THE ALGEBRAIC THEORY OF THE MASLOV INDEX

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Contents

1. Various functions

Definition 1.1. (i) The **jump** of a function $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$ is

$$j_f(x) \;=\; \varinjlim_{\epsilon}(f(x+\epsilon) - f(x-\epsilon)) \in \mathbb{R}\ .$$

(ii) Given a real number $x \in \mathbb{R}$ let $[x] \in \mathbb{Z}$ be the **integral part** and let $\{x\} \in [0,1)$ be the **fractional part**, so that

$$x \;=\; [x] + \{x\} \in \mathbb{R}\ .$$

(iii) The **sawtooth function** of $x \in \mathbb{R}$ is

$$((x)) \;=\; \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \textit{if } x \in \mathbb{Z} \end{cases} \in (-1/2, 1/2)\ .$$

(iv) The **reverse sawtooth function** of $x \in \mathbb{R}$ is

$$\mu(x) \;=\; 1 - 2\{x\} \in (-1, 1]$$

$\square$

Proposition 1.2. (I) *The fractional function*

$$\{\ \}\ :\ \mathbb{R} \to [0,1)\ ;\ x \mapsto \{x\} \;=\; x - [x]$$

has the following properties:

(i) $\{\ \}$ *is continuous on* $\mathbb{R}\backslash\mathbb{Z}$, *with jump function*

$$j_{\{\ \}}(x) \;=\; \begin{cases} -1 & \text{if } \in \mathbb{Z} \\ 0 & \text{otherwise}\ . \end{cases}$$

(ii) $\{x\}+\{y\}-\{x+y\} \;=\; [x+y]-[x]-[y] \;=\; \begin{cases} 0 & \textit{if } 0 \leqslant \{x\} + \{y\} < 1 \\ 1 & \textit{if } 1 \leqslant \{x\} + \{y\} < 2. \end{cases}$

(iii) $\{x+1\} \;=\; \{x\}$.

(iv) $\{x\} + \{-x\} \;=\; \{x\} + \{1-x\} \;=\; \begin{cases} 1 & \textit{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \textit{if } x \in \mathbb{Z}. \end{cases}$

(v) $\{x+1/2\} - \{x\} \;=\; \begin{cases} 1/2 & \textit{if } 0 \leqslant \{x\} < 1/2 \\ -1/2 & \textit{if } 1/2 \leqslant \{x\} < 1. \end{cases}$

(vi) $n\{x\} - \{nx\} = [nx] - n[x] \in \mathbb{Z} \subset \mathbb{R}$ $(n \in \mathbb{Z})$.

(II) *The sawtooth function*

$$((\ ))\ :\ \mathbb{R} \to (-1/2, 1/2)\ ;\ x \mapsto ((x)) \;=\; \begin{cases} \{x\} - 1/2 & \textit{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \textit{if } x \in \mathbb{Z} \end{cases}$$

has the following properties:

(i) $((\ ))$ *is continuous on* $\mathbb{R}\backslash\mathbb{Z}$, *with jump function*

$$j_{((\ ))}(x) \ = \ \begin{cases} -1 & \textit{if } x \in \mathbb{Z} \\ 0 & \textit{otherwise} \ . \end{cases}$$

(ii) $((x+1)) = ((x))$, $((-x)) = -((x))$.

(iii) $((0)) = ((1/2)) = 0$, $((1/4)) = -1/4$, $((3/4)) = 1/4$.

(iv) $((x)) = x + ([-x]-[x])/2 = (\{x\}-\{-x\})/2$.

(v) $$((x+1/2)) \ = \ \begin{cases} \{x\} & \textit{if } 0 \leqslant \{x\} < 1/2 \\ 0 & \textit{if } \{x\} = 1/2 \\ \{x\}-1 & \textit{if } 1/2 < \{x\} < 1. \end{cases}$$

(vi)

$$\begin{aligned} 2(\,((x))+((y))-((x+y))\,) \ &= -\mathrm{sgn}(\sin \pi x \sin \pi y \sin \pi(x+y)) \\ &= \begin{cases} 0 & \textit{if } x \in \mathbb{Z} \textit{ or } y \in \mathbb{Z} \textit{ or } x+y \in \mathbb{Z} \\ -1 & \textit{if } 0 < \{x\}+\{y\} < 1 \\ 1 & \textit{if } 1 < \{x\}+\{y\} < 2. \end{cases} \end{aligned}$$

In particular, for $x = y$

$$2(\,2((x))-((2x))\,) = -\mathrm{sgn}(\sin 2\pi x) \in \mathbb{Z} \subset \mathbb{R}.$$

(vii) *For any* $n \in \mathbb{Z}$

$$2(\,n((x))-((nx))\,) = -\sum_{k=1}^{n-1} \mathrm{sgn}(\sin \pi x \sin k\pi x \sin{(k+1)\pi x}) \in \mathbb{Z} \subset \mathbb{R}.$$

(viii) *For any* $\theta_1, \theta_2 \in \mathbb{R}$

$$\begin{aligned} &\mathrm{sgn}(\sin\theta_1 \sin\theta_2 \sin{(\theta_1+\theta_2)}) - 2\,\mathrm{sgn}(\,(\sin\theta_1/2)(\sin\theta_2/2)(\sin{(\theta_1+\theta_2)/2})\,) \\ &= 2(2((\theta_1/2\pi))-((\theta_1/\pi))) + 2(2((\theta_2/2\pi))-((\theta_2/\pi))) \\ &\qquad -2(2(((\theta_1+\theta_2)/2\pi))-(((\theta_1+\theta_2)/\pi))) \\ &= -\mathrm{sgn}(\sin\theta_1) - \mathrm{sgn}(\sin\theta_2) + \mathrm{sgn}(\sin{(\theta_1+\theta_2)}) \in \mathbb{Z} \ . \end{aligned}$$

(III) *The reverse sawtooth function*

$$\mu \ : \ \mathbb{R} \to (-1,1] \ ; \ x \mapsto 1-2\{x\}$$

has the following properties:

(i) $$\mu(x) \ = \ \begin{cases} -2((x)) & \textit{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 1 & \textit{if } x \in \mathbb{Z} \ . \end{cases}$$

(ii) *μ is continuous at $x \in \mathbb{R}\backslash\mathbb{Z}$, with jump function*

$$j_\mu(x) = \begin{cases} 2 & \text{if } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} .$$

(iii) $\mu(x)+\mu(y)-\mu(x+y) = \begin{cases} +1 & \text{if } 0 \leqslant \{x\}+\{y\} < 1 \\ -1 & \text{if } 1 \leqslant \{x\}+\{y\} < 2. \end{cases}$

(iv) $\mu(0) = 1$, $\mu(1/2) = 0$.

(v) $\mu(x+1) = \mu(x)$ *for* $x \in \mathbb{R}$.

(vi) $\mu(x)+\mu(-x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 2 & \text{if } x \in \mathbb{Z}. \end{cases}$

(vii) $\mu(x)-\mu(x+1/2) = 2\mu(x)-\mu(2x) = \begin{cases} +1 & \text{if } 0 \leqslant \{x\} < 1/2 \\ -1 & \text{if } 1/2 \leqslant \{x\} < 1. \end{cases}$

□

Remark 1.3. (i) Fourier expansion of sawtooth function

$$((x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin\left(2\pi n(x-1/2)\right) .$$

(ii) Eisenstein's formula for $x = p/q \in \mathbb{Q}$

$$((x)) = \frac{i}{2q} \sum_{j=1}^{q-1} \cot\frac{\pi j}{q} e^{2\pi i j x} .$$

□

Remark 1.4. The E**-function** of Barge and Ghys [5, p.239]

$$E : \mathbb{R} \to \mathbb{Z}[1/2] \subset \mathbb{R} ;$$

$$x \mapsto x - ((x)) = ([x]-[-x])/2 = \begin{cases} [x]+1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ x & \text{if } x \in \mathbb{Z} \end{cases}$$

is such that

$$E(-x) = -E(x) \ (x \in \mathbb{R})$$

and for any $x, y \in \mathbb{R}$

$$\begin{aligned} E(x+y)-E(x)-E(y) &= ((x))+((y))-((x+y)) \\ &= -\mathrm{sgn}(\sin\pi x \sin\pi y \sin\pi(x+y))/2 \\ &= \begin{cases} 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x+y \in \mathbb{Z} \\ -1/2 & \text{if } 0 < \{x\}+\{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\}+\{y\} < 2. \end{cases} \end{aligned}$$

□

Proposition 1.5. (Rademacher [45], Walsh [60], Paley [41])
(I) *The* **Rademacher functions**

$$\phi_n \;:\; \mathbb{R} \to \{-1,0,1\}\;;\; x \mapsto \mathrm{sgn}(\sin 2^{n+1}\pi x)\ (n \geqslant 0)$$

are such that

(i) $\phi_0(x) \;=\; \mathrm{sgn}(\sin 2\pi x) \;=\; \begin{cases} +1 & \text{if } 0 < \{x\} < 1/2 \\ -1 & \text{if } 1/2 < \{x\} < 1 \\ 0 & \text{if } \{x\} = 0 \text{ or } 1/2\ . \end{cases}$

(ii) $\phi_n(x) = \phi_0(2^n x) = 2((2^{n+1}x)) - 2^{n+2}((x))$.

(iii) $\phi_n(x+1) = \phi_n(x)$, $\phi_n(x+1/2) = \phi_n(-x) = -\phi_n(x)$.

(iv) $\phi_n(0) = \phi_n(1/2) = 0$.

(v) $\mu(x) - \mu(x+1/2) = 2\mu(x) - \mu(2x) = \phi_0(x)$ *for* $2x \in \mathbb{R}\backslash\mathbb{Z}$.

(II) *The* **Walsh functions** $\psi_n : \mathbb{R} \to \{-1,0,1\}$ $(n \geqslant 0)$ *are defined by*

$$\begin{aligned} \psi_0(x) \;=\; 1\ ,\ \psi_n(x) \;&=\; \phi_{n_1}(x)\phi_{n_2}(x)\dots\phi_{n_k}(x) \\ &(n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_k})\ . \end{aligned}$$

In particular

$$\psi_{2^n}(x) \;=\; \phi_n(x) \;=\; \phi_0(2^n x) \;=\; \mathrm{sgn}(\sin 2^{n+1}\pi x)\ .$$

The Walsh functions constitute a complete orthonormal set with respect to

$$(f,g) \mapsto \int_0^1 f(x)g(x)dx$$

behaving like trigonometric series on I*:*

$$\int_0^1 \psi_m(x)\psi_n(x)dx \;=\; \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n\ . \end{cases}$$

Every Lebesgue integrable function $f : I \to \mathbb{R}$ *has a Walsh-Fourier expansion*

$$F(x) \;=\; \sum_{n=0}^{\infty} c_n\psi_n(x) \text{ with } c_n \;=\; \int_0^1 f(x)\psi_n(x)dx \in \mathbb{R}$$

with $F(x) = f(x)$ *if* f *is continuous at* $x \in (0,1)$.

□

Example 1.6. The Fourier-Walsh expansion of the reverse sawtooth function $\mu(x)$ is

$$\mu(x) \;=\; \sum_{k=0}^{\infty} \psi_{2^k}(x)/2^{k+1} \;=\; \sum_{k=0}^{\infty} \phi_0(2^k x)/2^{k+1} \;,$$

with

$$\int_0^1 \mu(x)\psi_n(x)dx \;=\; \begin{cases} 1/2^{k+1} & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k. \end{cases}$$

□

Definition 1.7. The η**-invariant** function is

$$\begin{aligned} &\eta \;:\; \mathbb{R} \to (-1,1] \;; \\ &\theta \mapsto \eta(\theta) \;=\; -2((\theta/\pi)) \;=\; \begin{cases} \mu(\theta/\pi) = 1 - 2\{\theta/\pi\} & \text{if } \theta/\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } \theta/\pi \in \mathbb{Z}\;. \end{cases} \end{aligned}$$

□

Proposition 1.8. *The η-invariant function $\eta : \mathbb{R} \to (-1,1]$ has the following properties:*
(i) *η is continuous at $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, jumping by 2 at $\theta \in \pi\mathbb{Z}$.*
(ii) *$\eta(\pi n/2) = 0$ ($n \in \mathbb{Z}$).*
(iii) *$\eta(\theta + \pi) = \eta(\theta)$, $\eta(-\theta) = -\eta(\theta)$,*
$2\eta(\theta) - \eta(2\theta) = \eta(\theta) + \eta(\pi/2 - \theta) = \operatorname{sgn}(\sin 2\theta)$.
(iv)

$$\begin{aligned} \eta(\theta) \;&=\; -2((\theta/\pi)) \;=\; -2\theta/\pi + ([\theta/\pi] - [-\theta/\pi]) \\ &=\; 2E(\theta/\pi) - 2\theta/\pi \;=\; \begin{cases} 1 - 2\{\theta/\pi\} & \textit{if } \theta/\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \textit{if } \theta/\pi \in \mathbb{Z}\;. \end{cases} \end{aligned}$$

(v)

$$\begin{aligned} &\eta(\theta) + \eta(\phi) - \eta(\theta + \phi) \\ &=\; 2\big(((\,(\theta+\phi)/\pi\,)) - ((\theta/\pi)) - ((\phi/\pi))\big) \\ &=\; -2\big(E(\,(\theta+\phi)/\pi\,) - E(\theta/\pi) - E(\phi/\pi)\big) \\ &=\; \operatorname{sgn}\big(\sin(\theta)\sin(\phi)\sin(\theta+\phi)\big) \\ &=\; \begin{cases} 1 & \textit{if } \theta/\pi, \phi/\pi, (\theta+\phi)/\pi \in \mathbb{R}\backslash\mathbb{Z} \textit{ and } 0 \leqslant \{\theta/\pi\} + \{\phi/\pi\} < 1\;, \\ -1 & \textit{if } \theta/\pi, \phi/\pi, (\theta+\phi)/\pi \in \mathbb{R}\backslash\mathbb{Z} \textit{ and } 1 \leqslant \{\theta/\pi\} + \{\phi/\pi\} < 2\;, \\ 0 & \textit{otherwise}\;. \end{cases} \end{aligned}$$

In view of the identity

$$\sin(2\theta) + \sin(2\phi) - \sin(2(\theta+\phi)) \;=\; 4\sin(\theta)\sin(\phi)\sin(\theta+\phi)$$

we also have

$$\begin{aligned}\eta(\theta)+\eta(\phi)-\eta(\theta+\phi) &= 2\big((((\theta+\phi)/\pi))-((\theta/\pi))-((\phi/\pi))\big)\\ &= \mathrm{sgn}\big(\sin(2\theta)+\sin(2\phi)-\sin(2(\theta+\phi))\big)\\ &= \mathrm{sgn}\big(\sin(\theta)\sin(\phi)\sin(\theta+\phi)\big)\in\{-1,0,1\}\ .\end{aligned}$$

□

The **exponential** function $e:\mathbb{C}\to\mathbb{C}\backslash\{0\}$ is defined as usual by

$$e^z = \sum_{j=0}^{\infty}\frac{z^j}{j!}\in\mathbb{C}\backslash\{0\}$$

such that

(i) $z\mapsto e^z$ is continuous
(i) $e^0=1$, $e^{z+w}=e^ze^w\in\mathbb{C}$
(ii) $e^z=e^w\in\mathbb{C}\backslash\{0\}$ if and only if $z-w=2\pi ik$ for some $k\in\mathbb{Z}$.

The **principal logarithm** function

$$\log\ :\ \mathbb{C}\backslash\{0\}\to\mathbb{R}+i(-\pi,\pi]\subset\mathbb{C}$$

is defined as usual by

$$\log(z) = \log(|z|)+i\arg(z)\ (\arg(z)\in(-\pi,\pi])$$

such that

(i) $z\mapsto\log(z)$ is continuous on $\mathbb{C}\backslash\{(-\infty,0]\}$.
(ii) $\log(1)=0$, $\log(-1)=\pi i$, $\log(\pm i)=\pm\pi i/2$.
(iii) If

$$z = re^{i\theta}\in\mathbb{C}\backslash\{0\}\ (r>0,\theta\in\mathbb{R})$$

then

$$\begin{aligned}
\log(z) &= \log(r) + \pi i \mu(\frac{\pi-\theta}{2\pi}) \\
&= \log(r) + \pi i(1-2\{\frac{\pi-\theta}{2\pi}\}) \\
&= \begin{cases} \log(r) + 2\pi i\big((\frac{\theta+\pi}{2\pi})\big) & \text{for } \theta/\pi \in \mathbb{R}\backslash(2\mathbb{Z}+1), \\ & \qquad \text{with } 2\pi\big((\frac{\theta+\pi}{2\pi})\big) \in (-\pi,\pi) \\ \log(r) + \pi i & \text{for } \theta/\pi \in 2\mathbb{Z}+1, \text{ with } z=-r \end{cases} \\
&= \begin{cases} \log(r) - \pi i \eta(\frac{\theta+\pi}{2}) & \text{for } \theta/\pi \in \mathbb{R}\backslash(2\mathbb{Z}+1), \\ \log(r) + \pi i & \text{for } \theta/\pi \in 2\mathbb{Z}+1, \text{ with } z=-r. \end{cases}
\end{aligned}$$

Proposition 1.9. *The exponential and principal logarithm functions have the following properties:*
(i) $e^{\log(z)} = z \in \mathbb{C}\backslash\{0\}$ *for all* $z \in \mathbb{C}\backslash\{0\}$.
(ii) *For* $z = x+iy \in \mathbb{C}$

$$\log(e^z) \;=\; z - 2\pi i k \in \mathbb{C}\ ((2k-1)\pi < y \leqslant (2k+1)\pi)\ ,$$

that is

$$\log(e^{x+iy}) \;=\; x + \pi i(1-2\{\frac{\pi-y}{2\pi}\}) \in \mathbb{C}\backslash\{0\}\ .$$

(iii) *If* $z \in \mathbb{C}\backslash\{(-\infty,0]\}$ *then*

$$\log(z) \;=\; \int_{-\infty}^{0} (\frac{1}{x-z} - \frac{1}{x-1})dx \in \mathbb{C}\ .$$

(iv) *For* $z_1, z_2 \in \mathbb{C}\backslash\{0\}$

$$\begin{aligned}
\log(z_1z_2) - \log(z_1) - \log(z_2) &= i(\arg(z_1z_2) - \arg(z_1) - \arg(z_2)) \\
&= \begin{cases} 2\pi i & \textit{if } -2\pi < \arg(z_1)+\arg(z_2) \leqslant -\pi \\ 0 & \textit{if } -\pi < \arg(z_1)+\arg(z_2) \leqslant \pi \\ -2\pi i & \textit{if } \pi < \arg(z_1)+\arg(z_2) \leqslant 2\pi\ . \end{cases}
\end{aligned}$$

(v) *For* $\theta_1, \theta_2 \in \mathbb{R}$

$$\begin{aligned}
&\log(e^{i(\theta_1+\theta_2)}) - \log(e^{i\theta_1}) - \log(e^{i\theta_2}) \\
&\qquad = \begin{cases} -2\pi i & \textit{if } 0 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 1/2 \\ 0 & \textit{if } 1/2 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 3/2 \\ 2\pi i & \textit{if } 3/2 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 2\ . \end{cases}
\end{aligned}$$

$\square$

Example 1.10. For any $\theta \in \mathbb{R}$

$$\begin{aligned} \eta(\theta) &= -2((\theta/\pi)) \\ &= \begin{cases} \dfrac{1}{\pi i}\log(-e^{-2i\theta}) \;=\; 1-2\{\dfrac{\theta}{\pi}\} & \text{if } e^{i\theta} \neq \pm 1 \\ 0 & \text{if } e^{i\theta} = \pm 1 \; . \end{cases} \end{aligned}$$

$\square$

Given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ let

$$D(\lambda_1, \lambda_2, \ldots, \lambda_n) \;=\; \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \in M_n(\mathbb{C})$$

be the diagonal $n \times n$ matrix.

The **exponential** of a complex $n \times n$ matrix $A \in M_n(\mathbb{C})$ is the invertible complex $n \times n$ matrix defined by

$$\exp(A) \;=\; \sum_{j=0}^{\infty} \frac{A^j}{j!} \in GL_n(\mathbb{C})$$

and satisfies

(i) $\exp(D(\lambda_1, \lambda_2, \ldots, \lambda_n)) = D(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}) \in GL_n(\mathbb{C})$ for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. In particular, $\exp(0_n) = I_n$.
(ii) $\exp(A_1 + A_2) = \exp(A_1)\exp(A_2) \in GL_n(\mathbb{C})$.
(iii) $\det(I_n - zA) = \exp(-\sum\limits_{j=1}^{\infty} \dfrac{\mathrm{tr}(A^j)}{j} z^j) \in \mathbb{C}[z]$.

The eigenvalues of $A \in M_n(\mathbb{C})$ are the roots $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ of the characteristic polynomial

$$\mathrm{ch}_A(z) \;=\; \det(zI_n - A : \mathbb{C}^n[z] \to \mathbb{C}^n[z]) \;=\; \prod_{j=1}^{n} (z - \lambda_j) \in \mathbb{C}[z] \; .$$

A is diagonalizable if and only if $\mathbb{C}^n$ has a basis $\{b_1, b_2, \ldots, b_n\}$ consisting of eigenvectors of A, in which case $B = (b_1 \; b_2 \; \ldots \; b_n) \in GL_n(\mathbb{C})$ is such that

$$A \;=\; BD(\lambda_1, \lambda_2, \ldots, \lambda_n)B^{-1} \in M_n(\mathbb{C}) \; .$$

A is invertible if and only if each eigenvalue $\lambda_j \neq 0 \in \mathbb{C}$.

The **logarithm** of a diagonalizable $A \in GL_n(\mathbb{C})$ is the complex $n \times n$ matrix defined by

$$\log(A) \;=\; BD(\log(\lambda_1), \log(\lambda_2), \dots, \log(\lambda_n))B^{-1} \in M_n(\mathbb{C})$$

Proposition 1.11. *The logarithm function satisfies:*
(i) $\exp(\log(A)) = A \in GL_n(\mathbb{C})$.
(ii) $\log(D(\lambda_1, \lambda_2, \dots, \lambda_n)) = D(\log(\lambda_1), \log(\lambda_2), \dots, \log(\lambda_n)) \in GL_n(\mathbb{C})$ $(\lambda_j \in \mathbb{C}\backslash\{0\})$.
(iii) $\exp(\mathrm{tr}(\log(A))) = \det(A) = \lambda_1\lambda_2 \dots \lambda_n \in \mathbb{C}\backslash\{0\}$.
(iv) *For diagonalizable* $A_1, A_2, A_1A_2 \in GL_n(\mathbb{C})$

$$\mathrm{tr}(\log(A_1A_2)) - \mathrm{tr}(\log(A_1)) - \mathrm{tr}(\log(A_2)) \in 2\pi i\mathbb{Z} \subset \mathbb{C} \ .$$

(v) *If* $A \in GL_n(\mathbb{C})$ *is diagonalizable and the eigenvalues are not in* $(-\infty, 0]$ *then*

$$\log(A) \;=\; \int_{-\infty}^{0} (\frac{1}{x - A} - \frac{1}{x - I_n})dx \in GL_n(\mathbb{C}) \ .$$

(vi) *If* $A : I \to GL_n(\mathbb{C})$ *is a continuous function such that each* $A(s) \in GL_n(\mathbb{C})$ *is diagonalizable and the eigenvalues are not in* $(-\infty, 0]$ *then*

$$\log(A) \;:\; I \to GL_n(\mathbb{C}) \ ; \ s \mapsto \log(A(s))$$

is a continuous function.

$\square$

2. Group cohomology

The cohomology groups $H^n(G; A)$ are abelian groups defined for any discrete groups G, A and $n \geqslant 0$. In terms of the classifying space BG

$$H^n(G; A) \;=\; H^n(BG; A) \ (n \geqslant 0) \ .$$

We shall only be concerned with the cases $n = 1, 2$, with A abelian.

The **first cohomology group** of a discrete group G with coefficients in an abelian group A is

$$H^1(G; A) \;=\; \mathrm{Hom}(G, A) \ ,$$

the abelian group of group morphisms $f : G \to A$. The function

$$H^1(G; A) \to H^1(BG; A) \;=\; [BG, BA] \ ; \ f \mapsto Bf$$

is an isomorphism.

An A-valued **cocycle** on G is a function

$$\tau \ :\ G\times G\to A$$

such that

$$\tau(x,y)+\tau(xy,z) \ =\ \tau(x,yz)+\tau(y,z)\in A\ (x,y,z\in G)\ .$$

The **coboundary** of a function $\alpha : G\to A$ is the cocycle

$$\delta\alpha \ :\ G\times G\to A\ ;\ (x,y)\mapsto \delta\alpha(x,y) \ =\ \alpha(x)+\alpha(y)-\alpha(xy)\ .$$

The **second cohomology group** is

$$H^2(G;A) \ =\ \{\text{cocycles } \tau\}/\{\text{coboundaries } \delta\alpha\}\ ,$$

i.e. the abelian group of equivalences of cocycles τ, with the cocycles τ,τ' equivalent if there exists a function $\alpha : G\to A$ such that

$$\tau(x,y)-\tau'(x,y) \ =\ \delta\alpha(x,y) \ =\ \alpha(x)+\alpha(y)-\alpha(xy)\in A\ ,$$

and addition by

$$(\tau_1+\tau_2)(x,y) \ =\ \tau_1(x,y)+\tau_2(x,y)\in A\ .$$

Proposition 2.1. *$H^2(G;A)$ classifies the central group extensions of G by A*

$$0\longrightarrow A\xrightarrow{\ q\ }\widetilde{G}\xrightarrow{\ p\ }G\longrightarrow 0\ .$$

Proof. A cocycle τ determines the group extension

$$0\longrightarrow A\xrightarrow{\ q_\tau\ }G\times_\tau A\xrightarrow{\ p_\tau\ }G\longrightarrow 0$$

where $G\times_\tau A=G\times A$ (as sets) with group law

$$(x,a)(y,b) \ =\ (xy,a+b+\tau(x,y))\ ,$$

and

$$\begin{aligned} p_\tau &\ :\ G\times_\tau A\to G\ ;\ (x,a)\mapsto x\ ,\\ q_\tau &\ :\ A\to G\times_\tau A\ ;\ a\mapsto (1,a)\ .\end{aligned}$$

Conversely, given a central group extension

$$0\longrightarrow A\xrightarrow{\ q\ }\widetilde{G}\xrightarrow{\ p\ }G\longrightarrow 0$$

and a section $s : G\to\widetilde{G}$ of p such that

$$p(s(x)) \ =\ x\ (x\in G)$$

the function $\tau : G\times G\to A$ determined by the identity

$$q\tau(x,y) \ =\ s(x)s(y)s(xy)^{-1}\in \ker(p:\widetilde{G}\to G) \ =\ \mathrm{im}(q:A\to\widetilde{G})$$

is a cocycle such that there is defined an isomorphism of central group extensions

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "q"] \arrow[d, equal] & \widetilde{G} \arrow[r, "p"] \arrow[d, "\cong"'] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "q_\tau"] & G \times_\tau A \arrow[r, "p_\tau"] & G \arrow[r] & 0 \end{tikzcd}$$

with

$$\widetilde{G} \xrightarrow{\cong} G \times_\tau A \ ; \ x \mapsto (p(x), a) \ , \ q(a) = s(p(x))x^{-1} \ .$$

For any function $\alpha : G \to A$ the coboundary

$$\delta\alpha \ : \ G \times G \to A \ ; \ (x, y) \mapsto \delta\alpha(x, y) \ = \ \alpha(x) + \alpha(y) - \alpha(xy)$$

is such that there is defined an isomorphism of central group extensions

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "q"] \arrow[d, equal] & G \times A \arrow[r, "p"] \arrow[d, "\cong"'] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "q_{\delta\alpha}"] & G \times_{\delta\alpha} A \arrow[r, "p_{\delta\alpha}"] & G \arrow[r] & 0 \end{tikzcd}$$

with

$$G \times A \xrightarrow{\cong} G \times_{\delta\alpha} A \ ; \ (x, a) \mapsto (x, a + \alpha(x)) \ .$$

$\square$

Remark 2.2. $H^2(G; A)$ is isomorphic to $H^2(BG; A)$.

$\square$

A group morphism $f : G \to H$ induces group morphism

$$f^* \ : \ H^2(H; A) \to H^2(G; A) \ ; \ [\tau] \mapsto [f^*\tau]$$

by sending a cocycle $\tau : H \times H \to A$ to the cocycle

$$f^*\tau \ : \ G \times G \to A \ ; \ (x, y) \mapsto \tau(f(x), f(y))$$

with a morphism of central extensions

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "q_{f^*\tau}"] \arrow[d, equal] & G \times_{f^*\tau} A \arrow[r, "p_{f^*\tau}"] \arrow[d, "f \times 1"] & G \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r, "q_\tau"] & H \times_\tau A \arrow[r, "p_\tau"] & H \arrow[r] & 0 \end{tikzcd}$$

A short exact sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

induces a long exact sequence of cohomology groups

$$H^1(G;A) \longrightarrow H^1(G;B) \longrightarrow H^1(G;C) \xrightarrow{\delta}$$
$$H^2(G;A) \longrightarrow H^2(G;B) \longrightarrow H^2(G;C)\ .$$

If $\sigma : C \to B$ is a section of $B \to C$ the connecting map

$$\delta\ :\ H^1(G;C) \to H^2(G;A)\ ;\ [f] \mapsto [\tau]$$

sends a group morphism $f : G \to C$ to the A-valued cocycle

$$\begin{aligned}\tau(x,y)\ &=\ \sigma(f(x)) + \sigma(f(y)) - \sigma(f(xy))\\ &\in \ker(B \to C)\ =\ \mathrm{im}(A \to B)\ .\end{aligned}$$

For $G = C$ the connecting map sends $1 \in H^1(C;C) = \mathrm{Hom}(C,C)$ to the class $\delta(1) = [\tau] \in H^2(C;A)$ classifying the extension $0 \to A \to B \to C \to 0$ itself. If $\sigma, \sigma' : C \to B$ are two sections of $B \to C$ then

$$\tau - \tau'\ =\ \delta\alpha\ :\ C \times C \to A$$

with

$$\alpha\ :\ C \to A\ ;\ x \mapsto \sigma(x) - \sigma'(x)\ ,$$

and $\delta(1) = [\tau] = [\tau'] \in H^2(C;A)$.

The infinite cyclic covers of a space X

$$\mathbb{Z} \longrightarrow \overline{X} \xrightarrow{\ p\ } X$$

are classified by cohomology classes $f \in H^1(X) = [X, B\mathbb{Z}] = [X, S^1]$, with

$$\overline{X}\ =\ f^*\mathbb{R}\ =\ \{(x,t) \in X \times \mathbb{R}\,|\, f(x) = e^{2\pi i t} \in S^1\}$$

the pullback of the universal cover $p : \mathbb{R} \to S^1; t \mapsto e^{2\pi i t}$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "q"] \arrow[d, equal] & \overline{X} \arrow[r, "p"] \arrow[d, "\overline{f}"] & X \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "q"] & \mathbb{R} \arrow[r, "p"] & S^1 \end{tikzcd}$$

with $q : \mathbb{Z} \to \mathbb{R}$ the inclusion, and $\overline{f} : \overline{X} \to \mathbb{R}; (x,t) \mapsto t$.

For any topological group G let G^δ denote the discrete group underlying G. The identity function $j : G^\delta \to G$ is continuous, so that it induces a map of classifying spaces $j : BG^\delta \to BG$ and hence morphisms

$$j^*\ :\ H^*(BG) \to H^*(BG^\delta)\ =\ H^*(G^\delta;\mathbb{Z})\ .$$

For the classifying space $X = BG$ of a discrete group G the forgetful map $H^2(BG) \to H^2(G;\mathbb{Z})$ is an isomorphism. For a topological group G with $\pi_1(G) = \mathbb{Z}$ and universal cover $\widetilde{G}$ the morphism

$$j^* \ : \ H^2(BG) = \mathbb{Z} \to H^2(BG^\delta) \ = \ H^2(G^\delta;\mathbb{Z})$$

sends the generator $1 \in H^2(BG) = \mathbb{Z}$ to the central extension

$$0 \to \mathbb{Z} \to \widetilde{G}^\delta \to G^\delta \to \{1\} \ .$$

(See Rawnsley [49] for the construction of $\widetilde{G}$).

The classifying space of S^1 (regarded as a topological group) is the infinite-dimensional complex projective space

$$BS^1 \ = \ BU(1) \ = \ \mathbb{CP}^\infty \ .$$

For any space X the cohomology group $[X, \mathbb{CP}^\infty]$ classifies $\mathbb{C}$-bundles over X

$$\alpha \ : \ \mathbb{C} \to E(\alpha) \to X$$

with the first Chern class defining an isomorphism

$$[X, \mathbb{CP}^\infty] \to H^2(X) \ ; \ \alpha \mapsto c_1(\alpha) \ .$$

The generator

$$1 \in H^2(BS^1) \ = \ H^2(\mathbb{CP}^\infty) \ = \ \mathbb{Z}$$

classifies the canonical $\mathbb{C}$-bundle

$$\lambda \ : \ \mathbb{C} \to \text{Hopf bundle} \to \mathbb{CP}^\infty$$

with $c_1(\lambda) = 1 \in H^2(\mathbb{CP}^\infty) = \mathbb{Z}$.

