow{\varphi_{T(\nu_X)^*}} Q^{N-k}(\dot{C}(\widetilde{T(\nu_X)})^{N-*})$$

hyperquadratic constr^n

γ = image of Thom class U_{ν_X} $\qquad = \hat{Q}^0(C(\hat{X})^{-*})$ $\quad$ Wu classes live here

$$= \hat{Q}^n(\dot{C}(\hat{X})^{n-*})$$

$$\downarrow (\varphi_0)^{\%}$$

Class $[X]$ = Hurewicz image$(\rho_X) \in \dot{H}_{n+k}(T(\nu_X)) \cong H_n(X)$

$$H_n(X) \xrightarrow{\varphi_X} Q^n(C(\hat{X})) \xrightarrow{J} \hat{Q}^n(C(\hat{X}))$$

Assembly $(ii) \longrightarrow (i)$ $\qquad (\mathbb{Z}, X) \longrightarrow \mathbb{Z}[\pi_1 X]$

If (X, ν_X, ρ_X) is an n-dim^l geo normal cx then

(i) $\varphi_0 = -\cap[X] : C(\hat{X})^{n-*} \longrightarrow C(\hat{X})$ is a $\mathbb{Z}[\pi_1 X]$-chain eqv^{ce}

but (ii) $\varphi_0^{(\mathbb{Z},X)} : C(X)^{n-*} \longrightarrow C(X')$ is a $(\mathbb{Z}, X)$-chain eqv^{ce} iff

X is an n-dim^l homology mfd $\Rightarrow X \simeq$ TOP mfd (for $n \geq 5$)

<u>Theorem</u> 1 ~~(i)~~ An n-dim^l sym ch cx (C, φ) (in $\mathbb{A}$ with chain duality) has a normal structure ~~iff~~ (γ, χ) iff the bdy $(n-1)$-dim^l sym P. cx $\partial(C, \varphi) = ,\ \partial C = \mathcal{C}(\varphi_0 : C^{n-*} \longrightarrow C)_{*+1}$ has a quadratic structure ψ.

(Eg. If (C, φ) is Poincaré then $\partial C \simeq 0$ and have alg Spivak normal structure (γ, χ))

Proof

alg suspension

Let $e : C \longrightarrow S\partial C = (\partial C)_{*-1} = \mathcal{C}(\varphi_0 : C^{n-*} \to C)$ be the inclusion

so $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \longrightarrow C_r \oplus C^{n-r+1}$

There are exact seq^{n}s of Q-groups

$$Q_{n-1}^{n-1}(\partial C) \xrightarrow{1+T} Q^{n-1}(\partial C) \xrightarrow{J} \hat{Q}^{n-1}(\partial C) \cong_S \hat{Q}^n((\partial C)_{*-1})$$

$$\exists \psi\ ? \quad \longmapsto \quad \partial\varphi \quad \longmapsto J(\partial\varphi)$$

and

$$\hat{Q}^n(C^{n-*}) \xrightarrow{(\hat{\varphi}_0)^{\%}} \hat{Q}^n(C) \xrightarrow{\hat{e}^{\%}} \hat{Q}(\mathcal{C}(\varphi_0))$$

$S \cong$, $\hat{Q}^0(C^{-*})$ $\quad \exists\gamma? \quad \longmapsto J\varphi \longmapsto \hat{e}^{\%}(J\varphi) = SJ(\partial\varphi)$

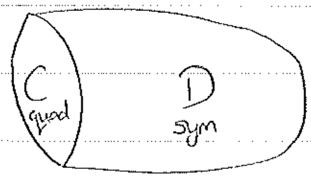
So $\exists \psi$ st $(1+T)\psi = \partial\varphi \iff \exists$ chain bdl. $\gamma \in \hat{Q}^0(C^{-*})$ st $(\hat{\varphi}_0)^{\%}(\gamma) = J\varphi$

Corollary. There is a natural 1-1 correspondence between the htpy eqvce classes of n-dim^l sym P. cxs in $\mathbb{A}$ and n-dim^l alg normal cxs $(C, \varphi, \gamma, \chi)$ with $\varphi_0 : C^{n-*} \longrightarrow C$ a chain eqvce.
(This is the algebraic Spivak normal cx of (C, φ))

Corollary. There is a natural 1-1 correspondence between htpy eqvce classes of n-dim^l quad cxs (C, ψ) and normal cxs $(C, \varphi, \gamma, \chi)$ with $\gamma = 0$.
($\varphi = (1+T)\psi$ so $J(\varphi) = J \circ (1+T)\psi = 0$)

Def^n. An n-dim^l (sym, quad) Poincaré pair $(f : C \longrightarrow D, (\delta\varphi, \psi))$ in $\mathbb{A}$ is an $(n-1)$-dim^l quad cx (C, ψ) and n-dim^l cx D with $\delta\varphi \in (W^{\%} D)_n$ st

$$f^{\%}((1+T)\psi) = d(\delta\varphi)$$



hyperquadratic L-gps

$$\longrightarrow L_n(A) \xrightarrow{1+T} L^n(A) \longrightarrow \hat{L}^n(A) = \text{cobordisms of such pairs} \longrightarrow L_{n-1}(A) \longrightarrow$$

Eg

$$\begin{array}{ccccccc} L_0\mathbb{Z} & \longrightarrow & L^0\mathbb{Z} & \longrightarrow & \hat{L}^0\mathbb{Z} & \longrightarrow & L_{-1}\mathbb{Z} \\ \| & & \| & & \| & & \\ \mathbb{Z} & \xrightarrow{8} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_8 & & \end{array}$$

NOTE: If (C, φ) is a sym cx then get n-dim^l sym P. pair
$((0\ 1) : \partial C \longrightarrow C^{n-*}, (0, \partial\varphi))$

Theorem 2. There is a natural 1-1 correspondence between htpy eqvce classes of n-dim^l alg normal cxs $(C, \varphi, \gamma, \chi)$ in $\mathbb{A}$ and n-dim^l (sym, quad) Poincaré pairs $(f : D \longrightarrow E, (\delta\varphi, \psi))$ in $\mathbb{A}$.

In particular, we will be able to define
$\partial(C, \varphi, \gamma, \chi) = (D, \psi)$, the quad Poincaré boundary.

Proof

Given $(C, \varphi, \gamma, \chi)$ define $f = (0\ 1) : D = \mathcal{C}(\varphi_0)_{*+1} \longrightarrow E = C^{n-*}$

and $\psi_0 = \begin{pmatrix} \chi_0 & 0 \\ 1+\gamma_{-n}\varphi_0^* & \gamma_{-n-1}^* \end{pmatrix}$, $\psi_s = \begin{pmatrix} \chi_{-s} & 0 \\ \gamma_{-n-s}\varphi_0^* & \gamma_{-n-1-s}^* \end{pmatrix}$ $s \geq 1$

and $(\delta\varphi)_s = \gamma_{-n-s}$

where $\psi_s : D^r = C^{r+1} \oplus C_{n-r} \longrightarrow D_{n-r-1-s} = C_{n-r-1-s} \oplus C^{r+1+s}$

$\delta\varphi_s : (C^{n-*})^r = C_{n-r} \longrightarrow (C^{n-*})_{r-s} = C^{n-r+s}$

Conversely, given an n-dim^l (sym, quad) Poincaré pair $(f : D \to E, (\delta\varphi, \psi))$ define an n-dim^l normal cx $(\mathcal{C}(f), \varphi, \gamma, \chi)$ by

$\varphi_0 = \begin{pmatrix} \delta\varphi_0 & 0 \\ (1+T)\psi_0 f^* & 0 \end{pmatrix}$, $\varphi_1 = \begin{pmatrix} \delta\varphi_1 & 0 \\ 0 & (1+T)\psi_0 \end{pmatrix}$, $\varphi_s = \begin{pmatrix} \delta\varphi_s & 0 \\ 0 & 0 \end{pmatrix}$ $s \geq 2$

$$\begin{array}{ccccccc} & (\varphi) & \hspace{-3em}\mapsto (\delta\varphi\,(1+T)\psi_0)^{\%} & & & & \\ Q^n(\mathcal{C}(f)) & \xleftarrow{\ \cong\ } & Q^n(D^{n-*}) & \longrightarrow & \hat{Q}^n(D^{n-*}) & = & \hat{Q}^0(D^{-*}) \\ \delta\varphi/(1+T)\psi & & \text{Poincaré -Lefschetz} & & & & \gamma \end{array}$$