Definition 2.3. The **Chern cocycle** is

$$\begin{aligned} u \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) &\mapsto [x+y] - [x] - [y] \\ = \ \{x\} + \{y\} - \{x+y\} \ &= \ \begin{cases} 0 & \text{if } 0 \leqslant \{x\} + \{y\} < 1 \\ 1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2. \end{cases} \end{aligned}$$

□

Proposition 2.4. *The Chern cocycle u corresponds to the central extension*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q} \mathbb{R}^\delta \xrightarrow{p} S^{1\delta} \longrightarrow 0 \ .$$

determined by the universal cover

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q} \mathbb{R} \xrightarrow{p} S^1 \longrightarrow 0$$

with

$$p \ : \ \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x} \ ,$$

$$q \ : \ \mathbb{Z} \to \mathbb{R} \ ; \ 1 \mapsto 1 \ .$$

The cohomology class

$$[u] \ = \ Bj^*(1) \ = \ j^*B(1) \ = \ c_1(j^*\lambda) \in H^2(BS^{1\delta}) \ = \ H^2(S^{1\delta};\mathbb{Z})$$

is the first Chern class of the pullback $j^*\lambda : BS^{1\delta} \to BS^1$*.*

Proof. The projection p has section

$$v \ : \ S^{1\delta} \to \mathbb{R}^\delta \ ; \ z \ = \ e^{2\pi i x} \mapsto \{x\} \ (0 \leqslant x < 1)$$

with corresponding cocycle

$$\begin{aligned} & u \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) \mapsto \\ & u(e^{2\pi i x}, e^{2\pi i y}) \ = \ v(x) + v(y) - v(x+y) \\ & = \ \{x\} + \{y\} - \{x+y\} \ = \ \begin{cases} 0 & \text{if } 0 \leqslant \{x\} + \{y\} < 1 \\ 1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2 \end{cases} \end{aligned}$$

such that

$$\begin{aligned} [u] \ = \ \delta(1) \in \operatorname{im}(\delta : H^1(S^{1\delta};S^{1\delta}) \to H^2(S^{1\delta};\mathbb{Z})) \\ = \ker(H^2(S^{1\delta};\mathbb{Z}) \to H^2(S^{1\delta};\mathbb{R}^\delta)) \ . \end{aligned}$$

The function

$$f \ : \ \mathbb{R}^\delta \to S^{1\delta} \times_u \mathbb{Z} \ ; \ x \mapsto (e^{2\pi i x}, [x])$$

is an isomorphism of groups, with inverse

$$f^{-1} \ : \ S^{1\delta} \times_u \mathbb{Z} \to \mathbb{R}^\delta \ ; \ (e^{2\pi i x}, m) \mapsto \{x\} + m \ .$$

The isomorphism f defines an isomorphism of central group extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "q"] \arrow[d, equal] & \mathbb{R}^\delta \arrow[r, "p"] \arrow[d, "f", "\cong"'] & S^{1\delta} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "q"] & S^{1\delta} \times_u \mathbb{Z} \arrow[r, "p"] & S^{1\delta} \arrow[r] & 0 \end{tikzcd}$$

□

Proposition 2.5. (i) *For any* $n \in \mathbb{Z}$ *the cocycles*

$$\begin{aligned} & nu \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \\ & (e^{2\pi i x}, e^{2\pi i y}) \mapsto nu(e^{2\pi i x}, e^{2\pi i y}) = n(\{x\} + \{y\} - \{x+y\}) \ , \\ & u(n \times n) \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \\ & (e^{2\pi i x}, e^{2\pi i y}) \mapsto u(e^{2\pi i n x}, e^{2\pi i n y}) = \{nx\} + \{ny\} - \{n(x+y)\} \end{aligned}$$

are such that

$$nu - u(n \times n) \ = \ \delta\alpha \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z}$$

with

$$\alpha \ : \ S^{1\delta} \to \mathbb{Z} \ ; \ e^{2\pi i x} \mapsto n\{x\} - \{nx\} = [nx] - n[x] \ .$$

The group morphism $n : S^{1\delta} \to S^{1\delta}; z \mapsto z^n$ *is thus such that the cohomology class*

$$n^*[u] \ = \ [u(n \times n)] \ = \ [nu] \ = \ n[u] \in H^1(S^{1\delta};\mathbb{Z})$$

corresponds to the central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q_n} \mathbb{R}^\delta \times \mathbb{Z}_n \xrightarrow{p_n} S^{1\delta} \longrightarrow 0$$

with

$$\begin{aligned} p_n \ &: \ \mathbb{R}^\delta \times \mathbb{Z}_n \to S^{1\delta} \ ; \ (x,r) \mapsto e^{2\pi i(x-r/n)} \ , \\ q_n \ &: \ \mathbb{Z} \to \mathbb{R}^\delta \times \mathbb{Z}_n \ ; \ 1 \mapsto (1/n, 1) \ . \end{aligned}$$

(ii) *For* $p,q,r \in S^1$ *define the* **cyclic order**

$$\begin{aligned} \mathrm{ord}(p,q,r) \ &= \ \begin{cases} 0 & \textit{if two of } p,q,r \textit{ coincide} \\ 1 & \textit{if } q \in (r,p) \\ -1 & \textit{if } q \in (p,r) \end{cases} \\ &= \ \{\textit{area of the ideal triangle in the Poincaré disc} \\ &\qquad \textit{with vertices at } p,q,r\}/\pi \ . \end{aligned}$$

as in Barge and Ghys [5, p.238], *such that for* $x,y,z \in \mathbb{R}$

$$\begin{aligned} \mathrm{ord}(e^{2\pi i x}, e^{2\pi i y}, e^{2\pi i z}) \ &= \ 2(E(x-y) + E(y-z) + E(z-x)) \\ &= \ 2(((y-x)) + ((z-y)) + ((x-z))) \\ &= \ \mathrm{sgn}(\sin \pi(x-y) \sin \pi(y-z) \sin \pi(z-x)) \ . \end{aligned}$$

The function

$$\begin{aligned} \epsilon \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) \mapsto \ & \epsilon(e^{2\pi i x}, e^{2\pi i y}) \\ &= \mathrm{ord}(1, e^{2\pi i x}, e^{2\pi i(x+y)}) \\ &= 2\big(((x)) + ((y)) - ((x+y))\big) \\ &= -\mathrm{sgn}(\sin\pi x \sin\pi y \sin\pi(x+y)) \end{aligned}$$

is the **area cocycle** *of Kirby and Melvin* [22]*such that*

$$[\epsilon] \ = \ 2[u] \in H^2(S^{1\delta};\mathbb{Z})$$

and

$$\begin{aligned}
&2u(e^{2\pi ix},e^{2\pi iy})-\epsilon(e^{2\pi ix},e^{2\pi iy})\\
&\qquad =\ 2\big(\{x\}+\{y\}-\{x+y\}\big)-2\big(((x))+((y))-((x+y))\big)\\
&\qquad =\ \beta(x)+\beta(y)-\beta(x+y)\ ,
\end{aligned}$$

$$\beta(e^{2\pi ix})\ =\ 2\{x\}-2((x))\ =\ \{x\}+\{-x\}\ =\ \begin{cases}0 & \textit{if } x\in\mathbb{Z}\\ 1 & \textit{if } x\notin\mathbb{Z}\ .\end{cases}$$

Proof. (i) The section of p_n

$$v_n\ :\ S^{1\delta}\to\mathbb{R}^\delta\times\mathbb{Z}_n\ ;\ e^{2\pi ix}\mapsto(\{x\},0)$$

is such that

$$v_n(g)+v_n(h)-v_n(g+h)\ =\ q_nu_n(g,h)\in\mathbb{R}^\delta\times\mathbb{Z}_n$$

with cocycle

$$nu\ :\ S^{1\delta}\times S^{1\delta}\to\mathbb{Z}\ ;\ (e^{2\pi ix},e^{2\pi iy})\mapsto n(\{x\}+\{y\}-\{x+y\})\ .$$

The function

$$f_n\ :\ \mathbb{R}^\delta\times\mathbb{Z}_n\to S^{1\delta}\times_{nu}\mathbb{Z}\ ;\ (x,r)\mapsto(e^{2\pi ix},r+n[x-r/n])$$

is an isomorphism of groups, with inverse

$$f_n^{-1}\ :\ S^{1\delta}\times_{nu}\mathbb{Z}\to\mathbb{R}^\delta\times\mathbb{Z}_n\ ;\ (z,m)\mapsto(x,m)$$

with $x\in\mathbb{R}$ the unique real number such that $e^{2\pi i\{x\}}=z\in S^1$ and

$$[x]\ =\ \begin{cases}[m/n] & \text{if } \{m/n\}\leqslant\{x\}\\ [m/n]+1 & \text{if } \{m/n\}>\{x\}.\end{cases}$$

The isomorphism f_n defines an isomorphism of central group extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "q_n"] \arrow[d, equal] & \mathbb{R}^\delta\times\mathbb{Z}_n \arrow[r, "p_n"] \arrow[d, "f_n", "\cong"'] & S^{1\delta} \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \mathbb{Z} \arrow[r, "q_{nu}"] & S^{1\delta}\times_{nu}\mathbb{Z} \arrow[r, "p_{nu}"] & S^{1\delta} \arrow[r] & 0
\end{tikzcd}$$

(iii) By construction. □

Example 2.6. Specialize to the case $n=2$, and consider the extension

$$0\longrightarrow\mathbb{Z}\xrightarrow{q_2}\mathbb{R}^\delta\times\mathbb{Z}_2\xrightarrow{p_2}S^{1\delta}\longrightarrow 0$$

classified by $2u\in H^2(S^{1\delta};\mathbb{Z})$, with

$$\begin{aligned}
p_2\ &:\ \mathbb{R}^\delta\times\mathbb{Z}_2\to S^{1\delta}\ ;\ (x,r)\mapsto e^{2\pi i(x-r/2)}\ ,\\
q_2\ &:\ \mathbb{Z}\to\mathbb{R}^\delta\times\mathbb{Z}_2\ ;\ n\mapsto(n/2,n(\text{mod } 2))\ .
\end{aligned}$$

The two sections of p_2

$$v_2 \;:\; S^{1\delta} \to \mathbb{R}^\delta \times \mathbb{Z}_2 \;;\; e^{2\pi i x} \mapsto (\{x\},0)\ ,$$

$$v'_2 \;:\; S^{1\delta} \to \mathbb{R}^\delta \times \mathbb{Z}_2 \;;\; e^{2\pi i x} \mapsto \begin{cases} (((x)),0) = (0,0) & \text{if } e^{2\pi i x} = 1 \\ (((x)),1) = (\{x\}-1/2,1) & \text{if } e^{2\pi i x} \neq 1 \end{cases}$$

differ by

$$v_2 - v'_2 \;=\; q_2\alpha \;:\; S^{1\delta} \to \mathbb{Z} \to \mathbb{R}^\delta \times \mathbb{Z}_2$$

with

$$\alpha \;:\; S^{1\delta} \to \mathbb{Z} \;;\; e^{2\pi i x} \mapsto \begin{cases} 0 & \text{if } e^{2\pi i x} = 1 \\ 1 & \text{if } e^{2\pi i x} \neq 1. \end{cases}$$

The corresponding cocycles

$$\begin{aligned} &v_2 \;:\; S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \\ &(e^{2\pi i x}, e^{2\pi i y}) \mapsto (q_2)^{-1}(v_2(e^{2\pi i x}) + v_2(e^{2\pi i y}) - v_2(e^{2\pi i(x+y)})) \\ &\qquad\qquad\qquad\qquad\qquad = \; 2(\{x\}+\{y\}-\{x+y\}) \\ &v'_2 \;:\; S^{1\delta} \times S^{1\delta} \to \mathbb{Z}\ ; \\ &(e^{2\pi i x}, e^{2\pi i y}) \mapsto (q_2)^{-1}(v'_2(e^{2\pi i x}) + v'_2(e^{2\pi i y}) - v'_2(e^{2\pi i(x+y)})) \\ &\qquad\qquad\qquad\qquad\qquad = \; 2(((x))+((y))-((x+y))) \end{aligned}$$

differ by the coboundary

$$v_2 - v'_2 \;=\; \delta\alpha\ .$$

v'_2 is the **signature cocycle** τ of Atiyah [3, (2.7)] (= the area cocycle ϵ of Proposition 2.5 (iii))

$$\begin{aligned} v'_2(e^{2\pi i x}, e^{2\pi i y}) \;&=\; v'_2(e^{2\pi i x}) + v'_2(e^{2\pi i y}) - v'_2(e^{2\pi i(x+y)}) \\ &=\; 2\big(((x))+((y))-((x+y))\big) \\ &=\; \epsilon(e^{2\pi i x}, e^{2\pi i y}) \\ &=\; -\mathrm{sgn}(\sin\pi x \sin\pi y \sin\pi(x+y)) \\ &=\; -2\big(E(x)+E(y)-E(x+y)\big) \\ &=\; \begin{cases} 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x+y \in \mathbb{Z} \\ -1 & \text{if } 0 < \{x\}+\{y\} < 1 \\ 1 & \text{if } 1 < \{x\}+\{y\} < 2. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} [v'_2] \;=\; [v_2] \;=\; 2[u] \;=\; \delta(2) \in\; &\ker(H^2(S^{1\delta};\mathbb{Z}) \to H^2(S^{1\delta};\mathbb{R}^\delta)) \\ =\; &\mathrm{im}(\delta : H^1(S^{1\delta};S^{1\delta}) \to H^2(S^{1\delta};\mathbb{Z}))\ . \end{aligned}$$

$\square$

Example 2.7. Specialize to the case $n = 4$, and consider the extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q_4} \mathbb{R}^\delta \times \mathbb{Z}_4 \xrightarrow{p_4} S^{1\delta} \longrightarrow 0$$

classified by $4u \in H^2(S^{1\delta};\mathbb{Z})$, corresponding to the extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{q_4} \mathbb{R}^\delta \times \mathbb{Z}_4 \xrightarrow{p_4} S^{1\delta} \longrightarrow 0$$

with

$$\begin{aligned} p_4 &: \mathbb{R}^\delta \times \mathbb{Z}_4 \to S^{1\delta} \ ; \ (x,r) \mapsto e^{2\pi i(x-r/4)} \ , \\ q_4 &: \mathbb{Z} \to \mathbb{R}^\delta \times \mathbb{Z}_4 \ ; \ n \mapsto (n/4, n(\mathrm{mod}\, 4)) \ . \end{aligned}$$

The section of p_4

$$v_4 \ : \ S^{1\delta} \to \mathbb{R}^\delta \times \mathbb{Z}_4 \ ; \ e^{2\pi i x} \mapsto (\{x\}, 0)$$

is such that

$$v_4(g) + v_4(h) - v_4(g+h) \ = \ q_4 v_4(g,h) \in \mathbb{R}^\delta \times \mathbb{Z}_4$$

with cocycle

$$v_4 \ = \ 2v_2 \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) \mapsto 4(\{x\} + \{y\} - \{x+y\}) \ .$$

The function

$$f_4 \ : \ \mathbb{R}^\delta \times \mathbb{Z}_4 \to S^{1\delta} \times_{v_4} \mathbb{Z} \ ; \ (x,r) \mapsto (e^{2\pi i x}, r + 4[x - r/4])$$

is an isomorphism of groups, with inverse

$$f_4^{-1} \ : \ S^{1\delta} \times_{v_4} \mathbb{Z} \to \mathbb{R}^\delta \times \mathbb{Z}_4 \ ; \ (z,m) \mapsto (x,m)$$

with $x \in \mathbb{R}$ the unique real number such that $e^{2\pi i\{x\}} = z \in S^1$ and

$$[x] \ = \ \begin{cases} [m/n] & \text{if } \{m/4\} \leqslant \{x\} \\ [m/4]+1 & \text{if } \{m/4\} > \{x\}. \end{cases}$$

The isomorphism f_4 defines an isomorphism of central group extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "q_4"] \arrow[d, equal] & \mathbb{R}^\delta \times \mathbb{Z}_4 \arrow[r, "p_4"] \arrow[d, "f_4", "\cong"'] & S^{1\delta} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "f_4 q_4"] & S^{1\delta} \times_{v_4} \mathbb{Z} \arrow[r, "p_4(f_4)^{-1}"] & S^{1\delta} \arrow[r] & 0 \end{tikzcd}$$

□

3. Geometric cobordism constructions

3.1. **Unions of cobordisms.** Manifolds M are oriented, and $-M$ will denote the same manifold with the opposite orientation. Isomorphisms $f : M \to M'$ will be assumed to be orientation-preserving, unless stated otherwise.

The orientation convention for a cobordism $(W; M, M')$ is

$$\partial W \;=\; M \sqcup -M' \;.$$

As we shall be dealing with various ways of regarding the same closed manifold M as a boundary it is convenient to consider manifolds with boundary $(N, \partial N)$ and a particular isomorphism $i : M \cong \partial N$:

Definition 3.1. (i) An $(m+1)$-dimensional **coboundary** (N, M, i) consists of an $(m+1)$-dimensional manifold with boundary $(N, \partial N)$, a closed m-dimensional manifold M and an isomorphism $i : M \cong \partial N$. The cobooundary with the opposite orientation is

$$-(N, M, i) \;=\; (-N, -M, i) \;.$$

(ii) The **union** of $(m+1)$-dimensional coboundaries (N_0, M, i_0), (N_1, M, i_1) is the closed $(m+1)$-dimensional manifold

$$(N_0, M, i_0) \cup -(N_1, M, i_1) \;=\; (N_0 \sqcup -N_1)/(i_0(x) \sim i_1(x)\, \text{for}\, x \in M) \;.$$

with $M \subset (N_0, M, i_0) \cup -(N_1, M, i_1)$ a separating codimension 1 submanifold. When i_0, i_1 are clear this is written as

$$(N_0, M, i_0) \cup -(N_1, M, i_1) \;=\; N_0 \cup_M -N_1 \;.$$

□

We shall make frequent use of the fact that boundaries of manifolds are collared:

Proposition 3.2. *For any $(m+1)$-dimensional coboundary $(N, M, i : M \to \partial N)$ the isomorphism i extends to an embedding $e : M \times I \to N$ such that*

$$e(x, 0) \;=\; i(x) \in N \;\; (x \in M) \;.$$

The $(m+1)$-dimensional manifold with boundary

$$(N', \partial N') \;=\; (\mathrm{cl.}(N \backslash e(M \times I)), e(M \times \{1\}))$$

is such that there are defined isomorphisms

$$i' \;:\; M \to \partial N' \;;\; x \mapsto e(x, 1) \;,$$

$$(f, 1) \;:\; (N', M, i') \to (N, M, i)$$

with

$$f(e(x,1)) = e(x,0) = i(x) \in \partial N \ (x \in M) .$$

$\square$

Definition 3.3. The **collared union** of two $(m+1)$-dimensional coboundaries (N_0, M, i_0), (N_1, M, i_1) is the closed $(m+1)$-dimensional manifold

$$(N_0, M, i_0) \cup M \times I \cup -(N_1, M, i_1) = (N_0 \sqcup M \times I \sqcup -N_1)/\sim ,$$
$$i_0(x) \sim (x,0),\ i_1(x) \sim (x,1) \ (x \in M)$$

with $N_0, -N_1 \subset (N_0, M, i_0) \cup M \times I \cup -(N_1, M, i_1)$ disjoint codimension 1 submanifolds. When i_0, i_1 are clear this is written as

$$(N_0, M, i_0) \cup M \times I \cup -(N_1, M, i_1) = N_0 \cup M \times I \cup -N_1 .$$

$\square$

Definition 3.4. Let M be an m-dimensional manifold. The **data** for a **surgery** on M is an embedding $A \times \partial B \subset M$ for some manifolds with boundary $(A, \partial A)$, $(B, \partial B)$ with $m = \dim(A) + \dim(B) - 1$. The **effect** of the surgery on $A \times \partial B \subset M$ is the m-dimensional manifold

$$M' = \mathrm{cl.}(M \backslash A \times \partial B) \cup \partial A \times B .$$

The **trace** of the surgery is the cobordism $(W; M, M')$ with

$$W = M \times I \cup A \times B , \ \partial W = M \cup -M' .$$

$\square$

Remark 3.5. The surgery in Definition 3.4 is an evident generalization of the usual case of surgery on an m-dimensional manifold with

$$(A, \partial A) = (D^p, S^{p-1}) , \ (B, \partial B) = (D^q, S^{q-1}) \ (p + q - 1 = m) .$$

$\square$

3.2. **The twisted double, mapping torus and cyclic unions.** We consider geometric cobordism constructions related to the mapping torus and the twisted double, which will motivate the development of algebraic analogues.

Definition 3.6. (i) The **twisted double** of an $(m+1)$-dimensional manifold with boundary $(N, \partial N)$ and an automorphism $f : M = \partial N \to M$ is the closed $(m+1)$-dimensional manifold

$$U(f) = N \cup_f -N$$

with Euler characteristic

$$\chi(U(f)) = 2\chi(N) - \chi(M) = (1 + (-1)^{m+1})\chi(N) .$$

(ii) The **mapping torus** of an automorphism $f : M \to M$ of a closed m-dimensional manifold M is the closed $(m+1)$-dimensional manifold

$$T(f) \;=\; M \times [0,1]/\{(x,0) \sim (f(x),1) \,|\, x \in M\}$$

which is a fibre bundle over S^1 with monodromy f

$$M \to T(f) \to S^1 \ ,$$

and Euler characteristic $\chi(T(f)) = 0 \in \mathbb{Z}$.

□

Example 3.7. The mapping torus of $f : M \to M$ is a twisted double

$$T(f) \;=\; U(1 \sqcup f : M \sqcup M \to M \sqcup M) \ .$$

□

Definition 3.8. Given an $(m+1)$-dimensional manifold with boundary $(N, \partial N)$ and an automorphism $f : M = \partial N \to M$ use a collaring of M in N to identify the twisted double $U(f)$ (3.6) with

$$\begin{aligned} U(f) \;=\; & (N \times \{0\} \sqcup M \times I \sqcup N \times \{1\})/ \\ & \qquad \{(x,0) \sim (i(x),0),\, (x,1) \sim (if(x),1) \,|\, x \in M\} \end{aligned}$$

with $i : M \to N$ the inclusion. The **twisted double cobordism** $(V(f); U(f), T(f))$ is the trace of the surgery on $N \times \{0,1\} \subset U(f)$, with

$$V(f) \;=\; U(f) \times I \cup_{N \times \{0,1\}} N \times I \ ,$$

noting that the effect of the surgery is

$$\mathrm{cl.}(U(f) \backslash N \times \{0,1\}) \cup_{1 \sqcup f} M \times I \;=\; M \times I \cup_{1 \sqcup f} M \times I \;=\; T(f) \ (3.7) \ .$$

The Euler characteristic of $V(f)$ is

$$\chi(V(f)) \;=\; \chi(U(f)) - \chi(N) \;=\; \chi(N) - \chi(M) \;=\; (-1)^{m+1}\chi(M) \in \mathbb{Z} \ .$$

□

Remark 3.9. The twisted double cobordism $(V(f); U(f), T(f))$ of $f : M \to M$ can be expressed more concisely as

$$\begin{aligned} V(f) \;&=\; (N \times [-1,2])/\{(x,t) \sim (f(x), 1-t) \,|\, x \in M,\, t \in [-1,0]\} \ , \\ U(f) \;&=\; N \times \{-1,2\}/\{(x,-1) \sim (f(x),2) \,|\, x \in M\} \ , \\ T(f) \;&=\; M \times [0,1]/\{(x,0) \sim (f(x),1) \,|\, x \in M\} \ . \end{aligned}$$

□

Proposition 3.10. (i) *For any (orientation-preserving) automorphism* $f : M \to M$ *there is defined an orientation-reversing isomorphism of the mapping tori of* f *and* $f^{-1} : M \to M$

$$T(f) \to T(f^{-1}) \ ; \ [x,t] \mapsto [x, 1-t] \ ,$$

giving an identification of (oriented) manifolds

$$T(f^{-1}) \ = \ -T(f) \ .$$

(ii) *For any automorphisms* $f, g : M \to M$ *there is defined an isomorphism*

$$T(gf) \to T(fg) \ ; \ [x,t] \mapsto [f(x), t] \ .$$

□

Definition 3.11. (i) Suppose that $(N_j, \partial N_j)$ $(0 \leqslant j \leqslant k-1)$ are k $(m+1)$-dimensional manifolds with the same boundary

$$M \ = \ \partial N_0 \ = \ \partial N_1 \ = \ \dots \ = \ \partial N_{k-1} \ .$$

The $(m+2)$-dimensional manifold with boundary

$$(P, \partial P) \ = \ (M \times D^2 \cup \bigcup_{j=0}^{k-1} N_j \times I, \coprod_{j=0}^{k-1} N_j \cup_M -N_{j+1}) \ (N_k = N_0)$$

is the **thickened union** of the stratified set

$$(\coprod_{j=0}^{k-1} N_j)/\{M \sim \partial N_0 \sim \partial N_1 \sim \dots \sim \partial N_{k-1}\}$$

which is a deformation retract of P, with

$$\begin{aligned} \chi(P) \ &= \ \sum_{j=0}^{k-1} \chi(N_j) - (k-1)\chi(M) \ , \\ \chi(\partial P) \ &= \ 2\sum_{j=0}^{k-1} \chi(N_j) - k\chi(M) \ . \end{aligned}$$

(ii) Given a closed m-dimensional manifold L and k automorphisms $f_0, f_1, \dots, f_{k-1} : L \to L$ such that

$$f_{k-1} f_{k-2} \dots f_1 f_0 \ = \ 1 \ : \ L \to L$$

define the **cyclic union** to be the thickened union $(m+2)$-dimensional manifold with boundary as in (i)

$$(Q, \partial Q) \ = \ ((L \times \{0,1\} \times D^2) \cup \bigcup_{j=0}^{k-1} L \times I \times I, \coprod_{j=0}^{k-1} T(f_j))$$

of the k null-cobordisms $N_j = L \times I$ of $M = L \times \{0,1\}$ determined by $f_0, f_1, \dots, f_{k-1}$, such that

$$\chi(Q) \;=\; (2-k)\chi(L)\;,\;\; \chi(\partial Q) \;=\; 0\;.$$

$\square$

The case $k = 3$ is of particular interest:

Definition 3.12. Let $f, g : M \to M$ be two automorphisms of a closed m-dimensional manifold M.
(i) The **double mapping torus** cobordism $(T(f,g); T(f)\sqcup T(g), T(gf))$ is the cyclic union (3.11 (ii)) with $f_0 = f$, $f_1 = g$, $f_2 = f^{-1}g^{-1}$ and

$$\chi(T(f,g)) \;=\; -\chi(M)\;.$$

(ii) For $M = \partial N$ define the **twisted double** cobordism $(U(f,g); U(f)\sqcup U(g), U(gf))$ is the thickened union (3.11 (i)) with $N_0 = N_1 = N_2 = N$

$$\chi(U(f,g)) \;=\; 3\chi(N) - 2\chi(M) \;=\; (1+2(-1)^{m+1})\chi(N)\;.$$

$\square$

Remark 3.13. Given an $(m+1)$-dimensional manifold with boundary (N, M) and two automorphisms $f, g : M \to M$ we have two cobordisms

$$(T(f,g); T(f)\sqcup T(g), T(gf))\;,\;\; (T'(f,g); T(f)\sqcup T(g), T(gf))$$

with $T(f,g)$ as in 3.12 (i) and

$$T'(f,g) \;=\; (V(f)\sqcup V(g)\sqcup V(gf))\cup_{U(f)\sqcup U(g)\sqcup U(gf)} U(f,g)\;,$$

with $U(f,g)$ as in 3.12 (ii), such that

$$\begin{aligned}
\chi(T(f,g)) \;&=\; -\chi(M) \;=\; (-1+(-1)^{m+1})\chi(N)\;,\\
\chi(T'(f,g)) \;&=\; \chi(U(f,g)) - 3\chi(U(f)) + 3\chi(V(f))\\
&=\; -2\chi(M) \;=\; 2(-1+(-1)^{m+1})\chi(N)\;.
\end{aligned}$$

Let F_k be the free group on k generators $x_0, x_1, \dots, x_{k-1}$.

Example 3.14. A pair of pants for a 3-dimensional animal with a total of k heads and legs is the 2-dimensional manifold with boundary

$$(P(k), \partial P(k)) \;=\; (\mathrm{cl.}(S^2\backslash \coprod_k D^2), \coprod_k S^1)$$

such that

$$P(k) \;\simeq\; \bigvee_{k-1} S^1\;,\;\; \pi_1(P(k)) \;=\; F_k/\langle x_0x_1\dots x_{k-1}\rangle \;=\; F_{k-1}$$

with $\chi(P(k)) = 2-k$, and $x_j \in \pi_1(P(k))$ represented by the jth boundary circle $S^1 \subset \partial P(k)$. $\square$

Example 3.15. The human pair of pants $P(k)$ for $k=3$ is the double torus cobordism with

$$\begin{aligned}
(N,M) &= (D^1,S^0)\ ,\ f\ =\ 1\ :\ M\to M\ ,\\
(V(f);U(f),T(f)) &= (P(3);S^1,S^1\sqcup S^1)\ .
\end{aligned}$$

It is also the thickened union (b) for $N_0=N_1=N_2=D^1$, with

$$\begin{aligned}
(P,\partial P) &= (M\times D^2\cup\bigcup_{j=0}^{2}N_j\times I,\coprod_{j=0}^{k-1}N_j\cup_M -N_{j+1})\\
&= (P(3),S^1\sqcup S^1\sqcup S^1)\\
P(3)\simeq S^1\vee S^1\ ,\ &\chi(P(3))=-1\ .
\end{aligned}$$

This is also the cyclic union (3.11 (ii)) for

$$f_0\ =\ f_1\ =\ f_2\ =\ 1\ :\ L\ =\ \{\text{pt.}\}\to L\ =\ \{\text{pt.}\}$$

with

$$(Q,\partial Q)\ =\ ((L\times\{0,1\}\times D^2)\cup\bigcup_{j=0}^{2}L\times I\times I,\coprod_{j=0}^{2}T(f_j))\ =\ (P(3),S^1\sqcup S^1\sqcup S^1)\ .$$

$\square$

Definition 3.16. Let $(N,M,i:M\to\partial N)$ be an n-dimensional coboundary, and let $f:M\to M$ be an automorphism. The **collared twisted double** is the collared union

$$\begin{aligned}
&(N,M,i)\cup M\times I\cup -(N,M,if)\ =\ ((N\times\{0\}\cup -N\times\{1\})\cup M\times I)/\sim\\
&(i(x),0)\sim(x,0),\ (if(x),1)\sim(x,1)\ (x\in M)\ .
\end{aligned}$$

$\square$

Again, in view of Proposition 3.2 the collared twisted double is isomorphic to the twisted double

$$\begin{aligned}
&(N,M,i)\cup -(N,M,if)\ =\ (N\times\{0\}\cup -N\times\{1\})/\sim\\
&(i(x),0)\sim(if(x),1)\ (x\in M)\ .
\end{aligned}$$

In cases when $i:M\to\partial N$ is clear we shall write the twisted doubles as

$$\begin{aligned}
(N,M,i)\cup -(N,M,if) &= N\cup_f -N\ ,\\
(N,M,i)\cup M\times I\cup -(N,M,if) &= N\cup M\times I\cup_f -N\ .
\end{aligned}$$

Also, in the untwisted case $f = 1 : M \to M$ write

$$\begin{aligned} N \cup_f -N &= N \cup_M -N \ , \\ (N, M, i) \cup M \times I \cup -(N, M, i) &= N \times \{0\} \cup M \times I \cup -N \times \{1\} \ . \end{aligned}$$

Example 3.17. For any n-dimensional coboundary $(N, M, i : M \to \partial N)$ there is defined an n-dimensional relative cobordism

$$(N \times I; N \times \{0\}, N \times \{1\}; M)$$

with boundary the collared untwisted double of (N, M, i)

$$\partial(N \times I) \ = \ N \times \{0\} \cup M \times I \cup -N \times \{1\} \ .$$

□

Proposition 3.18. (i) *The mapping torus of an automorphism* $f : M \to M$ *is a twisted double*

$$T(f) \ = \ (M \times I) \cup_{f \cup 1} -(M \times I)$$

with

$$\begin{aligned} &f \cup 1 \ : \ \partial(M \times I) \ = \ M \times \{0, 1\} \to M \times \{0, 1\} \ ; \\ &(x, 0) \mapsto (f(x), 0) \ , \ (x, 1) \mapsto (x, 1) \end{aligned}$$

and also a collared twisted double

$$T(f) \ = \ (M \times I) \times \{0\} \cup M \times \partial I \times I \cup_{f \cup 1} -(M \times I) \times \{1\} \ .$$

(ii) *Let* (N, M, i) *be a coboundary, and let* $f : M \to M$ *be an automorphism. The collared twisted double*

$$U(f) \ = \ (N \times \{0\}) \cup M \times I \cup_f -(N \times \{1\})$$

is related to the mapping torus $T(f)$ *by a canonical cobordism* $(V(f); U(f), T(f))$ *of collared twisted doubles*

$$V(f) \ = \ U(f) \times I \cup_{N \times \{0,1\} \times \{1\}} -N \times I$$

the trace of the surgery on $U(f)$ *replacing* $N \times \{0, 1\} \subset U(f)$ *by* $M \times I$. *The disjoint submanifolds*

$$\begin{aligned} V^+(f) &= N \times \{0\} \times I \cup_{N \times \{0\} \times \{1\}} N \times [0, 1/3] \ , \\ V^-(f) &= N \times \{1\} \times I \cup_{N \times \{1\} \times \{1\}} N \times [1/3, 1] \subset V(f) \end{aligned}$$

are related by the evident isomorphism, and

$$V(f) \ = \ V^+(f) \cup_1 (M \times I \times I \sqcup N \times [1/3, 2/3])_{(f \times 1) \sqcup 1} \cup V^-(f)$$

is a collared twisted double.

(iii) *The canonical cobordism of* (ii) *for the collared twisted double of* (i)

$$T(f) \ = \ U(f \cup 1) \ = \ (M \times I) \times \{0\} \cup M \times \partial I \times I \cup_{f \cup 1} -(M \times I) \times \{1\}$$

is

$$\begin{aligned}&(V(f\cup 1);T(f),T(f\cup 1))\\ &=\ (\mathrm{cl.}(T(f)\times I\backslash(M\times D^2));T(f)\times\{0\},T(f)\times\{1\}\cup(M\times S^1))\ .\end{aligned}$$

□

3.3. **Thickened union.**

Definition 3.19. Given a closed m-dimensional manifold M and k $(m+1)$-dimensional manifolds $(N_j,\partial N_j)$ $(j=0,1,\dots,k-1)$ with

$$\partial N_0\ =\ \partial N_1\ =\ \dots\ =\ \partial N_{k-1}\ =\ M$$

define the **thickened union** $(m+2)$-dimensional manifold with boundary

$$(P,\partial P)\ =\ (M\times D^2\cup\bigcup_{j=0}^{k-1}(N_j\times I),\coprod_{j=0}^{k-1}N_j\cup_M -N_{j+1})\ (N_k=N_0)\ .$$

The trace of the surgery on

$$\coprod_{j=0}^{k-1}N_j\times\partial I\subset\partial P$$

is the $(m+2)$-dimensional manifold with boundary

$$(P',\partial P')\ =\ (\partial P\times I\cup(\coprod_{j=0}^{k-1}N_j)\times I,M\times S^1)$$

such that

$$P\ =\ M\times D^2\cup_{M\times S^1}P'\ .$$

Let

$$i_j\ =\ \text{inclusion}\ :\ M\to N_j\ (0\leqslant j\leqslant k-1)$$

and set

$$i_k\ =\ i_0\ :\ M\to N_k\ =\ N_0\ .$$

If $M,N_0,N_1,\dots,N_{k-1}$ are connected then

$$N_0\cup_M -N_1\ ,\ N_1\cup_M -N_2\ ,\ \dots\ ,\ N_{k-1}\cup_M -N_0\ ,\ P$$

are connected, with fundamental groups

$$\begin{aligned}\pi_1(N_j\cup_M -N_{j+1})\ &=\ \pi_1(N_j)*\pi_1(N_{j+1})/\{i_j(y)=i_{j+1}(y)\,|\,y\in\pi_1(M)\}\ ,\\ \pi_1(P)\ &=\ \mathop{*}_{j=0}^{k-1}\pi_1(N_j)/\{i_j(y)=i_{j+1}(y)\,|\,y\in\pi_1(M)\}\ .\end{aligned}$$

Given a closed m-dimensional manifold L and k automorphisms $e_0, e_1, \dots, e_{k-1} : L \to L$ write

$$\begin{aligned} e_k &= e_0 : L \to L , \\ f_j &= (e_{j+1})^{-1} e_j : L \to L \ (0 \leqslant j \leqslant k-1) , \\ i_j &= e_j \cup e_{j+1} : M = L \times \partial I \to N_j = L \times I , \end{aligned}$$

noting that $f_k f_{k-1} \dots f_1 f_0 = 1 : L \to L$. The thickened union of the $(m+1)$-dimensional manifolds with boundary

$$(N_j, \partial N_j) = (L \times I, i_j(M))$$

is the $(m+2)$-dimensional manifold with boundary

$$(Q, \partial Q) = (M \times D^2 \cup \bigcup_{j=0}^{k-1} (N_j \times I), \coprod_{j=0}^{k-1} N_j \cup_M -N_{j+1})$$

with jth boundary component the mapping torus of f_j

$$\begin{aligned} N_j \cup_M -N_{j+1} &= T(f_j : L \to L) \\ &= L \times I / \{(x,0) \sim (f_j(x), 1) \,|\, x \in L\} \end{aligned}$$

If L is connected then so is Q, with fundamental group

$$\pi_1(Q) = \pi_1(L) * F_{k-1} / \{x_j y = f_j(y) x_j \,|\, y \in \pi_1(L)\} ,$$

and the inclusion $T(f_j) \to Q$ induces the evident morphism

$$\pi_1(T(f_j)) = \pi_1(L) * \langle x_j \rangle / \{x_j y = f_j(y) x_j \,|\, y \in \pi_1(L)\} \to \pi_1(Q) .$$

The projections

$$p_j : T(f_j) \to S^1 ; \ [x,t] \mapsto e^{2\pi i t}$$

extend to a map $p : Q \to P(k)$ inducing the surjection

$$p_* : \pi_1(Q) \to \pi_1(P(k)) = F_{k-1} ; \ x_j \mapsto x_j , \ y \mapsto \{1\}$$

which fits into an exact sequence

$$\{1\} \longrightarrow \pi_1(L) \longrightarrow \pi_1(Q) \xrightarrow{p_*} F_{k-1} \longrightarrow \{1\} .$$

As above, let $(K, \partial K = L)$ be an $(m+1)$-dimensional manifold with boundary, with k automorphisms $e_0, e_1, \dots, e_{k-1} : L \to L$, and let $M = L \times \{0,1\}$. The k $(m+1)$-dimensional manifolds

$$(N_j, \partial N_j) = (L \times I, (e_j \cup e_{j+1})(M))$$

is an $(m+2)$-dimensional manifold with boundary

$$(P, \partial P) = (M \times D^2 \cup \bigcup_{j=0}^{k-1} (N_j \times I), \coprod_{j=0}^{k-1} N_j \cup_M -N_{j+1})$$

with

$$N_j \cup_M N_{j+1} \ = \ T(f_j : L \to L) \ (f_j = (e_{j+1})^{-1} e_j)$$

In view of Proposition 3.2:

Proposition 3.20. (i) *For any coboundary $(N, M, i : M \to \partial N)$ there is defined an isomorphism of coboundaries*

$$(g, \partial g) \ : \ ((M \times I, -M \times \{1\}, i_1) \cup (N, M, i), M, i_0) \to (N, M, i)$$
$$i_0(x) = (x, 0), \ \ i_1(x) = (x, 1) \ \ (x \in M)$$

such that

$$\partial g(x, 0) \ = \ i(x) \in \partial N \ \ (x \in M) \ .$$

(ii) *The union and collared union of coboundaries (N_0, M, i_0), (N_1, M, i_1) are related by an isomorphism*

$$f \ = \ f_0 \cup f_1 \ : \ (N_0, M, i_0) \cup (-N_1, M, i_1) \to (N_0, M, i_0) \cup M \times I \cup (-N_1, M, i_1)$$

such that

$$f(x) \ = \ (x, 1/2) \in (N_0, M, i_0) \cup M \times I \cup (-N_1, M, i_1) \ \ (x \in M)$$

with the restrictions of f isomorphisms

$$\begin{aligned} f_0 \ &: \ N_0 \to N_0 \cup_{M \times \{0\}} M \times [0, 1/2] \ , \\ f_1 \ &: \ N_1 \to M \times [1/2, 1] \cup_{M \times \{1\}} N_1 \ . \end{aligned}$$

□

3.4. Relative cobordisms and k-cyclic unions.

Definition 3.21. (i) An n-dimensional **relative cobordism** $(P; N_0, N_1; M, i_0, i_1)$ is an n-dimensional manifold P with boundary

$$\partial P \ = \ (N_0, M, i_0) \cup M \times I \cup (-N_1, -M, i_1)$$

for given $(n-1)$-dimensional coboundaries (N_0, M, i_0), (N_1, M, i_1).
(ii) The **union** of relative cobordisms $(P_0; N_0, N_1; M, i_0, i_1)$, $(P_1; N_1, N_2; M, i_1, i_2)$ is the relative cobordism

$$\begin{aligned} &(P_0; N_0, N_1; M, i_0, i_1) \cup (P_1; N_1, N_2; M, i_1, i_2) \\ &\qquad = \ (P_0 \cup_{N_1} -P_1; N_1, N_2; M, i_0, i_2) \ . \end{aligned}$$

□

In view of Proposition 3.20 there is no essential difference between a relative cobordism $(P; N_0, N_1; M, i_0, i_1)$ as in 3.21 and an n-dimensional manifold with boundary $(P, \partial P)$ together with a codimension 1 separating submanifold $M \subset \partial P = N_0 \cup_M -N_1$.

For any relative cobordism $(P; N_0, N_1; M)$ there exists a real-valued Morse function $p : P \to I$ with

$$p| \ = \ \text{projection} \ : \ M \times I \to I \ , \ p^{-1}(0) \ = \ N_0 \ , \ p^{-1}(1) \ = \ N_0 \ .$$

If p has no critical values then

$$(P; N_0, N_1; M) \ = \ (N_0 \times I; N_0 \times \{0\}, -N_0 \times \{1\}; M) \ .$$

Definition 3.22. Let $k \geqslant 1$. The **k-cyclic union** of k n-dimensional relative cobordisms

$$(P_j; N_j, N_{j+1}; M) \ (j = 0, 1, \ldots, k-1)$$

along an isomorphism

$$(f, \partial f) \ : \ (N_k, \partial N_k = M) \to (N_0, \partial N_0 = M)$$

is the n-dimensional manifold with boundary

$$(P, \partial P) \ = \ (\coprod_{j=0}^{k-1} P_j / \sim, T(\partial f))$$

with

$$\begin{aligned} &(x \in N_{j+1} \subset P_j) \sim (x \in N_{j+1} \subset P_{j+1}) \ (j = 0, 1, \ldots, k-2) \ , \\ &(x \in N_k \subset P_{k-1}) \sim (f(x) \in N_0 \subset P_0) \ . \end{aligned}$$

The union of real-valued Morse functions

$$p_j \ : \ P_j \to [t_j, t_{j+1}] \ (t_0 < t_1 < \cdots < t_k = t_0 + 1)$$

such that

$$\begin{aligned} p_j^{-1}(t_j) \ &= \ N_j \ , \ p_j^{-1}(t_{j+1}) \ = \ N_{j+1} \subset P_j \ , \\ p(x, s) \ &= \ (1-s)t_j + s t_{j+1} \ (x \in M, s \in I) \end{aligned}$$

is a circle-valued Morse function

$$p \ : \ P \to S^1 \ ; \ x \mapsto e^{2\pi i p_j(x)} \ (x \in P_j)$$

such that

$$\begin{aligned} p^{-1}(e^{2\pi i t_j}) \ &= \ N_j \subset P \ , \\ p([x, t] \ &= \ e^{2\pi i t} \ (x \in M, t \in I) \ . \end{aligned}$$

□

Example 3.23. (i) The 1-cyclic union of an n-dimensional relative cobordism $(P_0; N_0, N_1; M)$ along an isomorphism

$$(f, \partial f) \ : \ (N_1, \partial N_1 = M) \to (N_0, \partial N_0 = M)$$

is an n-dimensional manifold with boundary

$$(P, \partial P) \ = \ (P_0/(x \sim f(x)), T(\partial f)) \ (x \in N_1)$$

A real-valued Morse function $p_0 : P_0 \to I$ such that

$$\begin{aligned} p_0^{-1}(0) &= N_0 \ , \ p_0^{-1}(1) \ = \ N_1 \subset P_0 \ , \\ p(x,s) &= s \ (x \in M, s \in I) \end{aligned}$$

determines a circle-valued Morse function

$$p \ : \ P \to S^1 \ ; \ x \mapsto e^{2\pi i p_0(x)} \ (x \in P_0)$$

such that

$$p^{-1}(1) \ = \ N_0 \subset P \ , \ p| \ = \ \text{projection} \ : \ \partial P \ = \ T(\partial f) \to S^1 \ .$$

(ii) Suppose given an $(n+1)$-dimensional manifold with boundary $(P, \partial P)$ and a map $(p, \partial p) : (P, \partial P) \to S^1$ such that $\partial p : \partial P \to S^1$ is the projection of a fibre bundle, with $\partial P = T(f : M \to M)$. Assume p is transverse regular at $1 \in S^1$. Cutting $(P, \partial P)$ along the codimension 1 framed submanifold

$$(p, \partial p)^{-1} \ = \ (N, M) \subset P$$

there is obtained an n-dimensional relative cobordism $(P_0; N_0, N_1; M)$ with an isomorphism

$$(f, \partial f) \ : \ (N_1, \partial N_1 = M) \to (N_0, \partial N_0 = M)$$

such that $(P, \partial P)$ is the 1-cyclic union.

$\square$

Proposition 3.24. *Let $(P, \partial P)$ be a k-cyclic union as in 3.22, and fix a sequence*

$$t_0 < t_1 < \cdots < t_{k-1} < t_k \ = \ t_0 + 1 \in \mathbb{R} \ .$$

The projection $p : T(\partial f) \to S^1$ extends to a Morse function $\delta p : P \to S^1$ such that the restrictions $\delta p| : P_j \to S^1$ lift to Morse functions

$$p_j \ : \ (P_j; N_j, N_{j+1}; M) \to [t_j, t_{j+1}] \ (0 \leqslant j \leqslant k-1)$$

and

$$\begin{aligned} \delta p(x) &= e^{2\pi i p_j(x)} \in S^1 \ (x \in P_j) \ , \\ p_j^{-1}(t_j) &= N_j \ , \ p_j^{-1}(t_{j+1}) \ = \ N_{j+1} \ . \end{aligned}$$

$\square$

Definition 3.25. (i) A submanifold $M^m \subset P^n$ is **framed** if the normal bundle is trivialized, so that M has a neighbourhood $M \times D^{n-m} \subset P$.
(ii) The **exterior** of a framed $M^m \subset P^n$ is the n-dimensional manifold with boundary

$$(P_M, \partial P_M) \ = \ (\mathrm{cl.}(P \backslash M \times D^{n-m}), M \times S^{n-m-1})$$

such that

$$P \;=\; M \times D^{n-m} \cup_{M \times S^{n-m-1}} P_M \ .$$

(iii) An **ambient coboundary** of a framed submanifold $M^m \subset P^n$ is a framed submanifold $N^{m+1} \subset P^n$ such that $\partial N = M$.
(iv) A framed submanifold $M \subset P$ **cobounds ambiently** if there exists an ambient coboundary $N \subset P$.

□

We shall be mainly concerned with framed submanifolds $M^m \subset P^n$ with codimension $n - m = 1$ or 2.

Definition 3.26. (i) A codimension 2 framed submanifold $M^{n-2} \subset P^n$ is **circular** if the exterior

$$(P_M, \partial P_M) \;=\; (\mathrm{cl.}(P \backslash M \times D^2), M \times S^1)$$

is such that the projection $\partial p : \partial P_M = M \times S^1 \to S^1$ extends to a map $p : P_M \to S^1$.
(ii) An $(n, n-1, n-2)$**-dimensional manifold triple** (P, N, M) is an n-dimensional manifold P^n together with a a codimension 2 framed submanifold $M^{n-2} \subset P$ and an ambient coboundary $N^{n-1} \subset P$.

□

Proposition 3.27. *Let $M^{n-2} \subset P^n$ be a codimension 2 framed submanifold.*
(i) *The connecting map δ in the cohomology exact sequence*

$$\dots \longrightarrow H^1(P_M) \longrightarrow H^1(M \times S^1) \xrightarrow{\delta} H^2(P_M, M \times S^1) \longrightarrow \dots$$

sends the class $\partial p \in H^1(M \times S^1) = [M \times S^1, S^1]$ to

$$\delta(\partial p) \;=\; [M] \in H^2(P_M, M \times S^1) \;=\; H_{n-2}(P_M) \ .$$

(ii) *The submanifold $M \subset P$ cobounds ambiently if and only if $M \subset P$ is circular.*
(iii) *For a circular $M \subset P$ it may be assumed that $p : P_M \to S^1$ is a Morse function, and:*

(a) *if $t \in \mathbb{R}$ is such that $e^{2\pi i t} \in S^1$ is a regular value of δp then*

$$N^{n-1} \;=\; p^{-1}(e^{2\pi i t}) \subset P_M$$

is a codimension 1 framed submanifold ambiently cobounding $\partial N = M$, so that (P, N, M) is an $(n, n-1, n-2)$-dimensional manifold triple,

(b) *if* $e^{2\pi i t_j} \in S^1$ $(0 \leqslant j \leqslant k-1)$ *are regular values of* p *with*

$$t_0 < t_1 < \cdots < t_{k-1} < t_k \ = \ t_0 + 1 \in \mathbb{R}$$

then

$$P_j \ = \ p^{-1}e^{2\pi i([t_j,t_{j+1}])} \ , \ N_j \ = \ p^{-1}e^{2\pi i t_j}$$

define k *relative cobordisms* $(P_j; N_j, N_{j+1}; M)$ *with* $N_k = N_0$ *and* k*-cyclic union*

$$\begin{aligned}&(P_M, \partial P_M) \ = \ (\coprod_{j=0}^{k-1} P_j/\sim, M \times S^1) \ , \\ &(x \in N_{j+1} \subset P_j) \sim (x \in N_{j+1} \subset P_{j+1}) \ (P_k = P_0)\end{aligned}$$

such that

$$P \ = \ M \times D^2 \cup_{M \times S^1} P_M \ .$$

The restrictions $p| : P_j \to S^1$ *lift to Morse functions*

$$p_j \ : \ (P_j; N_j, N_{j+1}; M) \to [t_j, t_{j+1}]$$

such that

$$\begin{aligned}p(x) \ &= \ e^{2\pi i p_j(x)} \in S^1 \ (x \in P_j) \ , \\ p_j^{-1}(t_j) \ &= \ N_j \ , \ p_j^{-1}(t_{j+1}) \ = \ N_{j+1} \ .\end{aligned}$$

□

Example 3.28. An m-knot is a codimension 2 framed submanifold

$$M \ = \ S^m \subset P \ = S^{m+2}$$

which is circular, and cobounds ambiently. The exterior

$$P_M \ = \ \mathrm{cl.}(S^{m+2}\backslash(S^m \times D^2))$$

is a (co)homology circle, and the generator $1 \in H^1(P_M) = \mathbb{Z}$ is represented by a Morse function $p : P_M \to S^1$ extending the projection

$$\partial p \ : \ \partial P_M \ = \ S^m \times S^1 \to S^1 \ .$$

For any regular value $e^{2\pi i t} \in S^1$ of p

$$N^{m+1} \ = \ p^{-1}(e^{2\pi i t}) \subset P_M$$

is an ambient coboundary for $M \subset P$, i.e. a Seifert surface for the knot with $\partial N = M \subset P$, and there is defined an $(m+2, m+1, m)$-dimensional manifold triple (P, N, M).

□

Definition 3.29. Let $(N_j, M_j, i_j : M_j \to \partial N_j)$ $(0 \leqslant j \leqslant k-1)$ be k n-dimensional coboundaries with common boundary

$$M_0 \;=\; M_1 \;=\; \dots \;=\; M_{k-1} \;=\; M \;.$$

Fix a sequence

$$t_0 < t_1 < \dots < t_k \;=\; t_0 + 1 \in \mathbb{R} \;,$$

and let

$$\begin{aligned}
A_j \;&=\; \{re^{2\pi it} \,|\, 1 \leqslant r \leqslant 2,\, t_j \leqslant t \leqslant t_{j+1}\} \;, \\
B_j^- \;&=\; \{2e^{2\pi it} \,|\, (t_{j-1} + 2t_j)/3 \leqslant t \leqslant t_j\} \;, \\
B_j^+ \;&=\; \{2e^{2\pi it} \,|\, t_j \leqslant t \leqslant (2t_j + t_{j+1})/3\} \;, \\
B_j \;&=\; B_j^- \cup B_j^+ \;=\; \{2e^{2\pi it} \,|\, (t_{j-1} + 2t_j)/3 \leqslant t \leqslant (2t_j + t_{j+1})/3\} \;, \\
I_j \;&=\; \{2e^{2\pi it} \,|\, (2t_j + t_{j+1})/3 \leqslant t \leqslant (t_j + 2t_{j+1})/3\} \;, \\
P_j \;&=\; N_j \times B_j^+ \cup_{M \times A_j \cap B_j^+} M \times A_j \cup_{M \times A_j \cap B_{j+1}^-} N_{j+1} \times B_{j+1}^- \;(N_k = N_0) \;.
\end{aligned}$$

The **thickened union** is the $(n+1, n, n-1)$-dimensional manifold triple (P, N_0, M) defined by the k-cyclic union

$$(P, \partial P) \;=\; (M \times D^2 \cup \bigcup_{j=0}^{k-1} P_j, \coprod_{j=0}^{k-1} N_j \cup M \times I \cup N_{j+1})$$

where

$$N_j \cup M \times I \cup N_{j+1} \;=\; N_j \times \{2e^{2\pi i(2t_j + t_{j+1})/3}\} \cup M \times I_j \cup N_{j+1} \times \{2e^{2\pi i(t_j + 2t_{j+1})/3}\} \;.$$

The exterior of $M \times D^2 \subset P$

$$(P_M, \partial P_M) \;=\; (\bigcup_{j=0}^{k-1} P_j, M \times S^1 \sqcup \coprod_{j=0}^{k-1} (N_j \cup M \times I \cup N_{j+1}))$$

is equipped with a Morse function $p : P_M \to S^1$ such that

$$p| \;=\; \text{projection} \;:\; M \times S^1 \to S^1 \;.$$

$\square$

Proposition 3.30. *Let $(N, M, i : M \to \partial N)$ be an $(m+1)$-dimensional coboundary. For any automorphisms $f, g : M \to M$ the thickened union of the coboundaries*

$$\begin{aligned}
(N_0, M_0, i_0) \;&=\; (N, M, i) \;, \\
(N_1, M_1, i_1) \;&=\; (N, M, if) \;, \\
(N_2, M_2, i_2) \;&=\; (N, M, ig)
\end{aligned}$$

is an $(m+2,m+1,m)$-dimensional manifold triple

$$\bigcup_{j=0}^{2}(N_j,i_j)\times I \;=\; (P,N_0,M)$$

with boundary the disjoint union of the twisted doubles

$$\partial P \;=\; (N\cup_f -N)\sqcup(N\cup_g -N)\sqcup(N\cup_{gf} -N)\ .$$

□

Definition 3.31. Let M be a closed m-dimensional manifold. The **double mapping torus** of automorphisms $f,g:M\to M$ is the cobordism $(T(f,g);T(f)\sqcup T(g),T(gf))$ with $T(f,g)$ the trace of the surgery on $T(f)\sqcup T(g)$ replacing $T_-(f)\sqcup T_+(g)=M\times I\times S^0$ by $M\times S^0\times I$, with

$$\begin{aligned}&T_+(f) \;=\; M\times[0,1/2]\ ,\ T_-(f) \;=\; M\times[1/2,1]\subset T(f)\ ,\\ &T(f) \;=\; T_+(f)\cup T_-(f)\ ,\ T_+(f)\cap T_-(f) \;=\; M\times S^0\ .\end{aligned}$$

The Euler characteristic of $T(f,g)$ is

$$\chi(T(f,g)) \;=\; -\chi(M)\ .$$

□

Proposition 3.32. *The double mapping torus $T(f,g)$ is the thickened union of the ??*

4. Forms, formations and triformations

We recall from [46] the basic definitions of forms and formations over a ring with involution.

In dealing with a noncommutative ring R we shall be working with left R-modules $K,L,\dots$, writing $\mathrm{Hom}_R(K,L)$ for the additive group of R-module morphisms $K\to L$. However, most of the rings here will be commutative.

For $p,q\geqslant 1$ let $M_{p,q}(R)$ be the additive group of $p\times q$ matrices

$$S \;=\; (s_{jk})_{1\leqslant j\leqslant p,1\leqslant j\leqslant q}$$

with entries $s_{jk}\in R$. Use the isomorphism

$$\begin{aligned}&M_{p,q}(R)\to \mathrm{Hom}_R(R^q,R^p)\ ;\ S=(s_{jk})\mapsto\\ &(R^q\to R^p;(x_1,x_2,\dots,x_q)\mapsto(\sum_{k=1}^{q}x_ks_{1k},\sum_{k=1}^{p}x_ks_{2k},\dots,\sum_{k=1}^{q}x_ks_{pk}))\end{aligned}$$

as an identification. The composition pairing

$$\mathrm{Hom}_R(R^q,R^p)\times \mathrm{Hom}_R(R^r,R^q)\to \mathrm{Hom}_R(R^r,R^p)\ ;\ (f,g)\mapsto fg$$

corresponds to the pairing given by matrix multiplication

$$M_{p,q}(R)\times M_{q,r}(R)\to M_{p,r}(R)\ ;\ (S,T)\mapsto ST=\{\sum_{j=1}^{q} s_{ij}t_{jk}\}_{1\leqslant i\leqslant p,1\leqslant k\leqslant r}$$

For $p=q$ write $M_{p,p}(R)=M_p(R)$.

Now suppose that R is equipped with an **involution**

$$R\to R\ ;\ a\mapsto \overline{a}\ .$$

Use the involution on R to define the **dual** of an R-module K to be the R-module

$$K^*\ =\ \mathrm{Hom}_R(K,R)\ ,\ R\times K^*\to K^*\ ;\ (a,f)\mapsto (x\mapsto f(x)\overline{a})\ .$$

The **dual** of an R-module morphism $f:K\to L$ is the R-module morphism

$$f^*\ :\ L^*\to K^*\ ;\ g\mapsto (x\mapsto g(f(x)))\ .$$

The natural R-module morphism

$$K\to K^{**}\ ;\ x\mapsto (f\mapsto \overline{f(x)})$$

is an isomorphism for f.g. free K, which will be used to identify $K=K^{**}$ for such K. For f.g. free K,L duality thus defines an isomorphism

$$*\ :\ \mathrm{Hom}_R(K,L)\to \mathrm{Hom}_R(L^*,K^*)\ ;\ f\mapsto f^*\ .$$

The duality isomorphism for $K=R^p$, $L=R^q$ corresponds to the **conjugation** isomorphism

$$*\ :\ M_{q,p}(R)\to M_{p,q}(R)\ ;\ S=(s_{jk})\mapsto S^*=(\bar{s}_{kj})\ ,$$

via the isomorphism

$$R^p\to (R^p)^*\ ;\ (x_1,x_2,\dots,x_p)\mapsto ((y_1,y_2,\dots,y_p)\mapsto \sum_{i=1}^{p} y_i\overline{x}_i)\ .$$

4.1. **Forms and formations.**

Definition 4.1. Let $\epsilon=+1$ or -1.
(i) An **ϵ-symmetric form** over R (K,ϕ) is defined by a f.g. free R-module K and a function

$$\phi\ :\ K\times K\to R\ ;\ (x,y)\mapsto \phi(x,y)$$

such that

$$\begin{aligned}\phi(x+x',y+y') &= \phi(x,y)+\phi(x,y')+\phi(x',y)+\phi(x',y')\ ,\\ \phi(ax,by) &= b\phi(x,y)\overline{a}\ ,\\ \phi(y,x) &= \epsilon\overline{\phi(x,y)} \in R\ (x,x'\in K,\ y,y'\in K',\ a,b\in R).\end{aligned}$$

The **adjoint** of ϕ is the R-module morphism

$$\phi\ :\ K\to K^*\ ;\ x\mapsto (y\mapsto \phi(x,y))$$

such that $\phi^*=\epsilon\phi$. There is virtually no difference between the form and the adjoint.
(ii) An ϵ-symmetric form (K,ϕ) is **nonsingular** if $\phi:K\to K^*$ is an isomorphism, or equivalently if $\phi^*:K\to K^*$ is an isomorphism.
(iii) A **morphism** of ϵ-symmetric forms over R

$$f\ :\ (K,\phi)\to(K',\phi')$$

is an R-module morphism $f:K\to K'$ such that $f^*\phi'f=\phi$, or equivalently such that

$$\phi(x,y)\ =\ \phi'(f(x),f(y))\in R\ (x,y\in K)\ .$$

(iv) Given an ϵ-symmetric form (K,ϕ) over R and a submodule $L\subseteq K$ define the submodule

$$L^\perp\ =\ \{x\in K\,|\,\phi(x,y)=0\in R \text{ for all } y\in L\}\subseteq K\ .$$

For $L=K$ this is the **radical** $K^\perp=\ker(\phi:K\to K^*)$ of (K,ϕ).
(v) A **sublagrangian** for a ϵ-symmetric form (K,ϕ) is a submodule $L\subseteq K$ such that $L\subseteq L^\perp$. If (K,ϕ) is nonsingular and $L\subseteq K$ is a direct summand there is induced a nonsingular ϵ-symmetric form $(L^\perp/L,[\phi])$. A **lagrangian** is a sublagrangian L such that $L=L^\perp$. In particular, L is a lagrangian of $H_\epsilon(L)$.
(vi) A nonsingular ϵ-symmetric form is **hyperbolic** if it admits a lagrangian. The **standard hyperbolic** nonsingular ϵ-symmetric form is defined for any ϵ-symmetric form (K,ϕ)

$$H_\epsilon(K,\phi)\ =\ (K^*\oplus K,\theta)\ ,\ \theta\ =\ \begin{pmatrix}0&1\\ \epsilon&\phi\end{pmatrix}$$

with

$$\theta(f,x)(g,y)\ =\ f(y)+\epsilon\overline{g(x)}+\phi(x,y)\in R\ (x,y\in K,\ f,g\in K^*)\ .$$

The form $H_\epsilon(K,\phi)$ has lagrangian K^*.
(vii) An **ϵ-symmetric automorphism** (K,ϕ,A) is a nonsingular symplectic form (K,ϕ) over R with an automorphism $A:(K,\phi)\to(K,\phi)$.

(viii) An **ϵ-symmetric formation** $(K,\phi;L_1,L_2)$ is a nonsingular ϵ-symmetric form (K,ϕ) with two lagrangians L_1,L_2.
(ix) An **isomorphism** of formations

$$f\ :\ (K,\phi;L_1,L_2)\to(K',\phi';L'_1,L'_2)$$

is an isomorphism of forms $f:(K,\phi)\to(K',\phi)$ such that

$$f(L_1)\ =\ L'_1\ ,\ f(L_2)\ =\ L'_2\ .$$

(x) A **stable isomorphism** of formations

$$[f]\ :\ (K,\phi;L_1,L_2)\to(K',\phi';L'_1,L'_2)$$

is an isomorphism of the type

$$f\ :\ (K,\phi;L_1,L_2)\oplus(H_\epsilon(M);M,M^*)\to(K',\phi';L'_1,L'_2)\oplus(H_\epsilon(M');M',M'^*)$$

for some f.g. free R-modules M,M'.
(xi) The **boundary** of an ϵ-symmetric form (K,ϕ) over R is the $-\epsilon$-symmetric formation over R

$$\partial(K,\phi)\ =\ (H_{-\epsilon}(K);K,\Gamma_{(K,\phi)})$$

with

$$\Gamma_{(K,\phi)}\ =\ \{(x,\phi(x))\in K\oplus K^*\,|\,x\in K\}$$

the **graph lagrangian**.

$\square$

Terminology For $\epsilon=1$ (resp. -1) an ϵ-symmetric form is called a **symmetric** (resp. **symplectic**) **form**. Similarly for formations.

$\square$

Example 4.2. Given an ϵ-symmetric automorphism (K,ϕ,A) and a lagrangian L of (K,ϕ) there is defined an ϵ-symmetric formation $(K,\phi;L,A(L))$.

$\square$

Proposition 4.3. (i) *The inclusion of a sublagrangian $j:(L,0)\to(K,\phi)$ in a nonsingular ϵ-symmetric form extends to an isomorphism of ϵ-symmetric forms over R*

$$(L^\perp/L,[\phi])\oplus(L\oplus L^*,\begin{pmatrix}0&1\\ \epsilon&\nu\end{pmatrix})\xrightarrow{\cong}(K,\phi)$$

with (L^*,ν) an ϵ-symmetric form over R. In particular, the inclusion of a lagrangian $j:(L,0)\to(K,\phi)$ extends to an isomorphism

$$(j\ \widetilde{j})\ :\ (L\oplus L^*,\begin{pmatrix}0&1\\ \epsilon&\widetilde{j}^*\phi\widetilde{j}\end{pmatrix})\to(K,\phi)$$

for any morphism $\widetilde{j} : L^* \to K$ *such that* $j^*\phi\widetilde{j} = 1 : L^* \to L^*$.
(ii) *If* $(K,\phi;L_1,L_2)$ *is an* ϵ*-symmetric formation over* R *such that* $L = L_1 \cap L_2$ *is a direct summand of* K *(e.g. if* R *is a field) then* L *is a sublagrangian of* (K,ϕ)*, and the inclusion* $(L,0) \to (K,\phi)$ *extends to an isomorphism of* ϵ*-symmetric formations over* R

$$(L^\perp/L,[\phi];L_1/L,L_2/L) \oplus (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \epsilon & \nu \end{pmatrix}; L, L) \;\cong\; (K,\phi;L_1,L_2)\ .$$

(iii) *If* $1/2 \in R$ *then for any* ϵ*-symmetric form* (L^*,ν) *over* R *there is defined an isomorphism of* ϵ*-symmetric forms over* R

$$\begin{pmatrix} 1 & \nu^*/2 \\ 0 & 1 \end{pmatrix} \;:\; (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \epsilon & \nu \end{pmatrix}) \xrightarrow{\cong} H_\epsilon(L) \;=\; (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix})$$

which is the identity on L.
(iv) *If* $1/2 \in R$ *then for any* ϵ*-symmetric formation* $(K,\phi;L_1,L_2)$ *over* R *there exists an* ϵ*-symmetric automorphism* (K,ϕ,A) *with* $A(L_1) = L_2$.

□

Remark 4.4. The condition $1/2 \in R$ in Proposition 4.3 (iii),(iv) can be dropped for ϵ-symmetric formations $(K,\phi;L_1,L_2)$ such that

$$\phi \;=\; \psi + \epsilon\psi^* \;:\; K \to K^*$$

for some $\psi \in \mathrm{Hom}_R(K,K^*)$ such that

$$\psi(x_j)(x_j) \in \{r - \epsilon\bar{r} \,|\, r \in R\}\ (x_j \in L_j,\ j = 1,2)\ .$$

See Ranicki [46] for the theory of ϵ-quadratic forms and formations.