□

$\to L^n(\mathbb{A}) \longrightarrow \hat{L}^n(\mathbb{A}) := \begin{matrix}n\text{-dim}^l \text{ (sym, quad)} \\ \text{Poincaré pairs}\end{matrix} \longrightarrow L_{n-1}(\mathbb{A}) \longrightarrow L_{n-1}(\mathbb{A})$

thm $\to \cong$ n-dim^l normal complexes $\quad (C, \varphi, \gamma, \chi) \longmapsto \partial(\_) = (\mathcal{C}(\varphi_0)_{*+1}, \psi) =$ quad P. bdy of normal cx

Also $L_{n-1}(\mathbb{A}) := (n-1)$-dim^l quad P. cxs in $\mathbb{A}$

thm $\to \cong$ n-dim^l (normal, sym, P.) pairs

ie

If $(f : C \to D, (\delta\varphi, \varphi))$ is an n-dim^l sym P.pair with (C, φ) Poincaré and normal structure $(\delta\varphi, \gamma, \chi)$ on D iff $(1+T)\psi = \varphi \in Q^{n-1}(C)$, $\psi \in Q_{n-1}(C)$

A simple example. $\hat{L}^0(A) \xrightarrow{\partial} L_3(A)$.. Consider a 0-dim^l normal cx $(P, \varphi, \gamma, \chi)$ is st P is a module, $\varphi = \varphi^* : P \longrightarrow P^*$, $\gamma = \gamma^* : P^* \longrightarrow P$

$\varphi\gamma\varphi^* - \varphi = \chi + \chi^*$

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$$\begin{tikzcd} C^0 \arrow[r] \arrow[d] & C_3 = 0 \arrow[d] \\ C^1 = P \arrow[r, "\psi_0 = 1-\gamma\varphi"] \arrow[d, "d^* = \varphi^*"'] \arrow[dr, "\psi_1 = \chi"] & C_2 = P \arrow[d, "d = \varphi"] \\ C^2 = P^* \arrow[r, "\psi_0 = 0"'] \arrow[d] & C_1 = P^* \arrow[d] \\ C^2 \arrow[r] & C_0 = 0 \end{tikzcd}$$

If we think of $\varphi : P \times P \to A$

$\gamma : P \times P \to A$

then $\varphi(x,x) = \gamma(\varphi(x), \varphi(x)) \in A/\{a + \bar{a}\}$

$$\begin{tikzcd} P \arrow[r, "\psi_0 = 1-\gamma\varphi"] \arrow[d, "d = \varphi"'] & P \arrow[d, "d = \varphi"] \\ P^* \arrow[r, "\psi_0^* = 1-\varphi\gamma"'] \arrow[ur, "\gamma"] & P^* \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "1"] \arrow[d, "\varphi"'] & P \arrow[d, "\varphi"] \\ P \arrow[r, "1"] & P \end{tikzcd}$$

$$\hat{L}^0(A) \xrightarrow{\partial} L_3(A) \xrightarrow{1+T} L^3(A)$$

$(P, \varphi, \gamma, \chi) \longmapsto (C, \psi) \leftarrow$ 3-dim^l quad cx

X = finite, ordered, simplicial cx.

Let $x \in X$ be a sx. $x = (v_0, v_1, \_\_, v_r)$ where $r = |x|$.

The vxs satisfy $v_0 < v_1 < \ldots < v_r$.

For any $[X] \in H_n(X)$, we can push

$$\begin{tikzcd} & H_n(X) \arrow[d] \arrow[dl, "D"'] \\ & \end{tikzcd}$$

$$H_n(X) \xrightarrow{\partial} H_{n-r}(D(x), \partial D(x)) = H_n(\Sigma^r(D(x)/\partial D(x)))$$

$$[X] \longmapsto [D(x)]$$

so $C(x) = C(D(x), \partial D(x))$

For any $(\mathbb{Z}, X)$-module chain cx C (eg $C = C(X')$)

$$C_n = \sum_{x \in X} C(x)_n \xrightarrow{\text{proj}^n} C(v_0)_n \xrightarrow{d(v_0, v_0v_1)} C(v_0v_1)_{n-1} \xrightarrow{d(v_0v_1, v_0v_1v_2)} C(v_0v_1v_2)_{n-2}$$

$$\longrightarrow \ldots \longrightarrow C(v_0 \ldots v_r)_{n-r}$$

$$d : \sum_{x \in X} C(x)_r \xrightarrow{d(x,y)} \sum_{y \in X, y < x} C(y)_{r-1} \qquad (d^2 = 0)$$

so $\partial_x : C \to C(x)_{*-|x|}$

Eg. $X = \Delta^1$. $C_n = C(0)_n \oplus C(1)_n \oplus C(01)_n \longrightarrow C_n(0) \xrightarrow{*_1} C_{n-1}(01)$

$$d_C = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ *_1 & *_2 & d \end{pmatrix} : C_n \longrightarrow C_{n-1}$$

$$\overset{S^{r-1}}{\overset{\shortparallel}{\partial x}} \overset{\text{join}}{\overset{\downarrow}{*}} (D(x), \partial D(x)) = (\mathrm{star}_{X'}(\hat{x}), \mathrm{link}_{X'}(\hat{x}))$$

$$X \longrightarrow S^{r-1} * \left(D(x)/\partial D(x)\right) = \Sigma^r\left(D(x)/\partial D(x)\right)$$

$$\overset{\shortparallel}{X/X\backslash \mathrm{star}_{X'}(x)} = T(\varepsilon^r|_{D(x)})/T(\varepsilon^r|_{\partial D(x)})$$

$$\nu_{D(x)} = \varepsilon^r$$

$$D(x) \hookrightarrow X \longrightarrow BG(k)$$

$$\nu_{D(x)} = \nu_X|_{D(x)} \oplus \varepsilon^r$$

An n-dim^l normal cx $(X, \nu_X : X \to BG(k), \rho_X : S^{n+k} \to T(\nu_X))$
determines $(n-r)$-dim^l normal pairs $(D(x), \partial D(x))$

$$\rho_{D(x),\partial D(x)} \in \mathrm{im}(\rho : \pi_{n+k}(T(\nu_X)) \longrightarrow \pi_{n+k}(T(\nu_X), T(\nu_X|_{X\backslash\{\hat{x}\}})))$$

$$= \pi_{n+k}\left(T(\nu_{D(x)})/T(\nu_{\partial D(x)})\right)$$

$$\begin{array}{ccc}
H_n(X) \ni [X] \mapsto \xrightarrow{\partial} [D(x)] & \xrightarrow{\partial} & H_{n-r}(D(x), \partial D(x)) \\
\wr\| \text{Thom} & & \wr\| \\
H_{n+k}(T(\nu_X)) & \longrightarrow & H_{n+k}(T(\nu_X), T(\nu_X|_{X\backslash\{\hat{x}\}})) \\
\uparrow \text{Hurewicz} & & \uparrow \\
\rho_X \in \pi_{n+k}(T(\nu_X)) & \longrightarrow & \pi_{n+k}(T(\nu_X), T(\nu_X|_{X\backslash\{\hat{x}\}}))
\end{array}$$

So each $(D(x), \partial D(x))$ is an $(n-|x|)$-dim^l normal pair.

$$\left(\mathscr{C}([D(x)]\cap - : C(D(x))^{n-|x|-*} \longrightarrow C(D(x))_{*+1}, \psi\right) =: (C(\overset{x}{X}), \psi)$$

$$(C, \psi) \in L_{n-1}(\mathbb{Z}, X) = H_{n-1}(X; \mathbb{L}_\cdot)$$

Lecture 18 10.12.08

~~¥~~ If X is an n-dim^l geo P. cx that is htpy eqvt to an cl n-dim^l top^l mfd M, how does a htpy eqvce $f : M \xrightarrow{\simeq} X$ determine a ~~res~~ reason why $s(X) = 0 \in \mathcal{S}_n(X)$

k, need a $\mathbb{Z}[\pi_1(X)]$-contractible quadratic Poincaré null-cobordism of $(n-1)$-dim^l quadratic Poincaré complex $\partial\sigma^*(X)$ in $\mathbb{A}(\mathbb{Z}, X)$
where $\partial\sigma^*(X)$ = n-dim^l alg normal cx in $\mathbb{A}(\mathbb{Z}, X)$
with $A(\partial\sigma^*(X)) = 0$

$$\sigma^*(X) = \{(C(D(x), \partial D(x)), \varphi, \gamma, \chi) \mid x \in X\}$$
$$\partial\sigma^*(X) = \{(\mathcal{C}([D(x)]\cap - : C(D(x))^{n-|x|-*} \longrightarrow C(D(x), \partial D(x))_{*+1}), \Psi) \mid x \in X\}$$
$$A(\partial\sigma^*(X)) = \{(\mathcal{C}([X]\cap - : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})), \Psi(X))\}$$
$$\simeq 0 \text{ since } X \text{ is Poincaré}$$