□

Definition 4.5. Let R be a ring with involution, $\epsilon = \pm 1$.
(i) The **Witt group of ϵ-symmetric forms** $L^0(R,\epsilon)$ is the abelian group with one generator for each isomorphism class of nonsingular ϵ-symmetric forms (K,ϕ) over R, with relations

$$\begin{aligned}(K,\phi) + (K',\phi') \;&=\; (K \oplus K', \phi \oplus \phi')\ ,\\ (K,\phi) \;&=\; 0 \text{ if } (K,\phi) \text{ admits a lagrangian.}\end{aligned}$$

(ii) The **Witt group of ϵ-symmetric formations** $L^1(R,\epsilon)$ is the abelian group with one generator for each isomorphism class of nonsingular ϵ-symmetric formations $(K,\phi;L_1,L_2)$ over R, with relations

$$\begin{aligned}&(K,\phi;L_1,L_2) + (K',\phi';L_1',L_2') \;=\; (K \oplus K', \phi \oplus \phi'; L_1 \oplus L_1', L_2 \oplus L_2')\ ,\\ &(K,\phi;L_1,L_2) \oplus (K,\phi;L_2,L_3) \;=\; (K,\phi;L_1,L_3)\ ,\\ &(K,\phi;L_1,L_2) \;=\; 0 \text{ if } K \;=\; L_1 \oplus L_2\ .\end{aligned}$$

(iii) The ϵ**-symmetric automorphism Witt group** $L\mathrm{Aut}^0(R,\epsilon)$ is the abelian group with one generator for each isomorphism class of ϵ-symmetric automorphisms (K,ϕ,A) over R, with relations

$$\begin{aligned}(K,\phi,A)+(K',\phi',A') &= (K\oplus K',\phi\oplus\phi',A\oplus A')\ ,\\ (K,\phi,A) &= 0\in L\mathrm{Aut}^0(R,\epsilon)\\ &\qquad \text{if } (K,\phi) \text{ admits a lagrangian } L \text{ such that } A(L)=L\ .\end{aligned}$$

□

Warning 4.6. In general

$$(K,\phi,A)+(K,\phi,B)\neq(K,\phi,AB)+(K,\phi,1)\in L\mathrm{Aut}^0(R,\epsilon)\ .$$

□

Remark 4.7. (i) For any $(-\epsilon)$-symmetric form (K,ϕ) over R

$$\partial(K,\phi)\ =\ 0\in L^1(R,\epsilon)$$

since K^* is a lagrangian in $H_\epsilon(K)$ which is a direct complement to both the lagrangians K, $\Gamma_{(K,\phi)}$.
(ii) For any ϵ-symmetric automorphism (K,ϕ,A) the diagonal lagrangian of $(K\oplus K,\phi\oplus-\phi)$

$$\Delta_K\ =\ \{(x,x)\,|\,x\in K\}\subset K\oplus K$$

is such that

$$(A\oplus A)(\Delta_K)\ =\ \Delta_K\subset K\oplus K\ ,$$

so that

$$-(K,\phi,A)\ =\ (K,-\phi,A)\in L\mathrm{Aut}^0(R,\epsilon)\ .$$

(iii) The forgetful map

$$L\mathrm{Aut}^0(R,\epsilon)\to L^1(R,\epsilon)\ ;\ (K,\phi,A)\mapsto(K\oplus K,\phi\oplus-\phi;\Delta_K,(A\oplus 1)\Delta_K)$$

is onto, with kernel generated by the elements of type

$$(K,\phi,A)+(K,\phi,B)-(K,\phi,AB)-(K,\phi,1)\in L\mathrm{Aut}^0(R,\epsilon)\ .$$

□

4.2. **Algebraic surgery.**

Definition 4.8. (i) An ϵ**-symmetric** $n\times n$ **matrix form over** R is an $n\times n$ matrix $Z\in M_n(R)$ such that

$$Z^*\ =\ \epsilon Z\in M_n(R)$$

corresponding to the ϵ-symmetric form (R^n, Z). The **metabolic** ϵ-symmetric $2n \times 2n$ matrix form over R is

$$H_\epsilon(Z) \;=\; \begin{pmatrix} 0 & 1 \\ \epsilon & Z \end{pmatrix} \;.$$

(ii) An **ϵ-symmetric $n \times n$ matrix formation over R** (X, Y, Z) is a triple of $n \times n$ matrices $X, Y, Z \in M_n(R)$ such that Z is ϵ-symmetric and

$$\begin{pmatrix} X \\ Y \end{pmatrix} \;:\; (R^n, 0) \to (R^n \oplus R^n, H_\epsilon(Z))$$

is the inclusion of a lagrangian

$$L \;=\; \mathrm{im}(\begin{pmatrix} X \\ Y \end{pmatrix} : R^n \to R^n \oplus R^n) \;,$$

i.e. such that

$$X^*Y + \epsilon Y^*X + Y^*ZY \;=\; 0 \in M_n(R)$$

and the sequence

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} X \\ Y \end{pmatrix}} R^n \oplus R^n \xrightarrow{\begin{pmatrix} \epsilon Y^* & X^* + Y^*Z \end{pmatrix}} R^n \longrightarrow 0$$

is exact. Then $(R^n \oplus R^n, H_\epsilon(R^n); R^n, L)$ is an ϵ-symmetric formation over R, which is also denoted by (X, Y, Z).
(iii) An **isomorphism** of ϵ-symmetric $n \times n$ matrix formations over R

$$(F, G, H) \;:\; (X, Y, Z) \to (X', Y', Z')$$

is given by $F, G, H \in GL_n(R)$ such that

$$H + \epsilon H^* + F^{-1}Z(F^*)^{-1} \;=\; Z' \in M_n(R) \;.$$

Then

$$\begin{pmatrix} F & FH \\ 0 & (F^*)^{-1} \end{pmatrix} \;:\; (R^n \oplus R^n, H_\epsilon(Z)) \to (R^n \oplus R^n, H_\epsilon(Z'))$$

is an isomorphism of ϵ-symmetric forms sending L to L', i.e. such that the diagram

$$\begin{tikzcd} R^n \arrow[r, "G"] \arrow[d, "\begin{pmatrix} X \\ Y \end{pmatrix}"'] & R^n \arrow[d, "\begin{pmatrix} X' \\ Y' \end{pmatrix}"] \\ R^n \oplus R^n \arrow[r, "\begin{pmatrix} F & FH \\ 0 & (F^*)^{-1} \end{pmatrix}"] & R^n \oplus R^n \end{tikzcd}$$

commutes.
(iv) A **stable isomorphism** from an ϵ-symmetric $n \times n$ matrix formation over R to an $n' \times n'$ matrix formation over R

$$[F, G, H] \; : \; (X, Y, Z) \to (X', Y', Z')$$

is an isomorphism of the type

$$(F, G, H) \; : \; (X, Y, Z) \oplus (0_{n'}, I_{n'}, 0_{n'}) \to (X', Y', Z') \oplus (0_n, I_n, 0_n)$$

(v) The **boundary** of an ϵ-symmetric $n \times n$ matrix form Z is the $(-\epsilon)$-symmetric $n \times n$ matrix formation

$$\partial Z \;=\; (I_n, Z, 0) \; .$$

Proposition 4.9. (i) *An ϵ-symmetric $n \times n$ matrix formation (X, Y, Z) is stably isomorphic to $(0, 0)$ if and only if $Y \in M_n(R)$ is invertible. In particular, the boundary $\partial\Phi = (1, \Phi)$ of a matrix form Φ is stably isomorphic to $(0, 0)$ if and only if Φ is invertible (= (R^n, Φ) is nonsingular).*
(ii) *An ϵ-symmetric $n \times n$ matrix formation (X, Y, Z) is isomorphic to the boundary ∂Q of a $(-\epsilon)$-symmetric $n \times n$ matrix form Q if and only if $X \in M_n(R)$ is invertible, in which case there is defined an isomorphism*

$$(X^{-1}, 1, 0) \; : \; (X, Y, Z) \to \partial X^* Y \; .$$

□

Definition 4.10. There are three types of algebraic surgery on an ϵ-symmetric $n \times n$ matrix formation (X, Y, Z) over R.
(i) (ii) The **data** (P, Q) for a **type II algebraic surgery** is an $m \times n$ matrix $P \in M_{m,n}(R)$ together with a $m \times m$ matrix form $Q \in M_m(R)$ such that

$$\begin{aligned} P^* Z P \;&=\; Q - \epsilon Q^* \in M_m(R) \; , \\ \ker(\begin{pmatrix} Y & P \end{pmatrix} : R^n \oplus R^m \to R^n) \;&\cong\; R^k \end{aligned}$$

for some $k \geqslant 0$. The **effect** of the type II algebraic surgery is the ϵ-symmetric $(m+n) \times (m+n)$ matrix formation (X', Y', Z') with

$$X' \;=\; \begin{pmatrix} X & 0 \\ 0 & I_m \end{pmatrix} \; , \; Y' \;=\; \begin{pmatrix} Y & -\epsilon P \\ P^* X & Q \end{pmatrix} \; , \; Z' \;=\; \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \in M_{m+n}(R) \; .$$

The **trace** of the type II algebraic surgery is the $(-\epsilon)$-symmetric $\kappa \times \kappa$ matrix form form over R defined by the

$$\begin{aligned} U \;&=\; \begin{pmatrix} 0 & (X^* + Y^* Z^*) P \\ 0 & Q \end{pmatrix} \;=\; -\epsilon U^* \; : \\ \ker(\begin{pmatrix} Y & P \end{pmatrix} &: R^n \oplus R^m \to R^n) \;=\; R^k \to (R^k)^* \cong \; R^k \end{aligned}$$

with the boundary ∂U stably isomorphic to $(X, Y, Z) \oplus (-X', Y', Z')$.

□

Proposition 4.11. (i) *For any ϵ-symmetric matrix formation (X, Y, Z) there is defined a stable isomorphism*

$$(X, Y, Z) \oplus (Y, X, Z) \to \partial(X^*Y) \ .$$

The effect of the algebraic surgery on (X, Y, Z) with data $(I_n, 0)$ and trace X^*Y is an ϵ-symmetric matrix formation (X', Y', Z') with stable isomorphisms

$$(X', Y', Z') \to (-Y, X, ?) \ , \ (X, Y, Z) \oplus (-X', Y', ?) \to \partial(X^*Y) \ .$$

(ii) *Two ϵ-symmetric matrix formations (X, Y, Z), (X', Y', Z') are such that $(X, Y, Z) = (X', Y', Z') \in L^1(R, \epsilon)$ if and only if (X', Y', Z') is stably isomorphic to the effect of a sequence of algebraic surgeries on (X, Y, Z).*
(iii) *An ϵ-symmetric matrix formation (X, Y, Z) is stably isomorphic to the boundary ∂Q of a $(-\epsilon)$-symmetric form Q, if and only if there exists a type II algebraic surgery on (X, Y, Z) with data (P, Q) such that*

$$Y' \ = \ \begin{pmatrix} Y & -\epsilon P^* \\ PX & Q \end{pmatrix} \in M_{m+n}(R)$$

is invertible.

□

4.3. **Triformations.**

Definition 4.12. (i) An **ϵ-symmetric triformation** $(K, \phi; L_1, L_2, L_3)$ over R is a nonsingular ϵ-symmetric form (K, ϕ) over R with three lagrangians $L_1, L_2, L_3 \subset K$ such that the R-module

$$\begin{aligned} U \ &= \ \{(x_1, x_2, x_3) \in L_1 \oplus L_2 \oplus L_3 \,|\, x_1 + x_2 + x_3 = 0 \in K\} \\ &= \ \ker((j_1 \ j_2 \ j_3) : L_1 \oplus L_2 \oplus L_3 \to K) \end{aligned}$$

is f.g. free, with $j_i : L_i \to K$ $(i = 1, 2, 3)$ the inclusions.
(ii) The **union** of the triformation is the $(-\epsilon)$-symmetric form over R

$$U(K, \phi; L_1, L_2, L_3) \ = \ (U, \psi)$$

with

$$\psi \ : \ U \to U^* \ ; \ (x_1, x_2, x_3) \mapsto \big((y_1, y_2, y_3) \mapsto \phi(j_1(x_1))(j_2(y_2))\big) \ .$$

□

Remark 4.13. (i) It follows from

$$\begin{pmatrix} 0 & j_1^*\phi j_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -j_1^*\phi j_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j_2^*\phi j_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -j_2^*\phi j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ j_3^*\phi j_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -j_3^*\phi j_2 & 0 \end{pmatrix}$$

$$: U \to U^*$$

that $\psi^* = -\epsilon\psi : U \to U^*$. More directly, for any $(x_1, x_2, x_3), (y_1, y_2, y_3) \in U$

$$\begin{aligned} & \psi(x_1, x_2, x_3)(y_1, y_2, y_3) + \epsilon\psi(y_1, y_2, y_3)(x_1, x_2, x_3) \\ &= \phi(j_1(x_1))(j_2(y_2)) + \epsilon\phi(j_1(y_1))(j_2(x_2)) \\ &= \phi(j_1(x_1))(j_2(y_2)) + \phi(j_2(x_2))(j_1(y_1)) \\ &= \phi(j_1(x_1) + j_2(x_2), j_1(y_1) + j_2(y_2)) \\ &= \phi(j_3(x_3))(j_3(y_3)) \ = \ 0 \text{ (since } j_i^*\phi j_i = 0 \text{ for } i = 1, 2, 3) \ . \end{aligned}$$

(ii) The union construction of Definition 4.12 is a generalization to an arbitrary R of the constructions of Wall [57] and Kashiwara and Shapira [20] (for $R = \mathbb{Z}$ in the symplectic case $\epsilon = -1$).
(iii) If R is commutative and a principal ideal domain then every submodule of a f.g. free R-module is f.g. free, so that an ϵ-symmetric triformation $(K, \phi; L_1, L_2, L_3)$ over R is a nonsingular ϵ-symmetric form (K, ϕ) over R with three lagrangians $L_1, L_2, L_3 \subset K$.

$\square$

Proposition 4.14. *Let $(U, \psi) = U(K, \phi; L_1, L_2, L_3)$ be the union of an ϵ-symmetric triformation, as in Definition 4.12.*
(i) *The R-module isomorphism*

$$\begin{aligned} & \ker((j_3^*\phi j_1 \ \ j_3^*\phi j_2) : L_1 \oplus L_2 \to L_3^*) \xrightarrow{\cong} \\ & U \ = \ (\ker(j_1 \ j_2 \ j_3) : L_1 \oplus L_2 \oplus L_3 \to K) \ ; \\ & \qquad (x_1, x_2) \mapsto (x_1, x_2, -j_1(x_1) - j_2(x_2)) \end{aligned}$$

defines an isomorphism of forms

$$(\ker((j_3^*\phi j_1 \ \ j_3^*\phi j_2) : L_1 \oplus L_2 \to L_3^*), \begin{pmatrix} 0 & j_1^*\phi j_2 \\ 0 & 0 \end{pmatrix}) \cong (U, \psi) \ .$$

If $j_3^*\phi j_2 : L_2 \to L_3^*$ is an isomorphism, there is defined an isomorphism of forms

$$\begin{pmatrix} 1 \\ -(j_3^*\phi j_2)^{-1}(j_3^*\phi j_1) \end{pmatrix} \ : \ (L_1, -(j_1^*\phi j_2)(j_3^*\phi j_2)^{-1}(j_3^*\phi j_1)) \\ \to (\ker((j_3^*\phi j_1 \ j_3^*\phi j_2) : L_1 \oplus L_2 \to L_3^*), \begin{pmatrix} 0 & j_1^*\phi j_2 \\ 0 & 0 \end{pmatrix}) \ .$$

(ii) *The radical of (U,ψ) is*

$$\begin{aligned} (U,\psi)^\perp &= \{(x_1,x_2,x_3) \in U \mid \psi(x_1,x_2,x_3) = 0 \in U^*\} \\ &= \ker((j_1 \ j_2) : L_1 \oplus L_2 \to K) + \ker((j_2 \ j_3) : L_2 \oplus L_3 \to K) \\ &\qquad + \ker((j_1 \ j_3) : L_1 \oplus L_3 \to K) \\ &= \Big(\ker((j_1 \ j_2) : L_1 \oplus L_2 \to K) \oplus \ker((j_2 \ j_3) : L_2 \oplus L_3 \to K) \\ &\qquad \oplus \ker((j_1 \ j_3) : L_1 \oplus L_3 \to K)\Big) / \\ &\qquad \ker(\begin{pmatrix} j_1^*\phi \\ j_2^*\phi \\ j_3^*\phi \end{pmatrix} : K \to L_1^* \oplus L_2^* \oplus L_3^*) \\ &= \frac{(L_1 \cap L_2) \oplus (L_2 \cap L_3) \oplus (L_3 \cap L_1)}{L_1 \cap L_2 \cap L_3} \subseteq U \ . \end{aligned}$$

(iii) *If $L_1 + L_2 + L_3 = K$ the form (U,ψ) is nonsingular if and only if L_1, L_2, L_3 are pairwise complements in K, if and only if each of*

$$j_1^*\phi j_2 \ : \ L_2 \to L_1^* \ , \ j_3^*\phi j_2 \ : \ L_2 \to L_3^* \ , \ j_3^*\phi j_1 \ : \ L_1 \to L_3^*$$

is an isomorphism.

□

Example 4.15. Let $R = \mathbb{R}$, $\epsilon = -1$. The lagrangians of the symplectic form $(K,\phi) = H_-(\mathbb{R})$ are the 1-dimensional subspaces $L \subset K$, with

$$\begin{aligned} L &= L(\theta) = \mathrm{im}(j(\theta)) \subset K \ (\theta \in \mathbb{R}) \ , \\ j(\theta) &= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : \mathbb{R} \to K \ = \ \mathbb{R} \oplus \mathbb{R} \ , \\ j(\theta)^*\phi &= \begin{pmatrix} -\sin\theta & \cos\theta \end{pmatrix} : K \ = \ \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \ . \end{aligned}$$

For any $\theta_1, \theta_2 \in \mathbb{R}$

$$j(\theta_2)^*\phi j(\theta_1) \ = \ \sin(\theta_1 - \theta_2) \ : \ \mathbb{R} \to \mathbb{R} \ .$$

The following conditions are equivalent:

(1) $L(\theta_1) = L(\theta_2)$,

(2) $\theta_1 - \theta_2 = n\pi$ for some $n \in \mathbb{Z}$.
(3) $j(\theta_2)^*\phi j(\theta_1) = 0$.

For any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ the union of the symplectic triformation $(K, \phi; L(\theta_1), L(\theta_2), L(\theta_3))$ is the symmetric form

$$(U, \psi) \;=\; U(K, \phi; L(\theta_1), L(\theta_2), L(\theta_3))$$

given by Proposition 4.14 to be such that

$$\begin{aligned}
U \;&=\; \ker(\begin{pmatrix} \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \\ \sin\theta_1 & \sin\theta_2 & \sin\theta_3 \end{pmatrix} : \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}) \\
&\cong\; \ker(\begin{pmatrix} \sin(\theta_1 - \theta_3) & \sin(\theta_2 - \theta_3) \end{pmatrix} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R})\ , \\
\psi \;&:\; U \to U^*\ ;\ (x_1, x_2) \mapsto ((y_1, y_2) \mapsto \sin(\theta_2 - \theta_1)(x_1)(y_2))\ , \\
U^\perp \;&=\; \Big(\ker(\begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}) \\
&\qquad \oplus\ker(\begin{pmatrix} \cos\theta_2 & \cos\theta_3 \\ \sin\theta_2 & \sin\theta_3 \end{pmatrix} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}) \\
&\qquad \oplus\ker(\begin{pmatrix} \cos\theta_3 & \cos\theta_1 \\ \sin\theta_3 & \sin\theta_1 \end{pmatrix} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R})\Big)/ \\
&\qquad\qquad \ker(\begin{pmatrix} -\sin\theta_1 & \cos\theta_1 \\ -\sin\theta_2 & \sin\theta_2 \\ -\sin\theta_3 & \sin\theta_3 \end{pmatrix} : \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R})\ .
\end{aligned}$$

Thus

$$(U, \psi) \;\cong\; \begin{cases} (\mathbb{R}, \sin(\theta_1 - \theta_2)\sin(\theta_2 - \theta_3)\sin(\theta_3 - \theta_1)) & \text{if } \dim_{\mathbb{R}}(U) = 1 \\ (\mathbb{R}\oplus\mathbb{R}, 0) & \text{if } \dim_{\mathbb{R}}(U) = 2, \end{cases}$$

noting that $\dim_{\mathbb{R}}(U) = 2$ if and only if

$$\sin(\theta_1 - \theta_2) \;=\; \sin(\theta_2 - \theta_3) \;=\; \sin(\theta_3 - \theta_1) \;=\; 0\ ,$$

if and only if

$$L(\theta_1) \;=\; L(\theta_2) \;=\; L(\theta_3) \subset \mathbb{R}^2\ .$$

□

Example 4.16. Let N_1, N_2, N_3 be three $(k-1)$-connected $(2k+1)$-dimensional manifolds with the same $(k-1)$-connected $2k$-dimensional boundary

$$M \;=\; \partial N_1 \;=\; \partial N_2 \;=\; \partial N_3\ ,$$

and let

$$(P,\partial P) \;=\; (M\times D^2\cup\bigcup_{j=1}^{3}N_j\times I,\coprod_{j=1}^{3}N_j\cup_M N_{j+1})\;(N_4=N_1),$$

be the thickened union, the k-connected $(2k+2)$-dimensional manifold with boundary constructed in §3. Each of

$$\begin{aligned}K &= H_k(M)\ ,\ L_i \;=\; \ker(H_k(M)\to H_k(N_i))\ (i=1,2,3)\ ,\\ U &= \ker((j_1\ j_2\ j_3):L_1\oplus L_2\oplus L_3\to K)\ (j_i=\text{inclusion})\end{aligned}$$

is f.g. free, so that there is defined a $(-1)^k$-symmetric triformation $(K,\phi;L_1,L_2,L_3)$ over $\mathbb{Z}$ with ϕ the intersection form, such that the union $U(K,\phi;L_1,L_2,L_3)=(U,\psi)$ is the intersection form on

$$\begin{aligned}U \;=\; H_{k+1}(P) &= \ker((j_1\ j_2\ j_3):L_1\oplus L_2\oplus L_3\to K)\\ &\cong \ker((j_3^*\phi j_1\ j_3^*\phi j_2):L_1\oplus L_2\to L_3^*)\ .\end{aligned}$$

4.4. **The algebraic mapping torus.**

Definition 4.17. (i) An ϵ-symmetric automorphism (K,ϕ,A) determines the inclusion of a lagrangian

$$j(A) \;=\; \begin{pmatrix}1\\A\end{pmatrix}\ :\ (K,0)\to(K\oplus K,\phi\oplus-\phi)\ .$$

(ii) The **mapping torus** of an ϵ-symmetric automorphism (K,ϕ,A) over R is the ϵ-symmetric formation over R

$$\begin{aligned}T(A) &= (K\oplus K,\phi\oplus-\phi;j(1)(K),j(A)(K))\\ &= (K\oplus K,\phi\oplus-\phi;\Delta_K,(1\oplus A)\Delta_K)\end{aligned}$$

with $\Delta_K=\{(x,x)\,|\,x\in K\}\subset K\oplus K$.

□

Proposition 4.18. *Let (K,ϕ,A_1), (K,ϕ,A_2), (K,ϕ,A_3) be ϵ-symmetric automorphisms over R.*
(i) *The isomorphism of ϵ-symmetric forms*

$$1\oplus A_1\ :\ (K\oplus K,\phi\oplus-\phi)\to(K\oplus K,\phi\oplus-\phi)$$

defines an isomorphism of ϵ-symmetric formations

$$T(A_2^{-1}A_1)\;\cong\;(K\oplus K,\phi\oplus-\phi;j(A_1)(K),j(A_2)K)\ .$$

(ii) *The R-module morphism*

$$\begin{aligned}j(A_2)^*(\phi\oplus-\phi)j(A_1) &= \begin{pmatrix}1 & A_2^*\end{pmatrix}\begin{pmatrix}\phi & 0\\0 & -\phi\end{pmatrix}\begin{pmatrix}1\\A_1\end{pmatrix}\\ &= \phi(1-A_2^{-1}A_1)\ :\ K\to K^*\ ,\end{aligned}$$

is an isomorphism if and only if $1 - A_2^{-1}A_1 : K \to K$ *is an automorphism.*

(iii) *The union of the* ϵ*-symmetric triformation*

$$(K \oplus K, \phi \oplus -\phi; j(A_1)(K), j(A_2)(K), j(A_3)(K))$$

is (isomorphic to) the $(-\epsilon)$*-symmetric form*

$$(U,\psi) \ = \ (\ker(\begin{pmatrix}(A_1 - A_3) & (A_2 - A_3)\end{pmatrix} : K \oplus K \to K), \begin{pmatrix} 0 & \phi(1 - A_2^{-1}A_1) \\ 0 & 0 \end{pmatrix}) \ .$$

The union is the trace of an algebraic surgery on $T(A_3^{-1}A_1) \oplus T(A_2^{-1}A_3)$ *with effect* $T(A_2^{-1}A_1)$*, with a stable isomorphism*

$$\partial(U,\psi) \ \cong \ T(A_3^{-1}A_1) \oplus T(A_2^{-1}A_3) \oplus -T(A_2^{-1}A_1)$$

so that

$$T(A_3^{-1}A_1) + T(A_2^{-1}A_3) \ = \ T(A_2^{-1}A_1) \in L^1(R,\epsilon) \ .$$

(iv) *If* $A_2 - A_3 : K \to K$ *is an automorphism the* R*-module isomorphism*

$$K \xrightarrow{\cong} U \ ; \ x \mapsto (x, -(A_2 - A_3)^{-1}(A_1 - A_3)(x))$$

defines an isomorphism of forms

$$(U,\psi) \ \cong \ (K, \phi(1 - A_2^{-1}A_1)(1 - A_3^{-1}A_2)^{-1}(1 - A_3^{-1}A_1)) \ .$$

(v) *For* $A_3 = A_1A_2$ *there is defined a commutative square*

$$\begin{array}{ccc}
K \oplus K & \xrightarrow{(A_1 - A_3 \ \ A_2 - A_3)} & K \\
{\scriptstyle 1 \oplus A_1^{-1}A_2} \big\downarrow \cong & & \cong \big\downarrow {\scriptstyle A_1^{-1}} \\
K \oplus K & \xrightarrow{(1 - A_2 \ \ 1 - A_1)} & K
\end{array}$$

inducing an isomorphism of $(-\epsilon)$*-symmetric forms*

$$(U,\psi) \cong (\ker(\begin{pmatrix} 1 - A_2 & 1 - A_1 \end{pmatrix} : K \oplus K \to K), \begin{pmatrix} 0 & -\phi(1 - A_2^{-1}A_1) \\ 0 & 0 \end{pmatrix}) \ .$$

If $1 - A_1 : K \to K$ *is an automorphism the* R*-module isomorphism*

$$K \xrightarrow{\cong} U \ ; \ x \mapsto (x, -(A_2 - A_1A_2)^{-1}(A_1 - A_1A_2)(x))$$

defines an isomorphism of forms

$$(U,\psi) \ \cong \ (K, \phi(1 - A_2^{-1}A_1)(1 - A_2^{-1}A_1^{-1}A_2)^{-1}(1 - A_2^{-1})) \ .$$

Proof. (i)-(iv) Apply Proposition 4.14.
(v) Apply (iii)+(iv). □

Example 4.19. Let $R = \mathbb{R}$, $\epsilon = -1$. For $\theta \in \mathbb{R}$ define the rotation automorphism

$$R(\theta) \;=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;:\; (K,\phi) \;=\; H_-(\mathbb{R}) \to (K,\phi)$$

corresponding to $e^{i\theta} = \cos\theta + i\sin\theta \in \mathbb{C}$, so that $R(\theta) = R(\theta')$ if and only if $\theta' - \theta = 2n\pi$ for some $n \in \mathbb{Z}$. It is immediate from the identities

$$\begin{aligned} e^{-i\theta} &= (e^{i\theta})^{-1} \;,\; e^{i\theta_1}e^{i\theta_2} \;=\; e^{i(\theta_1+\theta_2)} \;, \\ e^{i\theta_1} - e^{i\theta_2} &= (2\sin(\theta_1-\theta_2)/2)\, e^{i(\pi+\theta_1+\theta_2)/2} \end{aligned}$$

that

$$\begin{aligned} & R(\theta)^{-1} \;=\; R(-\theta) \;,\; R(\theta_1)R(\theta_2) \;=\; R(\theta_1+\theta_2) \;, \\ & R(\theta_1) - R(\theta_2) \;=\; (2\sin(\theta_1-\theta_2)/2)R((\pi+\theta_1+\theta_2)/2) \;, \\ & R(\theta) - 1 \;=\; 2\sin\theta/2\, R((\pi+\theta)/2) \;. \end{aligned}$$

As in Proposition 4.18 (iii) for any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ there is defined a symplectic triformation over $\mathbb{R}$

$$(K \oplus K, \phi \oplus -\phi; L(\theta_1), L(\theta_2), L(\theta_3))$$

with the lagrangian of (K,ϕ)

$$L(\theta_i) \;=\; j(R(\theta_i))(K) \;=\; \{(x, R(\theta_i)(x)) \,|\, x \in K\}$$

depending only on $2\theta_i/\pi \in \mathbb{R}/\mathbb{Z}$. The union is the symmetric form over $\mathbb{R}$

$$\begin{aligned} (U,\psi) \;&=\; (\ker(\begin{pmatrix} (R(\theta_1)-R(\theta_3)) & (R(\theta_2)-R(\theta_3)) \end{pmatrix} : K\oplus K \to K), \\ &\qquad\qquad \begin{pmatrix} 0 & \phi(1-R(\theta_2)^{-1}R(\theta_1)) \\ 0 & 0 \end{pmatrix}) \\ &\cong\; \begin{cases} (K\oplus K, 0) \text{ if } R(\theta_1) = R(\theta_2) = R(\theta_3) \\ (K, \sin(\theta_1-\theta_2)/2\sin(\theta_2-\theta_3)/2\sin(\theta_3-\theta_1)/2) \text{ otherwise .} \end{cases} \end{aligned}$$

If $R(\theta_1) = R(\theta_2) = R(\theta_3)$ then

$$(U,\psi) \;=\; (K\oplus K, 0) \;,\; \sigma(U,\psi) \;=\; 0 \;.$$

If $R(\theta_1), R(\theta_2), R(\theta_3)$ are not all the same the $\mathbb{R}$-module isomorphism

$$f \;:\; K \to U \;;\; x \mapsto ((R(\theta_2)-R(\theta_3))(x), (R(\theta_3)-R(\theta_1))(x)))$$

defines an isomorphism of symmetric forms over $\mathbb{R}$

$$f \;:\; (K, \sin(\theta_1-\theta_2)/2\sin(\theta_2-\theta_3)/2\sin(\theta_3-\theta_1)/2) \to (U,\psi)$$

so that

$$\sigma(U,\psi) \ = \ 2\,\mathrm{sgn}(\sin{(\theta_1-\theta_2)/2}\sin{(\theta_2-\theta_3)/2}\sin{(\theta_3-\theta_1)/2})) \in \mathbb{Z}\ .$$

□

Proposition 4.20. (i) *If (K,ϕ,A) is an ϵ-symmetric automorphism over R there is defined an isomorphism of ϵ-symmetric formations over R*

$$\begin{pmatrix} 0 & \phi \\ -1 & 1 \end{pmatrix} \ : \ T(A) \to T'(A)\ ,$$

with

$$T'(A) \ = \ (K^*\oplus K, \begin{pmatrix} 0 & 1 \\ \epsilon & \phi \end{pmatrix}, K^*, \mathrm{im}(\begin{pmatrix} \phi A \\ I-A \end{pmatrix} : K \to K^*\oplus K))\ ,$$

so that

$$T(A) \ = \ T'(A) \in L^1(R,\epsilon)\ .$$

(ii) *If $1/2\in R$ and $(K,\phi;L_1,L_2)$ is an ϵ-symmetric formation over R there exists an automorphism $A:(K,\phi)\to(K,\phi)$ such that $A(L_1)=L_2$, and for any such A*

$$T(A) \ = \ (K,\phi;L_1,L_2) \in L^1(R,\epsilon)\ .$$

(iii) *If $1/2\in R$, (K,ϕ,A) is an ϵ-symmetric automorphism over R and L is a lagrangian of (K,ϕ) then*

$$T(A) \ = \ (K,\phi;L,A(L)) \in L^1(R,\epsilon)\ .$$

Proof. (i) By construction.
(ii) The inclusions $L_j\to K$ $(j=1,2)$ extend to isomorphisms of forms

$$a_j \ : \ H_\epsilon(L_j) \to (K,\phi)\ .$$

Now L_1, L_2 are f.g. free R-modules with

$$\dim_R(L_1) \ = \ \dim_R(K)/2 \ = \ \dim_R(L_2)\ ,$$

so there exists an isomorphism $a_3 : L_1 \to L_2$. The automorphism

$$A \ = \ a_2\begin{pmatrix} a_3 & 0 \\ 0 & (a_3^*)^{-1} \end{pmatrix}(a_1)^{-1} \ : \ (K,\phi)\to(K,\phi)$$

is such that $A(L_1)=L_2$.

Let L_3 be a lagrangian of (K,ϕ) which is a direct complement of L_1, so that $L_1 \oplus L_3$ is a lagrangian of $(K \oplus K, \phi \oplus -\phi)$ which is a direct complement of Δ_K. Then

$$\begin{aligned} T(A) &= (K \oplus K, \phi \oplus -\phi; \Delta_K, (A \oplus 1)\Delta_K) \\ &= (K \oplus K, \phi \oplus -\phi; L_1 \oplus L_3, (A \oplus 1)(L_1 \oplus L_3)) \\ &= (K \oplus K, \phi \oplus -\phi; L_1 \oplus L_3, L_2 \oplus L_3) \\ &= (K, \phi; L_1, L_2) \in L^1(R, \epsilon) \ . \end{aligned}$$

(iii) Immediate from (ii). □

Remark 4.21. The condition $1/2 \in R$ in Proposition 4.20 (ii) is not necessary, in the sense that for every ϵ-symmetric formation $(K, \phi; L_1, L_2)$ over any ring with involution R there exists an ϵ-symmetric automorphism (K', ϕ', A') over R such that

$$(K, \phi; L_1, L_2) \ = \ T(A') \in L^1(R, \epsilon)$$

by Ranicki [47, Remark 30.29].