We work in PL category, for the full TOP category we have to work much harder! ($TOP/PL \simeq K(\mathbb{Z}_2, 3)$ by Rochlin's th^m)

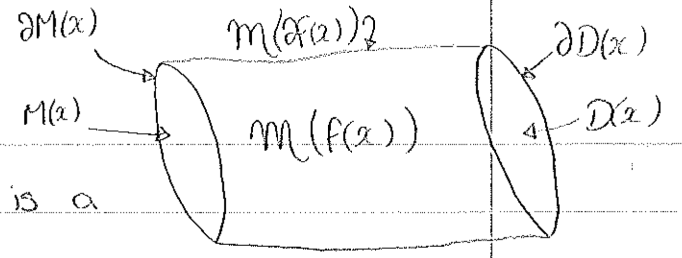
Th^m (M. Cohen) If M is an n-dim^l PL mfd and X is a simplicial complex then for any simplicial map $f: M \longrightarrow X'$, the inverse images $M(x) := f^{-1}(D(x)) \subseteq M$ are $(n-|x|)$-dim^l submfds with $\partial M(x) = f^{-1}(\partial D(x)) = \bigcup_{y > x} M(y)$

$(M, f) \in H_n(X; \Omega^{PL}) \cong \Omega^{PL}_n(X)$

Proposition 1 A normal map $(f, b): M \longrightarrow X'$, from an n-dim^l (PL) mfd M to an n-dim^l P. cx X, determines a quadratic Poincaré null-cobordism of $\partial\sigma^*(X)$ = $(n-1)$-dim^l quad P. cx in $\mathbb{A}(\mathbb{Z}, X)$ representing $s(X) = \partial\sigma^*(X) \in \mathbb{S}_n(X)$

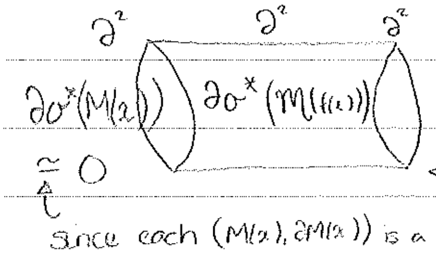
Proof

Each $(f(x), b(x)) = (f, b)|_{M(x)}$ $\underset{\text{mfd}}{M(x)} = f^{-1}(D(x)) \longrightarrow \underset{\text{normal, not P. in general}}{D(x)}$ is a normal map from an $(n-|x|)$-dim^l (PL) mfd with boundary to an $(n-|x|)$-dim^l normal cx (with bdy).



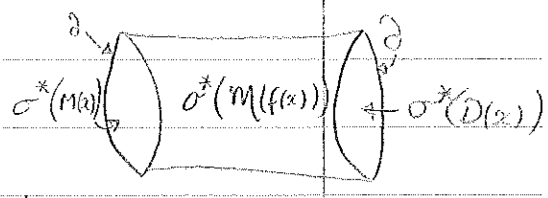
The mapping cylinder $\mathcal{M}$ of $(f(x), b(x))$ is a normal cx cobordism.

Such a geo cobordism determines a cobord of the alg normal ~~cx~~ cobordisms (cxs?) and also for the quadratic boundaries



failure of $\partial D(x)$ to be an $S^{n-|x|-1}$
$\parallel$
$\partial\sigma^*(D(x))$

quad. P. null-cobordism in $\mathbb{A}(\mathbb{Z}, X)$



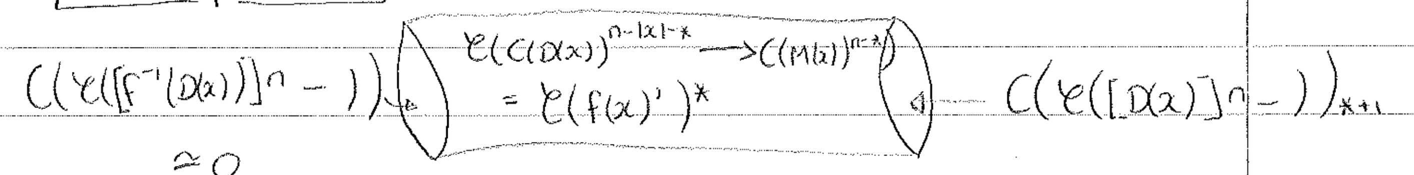
$$L_n(\mathbb{Z}[\pi_1 X]) \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X; \mathbb{L}_\cdot)$$

surgery obstruction $= \sigma_*(f, b) \longmapsto s(X) = \partial\sigma^*(X) \longmapsto 0$ □

<u>Proposition 2</u>: The normal map $(f, b): M \to X'$ in Prop^n 1 has $f: M \to X'$ being a htpy eqvce iff the assembly of the quad P. null-cobordism of $\partial\sigma^*(X)$ is $\mathbb{Z}[\pi_1(X)]$-contractible.

<u>Proof</u>

$$\underbrace{A(\partial\sigma^*(\mathcal{M}(f(x))))}_{1} \overset{?}{=} \underbrace{\mathcal{C}(f: C(\tilde{M}) \longrightarrow C(\tilde{X}))}_{2}{}^{n-*-1}$$



where $[f^{-1}(D(x))]^{n} - : C(M(x))^{n-*} \longrightarrow C(M(x), \partial M(x))$
$[D(x)]^{n} - : C(D(x))^{n-|x|-*} \longrightarrow C(D(x), \partial D(x))$

We need the following lemma...

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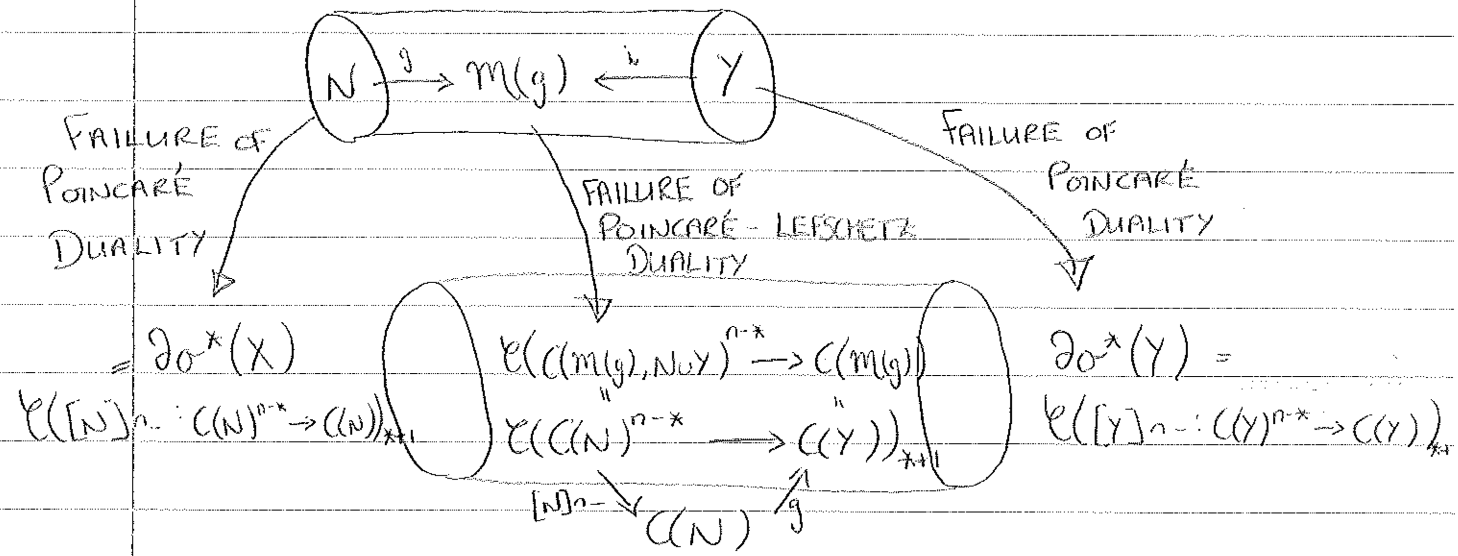
<u>Lemma</u> A normal map $(g,c): N \longrightarrow Y$ of n-dim^l normal cxs determines a normal bordism



Back to proof of prop^n:

Apply lemma with $g = f(x)$, $N = M(x)$, $Y = D(x)$

$$M(x) \xrightarrow{f(x)} \underset{\simeq D(x)}{\mathcal{M}(f(x))} \xleftarrow{i} D(x) \longrightarrow \underset{\substack{\simeq 0 \\ \because M(x) \text{ a mfd}}}{\partial\sigma^*(M(x))} \quad A(\mathcal{C}(C(M(x))^{n-*} \to C(D(x)))) \longleftarrow A(\partial\sigma^*(X))$$

And so...