□

Definition 4.22. Let (K, ϕ, A) be an ϵ-symmetric automorphism over R. As in Proposition 4.20 (i) if $K = R^n$ the algebraic mapping torus $T(A)$ is isomorphic to the ϵ-symmetric $n \times n$ matrix formation over R (Definition 4.8 (ii))

$$T'(A) \ = \ (X, Y, Z) \ = \ (\phi A, I - A, \phi) \ .$$

□

Definition 4.23. Let (K, ϕ, A) be an ϵ-symmetric automorphism over R, with $K = R^n$. If coker$(I - A)$ is a f.g. free R-module there is ???
□

Definition 4.24. For ϵ-symmetric automorphisms (K, ϕ, A_1), (K, ϕ, A_2) the **canonical algebraic surgery** on

$$T'(A_1) \oplus -T'(A_2) \ = \ (\phi A_1, I - A_1, \phi) \oplus (-\phi A_2, I - A_2, -\phi)$$

has data, effect $T'(A_1 A_2)$ and trace (4.20 (vi))

$$(P, Q) \ = ??$$

□

4.5. **The Laurent polynomial extension.**

Definition 4.25. (i) The **Laurent polynomial extension** $R[z,z^{-1}]$ of a ring with involution R is the ring of finite polynomials

$$p(z) \;=\; \sum_{j=-N}^{N} a_j z^j \;(a_j \in R,\; N \geqslant 0)$$

with involution by $\overline{z} = z^{-1}$, that is

$$\overline{p(z)} \;=\; \sum_{j=-N}^{N} a_j z^{-j} \in R[z,z^{-1}]\;.$$

(ii) Let K, L be f.g. free R-modules. An $R[z,z^{-1}]$-module morphism $f : K[z,z^{-1}] \to L[z,z^{-1}]$ is **Fredholm** if f is injective and $\mathrm{coker}(f)$ is a f.g. free R-module.

□

Example 4.26. For any automorphism $A : K \to K$ of a f.g. free R-module the $R[z,z^{-1}]$-module morphism

$$1 - zA \;:\; K[z,z^{-1}] \to K[z,z^{-1}]$$

is Fredholm, with

$$\mathrm{coker}(1 - zA) \to K \;;\; \sum_{j=-\infty}^{\infty} z^j x_j \mapsto \sum_{j=-\infty}^{\infty} A^{-j}(x_j)$$

an R-module isomorphism.

□

Definition 4.27. For an integral domain R let $R(z)$ be the quotient field of $R[z,z^{-1}]$, the localization of $R[z,z^{-1}]$ inverting all the polynomials $p(z) \neq 0 \in R[z,z^{-1}]$.

□

Proposition 4.28. *Let R be an integral domain, and let $f : K[z,z^{-1}] \to L[z,z^{-1}]$ be an $R[z,z^{-1}]$-module morphism, with K, L f.g. free R-modules, and let*

$$p(z) \;=\; \det(f) \in R[z,z^{-1}]$$

for any bases of K, L.
(i) *The following conditions on f are equivalent:*

(a) *f is Fredholm,*
(b) *$p(z) = \sum\limits_{j=M}^{N} a_j z^j \in R[z,z^{-1}]$ with $a_M, a_N \in R$ units and $M < N$,*

(c) *the induced $R(z)$-module morphism*

$$1\otimes f \;:\; R(z)\otimes_{R[z,z^{-1}]}K[z,z^{-1}]\to R(z)\otimes_{R[z,z^{-1}]}L[z,z^{-1}]$$

is an isomorphism.

(ii) *If R is a field then f is Fredholm if and only if $p(z)\in R[z,z^{-1}]$ is not constant.*

Proof. See Ranicki [47]. □

Definition 4.29. (i) An ϵ-symmetric form $(K[z,z^{-1}],\phi)$ over $R[z,z^{-1}]$ is **nondegenerate** if the $R[z,z^{-1}]$-module morphism

$$\phi \;:\; K[z,z^{-1}]\to K^*[z,z^{-1}]$$

is Fredholm.
(ii) The **boundary** of a nondegenerate ϵ-symmetric form $(K[z,z^{-1}],\phi)$ over $R[z,z^{-1}]$ is the nonsingular $(-\epsilon)$-symmetric form $(\partial K,\partial\phi)$ over R

$$\begin{aligned}\partial K \;&=\; \mathrm{coker}(\phi : K[z,z^{-1}]\to K^*[z,z^{-1}])\ ,\\ \partial\phi \;&:\; \partial K\times\partial K\to R\ ;\ (x,y)\mapsto y(\phi^{-1}(x))_{-1}\end{aligned}$$

with the automorphism

$$A \;:\; (\partial K,\partial\phi)\to(\partial K,\partial\phi)\ ;\ x\mapsto zx\ .$$

□

Proposition 4.30. (Ranicki [47]) *Let R be an integral domain with involution.*
(i) *Every nonsingular ϵ-symmetric form over $R(z)$ is isomorphic to $R(z)\otimes_{R[z,z^{-1}]}(K[z,z^{-1}],\phi)$ for some nondegenerate ϵ-symmetric form $(K[z,z^{-1}],\phi)$ over $R[z,z^{-1}]$.*
(ii) *There is defined a long exact sequence*

$$\dots\longrightarrow L^0(R[z,z^{-1}],\epsilon)\longrightarrow L^0(R(z),\epsilon)\xrightarrow{\partial}$$

$$LAut^0(R,-\epsilon)\xrightarrow{T} L^1(R[z,z^{-1}],-\epsilon)\longrightarrow L^1(R(z),-\epsilon)\longrightarrow\dots$$

with $L^1(R(z),-\epsilon)=0$ and

$$\begin{aligned}T \;&:\; LAut^0(R,\epsilon)\to L^1(R[z,z^{-1}],\epsilon)\ ;\\ &\qquad (K,\phi,A)\mapsto T(zA:(K,\phi)[z,z^{-1}]\to(K,\phi)[z,z^{-1}])\ ,\\ \partial \;&:\; L^0(R(z),\epsilon)\to LAut^0(R,-\epsilon)\ ;\\ &\qquad R(z)\otimes_{R[z,z^{-1}]}(K[z,z^{-1}],\phi)\mapsto(\partial K,\partial\phi,A)\ .\end{aligned}$$

(iii) *If R is a field then*

$$\begin{aligned} L^1(R,\epsilon) &= 0 , \\ L^0(R[z,z^{-1}],\epsilon) &= L^0(R,\epsilon)\oplus L^1(R,-\epsilon) = L^0(R,\epsilon) , \\ L^1(R[z,z^{-1}],\epsilon) &= L^1(R,\epsilon)\oplus L^0(R,\epsilon) = L^0(R,\epsilon) . \end{aligned}$$

(iii) *If R is a field of characteristic of $\neq 2$ there is defined a long exact sequence*

$$\begin{aligned} 0 \longrightarrow L^0(R,\epsilon) \longrightarrow L^0(R(z),\epsilon) &\xrightarrow{\partial} \\ L\mathrm{Aut}^0(R,-\epsilon) &\xrightarrow{T} L^0(R,-\epsilon) \longrightarrow 0 \end{aligned}$$

with

$$L^0(R[z,z^{-1}],\epsilon) = L^0(R,\epsilon) ,\ L^1(R[z,z^{-1}],-\epsilon) = L^0(R,-\epsilon) .$$

□

5. Signatures

5.1. **The triple signature in algebra.** The Witt class of a symmetric form (K,ϕ) over $\mathbb{R}$ is the signature of the nonsingular symmetric form $(K/K^{\perp},[\phi])$

$$\sigma(K,\phi) = \sigma(K/K^{\perp},[\phi]) \in L^0(\mathbb{R}) = \mathbb{Z} .$$

If L is a sublagrangian of (K,ϕ) then

$$(K,\phi) = (L^{\perp}/L,[\phi]) \in L^0(\mathbb{R}) = \mathbb{Z} .$$

Definition 5.1. Let $(K,\phi;L_1,L_2,L_3)$ be a symplectic triformation over $\mathbb{R}$, as defined by a nonsingular symplectic form (K,ϕ) over $\mathbb{R}$ and the inclusions of three lagrangians

$$j_i\ :(L_i,0)\to(K,\phi)\ (i=1,2,3) .$$

(i) (Wall [57]) The **Wall nonadditivity invariant** of $(K,\phi;L_1,L_2,L_3)$ is the signature

$$\sigma^W(K,\phi;L_1,L_2,L_3) = \sigma(U(K,\phi;L_1,L_2,L_3)) \in \mathbb{Z}$$

of the union symmetric form over $\mathbb{R}$ (4.12)

$$\begin{aligned} &U(K,\phi;L_1,L_2,L_3) = (U,\psi) \\ &= (\ker((j_1\ j_2\ j_3):L_1\oplus L_2\oplus L_3\to K), \begin{pmatrix} 0 & j_1^*\phi j_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) \end{aligned}$$

(ii) (Kashiwara and Shapira [20, p.486]) The **Maslov index** of $(K,\phi;L_1,L_2,L_3)$ is the signature

$$\sigma^M(K,\phi;L_1,L_2,L_3) \;=\; \sigma(L_1\oplus L_2\oplus L_3,\Phi_{123})\in\mathbb{Z}$$

of the symmetric form $(L_1\oplus L_2\oplus L_3,\Phi_{123})$ over $\mathbb{R}$ defined by

$$\Phi_{123} \;=\; \begin{pmatrix} 0 & j_1^*\phi j_2 & j_1^*\phi j_3 \\ -j_2^*\phi j_1 & 0 & j_2^*\phi j_3 \\ -j_3^*\phi j_1 & -j_3^*\phi j_2 & 0 \end{pmatrix} \;:\; L_1\oplus L_2\oplus L_3\to L_1^*\oplus L_2^*\oplus L_3^*.$$

□

Proposition 5.2. *$\sigma^W(K,\phi;L_1,L_2,L_3)=\sigma^M(K,\phi;L_1,L_2,L_3)\in\mathbb{Z}$.*

Proof. The sublagrangian $L_1\subset L_1\oplus L_2\oplus L_3$ of $(L_1\oplus L_2\oplus L_3,\Phi_{123})$ is such that the linear map

$$U\to L_1\oplus L_2\oplus L_3\ ;\ (x_1,x_2,x_3)\mapsto(0,x_2,x_3)/\sqrt{2}$$

induces an isomorphism of symmetric forms over $\mathbb{R}$

$$(U,\psi)\cong(L_1^\perp/L_1,[\Phi_{123}])\ ,$$

so that

$$\begin{aligned}\sigma^M(K,\phi;L_1,L_2,L_3) &= \sigma(L_1\oplus L_2\oplus L_3,\Phi_{123})\\ &= \sigma(L_1^\perp/L_1,[\Phi_{123}]) \;=\; \sigma(U,\psi)\\ &= \sigma^W(K,\phi;L_1,L_2,L_3)\in\mathbb{Z}\ .\end{aligned}$$

□

Definition 5.3. The **triple signature** of a symplectic triformation $(K,\phi;L_1,L_2,L_3)$ over $\mathbb{R}$ is the signature of the union symmetric form over $\mathbb{R}$

$$\begin{aligned}\sigma(K,\phi;L_1,L_2,L_3) &= \sigma(U(K,\phi;L_1,L_2,L_3))\\ &= \sigma^W(K,\phi;L_1,L_2,L_3)\\ &= \sigma^M(K,\phi;L_1,L_2,L_3)\in\mathbb{Z}\ .\end{aligned}$$

□

Definition 5.4. Let $(K,\phi)=H_-(\mathbb{R})$.
(i) The **triple signature** of $\theta_1,\theta_2,\theta_3\in\mathbb{R}$ is the triple signature 5.3

$$\sigma(\theta_1,\theta_2,\theta_2) \;=\; \sigma(K,\phi;L(\theta_1),L(\theta_2),L(\theta_3))\in\mathbb{Z}\ .$$

of the symplectic triformation $(K,\phi;L(\theta_1),L(\theta_2),L(\theta_3))$, with $L(\theta_i)\subset K$ the lagrangians of (K,ϕ) given (as in Example 4.15) by

$$L(\theta_i) \;=\; \mathrm{im}(\begin{pmatrix}\cos\theta_i\\ \sin\theta_i\end{pmatrix}:\mathbb{R}\to\mathbb{R}\oplus\mathbb{R})\ .$$

is the signature of the symplectic triformation over $\mathbb{R}$

$$(K\oplus K,\phi\oplus-\phi;j(R(\theta_1))(K),j(R(\theta_2))(K),j(R(\theta_3))(K))$$

given (as in Example 4.15) by the images of the diagonal lagrangian $\Delta\subset K\oplus K$ of $(K\oplus K,\phi\oplus-\phi)$ under the automorphisms

$$j(R(\theta_i)) ~=~ \begin{pmatrix} 1 \\ R(\theta_i)\end{pmatrix} ~:~ (K,0)\to(K\oplus K,\phi\oplus-\phi)~(i=1,2,3)~.$$

$\square$

Proposition 5.5. (i) *The triple signature of* $\theta_1,\theta_2,\theta_3\in\mathbb{R}$ *is*

$$\begin{array}{rcl}\sigma(\theta_1,\theta_2,\theta_3) & = & \sigma(\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix},\begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix},\begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix}) \\ & = & \mathrm{sgn}(\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)) \\ & = & -\mathrm{sgn}(\sin 2(\theta_2-\theta_1)+\sin 2(\theta_3-\theta_2)+\sin 2(\theta_1-\theta_3)) \\ & = & 2(\,(((\theta_2-\theta_1)/\pi))+(((\theta_3-\theta_2)/\pi))+(((\theta_1-\theta_3)/\pi))\,) \\ & = & v_2'(e^{2i(\theta_2-\theta_1)},e^{2i(\theta_3-\theta_2)})~(v_2'~\textit{as in Example}~2.6) \\ & = & \begin{cases}-\epsilon(ijk)\in\{\pm 1\} & \textit{if}~\{\theta_i/2\pi\}<\{\theta_j/2\pi\}<\{\theta_k/2\pi\} \\ 0 & \text{otherwise}\end{cases}\end{array}$$

with $\epsilon(ijk)$ *the sign of the permutation* $(123)\to(ijk)$.
(ii) *The signature cocycle* v_2' *of Example 2.6 representing* $v_2'=2[u]\in H^1(S^{1\delta};\mathbb{Z})$ *is expressed in terms of the triple signature by*

$$\begin{array}{rl} & v_2' ~:~ S^{1\delta}\times S^{1\delta}\to\mathbb{Z}~;~(e^{2\pi ix},e^{2\pi iy})\mapsto\sigma(0,\pi x,\pi(x+y)) \\ = & \sigma(\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix};\begin{pmatrix}1\\ 0\end{pmatrix}\mathbb{R},\begin{pmatrix}\cos\pi x\\ \sin\pi x\end{pmatrix}\mathbb{R},\begin{pmatrix}\cos\pi(x+y)\\ \sin\pi(x+y)\end{pmatrix}\mathbb{R}) \\ = & -\mathrm{sign}(\sin\pi x\sin\pi y\sin\pi(x+y)) \\ = & 2(\,((x))+((y))-((x+y))\,)~.\end{array}$$

(iii) *The mapping torus triple signature of* $\theta_1,\theta_2,\theta_3\in\mathbb{R}$ *is expressed in terms of the triple signature by*

$$\begin{array}{rl} & \sigma(K\oplus K,\phi\oplus-\phi;j(R(\theta_1))(K),j(R(\theta_2))(K),j(R(\theta_3))(K)) \\ = & 2\sigma(\theta_1/2,\theta_2/2,\theta_3/2) \\ = & -2\,\mathrm{sgn}(\sin(\theta_2-\theta_1)/2\sin(\theta_3-\theta_2)/2\sin(\theta_1-\theta_3)/2)\in\mathbb{Z}~.\end{array}$$

(iv) *The triple signature determines the* **mapping torus signature cocycle**

$$2\sigma = -2v_2' \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) \mapsto$$
$$\sigma(K \oplus K, \phi \oplus -\phi; j(R(0))(K), j(R(2\pi x))(K), j(R(2\pi(x+y)))(K))$$
$$= \ 2\,\mathrm{sgn}(\sin \pi x \sin \pi y \sin \pi(x+y))$$

with cohomology class

$$[2\sigma] \ = \ -4[u] \in H^2(S^{1\delta};\mathbb{Z}) \ .$$

Proof. (i) As in Definition 5.1

$$\sigma(\theta_1,\theta_2,\theta_3) \ = \ \sigma(\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}, \Phi_{123}) \in \mathbb{Z} \ .$$

The symmetric matrix

$$\Phi_{123} \ = \ \begin{pmatrix} 0 & \sin(\theta_2-\theta_1) & -\sin(\theta_1-\theta_3) \\ \sin(\theta_2-\theta_1) & 0 & \sin(\theta_3-\theta_2) \\ -\sin(\theta_1-\theta_3) & \sin(\theta_3-\theta_2) & 0 \end{pmatrix}$$

has principal minors

$$0 \ , \ -\sin^2(\theta_2-\theta_1) \ , \ -2\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)$$

and the signature is

$$\begin{aligned}
&\sigma(\theta_1,\theta_2,\theta_3) \\
&= \ 3 - 2\,\text{variation of signs}(1, 0, -\sin^2(\theta_2-\theta_1) \ , \\
&\qquad\qquad -2\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)) \\
&= \ \begin{cases} \mathrm{sgn}(\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)) & \text{if } \sin(\theta_2-\theta_1) \neq 0 \\ 0 & \text{if } \sin(\theta_2-\theta_1) = 0 \ . \end{cases}
\end{aligned}$$

The other expressions are immediate from the identity

$$\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_1-\theta_3)$$
$$= \ -(\sin 2(\theta_2-\theta_1) + \sin 2(\theta_3-\theta_2) + \sin 2(\theta_1-\theta_3))/4 \ .$$

(ii) By Proposition 2.5 (i) the function

$$\epsilon \ : \ S^{1\delta} \times S^{1\delta} \to \mathbb{Z} \ ; \ (e^{2\pi i x}, e^{2\pi i y}) \mapsto 2\big(((x)) + ((y)) - ((x+y)) \big)$$

is a cocycle representing $2[u] \in H^2(S^{1\delta};\mathbb{Z})$. The group morphism

$$2 \ : \ S^{1\delta} \to S^{1\delta} \ ; \ e^{i\theta} \mapsto e^{2i\theta}$$

induces

$$2^* \ = \ 2 \ : \ H^2(S^{1\delta};\mathbb{Z}) \to H^2(S^{1\delta};\mathbb{Z}) \ ,$$

so that

$$\sigma = 2^*\epsilon : S^{1\delta}\times S^{1\delta} \to \mathbb{Z} ; (e^{2\pi ix}, e^{2\pi iy}) \mapsto 2\big(((2x))+((2y))-((2(x+y))) \big)$$

is a cocycle representing $4[u] \in H^2(S^{1\delta};\mathbb{Z})$.
(iii) By Example 4.19.
(iv) By Example 2.6 □

Summary: the image of $1 \in H^2(BS^1;\mathbb{Z}) = \mathbb{Z}$ is the cohomology class $[u] \in H^2(BS^{1\delta};\mathbb{Z})$ represented by the cocycle

$$u : S^{1\delta} \times S^{1\delta} \to \mathbb{Z} ; (e^{2\pi ix}, e^{2\pi iy}) \mapsto \{x\} + \{y\} - \{x + y\} .$$

The area cocycle

$$\epsilon : S^{1\delta} \times S^{1\delta} \to \mathbb{Z} ; (e^{2\pi ix}, e^{2\pi iy}) \mapsto -\mathrm{sgn}(\sin \pi x \sin \pi y \sin \pi(x + y))$$

represents

$$[\epsilon] = 2[u] \in H^2(S^{1\delta};\mathbb{Z}) .$$

The mapping torus signature cocycle

$$\sigma : S^{1\delta} \times S^{1\delta} \to \mathbb{Z} ; (e^{2\pi ix}, e^{2\pi iy}) \mapsto 2\mathrm{sgn}(\sin\pi x \sin\pi y \sin\pi(x + y))$$

represents

$$[\sigma] = -4[u] \in H^2(S^{1\delta};\mathbb{Z}) .$$

5.2. **The triple signature in topology.** The signature of $4k$-dimensional manifold with boundary $(W, \partial W)$ is the signature of the symmetric intersection form ϕ_W on $H^{2k}(W, \partial W;\mathbb{R})$

$$\sigma(W) = \sigma(H^{2k}(W, \partial W;\mathbb{R}), \phi_W) \in \mathbb{Z} .$$

Novikov [40] proved that if $(W_1, \partial W_1)$, $(W_2, \partial W_2)$ are cobordisms with $\partial W_1 = -\partial W_2$ then

$$\sigma(W_1 \cup_\partial W_2) = \sigma(W_1) + \sigma(W_2) \in \mathbb{Z} .$$

Definition 5.6. (i) The **signature** of a $4k$-dimensional relative cobordism $(W; N_1, N_2; M)$ is

$$\sigma(W; N_1, N_2; M) = \sigma(W) \in \mathbb{Z} .$$

(ii) The **triple signature** of a closed $(4k - 2)$-dimensional manifold M with three null-cobordisms N_1, N_2, N_3 is

$$\sigma(M; N_1, N_2, N_3) = \sigma(K, \phi; L_1, L_2, L_3) \in \mathbb{Z}$$

with $(K, \phi) = (H_{2k-1}(M;\mathbb{R}), \phi_M)$ the nonsingular intersection symplectic form over $\mathbb{R}$, and $L_j = \ker(H_{2k-1}(M;\mathbb{R}) \to H_{2k-1}(N_j;\mathbb{R}))$ the three lagrangians.

□

Proposition 5.7. (Wall [57]) *The signature of a union of* $4k$*-dimensional relative cobordisms is*

$$\begin{aligned}&\sigma((W;N_1,N_2;M)\cup(W';N_2,N_3;M))\\&=\ \sigma(W;N_1,N_2;M)+\sigma(W';N_2,N_3;M)+\sigma(M;N_1,N_2,N_3)\in\mathbb{Z}\ .\end{aligned}$$

□

If $(W^{4k},\partial W)$ is a $4k$-dimensional manifold with boundary and $M^{4k-2}\subset W\backslash\partial W$ is a codimension 2 framed submanifold with $\delta(p)=0\in H^2(W,M)$ then for any codimension 1 framed submanifold $N^{4k-1}\subset W\backslash\partial W$ the $4k$-dimensional manifold with boundary

$$(W',\partial W')\ =\ (P\cup_{\partial P}N\times S^1,\partial W)$$

is cobordant rel ∂ to $(W,\partial W)$ and

$$\sigma(W)\ =\ \sigma(W')\in\mathbb{Z}\ .$$

Remark 5.8. For $R=\mathbb{R}$ there is defined a split exact sequence

$$\begin{aligned}0\longrightarrow L^0(\mathbb{R})=\mathbb{Z}\longrightarrow L^0(\mathbb{R}(z))=\bigoplus_{\infty}\mathbb{Z}\xrightarrow{\partial}\\ L\mathrm{Aut}^0(\mathbb{R},-1)=\bigoplus_{\infty-1}\mathbb{Z}\longrightarrow 0\end{aligned}$$

Let $(W^{4k},\partial W)$ be a $4k$-dimensional manifold with boundary, and let $(M^{4k-2},\partial M)\subset(W,\partial W)$ be a codimension 2 framed submanifold with neighbourhood

$$(M^{4k-2},\partial M)\times D^2\subset(W,\partial W)\ .$$

The projection $M\times S^1\to S^1$ extends on the exterior

$$(P,\partial P)\ =\ (\mathrm{cl.}(W\backslash M\times D^2),M\times S^1\cup\mathrm{cl.}(\partial W\backslash\partial M\times D^2))$$

to a map $(p,\partial p):(P,\partial P)\to S^1$. Let $\overline{P}=p^*\mathbb{R}$ be the pullback infinite cyclic cover of P, with generating covering translation $\zeta:\overline{P}\to\overline{P}$. If $\overline{\partial P}=(\partial p)^*\mathbb{R}$ of ∂P has finite-dimensional homology $H_*(\overline{\partial P};\mathbb{R})$ there is defined an element

$$\sigma^*(P,p)\in L^0(\mathbb{R}(z))$$

with images

$$\begin{aligned}\sigma(P)\ &=\ \sigma(W)\in L^0(\mathbb{R})\ =\ \mathbb{Z}\ ,\\ \partial\sigma^*(P,p)\ &=\ (H_{2k-1}(\overline{\partial P};\mathbb{R}),\zeta|)\in L\mathrm{Aut}^0(\mathbb{R},-1)\end{aligned}$$

Three important special cases:
(i) If $W = D^{4k}$ and

$$\partial M \;=\; S^{4k-3} \subset \partial W \;=\; S^{4k-1}$$

is a knot with $M^{4k-2} \subset W$ a pushed in Seifert surface, then $\sigma_*(P,p) \in L^0(\mathbb{R}(z))$ is the high-dimensional knot cobordism class.
(ii) If

$$\partial M \;=\; \emptyset \;,\; \partial W \;=\; \emptyset$$

then $\partial P = M \times S^1$ and

$$\partial\sigma^*(P,p) \;=\; (H_{2k-1}(M;\mathbb{R}),1) \;=\; 0 \in L\mathrm{Aut}^0(\mathbb{R},-1) \;.$$

(iii) Let M be a closed $(4k-2)$-dimensional manifold. The double mapping torus (Definiton 3.12 (i)) of automorphisms $A,B : M \to M$ is a $4k$-dimensional manifold

$$T(A,B) \;=\; M \times I \times I \cup M \times I \times I \cup M \times I \times I$$

with boundary

$$\partial T(A,B) \;=\; T(A) \sqcup T(B) \sqcup T(AB) \;.$$

Let

$$(P(A,B),\partial P(A,B)) \;=\; (\mathrm{cl.}(T(A,B)\backslash M \times D^2), M \times S^1 \cup \partial T(A,B))$$

be the exterior of the codimension 2 framed submanifold $M \times D^2 \subset T(A,B)$. The projection $M\times S^1 \to S^1$ extends to $(P(A,B),\partial P(A,B)) \to S^1$ and

$$\begin{aligned}
&\sigma(P(A,B)) \;=\; \sigma(T(A,B))\\
&=\; \text{triple signature}\, \sigma(M \cup M; M \times I, M \times I, M \times I) \in L^0(\mathbb{R}) \;=\; \mathbb{Z} \;,\\
&\partial\sigma^*(P(A,B),p)\\
&=\; (M,A) \oplus (M,B) \oplus (M,-AB) \oplus (M,-1) \in L\mathrm{Aut}^0(\mathbb{R},-1) \;.
\end{aligned}$$

□

Proposition 5.9. *Given ϵ-symmetric automorphisms (K,ϕ,A), (K,ϕ,B) there is an algebraic surgery on $T(zA) \oplus T(zB)$ over $R[z,z^{-1}]$ with* **effect** *$T(z^2AB)$. The trace is the nondegenerate $(-\epsilon)$-symmetric form over $R[z,z^{-1}]$*

$$\begin{aligned}
&T_z(A,B)\\
&=\; (\ker(((I-Az \;\; I-Bz) : K[z,z^{-1}] \oplus K[z,z^{-1}] \to K[z,z^{-1}]),\phi_{A,B})\\
&\phi_{A,B}((x_1,y_1),(x_2,y_2)) \;=\; \phi(x_1,y_2)
\end{aligned}$$

with a stable isomorphism

$$\partial T_z(A,B) \;\cong\; T(zA) \oplus T(zB) \oplus -T(z^2AB) \;,$$

so that

$$\begin{aligned}(K,\phi,A)+(K,\phi,B)-(K,\phi,AB)-(K,\phi,1) &= \partial T_z(A,B)-\partial T_z(AB,1)\\ &\in \mathrm{im}(\partial : L^0(R(z),-\epsilon)\to L\mathrm{Aut}^0(R,\epsilon))\\ &= \ker(T : L\mathrm{Aut}^0(R,\epsilon)\to L^1(R[z,z^{-1}],\epsilon))\ .\end{aligned}$$

Proof. ??? □

Definition 5.10. The **double algebraic mapping torus** $T(A,B)$ of ϵ-symmetric automorphisms (K,ϕ,A), (K,ϕ,B) is the trace of the algebraic surgery on $T(A)\oplus T(B)$ with effect $T(AB)$, i.e. the nondegenerate $(-\epsilon)$-symmetric form over R

$$\begin{aligned}T(A,B) &= (\ker(((I-A\ \ I-B) : K\oplus K\to K),\phi_{A,B})\\ \phi_{A,B}((x_1,y_1),(x_2,y_2)) &= \phi(x_1,y_2)\end{aligned}$$

(assuming $\ker((I-A)\ (I-B))$ is a f.g. free R-module) with a stable isomorphism

$$\partial T(A,B)\ \cong\ T(A)\oplus T(B)\oplus -T(AB)\ .$$

Definition 5.11. (Meyer [32]) Let $R=\mathbb{R}$, $(K,\phi)=H_-(\mathbb{R}^g)$, so that $\mathrm{Aut}_{\mathbb{R}}(K,\phi)=\mathrm{Sp}(2g,\mathbb{R})$ is the discrete symplectic group. The **Meyer cocycle** is given by

$$\tau\ :\ \mathrm{Sp}(2g,\mathbb{R})\times\mathrm{Sp}(2g,\mathbb{R})\to\mathbb{Z}\ ;\ (A,B)\mapsto\sigma(T(A,B^{-1}))$$

with

$$\begin{aligned}\tau(A,B) &= \sigma\big(\{(x,y)\in\mathbb{R}^{2g}\oplus\mathbb{R}^{2g}\,|\,(A^{-1}-1)(x)+(B-1)(y)=0\},\\ &\qquad ((x_1,y_1),(x_2,y_2))\mapsto\phi(x_1+y_2,(B-1)(y_2))\big)\ .\end{aligned}$$

Proposition 5.12. *The Meyer cocycle satisfies*

$$\begin{aligned}\tau(x,y) &= \tau(y,z) = \tau(z,x)\ (\text{if } xyz=1)\ ,\\ \tau(x,1) &= \tau(x,x^{-1}) = 0\ ,\\ \tau(x,y) &= \tau(y,x)\ ,\\ \tau(x^{-1},y^{-1}) &= -\tau(x,y)\ ,\\ \tau(wxw^{-1},wyw^{-1}) &= \tau(x,y)\in\mathbb{Z}\ .\end{aligned}$$

□

Example 5.13. The case $g=1$: the isomorphism

$$S^{1\delta}\to\mathrm{Sp}(2,\mathbb{R})\ ;\ e^{i\theta}\mapsto R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

induces an isomorphism

$$H^2(\mathrm{Sp}(2,\mathbb{R});\mathbb{Z}) \cong H^2(S^{1\delta};\mathbb{Z}) \ .$$

By Example 4.19 for any $\theta_1, \theta_2 \in \mathbb{R}$

$$\tau(R(\theta_1), R(\theta_2)) \ = \ 2\,\mathrm{sgn}(\sin\theta_1/2\sin\theta_2/2\sin(\theta_1-\theta_2)/2) \in \mathbb{Z} \ ,$$

so that by Definition 5.4 and Proposition 5.5 the cocycle

$$\tau \ = \ 2v_2' \ = \ 4u \in H^2(\mathrm{Sp}(2g,\mathbb{R});\mathbb{Z}) \ = \ H^2(S^1\delta;\mathbb{Z}) \ .$$

6. The space of lagrangians $\Lambda(n)$

Definition 6.1. Let $\Lambda(n)$ be the space of lagrangians in the standard hyperbolic symplectic form over $\mathbb{R}$

$$H_-(\mathbb{R}^n) \ = \ (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

and let $q : \widetilde{\Lambda}(n) \to \Lambda(n)$ be the universal cover.

□

It is known from the work of Arnold [1, 2], Leray [25], Lion and Vergne [28], Souriau [54], Turaev [56], deGosson [14, 15], Cappell, Lee and Miller [10], Bunke [8], … that there are unique functions

$$\begin{aligned} \widetilde{m} \ &: \ \widetilde{\Lambda}(n) \times \widetilde{\Lambda}(n) \to \mathbb{Z} \ , \\ m \ &: \ \Lambda(n) \times \Lambda(n) \to \mathbb{R} \end{aligned}$$

such that

$$\begin{aligned} \widetilde{m}(\widetilde{L}_1,\widetilde{L}_2) + \widetilde{m}(\widetilde{L}_2,\widetilde{L}_3) + \widetilde{m}(\widetilde{L}_3,\widetilde{L}_1) \ &= \ \sigma(q(\widetilde{L}_1), q(\widetilde{L}_2), q(\widetilde{L}_3)) \ , \\ m(L_1,L_2) + m(L_2,L_3) + m(L_3,L_1) \ &= \ \sigma(L_1,L_2,L_3) \\ m(A(L_1),A(L_2)) \ &= \ m(L_1,L_2) \in \mathbb{Z} \subset \mathbb{R} \ (A \in Sp(2n)) \ . \end{aligned}$$

In the first instance, consider the case $n = 1$. The function

$$S^1 \to \Lambda(1) \ ; \ z = e^{i\psi} \mapsto \sqrt{z} = L(\psi/2)$$

is a diffeomorphism.

Example 6.2. By Example 2.6 and Proposition 5.5 the triple signature of lagrangians $L(\theta_1), L(\theta_2), L(\theta_3) \in \Lambda(1)$ is

$$\begin{aligned} \sigma(\theta_1,\theta_2,\theta_3) \ &= \ \sigma(H_-(\mathbb{R}); L(\theta_1), L(\theta_2), L(\theta_3)) \\ &= \ \mathrm{sgn}(\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_3-\theta_1)) \in \mathbb{Z} \ . \end{aligned}$$

$\square$

Proposition 6.3. *The triple signature function*

$$\begin{aligned}
\sigma \ &: \ \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \{-1,0,1\} \ ; \\
(\theta_1,\theta_2,\theta_3) \mapsto \sigma(\theta_1,\theta_2,\theta_3) \ &= \ \mathrm{sgn}(\sin(\theta_2-\theta_1)\sin(\theta_3-\theta_2)\sin(\theta_3-\theta_1))
\end{aligned}$$

has the following properties.
(i) $\sigma(0,\theta,\pi/2) = \mathrm{sgn}(\sin\theta\cos\theta) = \mathrm{sgn}(\sin 2\theta)$.
(ii) *The η-function is the average triple signature*

$$\begin{aligned}
\eta(\theta) \ &= \ \int_{\ell\in\Lambda(1)}\sigma(\ell,L(0),L(\theta))d\ell \\
&= \ \int_{z\in S^1}\sigma(\sqrt{z},L(0),L(\theta))dz \\
&= \ \frac{1}{2\pi}\int_0^{2\pi}\sigma(\psi/2,0,\theta)d\psi \in \mathbb{R} \ .
\end{aligned}$$

(iii)

$$\begin{aligned}
\sigma(\theta_1,\theta_2,\theta_3) \ &= \ \sigma(\theta_1+\psi,\theta_2+\psi,\theta_3+\psi) \\
&= \ \eta(\theta_2-\theta_1)+\eta(\theta_3-\theta_2)+\eta(\theta_1-\theta_3) \\
&= \ 2(E((\theta_1-\theta_2)/\pi)+E((\theta_2-\theta_3)/\pi)+E((\theta_3-\theta_1)/\pi)) \\
&= \ -2\big((((\theta_1-\theta_2)/\pi))+(((\theta_2-\theta_3)/\pi))+(((\theta_3-\theta_1)/\pi))\big) \\
&\in \{-1,0,1\}\subset\mathbb{R} \ .
\end{aligned}$$

(iv)

$$\eta(\theta_1)+\eta(\theta_2)-\eta(\theta_1+\theta_2) \ = \ \sigma(0,\theta_1,-\theta_2) \ = \ \sigma(0,\theta_2,-\theta_1)$$

$$= \begin{cases} +1 & \text{if } 0<\{\theta_1/\pi\}+\{\theta_2/\pi\}<1, \ \theta_1/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z} \\ -1 & \text{if } 1<\{\theta_1/\pi\}+\{\theta_2/\pi\}<2, \ \theta_1/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \in \{-1,0,1\}\subset\mathbb{R} \ .$$

For $\theta_1=\theta_2=\theta$

$$2\eta(\theta)-\eta(2\theta) \ = \ \sigma(0,\theta,-\theta) \ = \ \phi_0(\theta/\pi)$$

$$= \ \mathrm{sign}(\sin 2\theta) \ = \ \begin{cases} +1 & \text{if } 0<\{\theta/\pi\}<1/2 \\ -1 & \text{if } 1/2<\{\theta/\pi\}<1 \\ 0 & \text{otherwise} \end{cases} \in \{-1,0,1\}\subset\mathbb{R} \ .$$

(v)

$$\sigma(\theta_1,\theta_2,\theta_1+\theta_2) \ = \ \eta(\theta_1)-\eta(\theta_2)+\eta(\theta_2-\theta_1) \ = \ \sigma(0,\theta_2-\theta_1,\theta_2)$$

$$= \begin{cases} +1 & \text{if } 0<\{(\theta_2-\theta_1)/\pi\}+\{-\theta_2/\pi\}<1, \ (\theta_2-\theta_1)/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z} \\ -1 & \text{if } 1<\{(\theta_2-\theta_1)/\pi\}+\{-\theta_2/\pi\}<2, \ (\theta_2-\theta_1)/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{otherwise} \ . \end{cases}$$

(vi) $\sigma(\theta_{\pi(1)},\theta_{\pi(2)},\theta_{\pi(3)}) = \mathrm{sgn}(\pi)\sigma(\theta_1,\theta_2,\theta_3)$ *for any $\pi\in\Sigma_3$.*
(vii) $\sigma(-\theta_1,-\theta_2,-\theta_3) = -\sigma(\theta_1,\theta_2,\theta_3)$.

(viii) $\sigma(\theta_1,\theta_2,\theta_3)=0$ *if* $\theta_1=\theta_2$.
(ix) $\sigma(\theta_1,\theta_2,\theta_3)=\sigma(0,\theta_2-\theta_1,\theta_3-\theta_1)$.
(x) *For any* $\theta_1,\theta_2,\theta_3\in\mathbb{R}$ *define a loop in* $\Lambda(1)$ *from* $L(0)=\mathbb{R}\oplus 0$ *through* $L(\pi\eta(\theta_2-\theta_1))$ *and* $L(\pi\eta(\theta_2-\theta_1)+\pi\eta(\theta_3-\theta_2))$ *and then back to* $L(0)$

$$\begin{aligned}&\omega(\theta_1,\theta_2,\theta_3)\ :\ S^1\to\Lambda(1)\ ;\\ &e^{2\pi it}\mapsto\begin{cases}L(3\pi t\eta(\theta_2-\theta_1)) & \text{if } 0\leqslant t\leqslant 1/3\\ L(\pi\eta(\theta_2-\theta_1)+(3t-1)\pi\eta(\theta_3-\theta_2)) & \text{if } 1/3\leqslant t\leqslant 2/3\\ L(\pi\eta(\theta_2-\theta_1)+\pi\eta(\theta_3-\theta_2)+(3t-2)\pi\eta(\theta_1-\theta_3)) & \text{if } 2/3\leqslant t\leqslant 1\end{cases}\end{aligned}$$

with lift

$$\begin{aligned}&\widetilde{\omega}(\theta_1,\theta_2,\theta_3)\ :\ I\to\widetilde{\Lambda(1)}=\mathbb{R}\ ;\\ &e^{2\pi it}\mapsto\begin{cases}3t\eta(\theta_2-\theta_1) & \text{if } 0\leqslant t\leqslant 1/3\\ \eta(\theta_2-\theta_1)+(3t-1)\eta(\theta_3-\theta_2) & \text{if } 1/3\leqslant t\leqslant 2/3\\ \eta(\theta_2-\theta_1)+\eta(\theta_3-\theta_2)+(3t-2)\eta(\theta_1-\theta_3) & \text{if } 2/3\leqslant t\leqslant 1\end{cases}\end{aligned}$$

The Maslov index (= degree) of $\omega(\theta_1,\theta_2,\theta_3)$ *is the triple signature*

$$\begin{aligned}\mathrm{Mas}(\omega(\theta_1,\theta_2,\theta_3))\ &=\ \eta(\theta_2-\theta_1)+\eta(\theta_3-\theta_2)+\eta(\theta_1-\theta_3)\\ &=\ \sigma(\theta_1,\theta_2,\theta_3)\in\mathbb{Z}\end{aligned}$$

(xi) $\sigma(\theta_1,\theta_2,\theta_3)=\sigma(L(\theta_1),L(\theta_2),L(\theta_3))$.
(xii) (Bunke [8, p.404])

$$\frac{1}{\pi}\int_{\theta_1=0}^{\pi}\sigma(L(\theta_1),L(\theta_2),L(\theta_3))d\theta_1\ =\ \mu(\frac{\theta_3-\theta_2}{\pi})\ \textit{if}\ 0<\theta_2,\theta_3<\pi\ .$$

(xiii) (Meyer [32], Atiyah [3]) *The surface with 3 boundary components*

$$(X,\partial X)\ =\ (\mathrm{cl.}(S^2\backslash\bigcup_3 D^2),\bigcup_3 S^1)$$

has $\pi_1(X)=F_2=\{g_1,g_2\}$ *the free group on 2 generators* g_1,g_2. *Let* E *be the local coefficient system over* X *of flat hermitian vector spaces classified by the group morphism*

$$\pi_1(X)=F_2\to U(1)=S^1\ ;\ g_j\mapsto e^{i\theta_j}\ (j=1,2)\ .$$

The index of a first-order elliptic operator $\overline{\partial}$ *coupled to* E *is the signature of* $(\mathbb{C},i\phi)$, *with* $(H^1(X,\partial X;E)=\mathbb{C},\phi)$ *the skew-hermitian form* *over $\mathbb{C}$ defined by the cup-product and the hermitian form on E, and*

$$\begin{aligned}\sigma(\mathbb{C}, i\phi) &= 2(((\theta_1/2\pi)) + ((\theta_2/2\pi)) - (((\theta_1+\theta_2)/2\pi))) \\ &= \eta((\theta_1+\theta_2)/2) - \eta(\theta_1/2) - \eta(\theta_2/2) \\ &= -\sigma(0, \theta_1/2, -\theta_2/2) \\ &= \begin{cases} -1 & \text{if } 0 < \{\theta_1/2\pi\} + \{\theta_2/2\pi\} < 1 \\ +1 & \text{if } 1 < \{\theta_1/2\pi\} + \{\theta_2/2\pi\} < 2 \\ 0 & \text{otherwise .} \end{cases}\end{aligned}$$

The discontinuous measurable function

$$\begin{aligned}&U(1) \times U(1) \to \mathbb{R} ; \\ &(e^{i\theta_1}, e^{i\theta_2}) \mapsto \sigma(\mathbb{C}, i\phi)/2 \ = \ ((\theta_1/2\pi)) + ((\theta_2/2\pi)) - (((\theta_1+\theta_2)/2\pi))\end{aligned}$$

is a cocycle representing a generator of $H^2(U(1)) = \mathbb{Z}$, corresponding to the universal cover regarded as a central group extension

$$\mathbb{Z} \to \mathbb{R} \to U(1) = S^1 \ .$$

□

Now for $n \geqslant 1$.

Let $(\mathbb{C}^n, \langle\ ,\ \rangle)$ be the standard positive definite hermitian form over $\mathbb{C}$, with

$$\begin{aligned}&\langle\ ,\ \rangle \ : \ \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} ; \\ &\qquad (z, w) \ = \ ((z_1, z_2, \dots, z_n), (w_1, w_2, \dots, w_n)) \mapsto \langle z, w\rangle \ = \ \sum_{j=1}^{n} \overline{z}_j w_j \ .\end{aligned}$$

Given a complex $n \times n$ matrix $A = (a_{jk}) \in M_n(\mathbb{C})$ let

$$A^t \ = \ (a_{kj}) \ , \ A^* \ = \ (\overline{a}_{kj}) \in M_n(\mathbb{C}) \ .$$

Definition 6.4. (i) A **unitary matrix** is a complex $n \times n$ matrix

$$A \ = \ (a_{jk}) \in M_n(\mathbb{C})$$

such that $AA^* = I$, corresponding to an automorphism

$$A \ : \ (\mathbb{C}^n, \langle\ ,\ \rangle) \to (\mathbb{C}^n, \langle\ ,\ \rangle) \ .$$

(ii) The **unitary group** $U(n)$ is the group of $n \times n$ unitary matrices.

□

Let $(\mathbb{R}^n, \langle\ ,\ \rangle)$ be the standard positive definite symmetric form over $\mathbb{R}$, with

$$\begin{aligned}&\langle\ ,\ \rangle\ :\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\ ;\\ &\qquad (x,y)\ =\ ((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)) \mapsto \langle x,y\rangle\ =\ \sum_{j=1}^{n} x_j y_j\ .\end{aligned}$$

Definition 6.5. (i) An **orthogonal matrix** is a real $n \times n$ matrix

$$A\ =\ (a_{jk}) \in M_n(\mathbb{R})$$

such that $AA^t = I$, corresponding to an automorphism

$$A\ :\ (\mathbb{R}^n, \langle\ ,\ \rangle) \to (\mathbb{R}^n, \langle\ ,\ \rangle)\ .$$

(ii) The **orthogonal group** $O(n)$ is the group of orthogonal $n \times n$ matrices.

$\square$

Use the isomorphism of real vector spaces

$$\mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{C}^n\ ;\ (x,y) \mapsto x + iy$$

to identify $(\mathbb{C}^n, \phi)$ over $\mathbb{R}$ with $H_-(\mathbb{R}^n)$, where

$$\phi\ :\ \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}\ ;\ (w,z) \mapsto \mathrm{Im}(\langle z,w\rangle)\ .$$

Definition 6.6. (i) A **symplectic matrix** is a real $2n \times 2n$ matrix

$$A\ =\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2n}(\mathbb{R})$$
$$(a^t c\ =\ c^t a\ ,\ b^t d\ =\ d^t b\ ,\ a^t d - c^t b\ =\ I \in M_n(\mathbb{R}))$$

such that $A^t \phi A = \phi$, corresponding to an automorphism

$$A\ :\ H_-(\mathbb{R}^n) \to H_-(\mathbb{R}^n)\ .$$

(ii) The **symplectic group** $Sp(2n)$ is the group of symplectic $2n \times 2n$ matrices.
(iii) Let $ESp(2n) \subset Sp(2n)$ be the subgroup

$$\begin{aligned} ESp(2n)\ &=\ \{A \in Sp(2n) \,|\, A(\mathbb{R}^n \oplus \{0\}) = \mathbb{R}^n \oplus \{0\} \in \Lambda(n)\}\\ &=\ \{A \in Sp(2n) \,|\, c = 0 \in M_n(\mathbb{R})\}\\ &=\ \{\begin{pmatrix} a & b \\ 0 & (a^{-1})^t \end{pmatrix} \,|\, a \in GL_n(\mathbb{R}),\ b \in M_n(\mathbb{R}),\ ab^t = ba^t\}\ .\end{aligned}$$

$\square$

Proposition 6.7. (i) *A symplectic matrix $A \in Sp(2n)$ is an invertible $2n \times 2n$ matrix*

$$A \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_{2n}(\mathbb{R})$$

such that

$$a^tc \ = \ c^ta \ , \ b^td \ = \ d^tb \ , \ a^td - c^tb \ = \ 1 \in M_n(\mathbb{R}) \ ,$$

with

$$A(\mathbb{R}^n \oplus \{0\}) \ = \ \mathrm{im}(\begin{pmatrix} a \\ c \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n) \in \Lambda(n) \ .$$

The linear map

$$i \ : \ \ker(c : \mathbb{R}^n \to \mathbb{R}^n) \to \mathbb{R}^n \ ; \ x \mapsto a(x)$$

is an injection, with

$$M \ = \ \ker(i^* : \mathbb{R}^n \to \ker(c)^*) \subset \mathbb{R}^n$$

a direct complement of the image

$$i(\ker(c)) \oplus M \ = \ \mathbb{R}^n \ .$$

Likewise, the linear map

$$j \ : \ \ker(b : \mathbb{R}^n \to \mathbb{R}^n) \to \mathbb{R}^n \ ; \ x \mapsto d(x)$$

is an injection, with

$$N \ = \ \ker(j^* : \mathbb{R}^n \to \ker(b)^*) \subset \mathbb{R}^n$$

a direct complement of the image

$$j(\ker(b)) \oplus N \ = \ \mathbb{R}^n \ .$$

The linear map

$$M \to N^* \ ; \ i^*(x) \mapsto (j^*(y) \mapsto \langle x, y \rangle)$$

is an isomorphism. The sublagrangian of $H_-(\mathbb{R}^n) = (\mathbb{R}^n \oplus \mathbb{R}^n, \phi)$

$$L \ = \ A(\mathbb{R}^n \oplus \{0\}) \cap (\mathbb{R}^n \oplus \{0\}) \ = \ i(\ker(c)) \oplus \{0\} \subset \mathbb{R}^n \oplus \mathbb{R}^n$$

is such that

$$\begin{aligned} & L^\perp \ = \ L \oplus M \oplus M^* \ , \ (L^\perp/L, [\phi]) \ = \ H_-(M) \ , \\ & H_-(\mathbb{R}^n) \ = \ H_-(L) \oplus H_-(M) \ . \end{aligned}$$

(ii) *A symplectic matrix $A \in Sp(2n)$ is such that $AJ = JA$ for the standard complex structure*

$$J \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ : \ \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n$$

if and only if and only if $A \in U(n) \subset Sp(2n)$*, with*

$$\begin{aligned} U(n) &= \{\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \in GL_{2n}(\mathbb{R}) \,|\, a^t c = c^t a,\ a^t a + c^t c = 1\} \\ &= Sp(2n) \cap O(2n) \subset Sp(2n) \ . \end{aligned}$$

Furthermore, the image of the injection

$$O(n) \to U(n) \ ; \ a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \ (a^t a = 1)$$

is

$$O(n) \ = \ U(n) \cap ESp(2n) \subset Sp(2n) \ .$$

(iii) *For every* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n)$ *there exists* $U_A \in U(n)$ *such that*

$$A(\mathbb{R}^n \oplus \{0\}) \ = \ U_A(\mathbb{R}^n \oplus \{0\}) \in \Lambda(n)$$

with $(U_A)^{-1}A \in ESp(2n)$*. The symmetric form* $(\mathbb{R}^n, a^t a + c^t c)$ *is positive definite, and*

$$U_A \ = \ ((a + ic)e^{-1}) \in U(n)$$

is such a unitary matrix for any $e \in GL_n(\mathbb{R})$ *such that*

$$a^t a + c^t c \ = \ e^t e \in GL_n(\mathbb{R}) \ .$$

If $A \in U(n) \subset Sp(2n)$ *then* $e = 1$ *will do, with* $U_A = A$*.*
(iv) *The inclusion* $U(n) = Sp(2n) \cap O(2n) \to Sp(2n)$ *induces a diffeomorphism*

$$U(n)/O(n) \ \cong \ Sp(2n)/ESp(2n)$$

to the set $Sp(2n)/ESp(2n)$ *of left cosets* $A.ESp(2n) \subset Sp(2n)$*, with inverse* $A \mapsto U_A$*. The function*

$$Sp(2n)/ESp(2n) \to \Lambda(n) \ ; \ A.ESp(2n) \mapsto A(\mathbb{R}^n \oplus \{0\})$$

is a diffeomorphism, with

$$Sp(2n)/ESp(2n) \to S^1 \ ; \ A.ESp(2n) \mapsto \det(A)^2$$

inducing the Maslov index isomorphism

$$\pi_1(Sp(2n)/ESp(2n)) \xrightarrow{\cong} \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

$\square$

Proposition 6.8. (Arnold [1])
(i) *Every lagrangian* $L \in \Lambda(n)$ *is the image*

$$L \ = \ A(\mathbb{R}^n \oplus 0) \subset \mathbb{C}^n$$

for some $A \in U(n)$. *The function*

$$U(n)/O(n) \to \Lambda(n) \ ; \ A \mapsto A(\mathbb{R}^n \oplus 0)$$

is a diffeomorphism.
(ii) *The function*

$$\mathrm{Mas} \ : \ \Lambda(n) \to S^1 \ ; \ L = A(\mathbb{R}^n \oplus 0) \mapsto \det(AA^t) = \det(A)^2$$

induces the **Maslov index** *isomorphism*

$$\mathrm{Mas}_* \ : \ \pi_1(\Lambda(n)) \xrightarrow{\cong} \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

The pullback of the universal cover

$$\mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is the universal cover of $\Lambda(n)$

$$\begin{aligned} q \ : \ \widetilde{\Lambda}(n) \ &= \ \mathrm{Mas}^*\mathbb{R} \ = \ \{(L,x) \in \Lambda(n) \times \mathbb{R} \,|\, \det(AA^t) = e^{2\pi i x}\} \\ &\qquad\qquad \to \Lambda(n) \ ; \ (L,x) \mapsto L \ . \end{aligned}$$

(iii) *The function*

$$\Lambda(n) \ = \ U(n)/O(n) \to U(n) \ ; \ A = (a_{jk}) \mapsto AA^t = (\sum_{\ell=1}^{n} a_{j\ell}a_{k\ell})$$

is an embedding, with image the symmetric unitary $n \times n$ *matrices. The composite*

$$\Lambda(n) \ = \ U(n)/O(n) \longrightarrow U(n) \xrightarrow{\det} S^1$$

induces

$$2 \ : \ \pi_1(\Lambda(n)) \ = \ \mathbb{Z} \to \pi_1(U(n)) \ \cong \ \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

□

Definition 6.9. (i) The **Maslov index** of a path $\alpha : I \to \Lambda(n)$ is

$$\mathrm{Mas}_*(\alpha) \ = \ p_2\widetilde{\alpha}(1) - p_2\widetilde{\alpha}(0) \in \mathbb{R}$$

for any lift $\widetilde{\alpha} : I \to \widetilde{\Lambda}(n)$ with

$$\begin{aligned} p_2 \ &: \ \widetilde{\Lambda}(n) \to \mathbb{R} \ ; \ (L,x) \mapsto x \ , \\ \mathrm{Mas}\,\alpha(x) \ &= \ L(\pi\widetilde{\alpha}(x)) \in S^1 = \Lambda(1) \ . \end{aligned}$$

(ii) The **Maslov index** of a loop $\omega : S^1 \to \Lambda(n)$ is the integer-valued Maslov index of the closed path

$$\alpha \ : \ I \to \Lambda(n) \ ; \ x \mapsto \omega(e^{2\pi i x}) \ ,$$

that is

$$\mathrm{Mas}_*(\omega) \;=\; \mathrm{Mas}_*(\alpha) \;=\; \widetilde{\alpha}(1)-\widetilde{\alpha}(0)\in\mathbb{Z}\subset\mathbb{R}$$

for any lift $\widetilde{\alpha}: I\to\mathbb{R}$ of α with

$$\mathrm{Mas}\,\omega(e^{2\pi ix}) \;=\; \mathrm{Mas}\,\alpha(x) \;=\; L(\pi\widetilde{\alpha}(x))\in S^1=\Lambda(1)\ .$$

□

Example 6.10. For $n=1$

$$U(1) \;=\; \{A(\theta)\}\subset O(2)$$

with

$$A(\theta) \;=\; \begin{pmatrix}\cos\,\theta & -\sin\,\theta\\ \sin\,\theta & \cos\,\theta\end{pmatrix}\in O(2)\ (0\leqslant\theta<2\pi)$$

such that

$$A(\theta)(\mathbb{R}\oplus\{0\}) \;=\; L(\theta) \;=\; \{(x\cos\,\theta, x\sin\,\theta)\,|\,x\in\mathbb{R}\}\subset\mathbb{R}\oplus\mathbb{R}\ .$$

a lagrangian of $H_-(\mathbb{R})$, and

$$O(1) \;=\; \{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\}\ .$$

The function

$$U(1)/O(1)\to\Lambda(1)\ ;\ A(\theta)\mapsto L(\theta)$$

is a diffeomorphism.
(i) The function

$$Sp(2)\to U(1)\times\mathbb{R}^2\ ;\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto\Big(\begin{pmatrix}a/\sqrt{a^2+c^2} & b/\sqrt{b^2+d^2}\\ c/\sqrt{a^2+c^2} & d/\sqrt{b^2+d^2}\end{pmatrix}, (\sqrt{a^2+c^2},\sqrt{b^2+d^2})\Big)$$

is a diffeomorphism.
(ii) The functions

$$\lambda:\ S^1\to\Lambda(1)\ ;\ e^{2\pi ix}\mapsto L(\pi x)\ ,$$
$$\mathrm{Mas}\ :\ \Lambda(1)\to S^1\ ;\ L(\theta)\mapsto e^{2i\theta}$$

are inverse diffeomorphisms, with

$$\mathrm{Mas}_*\ :\ \pi_1(\Lambda(1))\xrightarrow{\cong}\pi_1(S^1) \;=\; \mathbb{Z}\ .$$

In other words, every 1-dimensional subspace of $H_-(\mathbb{R})$ is a lagrangian, and $\Lambda(1)=\mathbb{RP}^1$.
(iii) The pullback along $\mathrm{Mas}:\Lambda(1)\to S^1$ of the universal cover of S^1

$$\mathbb{R}\to S^1\ ;\ x\mapsto e^{2\pi ix}$$

is the universal cover of $\Lambda(1)$

$$\begin{aligned}\widetilde{\Lambda}(1) \ &= \ \{(L,x) \in \Lambda(1)\times\mathbb{R}\,|\,L = L(\pi x)\in\Lambda(1)\} \\ &\qquad\qquad\qquad \to \Lambda(1)\ ;\ (L,x)\mapsto L(\pi x)\ .\end{aligned}$$

The projection

$$p_2\ :\ \widetilde{\Lambda}(1)\to\mathbb{R}\ ;\ (L,x)\mapsto x$$

is a diffeomorphism with inverse

$$(p_2)^{-1}\ :\ \mathbb{R}\to\widetilde{\Lambda}(1)\ ;\ x\mapsto (L(\pi x),x)$$

and there is defined a commutative square

$$\begin{tikzcd} \widetilde{\Lambda(1)} \arrow[r, "p_2", "\cong"'] \arrow[d] & \mathbb{R} \arrow[d] \\ \Lambda(1) \arrow[r, "\mathrm{Mas}", "\cong"'] & S^1 \end{tikzcd}$$

(iii) The Maslov index of a path $\alpha : I\to\Lambda(1)$ is

$$\mathrm{Mas}_*(\alpha)\ =\ \widetilde{\alpha}(1)-\widetilde{\alpha}(0)\in\mathbb{R}$$

for any lift $\widetilde{\alpha} : I\to\mathbb{R}$ with

$$\alpha(x)\ =\ L(\pi\widetilde{\alpha}(x))\in\Lambda(1)\ .$$

(iv) The Maslov index of a loop $\omega : S^1\to\Lambda(1)$ is the integer-valued Maslov index of the closed path

$$\alpha\ :\ I\to\Lambda(1)\ ;\ x\mapsto\omega(e^{2\pi ix})\ ,$$

that is

$$\mathrm{Mas}_*(\omega)\ =\ \mathrm{Mas}_*(\alpha)\ =\ \widetilde{\alpha}(1)-\widetilde{\alpha}(0)\in\mathbb{Z}\subset\mathbb{R}$$

for any lift $\widetilde{\alpha} : I\to\mathbb{R}$ of α with

$$\omega(e^{2\pi ix})\ =\ \alpha(x)\ =\ L(\pi\widetilde{\alpha}(x))\in\Lambda(1)\ .$$

(v) The loop of (i)

$$\lambda\ :\ S^1\to\Lambda(1)\ ;\ e^{2\pi ix}\mapsto L(\pi x)$$

has lift

$$\widetilde{\lambda}\ :\ [0,1]\to\mathbb{R}\ ;\ x\mapsto x$$

and Maslov index

$$\mathrm{Mas}_*(\lambda)\ =\ 1\in\mathbb{Z}\ .$$

(vi) For any $x_0,x_1\in\mathbb{R}$ the path from $L(x_0)$ to $L(x_1)\in\Lambda(1)$ defined by

$$\alpha\ :\ I\to\Lambda(1)\ ;\ t\mapsto L((1-t)\pi x_0+t\pi x_1)$$

has Maslov index

$$\mathrm{Mas}_*(\alpha) \;=\; x_1 - x_0 \in \mathbb{R}\ .$$

□

The symplectic group $Sp(2n)$ is a closed subgroup of $GL(2n,\mathbb{R})$. The unitary group is a maximal compact subgroup $U(n) \subset Sp(2n)$ such that

$$Sp(2n) \;=\; U(n) \times \mathbb{R}^{n(n+1)}\ ,$$

with the inclusion $U(n) \to Sp(2n)$ inducing a homotopy equivalence $BU(n) \to BSp(2n)$. The generator

$$1 \;=\; c_1(\lambda_n) \in H^2(BSp(2n)) \;=\; H^2(BU(n)) \;=\; \mathbb{Z}$$

is the first Chern class of the canonical $\mathbb{C}^n$-bundle

$$\mathbb{C}^n \to \lambda_n \;=\; \mathbb{C}^n \times_{U(n)} EU(n) \to BU(n) \;=\; EU(n)U(n)\ .$$

The pullback under the identity $j : Sp(2n)^\delta \to Sp(2n)$

$$u \;=\; j^*(1) \in H^2(BSp(2n)^\delta)$$

is the first Chern class of the pullback $\mathbb{C}^n$-bundle

$$\mathbb{C}^n \to j^*\lambda_n \;=\; \mathbb{C}^n \times_{Sp(2n)^\delta} ESp(2n)^\delta \to BSp(2n)^\delta = ESp(2n)^\delta/Sp(2n)^\delta\ ,$$

corresponding to the universal central group extension

$$0 \to \mathbb{Z} \to \widetilde{Sp(2n)}^{\,\delta} \to Sp(2n)^\delta \to \{1\}$$

determined by the universal cover

$$\mathbb{Z} \longrightarrow \widetilde{Sp(2n)} \xrightarrow{\;q\;} Sp(2n)\ \ ,$$

with

$$\widetilde{Sp(2n)} \;=\; \{(A,\theta) \in Sp(2n) \times \mathbb{R} \,|\, \det(A) = e^{i\theta} \in S^1\}\ .$$

BUT $Sp(2n)$ IS PERFECT, SO $\det_{\mathbb{R}} : Sp(2n) \to \{+1\} \subset \mathbb{Z}$.
HOWEVER, $U(n)$ IS NOT PERFECT (E.G. $U(1) = S^1$ IS ABELIAN), AND $\det_{\mathbb{C}} : U(n) \to S^1$ IS ONTO.

Remark 6.11. (i) The form $T(A,B)$ in Proposition 4.18 was first obtained by Meyer [32, 33], with the signature function

$$\sigma \;:\; Sp(2n)^\delta \times Sp(2n)^\delta \to \mathbb{Z}\ ;\ (A,B) \mapsto \sigma(T(A,B^{-1}))$$

a cocycle for a cohomology class $\sigma \in H^2(BSp(2n)^\delta)$ with $Sp(2n)^\delta = Aut\, H_-(\mathbb{R}^n)$. The signature of an orientable surface bundle

$$F^2 \to E^4 \to B^2$$

is given by

$$\sigma(E) \;=\; \langle \sigma, \rho_*[B]\rangle \in L^4(\mathbb{Z}) \;=\; \mathbb{Z}$$

with $[B] \in H_2(B)$ the fundamental class and

$\rho \,:\, B \to BAut(F) \to BAut(H^1(F;\mathbb{R}),\phi) \;=\; BSp(2n)^\delta \; (n = \dim_{\mathbb{R}} H^1(F;\mathbb{R})/2)$.