$$A\Big(\mathcal{C}(C(M(x))^{n-*-|x|} \longrightarrow C(D(x)))\Big)$$

$$= \mathcal{C}\big(C(\hat{M})^{n-*} \longrightarrow C(\hat{X})\big) = \underbrace{\mathcal{C}(\hat{f})}_{2}{}_{*+1}$$

$$C(\hat{M}) \xrightarrow{\hat{f}} C(\hat{X}), \quad C(\hat{M})^{n-*} \xrightarrow{[M]_{n} \cap} C(\hat{M})$$

Note that the assembly of the quad^l null-cobordism of $\partial\sigma^*(X)$ is $\mathbb{Z}[\pi_1(X)]$-contractible iff $A(\partial\sigma^*(X)) = 0$ □

Where do normal maps $(f,b): M \longrightarrow X$ come from?

Answer: The Browder-Novikov transversality construction

$T(\tilde{\nu}_X)$
$T(\nu_X)$
$\mathbb{R}$

If X is an n-dim^{l} normal space $(X, \nu_X : X \to BG(k),\ \rho_X \in \pi_{n+k}(T(\nu_X)))$

$$\left\{\begin{array}{l}\text{bordism classes of deg 1}\\ \text{normal maps } (f,b) \text{ in PL category}\end{array}\right\} \overset{\text{1-1 correspondence}}{\longleftrightarrow} \left\{\text{PL reductions } \begin{array}{ccc} & & BPL(k)\\ & \nearrow^{\tilde{\nu}_X} & \downarrow \\ X & \xrightarrow{\nu_X} & BG(k)\end{array}\right\}$$

$$\begin{array}{ccc}\nu_X & \xrightarrow{b} & \tilde{\nu}_X\\ \downarrow & & \downarrow\\ M & \xrightarrow{f} & X\end{array}, \quad \tilde{\nu}_X : X \longrightarrow BPL(k)$$

<u>Proposition 3</u>. If X is an n-dim^{l} normal cx then

$$\left\{\begin{array}{l}\text{normal maps } (f,b): M \to X\\ \text{in the PL category}\end{array}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{array}{l}\text{quad P. null-cobordisms}\\ \text{of } \partial\sigma^*(X) \text{ in } \mathbb{A}(\mathbb{Z}, X)\end{array}\right\}$$

ALMOST, need TOP for full correspondence, only have $\xrightarrow{\sigma_X}$, not $\longleftarrow$ $(\Delta^n, c) \mapsto \sigma_X(f,b)$

Note: $\exists$ quad P. null-cobord of $\partial\sigma^*(X)$ iff $\mathcal{C}([D(x)]\cap -)$
$s(X) \in \ker(\mathbb{S}_n(X) \longrightarrow H_{n-1}(X; \mathbb{L}_\cdot))$

In fact, except for $\mathbb{Z}_2$ in dim^n 4, there is a homotopy eqvce

$G(k)/PL(k) \longrightarrow \mathbb{L}_\cdot(\mathbb{Z})$ = Kan Δ-set of quad P. cxs (pairs) over $\mathbb{Z}$
(k large)

$\|$

$$\left\{\begin{array}{l}\text{Kan } \Delta\text{-set of singular}\\ \text{simplices } \Delta^n \xrightarrow{c} G(k)/PL(k)\end{array}\right\} = \text{htpy fibre } BPL(k) \longrightarrow BG(k)$$

= Kan Δ-set of $\mathbb{R}^k \times \Delta^n \xrightarrow[\mathbb{R}]{\sim} \mathbb{R}^k \times \Delta^n$ PL homeom respecting proj^n to Δ^n
$\searrow \Delta^n \swarrow$

$\wr\|$

$$\left\{\begin{array}{l}\text{singular sxs PL transverse at}\\ \text{zero section of canonical } PL(k)\text{-bundle over } G(k)/PL(k)\end{array}\right\}$$

$$\begin{array}{ccccc} \nu_M & \longrightarrow & \tilde{\nu}_{\Delta^n} & \longrightarrow & \\ & & \| & & \\ & & \nu_{\Delta^n \subseteq \mathbb{R}^{n+k}} \oplus c^*\xi & & \downarrow \xi(k)\\ \downarrow & & \downarrow & & \\ M^n & \xrightarrow{f} & \Delta^n & \longrightarrow & G(k)/PL(k)\\ \text{n-mfd with faces} & & \overset{\|}{X} \text{ with faces} & & \end{array}$$