(ii) Turaev [56] identified

$$\begin{aligned}\sigma \;=\; 4u \in \mathrm{im}(H^2(BSp(2n)) \to H^2(BSp(2n)^\delta)) \\ \cap \ker(q^* : H^2(BSp(2n)^\delta) \to H^2(B\widetilde{Sp(2n)}^\delta))\end{aligned}$$

with the Maslov class $u \in H^2(BSp(2n)^\delta))$ the image of the generator $1 \in H^2(BSp(2n)) = \mathbb{Z}$, and constructed a function

$$\Phi \;:\; \widetilde{Sp(2n)} \to \mathbb{Z}$$

such that

$$\sigma(q(\widetilde{A}), q(\widetilde{B})) \;=\; \Phi(\widetilde{A}) + \Phi(\widetilde{B}) - \Phi(\widetilde{A}\widetilde{B}) \in \mathbb{Z}\ .$$

The signature of an orientable surface bundle $F^2 \to E^4 \to B^2$ is thus

$$\sigma(E) \;\equiv\; 0 \mod 4$$

a special case of the mod 4 multiplicativity of the signature of fibre bundles obtained (much later) by Hambleton, Korzeniewski and Ranicki [18]. The universal cover

$$\mathbb{Z} \longrightarrow \widetilde{U(n)} \longrightarrow U(n)$$

is given by

$$\widetilde{U(n)} \;=\; \{(A,\theta) \in U(n) \times \mathbb{R} \,|\, \det(A) = e^{i\theta} \in S^1\}\ .$$

The composite

$$\widetilde{U(n)} \longrightarrow \widetilde{Sp(2n)} \xrightarrow{\Phi} \mathbb{Z}$$

is given by

$$\begin{aligned}\Phi(A,\theta) \;&=\; 4\sum_{j=1}^{n}(\theta_j/2\pi - ((\theta_j/2\pi))) \\ &=\; 4\sum_{j=1}^{n} E(\theta_j/2\pi) \in \mathbb{Z} \subset \mathbb{R}\end{aligned}$$

with $e^{i\theta_j} \in S^1$ the eigenvalues of A, θ_j chosen so that

$$\theta \;=\; \sum_{j=1}^{n} \theta_j \in \mathbb{R}\ ,$$

and E as in Barge and Ghys [5, 3.8] (cf. Remark 1.4). In particular, for $n=1$

$$\begin{aligned}\Phi(e^{i\theta},\theta) &= 4E(\theta/2\pi)\\ &= 2\theta/\pi - 4((\theta/2\pi))\\ &= \begin{cases} 4[\theta/2\pi]+2 & \text{if } \theta/2\pi\in\mathbb{R}\backslash\mathbb{Z}\\ 2\theta/\pi & \text{if } \theta/2\pi\in\mathbb{Z}\ .\end{cases}\end{aligned}$$

□

Remark 6.12. (i) The classical Dedekind sum is defined for a coprime pair of integers (a,c) with $c\neq 0$ to be

$$\begin{aligned}s(a,c) &= \frac{1}{4|c|}\sum_{k=1}^{|c|-1}\cot\left(\frac{k\pi}{c}\right)\cot\left(\frac{ka\pi}{c}\right)\\ &= \sum_{k=1}^{|c|-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ka}{c}\right)\right)\in\mathbb{Q}\ .\end{aligned}$$

(ii) The Rademacher ϕ-function is defined by

$$\begin{aligned}\phi\ :\ Sp(2,\mathbb{Z}) &= SL(2,\mathbb{Z})\to\mathbb{Z}\ ;\\ A &= \begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{cases}\dfrac{b}{d} & \text{if } c=0\\ \dfrac{a+d}{c}-12\mathrm{sgn}(c)s(a,c) & \text{if } c\neq 0\ .\end{cases}\end{aligned}$$

Define also

$$\begin{aligned}&\nu\ :\ SL(2,\mathbb{Z})\to\mathbb{Z}\ ;\\ &A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{cases}\mathrm{sgn}(b) & \text{if } c=0 \text{ and } a=1\\ \mathrm{sgn}(c(a+d-2)) & \text{otherwise}\ .\end{cases}\end{aligned}$$

The function

$$\varphi\ :\ SL(2,\mathbb{Z})\to\mathbb{Q}\ ;\ A\mapsto -\phi(A)/3+\nu(A)$$

is then such that

$$\begin{aligned}\sigma(U(A,B)) &= \varphi(A)+\varphi(B)+\varphi((AB)^{-1})\\ &= \nu(A)+\nu(B)-\nu(AB)-\mathrm{sgn}(c_Ac_Bc_{AB})\in\mathbb{Z}\subset\mathbb{Q}\end{aligned}$$

(Meyer [33], Kirby and Melvin [22], Barge and Ghys [5]). □

Proposition 6.13. (i) *For any $A\in U(n)$ the image*

$$L = A(\mathbb{R}^n\oplus\{0\})\subset\mathbb{R}^n\oplus i\mathbb{R}^n = \mathbb{C}^n$$

is a lagrangian of $H_-(\mathbb{R}^n)$ *and*

$$\ker(I_n - AA^t : \mathbb{C}^n \to \mathbb{C}^n) \;=\; (L\cap(\mathbb{R}^n\oplus\{0\}))\oplus i(L\cap(\mathbb{R}^n\oplus\{0\})) \subseteq \mathbb{C}^n \ ,$$

so that

$$\dim_{\mathbb{C}}(\ker(I_n - AA^t)) \;=\; \dim_{\mathbb{R}}(L\cap(\mathbb{R}^n\oplus\{0\})) \ .$$

The lagrangian L *is such that* $L\cap(\mathbb{R}^n\oplus\{0\}) = \{0\}$ *if and only if* $I_n - AA^t$ *is invertible.*

(ii) *For any* $A, A' \in U(n)$ *let*

$$L \;=\; A(\mathbb{R}^n\oplus\{0\}) \ , \ L' \;=\; A'(\mathbb{R}^n\oplus\{0\}) \subset \mathbb{R}^n\oplus i\mathbb{R}^n \;=\; \mathbb{C}^n \ .$$

The linear map

$$\ker(A'(A')^t - AA^t) \to (L\cap L')\oplus i(L\cap L') \ ; \ x \mapsto AA^t(x) = A'(A')^t(x)$$

is an isomorphism of complex vector spaces, so that

$$\dim_{\mathbb{C}}(\ker(A'(A')^t - AA^t)) \;=\; \dim_{\mathbb{R}}(L\cap L') \ .$$

The lagrangians L, L' *are such that* $L\cap L' = \{0\}$ *if and only if* $A'(A')^t - AA^t$ *is invertible.*

(iii) *The eigenvalues of* $A \in U(n)$ *are on the unit circle* S^1

$$e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n} \in S^1 \subset \mathbb{C} \ .$$

A *is invertible, with*

$$\det(A) \;=\; e^{i(\sum\limits_{j=1}^{n}\theta_j)} \in S^1 \ .$$

(iv) A *is diagonalizable: there exists a unitary matrix* $B \in U(n)$ *with columns an orthonormal basis for* $\mathbb{C}^n$ *consisting of eigenvectors for* A*, and*

$$A \;=\; BD(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})B^{-1} \in U(n)$$

with

$$\begin{aligned} L\cap(\mathbb{R}^n\oplus\{0\}) \;&\cong\; \sum_{j=1}^{n}(\mathbb{R}(\cos(\theta_j),\sin(\theta_j))\cap(\mathbb{R}\oplus\{0\})) \ , \\ \dim_{\mathbb{R}}(L\cap(\mathbb{R}^n\oplus\{0\})) \;&=\; \dim_{\mathbb{C}}(\ker(I_n - AA^t : \mathbb{C}^n \to \mathbb{C}^n)) \\ &=\; (\textit{no. of } \theta_j\textit{'s with } e^{i\theta_j} = \pm 1) \ . \end{aligned}$$

(v) *The logarithm of* $A \in U(n)$

$$\log(A) \;=\; BD(\log(e^{i\theta_1}), \log(e^{i\theta_2}), \dots, \log(e^{i\theta_n}))B^{-1} \in M_n(\mathbb{C})$$

has trace

$$\mathrm{tr}(\log(A)) \;=\; \sum_{j=1}^{n}\log(e^{i\theta_j}) \;=\; \pi i\sum_{j=1}^{n}(1 - 2\{\frac{\pi - \theta_j}{2\pi}\}) \in \mathbb{C} \ .$$

(vi) $A^2 \in U(n)$ *has*

$$\begin{aligned} A^2 &= BD(e^{2i\theta_1}, e^{2i\theta_2}, \dots, e^{2i\theta_n})B^{-1} , \\ I_n - A^2 &= BD(1-e^{2i\theta_1}, 1-e^{2i\theta_2}, \dots, 1-e^{2i\theta_n})B^{-1} \in U(n) , \\ \det(A^2) &= \det(A)^2 = e^{2i(\sum\limits_{j=1}^n \theta_j)} \in S^1 , \\ \mathrm{tr}(\log(A^2)) &= \sum_{j=1}^n \log(e^{2i\theta_j}) = \pi i \sum_{j=1}^n (1 - 2\{\frac{\pi - 2\theta_j}{2\pi}\}) \in \mathbb{C} . \end{aligned}$$

(vii) A^2 *and* $AA^t \in U(n)$ *are such that*

$$\begin{aligned} \det(AA^t) &= \det(A^2) = e^{2i(\sum\limits_{j=1}^n \theta_j)} \in S^1 , \\ \dim_{\mathbb{C}}(\ker(I_n - AA^t : \mathbb{C}^n \to \mathbb{C}^n)) &= \dim_{\mathbb{C}}(\ker(I_n - A^2 : \mathbb{C}^n \to \mathbb{C}^n)) \\ &= (\textit{no. of } \theta_j\textit{'s with } e^{i\theta_j} = \pm 1) \in \mathbb{Z} , \\ \mathrm{tr}(\log(AA^t)) = \mathrm{tr}(\log(A^2)) &= \pi i \sum_{j=1}^n (1 - 2\{\frac{\pi - 2\theta_j}{2\pi}\}) \in \mathbb{C} . \end{aligned}$$

Proof. (i) Write $A = X + iY$ with $X, Y \in M_n(\mathbb{R})$, so that

$$\begin{aligned} A^{-1} &= A^* = X^t - iY^t , \ A^t = X^t + iY^t , \\ \overline{A} &= (A^t)^{-1} = X - iY \in U(n) , \\ I_n - AA^t &= A(A^* - A^t) = -2iAY^t \in M_n(\mathbb{C}) . \end{aligned}$$

It follows from

$$\ker(X) \cap \ker(Y) = \{0\} , \ Y^tX = -X^tY , \ \dim_{\mathbb{R}}\ker(Y) = \dim_{\mathbb{R}}\ker(Y^t)$$

that the linear map of real vector spaces

$$\ker(Y) \to \ker(Y^t) \ ; \ w \mapsto X(w)$$

is an isomorphism. Thus

$$\begin{aligned} L &= A(\mathbb{R}^n \oplus \{0\}) = \{X(w) + iY(w) \,|\, w \in \mathbb{R}^n\} \subset \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n , \\ L \cap (\mathbb{R}^n \oplus \{0\}) &= X(\ker(Y)) = \ker(Y^t) \subset \mathbb{C}^n , \\ \ker(I_n - AA^t) &= \ker(Y^t) \oplus i\ker(Y^t) \subset \mathbb{C}^n . \end{aligned}$$

(ii) Let

$$A'' = (A')^{-1}A \in U(n) , \ L'' = A''(\mathbb{R}^n \oplus \{0\}) .$$

The linear map

$$L \cap L' \to L'' \cap (\mathbb{R}^n \oplus \{0\}) \ ; \ x \mapsto (A')^{-1}(x)$$

is an isomorphism of real vector spaces, and by (i)

$$\ker(I_n - A''(A'')^t : \mathbb{C}^n \to \mathbb{C}^n) = (L'' \cap (\mathbb{R}^n \oplus \{0\}) \oplus i(L'' \cap (\mathbb{R}^n \oplus \{0\}) \subseteq \mathbb{C}^n .$$

It follows from the identity

$$I_n - A''(A'')^t = (A')^{-1}(A'(A')^t - AA^t)((A')^t)^{-1}$$

that the linear map

$$\begin{aligned}(L'' \cap (\mathbb{R}^n \oplus \{0\})) \oplus i(L'' \cap (\mathbb{R}^n \oplus \{0\}) &= \ker(I_n - A''(A'')^t) \\ &\to \ker(A'(A')^t - AA^t) \; ; \; x \mapsto ((A')^t)^{-1}(x) .\end{aligned}$$

is an isomorphism of complex vector spaces.

(iii) For any $z, w \in \mathbb{C}^n$

$$\langle A(z), A(w) \rangle = \langle z, w \rangle \in \mathbb{C} .$$

If $z \neq 0 \in \mathbb{C}^n$ is an eigenvector with eigenvalue $\lambda \in \mathbb{C}$ then $A(z) = \lambda z$ and

$$\langle A(z), A(z) \rangle = \lambda\overline{\lambda}\langle z, z \rangle = \langle z, z \rangle \in \mathbb{C}$$

so that $\lambda\overline{\lambda} = 1 \in \mathbb{C}$.

(iv) Choose an eigenvector $b_1 \neq 0 \in \mathbb{C}^n$, so that A is an automorphism of the positive definite hermitian form

$$(\mathbb{C}^n, \langle\ ,\ \rangle) = (\mathbb{C}b_1, \langle\ ,\ \rangle) \oplus ((\mathbb{C}b_1)^{\perp}, \langle\ ,\ \rangle)$$

with

$$(\mathbb{C}b_1)^{\perp} = \{z \in \mathbb{C}^n \,|\, \langle z, b_1 \rangle = 0\} .$$

Now proceed by induction, obtaining an orthonormal basis $\{b_1, b_2, \dots, b_n\}$ for $\mathbb{C}^n$ consisting of eigenvectors of A, and $B = (b_1\ b_2\ \dots\ b_n) \in U(n)$ such that

$$B^{-1}AB = D(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in U(n) .$$

(v) Immediate from (iv).

(vi) Immediate from $A^2 = BD(e^{2i\theta_1}, e^{2i\theta_2}, \dots, e^{2i\theta_n})B^{-1}$ and (iv).

(vii) It follows from

$$e^{\mathrm{tr}(\log(AA^t))} = \det(AA^t) = \det(A^2) = e^{\mathrm{tr}(\log(A^2))} \in \mathbb{C}\backslash\{0\}$$

that

$$\mathrm{tr}(\log(AA^t)) - \mathrm{tr}(\log(A^2)) \in 2\pi i\mathbb{Z} \subset \mathbb{C}$$

As in [10, Proposition 6.3] choose a path

$$B : I \to U(n) \; ; \; s \mapsto B(s)$$

from $B(0) = I_n$ to $B(1) = B$, so that

$$A : I \to U(n) \; ; \; s \mapsto B(s)D(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})B(s)^{-1}$$

is a path from $A(0) = D(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ to the given $A(1) = A \in U(n)$. For $s \in I$ let

$$\begin{aligned}\Sigma_1(s) &= \text{spectrum}(A(s)A(s)^t)\ ,\\ \Sigma_2(s) &= \text{spectrum}(A(s)^2) \;=\; \{e^{2i\theta_1}, e^{2i\theta_2}, \dots, e^{2i\theta_n}\} \subset S^1\end{aligned}$$

By (vi)

$$\begin{aligned}\dim_{\mathbb{C}}(\ker(I_n - A(s)A(s)^t)) &= |\Sigma_1(s) \cap \{1\}|\\ &= \dim_{\mathbb{C}}(\ker(I_n - A(s)^2)) \;=\; |\Sigma_2(s) \cap \{1\}|\ .\end{aligned}$$

Since $\Sigma_1(s), \Sigma_2(s)$ vary continuously with s the function

$$h \;:\; I \to 2\pi i\mathbb{Z} \subset \mathbb{C}\ ;\ s \mapsto \text{tr}(\log(A(s)A(s)^t)) - \text{tr}(\log(A(s)^2))$$

is continuous, and hence constant with

$$\text{tr}(\log(AA^t)) - \text{tr}(\log(A^2)) \;=\; h(1) \;=\; h(0) \;=\; 0 \in 2\pi i\mathbb{Z}\ .$$

□

Definition 6.14. (i) The **η-invariant** function is

$$\begin{aligned}&\eta \;:\; U(n) \to \mathbb{R}\ ;\ A \mapsto\\ &\eta(A) = \frac{1}{\pi i}(\text{tr}(\log(-A^{-2}))) - \dim_{\mathbb{C}}(\ker(I_n - A^2 : \mathbb{C}^n \to \mathbb{C}^n))\ .\end{aligned}$$

If $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}$ are eigenvalues of A then

$$\eta(A) \;=\; \sum_{j=1}^{n} \eta(\theta_j) \;=\; \sum_{j=1}^{n}(-2\theta_j/\pi + [\theta_j/\pi] - [-\theta_j/\pi]) \in \mathbb{R}\ .$$

(ii) The **$\widetilde{\eta}$-invariant** function is

$$\begin{aligned}&\widetilde{\eta} \;:\; \widetilde{U(n)} \to \mathbb{Z}\ ;\ (A, \theta) \mapsto\\ &\widetilde{\eta}(A, \theta)\ ??? = \frac{1}{\pi i}(\text{tr}(\log(-A^{-2}))) - \dim_{\mathbb{C}}(\ker(I_n - A^2 : \mathbb{C}^n \to \mathbb{C}^n))\ (\det(A) = e^{i\theta})\end{aligned}$$

If $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}$ are eigenvalues of A and

$$\sum_{j=1}^{n} \theta_j \;=\; \theta \in \mathbb{R}$$

then

$$\widetilde{\eta}(A, \theta) \;=\; 2\sum_{j=1}^{n} E(\theta_j/\pi) \;=\; \sum_{j=1}^{n}([\theta_j/\pi] - [-\theta_j/\pi]) \in \mathbb{Z}\ .$$

□

Example 6.15. For any $\theta \in \mathbb{R}$ the 1-dimensional lagrangian

$$L(\theta) \; = \; \mathbb{R}(\cos(\theta), \sin(\theta)) \in \Lambda(1)$$

is such that $L(\theta) = A(\mathbb{R} \oplus \{0\})$ with $A = (e^{i\theta}) \in U(1)$. The η-invariant is

$$\begin{aligned} \eta(L(\theta)) \; &= \; \eta(A) \\ &= \begin{cases} \dfrac{1}{\pi i}(\log(-e^{-2i\theta})) = 1 - 2\{\theta/\pi\} & \text{if } \theta/\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } \theta/\pi \in \mathbb{Z} \end{cases} \\ &= \; -2((\theta/\pi)) \; = \; \eta(\theta) \in \mathbb{R} \end{aligned}$$

□

Remark 6.16. (Atiyah, Patodi and Singer [4, p. 411])
The ρ-invariant of S^1 with respect to the 1-dimensional unitary representation

$$\alpha \; : \; \pi_1(S^1) \; = \; \mathbb{Z} \to U(1) \; = \; S^1 \; ; \; 1 \mapsto e^{i\theta}$$

is

$$\rho_\alpha(S^1) \; = \; \eta(\theta/2) \; = \; \begin{cases} 1 - 2\{\theta/2\pi\} & \text{if } \theta/2\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } \theta/2\pi \in \mathbb{Z} \end{cases} \in \mathbb{R} \; .$$

Let ∇_θ $(0 \leqslant \theta < 2\pi)$ be an analytic family of flat connections on the trivial line bundle over S^1, such that the induced family of monodromy representations is given by α. The η-invariant of the corresponding operator of [4]

$$-i * \nabla_\theta \; : \; C^\infty \to C^\infty$$

is also given by

$$\eta(-i * \nabla_\theta) \; = \; \eta(\theta/2) \in \mathbb{R} \; .$$

(See also Farber and Levine [16, §8]).

□

Proposition 6.17. (i) *If $A \in U(n)$ has eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n} \in S^1$ then*

$$\begin{aligned} &\dim_{\mathbb{C}}(\ker(I_n - A^2 : \mathbb{C}^n \to \mathbb{C}^n)) \\ &\qquad\qquad = \; \textit{no. of eigenvalues with } e^{i\theta_j} = \pm 1 \; , \\ &\eta(A) \; = \; \sum_{j=1}^{n} \eta(\theta_j) \in \mathbb{R} \; . \end{aligned}$$

Define

$$\theta'_j = \begin{cases} \theta_j & \text{if } 0 < \theta_j < \pi \\ \pi + \theta_j & \text{if } -\pi < \theta_j < 0 \\ \pi/2 & \text{if } \theta_j = 0 \text{ or } \pi \ , \end{cases}$$

so that $0 \leqslant \theta'_j < \pi$ *and*

$$\begin{aligned} \eta(\theta_j) &= 1 - 2\{\frac{\theta_j}{\pi}\} = \eta(\theta'_j) = 1 - \frac{2\theta'_j}{\pi} \ , \\ \eta(A) &= \sum_{j=1}^{n} \eta(\theta'_j) = \sum_{j=1}^{n} (1 - \frac{2\theta'_j}{\pi}) \in \mathbb{R} \ . \end{aligned}$$

(ii) *The eigenvalues* $e^{i\theta_j} = \pm 1, \pm i$ *contribute* $\eta(\theta_j) = 0$ *to* $\eta(A)$.
(iii) *A complex conjugate pair* $\{e^{i\theta}, e^{-i\theta}\}$ *of eigenvalues contributes* $\eta(\theta) + \eta(-\theta) = 0 \in \mathbb{R}$ *to* $\eta(A)$. *Thus* $\eta(A) = 0$ *for* $A \in O(n) \subset U(n)$.
(iv) *The inverse* $A^{-1} = A^* \in U(n)$ *has eigenvalues* $e^{-i\theta_j}$, *so*

$$\eta(A^{-1}) = \eta(A^*) = \sum_{j=1}^{n} \eta(-\theta_j) = \sum_{j=1}^{n} -\eta(\theta_j) = -\eta(A) \in \mathbb{R} \ .$$

(v) *The* η*-invariant of* $A \in U(n)$ *depends only on the lagrangian* $L = A(\mathbb{R}^n \oplus \{0\}) \in \Lambda(n)$, *allowing the definition*

$$\eta \ : \ U(n)/O(n) = \Lambda(n) \to \mathbb{R} \ ; \ L = A(\mathbb{R}^n \oplus \{0\}) \mapsto \eta(L) = \eta(A) \ .$$

(vi) *The* η*-invariant is such that*

$$\eta(L) = \eta(A) = \frac{1}{\pi i}(\mathrm{tr}(\log(-(A^tA)^{-1})) - \dim_{\mathbb{R}}(L \cap (\mathbb{R}^n \oplus \{0\}))$$

for any $L \in \Lambda(n)$, $A \in U(n)$ *with* $L = A(\mathbb{R}^n \oplus 0)$.
(vii) *For any* $A_1, A_2 \in U(n)$

$$\eta(A_1) + \eta(A_2) - \eta(A_1A_2) = \sigma(L_0, L_1, L_2) \in \mathbb{Z} \subset \mathbb{R}$$

is the triple signature of the lagrangians

$$L_0 = \mathbb{R}^n \oplus 0 \ , \ L_1 = A_1(\mathbb{R}^n \oplus 0) \ , \ L_2 = A_1A_2(\mathbb{R}^n \oplus 0) \in \Lambda(n) \ .$$

(viii) *For any* $A \in U(n)$

$$2\eta(A) - \eta(A^2) = \sum_{j=1}^{n} \mathrm{sign}(\sin 2\theta_j) \in \mathbb{Z} \subset \mathbb{R}$$

with $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}$ *the eigenvalues of* A.

$\square$

Remark 6.18. (i) The expression of $\eta(A)$ as an integral goes back to the construction due to Souriau [54, p.126] and Lion-Vergne [28, p.96] of the Maslov index of $(L,\theta) \in \widetilde{\Lambda}(n)$ with $L \cap (\mathbb{R}^n \oplus \{0\}) = \{0\}$ as

$$\mathrm{Mas}(L,\theta) \;=\; \frac{1}{2\pi}(\theta + i\mathrm{tr}(\log(-AA^t))) \in \{m/2 \,|\, m \in \mathbb{Z}\} \subset \mathbb{R}$$

for $A \in U(n)$ such that $L = A(\mathbb{R}^n \oplus \{0\})$.
(ii) The definition of $\eta : \Lambda(n) \to \mathbb{R}$ is motivated by the η-invariant of Atiyah-Patodi-Singer [4, p.411]. See Neumann [38, p.150], Atiyah [3, 5.12], Turaev [56, Thm. 5], Cappell-Lee-Miller [10, p.152], Bunke [8, p.420], Lesch and Wojciechowski [26, Theorem 2.1], Kirk and Lesch [23, Theorem 4.4] and Müller [36, Theorem 2.21] etc.
(iii) ([4], [10, pp. 134-135, 151-154]) For any lagrangians $L_1, L_2 \in \Lambda(n)$ there is defined a first-order real self-adjoint elliptic operator $D(L_1,L_2) \;=\; -J\dfrac{d}{dt}$ on the Sobolev completion of the smooth functions $f : [0,1] \to \mathbb{R}^{2n}$ satisfying the boundary conditions $f(0) \in L_1$, $f(1) \in L_2$ with

$$J \;:\; \mathbb{R}^{2n} \;=\; \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n \;;\; (x,y) \mapsto (-y,x)$$

the standard complex structure. Without loss of generality can take

$$L_1 \;=\; \mathbb{R}^n \oplus \{0\} \;,\; L_2 \;=\; \sum_{j=1}^{n} \mathbb{R}e^{i\theta_j} \in \Lambda(n)$$

for some $\theta_1, \theta_2, \dots, \theta_n \in [0,\pi)$. The eigenvectors of $D(L_1,L_2)$ are

$$f_{j,m_j}(t) \;=\; e^{i(\pi m_j + \theta_j)t}(0,\dots,0,x_j,0,\dots,0)\;(1 \leqslant j \leqslant n,\; m_j \in \mathbb{Z},\; x_j \neq 0 \in \mathbb{R})$$

with eigenvalues $\lambda_{j,m_j} = \pi m_j + \theta_j$. The generalized ζ-function $\zeta(s,x)$ is defined for $x \in (0,1]$ by

$$\zeta(s,x) \;=\; \sum_{k=0}^{\infty}(x+k)^{-s}$$

with a unique meromorphic extension to $s=0$ with

$$\zeta(0,x) \;=\; 1/2 - x \;=\; \begin{cases} -((x)) & \text{if } x \in (0,1) \\ -1/2 & \text{if } x = 1 \end{cases} .$$

In terms of $\zeta(s,x)$ the η-invariant of $D(L_1,L_2)$ is determined by

$$\begin{aligned}\eta_{D(L_1,L_2)}(s) &= \sum_{\lambda_{j,m_j}\neq 0}\frac{\operatorname{sgn}(\lambda_{j,m_j})}{|\lambda_{j,m_j}|^s}\\ &= \sum_{m_j>0}\frac{1}{(\pi m_j+\theta_j)^s}-\sum_{m_j<0}\frac{1}{(\pi m_j+\theta_j)^s}\ (0\leqslant\theta_j<\pi)\\ &= \frac{1}{\pi^s}\sum_{j=1}^{n}(\zeta(s,\theta_j/\pi)-\zeta(s,-\theta_j/\pi))\end{aligned}$$

and

$$\begin{aligned}\eta(D(L_1,L_2)) &= \eta_{D(L_1,L_2)}(0)\\ &= -2\sum_{\theta_j}((\frac{\theta_j}{\pi}))\\ &= \sum_{\theta_j\neq 0}(1-\frac{2\theta_j}{\pi}) \ =\ \eta(A)\in\mathbb{R}\ .\end{aligned}$$

More generally, for any lagrangians L_1, L_2 in a symplectic form (K,ϕ) over $\mathbb{R}$ a choice of complex structure J on (K,ϕ) determines a unitary matrix $A\in U(n)$ ($n=\dim_{\mathbb{R}}(K)/2$) such that $A(L_1)=L_2$, and if A has eigenvalues $\pm e^{i\theta_1},\pm e^{i\theta_2},\dots,\pm e^{i\theta_n}$ then

$$\begin{aligned}\eta(D(L_1,L_2)) &= \eta(0) \ =\ \sum_{\theta_j\neq 0}(1-\frac{2\theta_j}{\pi})\\ &= \frac{1}{\pi i}(\operatorname{tr}(\log(-A^{-2})))-\dim_{\mathbb{C}}(\ker(I_n-A^2:\mathbb{C}^n\to\mathbb{C}^n))\\ &= \eta(A)\in\mathbb{R}\ .\end{aligned}$$

(iv) For any $A\in U(n)$ there is also defined a complex self-adjoint operator $D^\sharp(AA^t)=-i\dfrac{d}{dt}$ on the complex vector space of functions $\psi:[0,1]\to\mathbb{C}^n$ satisfying

$$\psi(1)\ =\ AA^t\psi(0)\in\mathbb{C}^n\ ,$$

with the same η-invariant as in A

$$\begin{aligned}\eta(-i\frac{d}{dt}) &= \frac{1}{\pi i}(\operatorname{tr}(\log(-(AA^t)^{-1}))-\dim_{\mathbb{C}}(\ker(I_n-AA^t:\mathbb{C}^n\to\mathbb{C}^n))\\ &= \eta(A)\in\mathbb{R}\ .\end{aligned}$$

□

The η-invariant $\eta(N)\in\mathbb{R}$ was defined by Atiyah, Patodi and Singer [4] for a 3-dimensional Riemannian manifold N. If $N=\partial P$ is the boundary of a 4-dimensional Riemanninan manifold P with a product metric near ∂P then

$$\eta(N) \ = \ \frac{1}{3}\int_N p_1 - \sigma(P) \ .$$

Example 6.19. (Atiyah, Patodi, Singer [4], Morifuji [35]) The η-invariant of the 3-dimensional lens space $L(p,q)$ with respect to the standard metric is

$$\eta(L(p,q)) \ = \ -\frac{1}{p}\sum_{k=1}^{p-1}\cot(\frac{\pi k}{p})\cot(\frac{\pi qk}{p}) \ = \ -4s(q,p) \in \mathbb{R}$$

with $s(x,y)$ the Dedekind sum defined for coprime integers x,y by

$$s(x,y) \ = \ \sum_{j=1}^{y}\big(\big(\frac{xj}{y}\big)\big)\big(\big(\frac{j}{y}\big)\big) \in \mathbb{R} \ .$$

$\square$

Example 6.20. (Atiyah [3, p.172]) Let Σ_n be a closed surface of genus n, with symplectic intersection pairing $(H^1(\Sigma_n;\mathbb{R}),\phi)$. The total space of a fibre bundle

$$\Sigma_n \to T^3 \to S^1$$

is the mapping torus of the monodromy automorphism $A:\Sigma_n \to \Sigma_n$

$$T \ = \ T(A) \ = \ \{\Sigma_n \times I \,|\, (x,0) = (A(x),1)\} \ .$$

The η-invariant of $T(A)$ is

$$\eta(T(A)) \ = \ \eta(A^* : (H^1(\Sigma_n;\mathbb{R}),\phi) \to (H^1(\Sigma_n;\mathbb{R}),\phi)) \in \mathbb{R}$$

with

$$A^* \ : \ H^1(\Sigma_n;\mathbb{R}) \to H^1(\Sigma_n;\mathbb{R}) \ .$$

$\square$

PROBLEM: Suppose given a 3-dimensional Riemannian manifold N with a decomposition $N = N_1 \cup_M N_2$ for a separating hypersurface $M \subset N$, such that the metric on N is a product in a neighbourhood of M. Given a complex structure J on the intersection form $(H^1(M),\phi)$ over $\mathbb{R}$ have lagrangians

$$L_j \ = \ \mathrm{im}(H^1(N) \to H^1(M)) \ (j=1,2) \ .$$

Explain the Lesch and Wojciechowski [26, Theorem 2.1], Bunke [8] Bunke [8, p.420] glueing formula

$$\eta(N) \ = \ \eta(N_1,M) + \eta(N_2,M) + \eta(H^1(M),\phi,J;L_1,L_2) \in \mathbb{R} \ ??$$

If $N = T(A : \Sigma_n \to \Sigma_n)$ take

$$M \;=\; \Sigma_n \cup \Sigma_n \;,\; N_1 \;=\; \Sigma_n \times I \;,\; N_2 \;=\; \Sigma_n \times I$$

A point $\widetilde{L} = (L, x) \in \widetilde{\Lambda}(n)$ of the universal cover of the space $\Lambda(n)$ consists of a lagrangian $L \in \Lambda(n)$ and a real number $x \in \mathbb{R}$ such that

$$\det A^2 \;=\; e^{2ix} \in S^1$$

for any $A \in U(n)$ with $L = A(\mathbb{R}^n \oplus \{0\})$.

Remark 6.21. Souriau [54], Leray [25], Arnold [2], Turaev [56] and de Gosson [14, 15] proved that there is a unique locally constant function

$$m \;:\; \widetilde{\Lambda}(n) \times \widetilde{\Lambda}(n) \to \mathbb{Z}$$

such that

$$m(\widetilde{L}_1, \widetilde{L}_2) + m(\widetilde{L}_2, \widetilde{L}_3) + m(\widetilde{L}_3, \widetilde{L}_1) \;=\; \sigma(L_1, L_2, L_3) \in \mathbb{Z} \;.$$

In the terminology of Remark 6.11

$$\begin{aligned} m((L_1, x_1), (L_2, x_2)) \;&=\; \Phi(A^2, 2(x_1 - x_2)) \\ &=\; 4\sum_{j=1}^{n} E(2\theta_j) \\ &=\; 4\sum_{j=1}^{n} (\theta_j/\pi - ((\theta_j/\pi))) \in \mathbb{Z} \subset \mathbb{R} \end{aligned}$$

with $e^{i\theta_j} \in S^1$ the eigenvalues of $A \in U(n)$ such that $A(L_2) = L_1$, and θ_j chosen so that

$$x_1 - x_2 \;=\; \sum_{j=1}^{n} \theta_j \in \mathbb{R} \;.$$

For $n = 1$

$$\begin{aligned} m((L(x), x), (L(x'), x')) \;&=\; \Phi(e^{2i(x-x')}, x - x') \\ &=\; E(x - x') \\ &=\; ((x - x')/\pi - (((x - x')/\pi))) \\ &=\; [(x - x')/\pi] \in \mathbb{Z} \;. \end{aligned}$$

?????

$$E \;:\; \mathbb{R} \to \mathbb{R} \;;\; x \mapsto \begin{cases} [x] + 1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ x & \text{if } x \in \mathbb{Z} \end{cases}$$

$\square$

Definition 6.22. The lagrangians $L_1, L_2 \in \Lambda(n)$ are **transverse** if

$$L_1 \cap L_2 \ = \ \{0\} \subset \mathbb{R}^n \oplus \mathbb{R}^n \ ,$$

i.e. if L_1, L_2 are direct complements in $\mathbb{R}^n \oplus \mathbb{R}^n$.

□

For transverse $L_1, L_2 \in \Lambda(n)$ there is defined a nonsingular symmetric form (L_1, ϕ_1) over $\mathbb{R}$ such that

$$\sigma(\mathbb{R}^n \oplus \{0\}, L_1, L_2) \ = \ \sigma(L_1, \phi_1) \in \mathbb{Z} \ .$$

and for $\widetilde{L}_1 = (L_1, x_1)$, $\widetilde{L}_2 = (L_2, x_2) \in \widetilde{\Lambda}(n)$

$$m(\widetilde{L}_1, \widetilde{L}_2) \ = \ \sigma(L_1, \phi_1) + (x_1 - x_2)/2\pi \in \mathbb{Z} + 1/2$$

such that for any $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3 \in \widetilde{\Lambda}(n)$

7. Complex structures

Definition 7.1. A **complex structure** on a real vector space K is an automorphism $J : K \to K$ such that

$$J^2 \ = \ -1 \ : \ K \to K \ .$$

Let (K, J) denote the corresponding complex vector space, with underlying real vector space K and $i = J : K \to K$.

□

Proposition 7.2. *The functor*

$$\{\textit{real vector spaces with a complex structure}\} \\ \to \{\textit{complex vector spaces}\} \ ; \ (K, J) \mapsto (K, J)$$

is an equivalence of categories, with $\dim_{\mathbb{C}}(K, J) = \dim_{\mathbb{R}}(K)/2$.

□

Definition 7.3. A **compatible complex structure** on a nonsingular symplectic form (K, ϕ) over $\mathbb{R}$ is a complex structure $J : K \to K$ such that

(i) $J^*\phi J = \phi : K \to K^*$,
(ii) $\phi J = (\phi J)^* : K \to K^*$ is a positive definite symmetric form over $\mathbb{R}$, with $\phi(x, Jx) > 0$ for all $x \neq 0 \in K$.

□

A compatible complex structure on (K, ϕ) is a positive definite symmetric form (K, c) such that $J = c^{-1}\phi : K \to K$ is a complex structure on K.

Proposition 7.4. *The functor*

$$\begin{aligned}&\{\textit{nonsingular symplectic forms over } \mathbb{R} \\ &\qquad\qquad\qquad\qquad \textit{with a compatible complex structure}\} \\ &\to \{\textit{positive definite hermitian forms over } \mathbb{C}\} \ ; \\ &(K, \phi, J) \mapsto ((K, J), (\phi, J)) \ , \ (\phi, J)(x, y) \ = \ \phi(x, Jy) + i\phi(x, y)\end{aligned}$$

is an equivalence of categories.

□

Example 7.5. (i) The standard complex structure

$$J_0 \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ : \ K \ = \ \mathbb{R}^n \oplus \mathbb{R}^n \to K \ = \ \mathbb{R}^n \oplus \mathbb{R}^n$$

is compatible with the standard symplectic form

$$(K, \phi) \ = \ H_-(\mathbb{R}^n) \ = \ (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$

corresponding to the positive definite hermitian form $(\mathbb{C}^n, \langle \ , \ \rangle)$.
(ii) If $J : K \to K$ is a complex structure compatible with (K, ϕ) then $-J_0 J : K \to K$ is an automorphism with a symmetric positive definite symplectic matrix.

□

Proposition 7.6. (McDuff and Salamon [31, Prop. 2.48])
For a fixed nonsingular symplectic form (K, ϕ) the space of compatible complex structures J is contractible, and is homeomorphic to the space of symmetric positive definite symplectic $2n \times 2n$ matrices.