$\nu_{\Delta^n \subseteq \mathbb{R}^{n+k}} \oplus c^*\xi$

$S^{n+k} \longrightarrow T(\nu_{\Delta^n \subseteq \mathbb{R}^{n+k}})$
$\mathbb{R}$
$T(\nu_{\Delta^n \subseteq \mathbb{R}^{n+k}} \oplus c^*\xi)$

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$$\mathcal{S}^{PL}_{\partial}(S^n) \longrightarrow \pi_n(G/PL) \xrightarrow[\cong]{\sigma_*} L_n(\mathbb{Z})$$

= PL exotic spheres — bordisms of PL normal maps $\longmapsto$ surgery obstruction

= 0 by Smale — $(f,b): M^n \longrightarrow S^n$ — $\sigma_*(f,b)$

$$\sigma_* : \pi_n(G/PL) \xrightarrow{\cong} L_n(\mathbb{Z}) \;\text{"}\pi_n(G/TOP)\text{"} \quad (\text{for } n \neq 4)$$

Proof of Propn 3.

The Eilenberg obstruction theory to constructing normal maps and null-cobordisms.

Do downward induction on $\dim(x) = |x| = n, n-1, \ldots, 0$ "□"

(We get a normal space from the dual cells $D(x)$ for $x \in X$, a simp cx)

Proposition 4. If X is an n-diml normal $\mathbb{R}$ cx then every quadratic Poincaré null-cobordism of $\partial\sigma^*(D(x))$, where

$$\partial\sigma^*(X)\left\{\left(\mathcal{C}([D(x)]\cap - : C(D(x),\partial D(x))^{n-|x|-*} \longrightarrow C(D(x))\right)_{*} =: C(x), \psi \right\} \Big| x \in X\Big\}$$

with $H_*(\delta C(x)) = 0$ for $2* \leq n-|x|$

For $n \geq 5$, $\left(C(x) \longrightarrow \delta C(x), (\delta\psi, \psi)\right)$ for $x \in X$, can be realised by a normal map $(f,b): M^n \longrightarrow X$ with

$\delta C(x) = \mathcal{C}(f(x): M(x) \longrightarrow D(x))_{*+1}$

If X is Poincaré, then f is a htpy eqvce iff

$A(\delta C(x)) = \mathcal{C}(\tilde{f} : C(\tilde{M}) \longrightarrow C(\tilde{X})) \simeq 0$ over $\mathbb{Z}[\pi_1(X)]$

and $f_*(\pi_1(M)) \cong \pi_1(X)$

Let $n \geq 5$.

<u>Theorem.</u> An n-dim^l geometric Poincaré complex X is homotopy equivalent to an n-mfd iff

$$s(X) = \partial\sigma^*(X) \in \mathcal{S}_n(X) \text{ is } 0.$$

Proof

Consider the exact sequence

$$\dots \longrightarrow H_n(X;\mathbb{L}_\cdot) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathcal{S}_n(X) \longrightarrow H_{n-1}(X;\mathbb{L}_\cdot) \to$$

1st Browder-Novikov obstruction:

$$\sigma_*(f,b) \longmapsto s(X) \longmapsto 0$$

$$s(X) = \{(C(x), \psi(x)) \mid x \in X\}$$

$s(X)$ has image 0 in $H_{n-1}(X;\mathbb{L}_\cdot)$

$\iff \exists$ quad P. null-cobordism of $s(X)$ in $\mathbb{A}(\mathbb{Z}, X)$

$\iff \exists$ normal map $(f,b): M \to X$ with $\delta C(x) = \mathcal{C}(f(x): M(x) \to D(x))'$

2nd B-N obstr^n

$s(X) = 0 \in \mathcal{S}_n(X)$ iff 1st obstr^n pushes $(C(x) \to \delta C(x), (\delta\psi(x), \psi(x)))$ into

$$\sigma_*(f,b) = A(\{\delta C(x), \delta\psi(x)) \mid x \in X\} \in L_n(\mathbb{Z}[\pi_1 X])$$

$$\simeq 0.$$

$\square$

Commonly used abbreviations

"cx" = complex "dim^l" = dimensional "P." = Poincaré

"sx" = simplex "simp" = simplicial "sym" = symmetric

"quad" = quadratic "geo" = geometric "alg" = algebraic

"seq" = sequence "eqvt" = equivalent "eqvce" = equivalence

"htpy" = homotopy "iff" = if and only if "ch" = chain

"top^l" = topological "mfd" = manifold