□

Proposition 7.7. *If J is a compatible complex structure on a nonsingular symplectic form (K, ϕ) over $\mathbb{R}$ and L is a lagrangian of (K, ϕ) then JL is a lagrangian complement of L, with an isomorphism of real vector spaces*

$$\phi' \ : \ JL \to L^* \ ; \ Jx \mapsto (y \mapsto \phi(x, Jy))$$

and a positive definite symmetric form

$$\lambda \ = \ \phi J|_L \ : \ L \to L^* \ ; \ x \mapsto (y \mapsto \phi(x, Jy))$$

such that

$$J = \begin{pmatrix} 0 & -\lambda^{-1}\phi' \\ \phi'^{-1}\lambda & 0 \end{pmatrix} : K = L \oplus JL \to K = L \oplus JL ,$$

$$\phi = \begin{pmatrix} 0 & \phi' \\ -(\phi')^* & 0 \end{pmatrix} : K = L \oplus JL \to K^* = L^* \oplus JL^* ,$$

$$\phi J = \begin{pmatrix} \lambda & 0 \\ 0 & (\phi')^*\lambda^{-1}\phi' \end{pmatrix} : K = L \oplus JL \to K^* = L^* \oplus JL^* .$$

The isomorphism of real vector spaces

$$\alpha_L : L \oplus L^* \to K = L \oplus JL ; (x,y) \mapsto (x, \phi'^{-1}(y))$$

defines an isomorphism of nonsingular symplectic forms over $\mathbb{R}$ *with compatible complex structure*

$$\alpha_L : (H_-(L), \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}) \to (K, \phi, J)$$

such that

$$\alpha_L(L) = L , \alpha_L(L^*) = JL .$$

A λ*-orthonormal basis* $\{b_1, b_2, \dots, b_n\}$ *for* L *extends to a* ϕJ*-orthonormal basis* $\{b_1, b_2, \dots, b_n, Jb_1, b_2, \dots, Jb_n\}$ *for* K*, corresponding to an isomorphism of positive definite symmetric forms over* $\mathbb{R}$

$$\beta : (\mathbb{R}^n, \langle\ ,\ \rangle) \to (L, \lambda)$$

and an extension of β *to an isomorphism of positive definite hermitian forms over* $\mathbb{C}$

$$\begin{aligned} &\alpha_{(L,\beta)} = \alpha_L \begin{pmatrix} \beta & 0 \\ 0 & (\beta^*)^{-1} \end{pmatrix} : \\ &(\mathbb{C}^n, \langle\ ,\ \rangle) = (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \to ((K,J),(\phi,J)) \end{aligned}$$

such that

$$\alpha_{(L,\beta)}(\mathbb{R}^n \oplus 0) = L , \alpha_{(L,\beta)}(0 \oplus \mathbb{R}^n) = JL \subset K .$$

(iii) *Let* (K,ϕ) *be a nonsingular symplectic form over* $\mathbb{R}$*, and let* L *be a lagrangian in* (K,ϕ)*. The compatible complex structures* J *on* (K,ϕ) *are in one-one correspondence with positive definite symmetric forms* (L',λ') *over* $\mathbb{R}$ *on lagrangians* L' *complementary to* L*. Given* J *let*

$$(L',\lambda') = (JL, (\phi J)|_{JL}) .$$

Conversely, given (L',λ') *note that the restriction of* ϕ *is an isomorphism*

$$\phi' = \phi| : L' \to L^*$$

such that

$$\phi \;=\; \begin{pmatrix} 0 & \phi' \\ -\phi'^* & 0 \end{pmatrix} \;:\; K^* \;=\; L \oplus L' \to K^* \;=\; L^* \oplus L'^* \;,$$

and

$$J \;=\; \begin{pmatrix} 0 & (-\phi'^*)^{-1}(\lambda')^{-1} \\ \lambda'\phi'^* & 0 \end{pmatrix} \;:\; K \;=\; L \oplus L' \to K \;=\; L \oplus L'$$

defines a compatible complex structure on (K, ϕ) *such that*

$$\phi J \;=\; \begin{pmatrix} \phi'(\lambda')^{-1}\phi'^* & 0 \\ 0 & \lambda' \end{pmatrix} \;:\; K \;=\; L \oplus L' \to K^* \;=\; L^* \oplus L'^* \;.$$

In particular, (K, ϕ) *admits a compatible complex structure* J*.*

□

8. Symplectic and hermitian automorphisms

Definition 8.1. (i) A **symplectic automorphism** (K, ϕ, A) is an automorphism $A : (K, \phi) \to (K, \phi)$ of a nonsingular symplectic form over $\mathbb{R}$, with

$$A^*\phi A \;=\; \phi \;:\; K \to K^*$$

or equivalently

$$\phi(Ax, Ay) \;=\; \phi(x, y) \in \mathbb{R} \;(x, y \in K) \;.$$

(ii) A **hermitian automorphism** (K, ϕ, J, A) is a symplectic automorphism (K, ϕ, A) with a compatible complex structure $J : K \to K$ such that $AJ = JA : K \to K$.

□

Remark 8.2. (i) The symplectic group

$$Sp(2n) \;=\; \{A \in GL_{2n}(\mathbb{R}) \mid A^t\phi A = \phi\}$$

is the group of symplectic automorphisms $(H_-(\mathbb{R}^n), A)$.
(ii) Given a nonsingular symplectic form (K, ϕ) over $\mathbb{R}$ choose complementary lagrangians L, L^* for (K, ϕ). Extend a basis $\{b_1, b_2, \ldots, b_n\}$ for L to a basis $\{b_1, b_2, \ldots, b_{2n}\}$ for K with $b_{n+i} = b_i^*$ and

$$\phi(b_i, b_j) \;=\; \begin{cases} 1 & \text{if } j = i + n \\ -1 & \text{if } i = j + n \\ 0 & \text{otherwise} \;, \end{cases} \quad \phi \;=\; \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R})$$

so that $(K, \phi) = H_-(\mathbb{R}^n)$ (up to isomorphism). The matrix of a symplectic automorphism (K, ϕ, A) with respect to such a basis $\{b_1, b_2, \ldots, b_{2n}\}$

for K is a symplectic matrix $A \in Sp(2n)$.
(iii) The unitary group $U(n)$ for $n \geqslant 1$ is the group of hermitian automorphisms $(\mathbb{C}^n, \langle\ ,\ \rangle, A)$, that is

$$U(n) \ = \ \{A \in GL_n(\mathbb{C}) \,|\, A^*A = I\} \ .$$

(iv) For any $A \in Sp(2n)$ there is defined a symplectic formation

$$(H_-(\mathbb{R}^n); \mathbb{R}^n \oplus \{0\}, A(\mathbb{R}^n \oplus \{0\})) \ .$$

As in Proposition 6.6 let $ESp(2n) \subset Sp(2n)$ be the subgroup of the symplectic $2n \times 2n$ matrices A such that $A(\mathbb{R}^n \oplus \{0\}) = \mathbb{R}^n \oplus \{0\}$.The forgetful map

$$\begin{gathered} U(n) \to Sp(2n) \ ; \ U = X + iY \mapsto A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \\ (X^tY = Y^tX,\ X^tX + Y^tY = I) \end{gathered}$$

induces a homeomorphism

$$U(n)/O(n) \to Sp(2n)/ESp(2n) \ ; \ U \mapsto A \ ,$$

namely the composite of the homeomorphism

$$U(n)/O(n) \to \Lambda(n) \ ; \ U \mapsto U(\mathbb{R}^n \oplus \{0\})$$

and the inverse of the homeomorphism

$$Sp(2n)/ESp(2n) \to \Lambda(n) \ ; \ A \mapsto A(\mathbb{R}^n \oplus \{0\}) \ .$$

□

Proposition 8.3. (i) *For any symplectic automorphism* (K, ϕ, A) *there is defined a symplectic formation over* $\mathbb{R}$

$$(K \oplus K, \phi \oplus -\phi; \Delta_K, (A \oplus 1)\Delta_K) \ .$$

(ii) *If* L *is any lagrangian of* (K, ϕ) *there is defined a symplectic formation* $(K, \phi; L, A(L))$.
(iii) *For any two lagrangians* L_1, L_2 *of a nonsingular symplectic form* (K, ϕ) *over* $\mathbb{R}$ *and any complex structure* $J : K \to K$ *compatible with* (K, ϕ) *there exists a hermitian automorphism* (K, ϕ, J, A) *such that* $A(L_1) = L_2$.

Proof. (i)+(ii) Standard.
(iii) As in Proposition 7.7 (ii) choose ϕJ-orthonormal bases

$$\{b_1, \dots, b_n, Jb_1, \dots, Jb_n\} \ , \ \{b'_1, \dots, b'_n, Jb'_1, \dots, Jb'_n\}$$

for K such that

$$\mathbb{R}b_1 \oplus \cdots \oplus \mathbb{R}b_n \ = \ L \ , \ \mathbb{R}b'_1 \oplus \cdots \oplus \mathbb{R}b'_n \ = \ L' \ .$$

The isomorphisms of 7.7 (ii)

$$\alpha_{(L,\beta)}\ ,\ \alpha_{(L',\beta')}\ :\ (\mathbb{C}^n,\langle\ ,\ \rangle)\to(K,\phi,J)$$

are such that

$$\alpha_{(L,\beta)}(\mathbb{R}^n\oplus 0)\ =\ L\ ,\ \alpha_{(L',\beta')}(\mathbb{R}^n\oplus 0)\ =\ L'\subset K\ .$$

The coefficients $x_{jk},y_{jk}\in\mathbb{R}$ in the linear combinations

$$b'_j\ =\ \sum_{k=1}^n x_{jk}b_k+\sum_{k=1}^n y_{jk}Jb_k\in K$$

are the real and complex parts of a unitary matrix

$$U\ =\ (x_{jk}+iy_{jk})\in U(n)\ ,$$

namely the matrix of

$$U\ =\ \alpha^{-1}_{(L',\beta')}\alpha_{(L,\beta)}\ :\ (\mathbb{C}^n,\langle\ ,\ \rangle)\to(\mathbb{C}^n,\langle\ ,\ \rangle)$$

with respect to the $\mathbb{C}$-basis $\{b_1,b_2,\dots,b_n\}$. □

Proposition 8.4. *Let (K,ϕ,A) be a symplectic automorphism.*
(i) *The characteristic polynomial*

$$\mathrm{ch}_A(z)\ =\ \det(zI-A:K[z]\to K[z])\in\mathbb{R}[z]$$

is a monic polynomial such that

$$\begin{aligned}&\mathrm{degree}(\mathrm{ch}_A(z))\ =\ \dim_{\mathbb{R}}(K)\ =\ 2n\ ,\\ &\mathrm{ch}_A(0)\ =\ \det(-A:K\to K)\ =\ \det(A:K\to K)\neq 0\in\mathbb{R}\ ,\\ &\mathrm{ch}_A(z)\ =\ z^{2n}\det(A)\mathrm{ch}_A(z^{-1})\in\mathbb{R}[z,z^{-1}]\end{aligned}$$

with roots $\lambda\in\mathbb{C}\backslash\{0\}$ the eigenvalues of A.
(ii) *If $x,y\in K$ are eigenvectors of A with eigenvalues $\lambda,\mu\in\mathbb{R}$ then*

$$\phi(Ax,Ay)\ =\ \lambda\mu\phi(x,y)\ =\ \phi(x,y)\in\mathbb{R}\ ,$$

so if $\lambda\mu\neq 1$ then $\phi(x,y)=0\in\mathbb{R}$.
(iii) *If $\lambda\in\mathbb{C}$ is an eigenvalue of A then $\lambda\neq 0$ and $\overline{\lambda},\lambda^{-1}\in\mathbb{C}\backslash\{0\}$ are also eigenvalues. The characteristic polynomial of A factors as*

$$\begin{aligned}&\mathrm{ch}_A(z)\\ &=\ (z-1)^{2p}(z+1)^{2q}\Big(\prod_{j=1}^n(z^2+a_jz+1)^{r_j}\Big)\Big(\prod_{k=1}^m(z^2+b_kz+1)^{s_k}\Big)\in\mathbb{R}[z]\\ &\qquad(p,q\geqslant 0,\ |a_j|<2,\ |b_k|>2,\ r_j,s_k\geqslant 1)\end{aligned}$$

with distinct $a_j\in(-2,2)$, $b_k\in(-\infty,-2)\cup(2,\infty)$. There are three types of eigenvalue :

(a) *the $2(p+q)$* **parabolic** *eigenvalues in $\{1,-1\}$,*

(b) *the* $2\sum\limits_{j=1}^{n} r_j$ **elliptic** *eigenvalues*

$$\lambda_j^+ \;=\; \frac{-a_j + i\sqrt{4-(a_j)^2}}{2}\;,\;\lambda_j^- \;=\; \frac{-a_j - i\sqrt{4-(a_j)^2}}{2} \in S^1\backslash\{-1,+1\}$$

with $\lambda_j^+ = (\lambda_j^-)^{-1} = \overline{\lambda_j^-} \neq \lambda_j^-$.

(c) *the* $2\sum\limits_{k=1}^{m} s_k$ **hyperbolic** *eigenvalues*

$$\mu_k^+ \;=\; \frac{-b_k + \sqrt{(b_k)^2-4}}{2}\;,\;\mu_k^- \;=\; \frac{-b_k - \sqrt{(b_k)^2-4}}{2} \in \mathbb{R}\backslash\{-1,+1\}$$

with $\mu_k^+ = (\mu_k^-)^{-1} \neq \mu_k^-$,

Proof. (i) It follows from

$$zI - A \;=\; -zA\phi^{-1}(z^{-1}I - A^*)\phi \;:\; K \to K^*$$

that

$$\begin{aligned}\mathrm{ch}_A(z) \;&=\; \det(-zA)\det(z^{-1}I - A^t)\\ &=\; z^{2n}\det(A)\mathrm{ch}_A(z^{-1}) \in \mathbb{R}[z,z^{-1}]\ .\end{aligned}$$

(ii) Standard.
(iii) Immediate from (i). □

Definition 8.5. Let (K,ϕ,A) be a symplectic automorphism.
(i) (K,ϕ,A) is **parabolic** if $K = K_{par}$, i.e. if every eigenvalue of A is parabolic, so that

$$\mathrm{ch}_A(z) \;=\; (z-1)^{2p}(z+1)^{2q}\ (p,q \geqslant 0)\ .$$

(iii) (K,ϕ,A) is **elliptic** if $K = K_{ell}$, i.e. if every eigenvalue of A is elliptic, so that

$$\mathrm{ch}_A(z) \;=\; \prod_{j=1}^{n}(z^2 + a_j z + 1)^{r_j}\ (a_j \in \mathbb{R},\ |a_j| < 2,\ r_j \geqslant 1)\ .$$

(ii) (K,ϕ,A) is **hyperbolic** if $K = K_{hyp}$, i.e. if every eigenvalue of A is hyperbolic, so that

$$\mathrm{ch}_A(z) \;=\; \prod_{k=1}^{m}(z^2 + b_k z + 1)^{s_k}\ (b_k \in \mathbb{R},\ |b_k| > 2,\ s_k \geqslant 1)\ .$$

□

Proposition 8.6. *Every symplectic automorphism (K, ϕ, A) has a canonical splitting as a sum of a parabolic, a hyperbolic and an elliptic automorphism*

$$(K, \phi, A) \;=\; (K_{par}, \phi_{par}, A_{par}) \oplus (K_{ell}, \phi_{ell}, A_{ell}) \oplus (K_{hyp}, \phi_{hyp}, A_{hyp})$$

with

$$\begin{aligned}
\mathrm{ch}_{A_{par}}(z) \;&=\; (z-1)^{2p}(z+1)^{2q}\ (p, q \geqslant 0)\ ,\\
\mathrm{ch}_{A_{ell}}(z) \;&=\; \prod_{j=1}^{n}(z^2 + a_j z + 1)^{r_j}\ (a_j \in \mathbb{R},\ |a_j| < 2,\ r_j \geqslant 1)\ ,\\
\mathrm{ch}_{A_{hyp}}(z) \;&=\; \prod_{k=1}^{m}(z^2 + b_k z + 1)^{s_k}\ (b_k \in \mathbb{R},\ |b_k| > 2,\ s_k \geqslant 1)\ .
\end{aligned}$$

The subspaces

$$\begin{aligned}
L^+ \;&=\; \sum_{k=1}^{m} \ker((\mu_k^+ I - A)^{s_k} : K \to K)\ ,\\
L^- \;&=\; \sum_{k=1}^{m} \ker((\mu_k^- I - A)^{s_k} : K \to K) \subset K_{hyp}
\end{aligned}$$

are complementary lagrangians in (K_{hyp}, ϕ_{hyp}).

Proof. Factorize the characteristic polynomial of (K, ϕ, A) as

$$\mathrm{ch}_A(z) \;=\; \mathrm{ch}_{A_{par}}(z)\mathrm{ch}_{A_{hyp}}(z)\mathrm{ch}_{A_{ell}}(z)$$

and let

$$K_{\#} \;=\; \ker(\mathrm{ch}_{A_{\#}}(A) : K \to K)\ (\# = \mathrm{par}, \mathrm{ell}, \mathrm{hyp})\ .$$

□

Definition 8.7. Let (K, ϕ, A) be symplectic automorphism.
(i) For $\omega \in S^1$ define the ω-**signature** of (K, ϕ, A) to be the signature

$$\sigma_\omega(K, \phi, A) \;=\; \sigma(\mathbb{C} \otimes_{\mathbb{R}} K, (\phi, A, \omega))$$

of the hermitian form over $\mathbb{C}$ given by

$$(\phi, A, \omega) \;=\; (1-\omega)(I-A^*)\phi + (1-\overline{\omega})\phi^*(I-A)\ :\ \mathbb{C}\otimes_{\mathbb{R}} K \to \mathbb{C}\otimes_{\mathbb{R}} K^*\ .$$

If $\omega = 1$ then $\sigma_\omega(K, \phi, A) = 0$.
(ii) The **signature** of (K, ϕ, A) is the signature

$$\sigma(K, \phi, A) \;=\; \sigma_{-1}(K, \phi, A) \;=\; \sigma(K, (\phi, A, -1)/2) \in \mathbb{Z}$$

of the symmetric form over $\mathbb{R}$ given by

$$(\phi, A, -1)/2 \;=\; (I - A^*)\phi + \phi^*(I - A) \;=\; \phi(A - A^{-1})\ :\ K \to K^*\ .$$

(iii) The $L^{(2)}$**-signature** of (K,ϕ,A) is

$$\rho(K,\phi,A) \;=\; \int_\omega \sigma_\omega(K,\phi,A) \in \mathbb{R}\ .$$

(iv) The η**-invariant** of (K,ϕ,A) is

$$\eta(K,\phi,A) \;=\; \sum_{j=1}^{n} \sigma(K_j,\phi_j(A_j-A_j^{-1}))\eta(\theta_j) \in \mathbb{R}$$

with

$$\begin{aligned}\mathrm{ch}_z(A_{ell}) \;&=\; \prod_{j=1}^{n}(z^2-2\cos(\theta_j)z+1)^{r_j}\ \ (\theta_j\in(0,\pi))\ ,\\ (K_j,\phi_j,A_j) \;&=\; (\{x\in K\,|\,(A^2-2A\cos(\theta_j)+1)^{r_j}(x)=0\},\phi|,A|)\ .\end{aligned}$$

□

Remark 8.8. (i) The parabolic and hyperbolic components of (K,ϕ,A) make zero contribution to the signature, $L^{(2)}$-signature and η-invariant:

$$\begin{aligned}\sigma(K,\phi,A) \;&=\; \sigma(K_{ell},\phi_{ell},A_{ell}) \in \mathbb{R}\ ,\\ \rho(K,\phi,A) \;&=\; \rho(K_{ell},\phi_{ell},A_{ell}) \in \mathbb{R}\ ,\\ \eta(K,\phi,A) \;&=\; \eta(K_{ell},\phi_{ell},A_{ell}) \in \mathbb{R}\ .\end{aligned}$$

(ii) The η-invariant is a function

$$\eta\ :\ \Lambda(n) \;=\; U(n)/O(n) \;=\; Sp(2n)/ESp(2n) \to \mathbb{R}\ .$$

□

Proposition 8.9. (i) *If (K,ϕ,A) is a symplectic automorphism such that $A(L)=L$ for a lagrangian L of (K,ϕ) then*

$$\eta(K,\phi,A) \;=\; 0 \in \mathbb{R}\ .$$

(ii) *The triple signature of lagrangians L_1,L_2,L_3 of (K,ϕ) is*

$$\sigma(K,\phi;L_1,L_2,L_3) \;=\; \eta(K,\phi,A_{12})+\eta(K,\phi,A_{23})+\eta(K,\phi,A_{31}) \in \mathbb{Z}\subset\mathbb{R}$$

for any symplectic automorphisms $A_{12},A_{23},A_{31}:(K,\phi)\to(K,\phi)$ such that

$$A_{12}(L_1) \;=\; L_2\ ,\ \ A_{23}(L_2) \;=\; L_3\ ,\ \ A_{31}(L_3) \;=\; L_1$$

with $A_{31}A_{23}A_{12}=1:L_1\to L_1$.
(iv) *For any symplectic automorphisms $A,B:(K,\phi)\to(K,\phi)$*

$$\begin{aligned}\eta(K,-\phi,A) \;&=\; \eta(K,\phi,A)\ ,\\ \eta(K,\phi,A^{-1}) \;&=\; -\eta(K,\phi,A)\ ,\\ \eta(K\oplus K,\phi\oplus\phi,A\oplus B) \;&=\; \eta(K,\phi,A)+\eta(K,\phi,B)\in\mathbb{R}\end{aligned}$$

and

$$\begin{aligned}
&\sigma(K \oplus K, \phi \oplus -\phi; \Delta_K, (1 \oplus B)\Delta_K, (1 \oplus AB)\Delta_K) \\
&= \ \eta(K, \phi, A) + \eta(K, \phi, B) + \eta(K, \phi, (AB)^{-1}) \\
&= \ \eta(K, \phi, A) + \eta(K, \phi, B) - \eta(K, \phi, AB) \in \mathbb{Z} \subset \mathbb{R} \ .
\end{aligned}$$

□

Definition 8.10. An automorphism $A : K \to K$ of a finite-dimensional real vector space is **fibred** if it satisfies any one of the following equivalent conditions:

(i) $A - A^{-1} : K \to K$ is an automorphism,
(ii) $1 - A, 1 + A : K \to K$ are automorphisms,
(iii) the characteristic polynomial

$$\mathrm{ch}_A(z) \ = \ \det(zI - A : K[z] \to K[z]) \in \mathbb{R}[z]$$

takes non-zero values for $z = \pm 1$.

□

Proposition 8.11. (i) *A symplectic automorphism (K, ϕ, A) is fibred if and only if A only has hyperbolic and elliptic eigenvalues, so that $K_{par} = 0$, $K = K_{hyp} \oplus K_{ell}$.*
(ii) *A symplectic automorphism (K, ϕ, A) is such that there exists complex structure $J : K \to K$ compatible with (K, ϕ) and $AJ = JA$ if and only if (K, ϕ, A) is fibred and the minimal polynomial of A is of the form*

$$\prod_{j=1}^{n} (z^2 + a_j z + 1) \ (a_j \in \mathbb{R}, \ |a_j| < 2)$$

for distinct a_j's. Such (K, ϕ, A) is elliptic and

$$\eta(A) \ = \ \eta(U_A) \in \mathbb{R}$$

with U_A a unitary matrix of the hermitian automorphism (K, ϕ, A, J) for any such J.

□

Definition 8.12. For any symplectic formation $(K, \phi; L_1, L_2)$ the intersection $L_1 \cap L_2$ is a sublagrangian of (K, ϕ), with

$$L_1 \cap L_2 \subseteq (L_1 \cap L_2)^{\perp} \ .$$

The **symplectic reduction** of $(K,\phi;L_1,L_2)$ is the symplectic formation $([K],[\phi];[L_1],[L_2])$ with

$$\begin{aligned}&[K] \ = \ \frac{(L_1\cap L_2)^{\perp}}{L_1\cap L_2}\ ,\ [L_i] \ = \ \frac{L_i}{L_1\cap L_2}\ ,\ [L_1]\oplus[L_2] \ = \ [K]\ ,\\ &(K,\phi;L_1,L_2) \ = \ ([K],[\phi];[L_1],[L_2])\oplus(H_-(L_1\cap L_2);L_1\cap L_2,L_1\cap L_2)\ .\end{aligned}$$

□

Proposition 8.13. *For any symplectic automorphism (K,ϕ,A) and lagrangian L of (K,ϕ) the symplectic reduction of $(K,\phi;L,A(L))$ is a symplectic formation $([K],[\phi];[L],[A(L)])$. The induced symplectic automorphism $([K],[\phi],[A])$ is fibred and elliptic, with*

$$\begin{aligned}&[A][L] \ = \ [A(L)]\ ,\ [L]\oplus[A(L)] \ = \ [K]\ ,\\ &\eta(K,\phi,A) \ = \ \eta([K],[\phi],[A])\in\mathbb{R}\ .\end{aligned}$$

□

Proposition 8.14. *Let (K,ϕ,A) be a fibred symplectic automorphism over $\mathbb{R}$.*
(i) *The direct sum splitting*

$$(K,\phi,A) \ = \ (K_{ell},\phi_{ell},A_{ell})\oplus(K_{hyp},\phi_{hyp},A_{hyp})$$

determines a factorization of the characteristic polynomial

$$\mathrm{ch}_A(z) \ = \ \mathrm{ch}_{A_{ell}}(z)\mathrm{ch}_{A_{hyp}}(z)$$

with

$$\begin{aligned}&\mathrm{ch}_{A_{ell}}(z) \ = \ \prod_{j=1}^{n}(z^2+a_jz+1)^{r_j}\ \ (|a_j|<2)\ ,\\ &\mathrm{ch}_{A_{hyp}}(z) \ = \ z^{\deg(p(z))}\det(A_{hyp})p(z)p(z^{-1})\\ &(p(z)\in\mathbb{R}[z]\ \textbf{with}\ p(1),p(-1)\neq 0)\end{aligned}$$

for distinct $a_1,a_2,\dots,a_n\in(-2,2)$.
(ii) *The elliptic component splits as*

$$(K_{ell},\phi_{ell},A_{ell}) \ = \ \sum_{j=1}^{n}(K_j,\phi_j,A_j)$$

with

$$(K_j,\phi_j,A_j) \ = \ (\{x\in K\,|\,(A^2+a_jA+1)^{r_j}(x)=0\},\phi|,A|)\ .$$

The symmetric forms $(K_j,\phi_j(A_j-A_j^{-1}))$ over $\mathbb{R}$ are nonsingular and

$$\sigma(K,\phi,A) \ = \ \sum_{j=1}^{n}\sigma(K_j,\phi_j(A_j-A_j^{-1}))\in\mathbb{Z}\ .$$

(iii) (Levine [27], Matumoto [30]) *Let* $\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}\} \subset S^1\backslash\{-1,1\}$ *be the eigenvalues of* A. *The function*

$$S^1 \to \mathbb{Z} \; ; \; \omega \mapsto \sigma_\omega(K,\phi,A)$$

is locally constant on each component of $S^1\backslash\{e^{i\theta_1}, \dots, e^{i\theta_n}\}$ *with jumps*

$$\begin{aligned}\varinjlim_{\epsilon} (\sigma_{e^{i(\theta_j+\epsilon)}}(K,\phi,A) - \sigma_{e^{i(\theta_j-\epsilon)}}(K,\phi,A)) \\ = \; 2\sigma(K_j,\phi_j(A_j - A_j^{-1})) \text{ for } j = 1,2,\dots,n \; .\end{aligned}$$

(iv) *Suppose that the minimal polynomial of* A_j *is* $z^2 - 2z\cos(\theta_j) + 1$, *so that* (K_j, ϕ_j, A_j) *is a sum of 2-dimensional symplectic automorphisms, with signature* ± 2. *Without loss of generality it may be assumed that* $\dim_{\mathbb{R}}(K_j) = 2$ *for each* j. *Choose* θ_j *such that*

$$\mathrm{sgn}(\sin\theta_j) \; = \; \sigma(K_j,\phi_j(A_j - A_j^{-1}))/2 \in \{-1,1\} \; .$$

The canonical complex structure on K_{ell}

$$J \; = \; \sum_{j=1}^{n} \frac{(A_j - (A_j)^{-1})}{2\sin(\theta_j)} \; : \; K_{ell} \; = \; \sum_{j=1}^{n} K_j \to K_{ell}$$

is compatible with (K_{ell}, ϕ_{ell}), *with* $\phi_{ell} J : K_{ell} \to K_{ell}^*$ *a positive definite symmetric form over* $\mathbb{R}$, *and* $A_{ell} J = J A_{ell} : K_{ell} \to K_{ell}$. *For any* $x \in K_j$

$$\begin{aligned} A_{ell}(x) \; &= \; \cos(\theta_j)(x) + ((A_j - A_j^{-1})/2)(x) \\ &= \; \cos(\theta_j)(x) + \sin(\theta_j)(Jx) \\ &= \; \cos(\theta_j)(x) + \sin(\theta_j)(Jx) \in K_j \end{aligned}$$

and the unitary matrix of A_{ell} *is diagonal*

$$U_{A_{ell}} \; = \; D(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in U(n) \; .$$

The η*-invariant is*

$$\eta(K,\phi,A) \; = \; \eta(U_{A_{ell}}) \; = \; \sum_{j=1}^{n} \eta(\theta_j) \in \mathbb{R} \; ,$$

$\square$

Proposition 8.15. (i) *The hermitian automorphism defined for any* $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$ *by*

$$(K,\phi,J,A) \; = \; (\mathbb{R}\oplus\mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix})$$

is such that

$$\begin{aligned}
(K,\phi(A-A^{-1})) &= (\mathbb{R}\oplus\mathbb{R},\begin{pmatrix} 2\sin\theta & 0 \\ 0 & 2\sin\theta\end{pmatrix})\ ,\\
\alpha &= \sigma(K,\phi(A-A^{-1}))/2 = \mathrm{sgn}(\sin\theta)\ ,\\
J &= \frac{A-A^{-1}}{2\sin\theta} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\ ,\\
\eta(K,\phi,A) &= \eta(\theta)\in\mathbb{R}\ .
\end{aligned}$$

(ii) *For any hermitian automorphism* (K,ϕ,J,A) *there exists a* (ϕ,J)*-orthonormal basis for* K *consisting of eigenvectors of* A*. Any such basis determines an isomorphism*

$$(K,\phi,J,A) \cong \sum_{j=1}^{n}(\mathbb{R}\oplus\mathbb{R},\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix},\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j\end{pmatrix})$$

with

$$\mathrm{ch}_A(z) = \prod_{j=1}^{n}(z^2-2\cos\theta_j\, z+1)\in\mathbb{R}[z]\ .$$

The matrix of A *with respect to such a basis is a diagonal unitary matrix*

$$U_A = D(e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_n})\in U(n)$$

and

$$\eta(K,\phi,A) = \sum_{j=1}^{n}\eta(\theta_j)\in\mathbb{R}\ .$$

□

Remark 8.16. (i) Let

$$S = S^* = \begin{pmatrix} p & r \\ r & q\end{pmatrix}\in GL_2(\mathbb{R})$$

be an invertible symmetric 2×2 matrix. The signature of the nonsingular symmetric form $(\mathbb{R}\oplus\mathbb{R},S)$ over $\mathbb{R}$ is

$$\sigma(\mathbb{R}\oplus\mathbb{R},S) = \begin{cases} 2\,\mathrm{sgn}(p) & \text{if } p\neq 0 \text{ and } pq-r^2>0 \\ 0 & \text{if } p=0 \text{ or if } pq-r^2<0\ . \end{cases}$$

(ii) Let

$$S = S^* = \begin{pmatrix} p=\overline{p} & r \\ \overline{r} & q=\overline{q}\end{pmatrix}\in GL_2(\mathbb{C})$$

be an invertible hermitian 2×2 matrix. The signature of the nonsingular hermitian form $(\mathbb{C} \oplus \mathbb{C}, S)$ over $\mathbb{C}$ is

$$\sigma(\mathbb{C} \oplus \mathbb{C}, S) \ = \ \begin{cases} 2\,\mathrm{sgn}(p) & \text{if } p \neq 0 \text{ and } pq - r\overline{r} > 0 \\ 0 & \text{if } p = 0 \text{ or if } pq - r\overline{r} < 0 \ . \end{cases}$$

□

Example 8.17. Let $(K, \phi) = H_-(\mathbb{R})$.
(i) The symplectic automorphisms (K, ϕ, A) are in one-one correspondence with 2×2 matrices

$$A \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$$

such that

$$ad - bc \ = \ 1 \in \mathbb{R} \ ,$$

i.e. $A \in Sp(2) = SL_2(\mathbb{R})$. The trace of A is

$$\mathrm{tr}(A) \ = \ a + d \ .$$

The characteristic polynomial is

$$\mathrm{ch}_A(z) \ = \ \begin{vmatrix} z - a & -b \\ -c & z - d \end{vmatrix} \ = \ z^2 - \mathrm{tr}(A)z + 1 \in \mathbb{R}[z]$$

with roots the eigenvalues

$$\lambda_1, \lambda_2 \ = \ \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4}}{2} \ .$$

$$A \text{ is } \begin{cases} \text{parabolic} \\ \text{hyperbolic} \\ \text{elliptic} \\ \text{fibred} \end{cases} \text{ if and only if } \begin{cases} |\mathrm{tr}(A)| = 2 \\ |\mathrm{tr}(A)| > 2 \\ |\mathrm{tr}(A)| < 2 \\ |\mathrm{tr}(A)| \neq 2. \end{cases}$$

(ii) Suppose that A is elliptic, so that

$$|a + d| < 2 \ , \ c \neq 0 \ .$$

Let $\theta \neq \pi \in (0, 2\pi)$ be such that

$$\cos\theta \ = \ \frac{a + d}{2} \ , \ \mathrm{sgn}(\sin\theta) \ = \ \mathrm{sgn}(c)$$

so that

$$\begin{aligned} \{\lambda_1, \lambda_2\} \ &= \ \{e^{i\theta}, e^{-i\theta}\} \subset S^1 \backslash \{-1, 1\} \ , \\ \mathrm{ch}_A(z) \ &= \ z^2 - 2z\cos\theta + 1 \in \mathbb{R}[z] \ . \end{aligned}$$

The signature of (K,ϕ,A) is the signature of the symmetric form $(K,\phi(A-A^{-1}))$ over $\mathbb{R}$

$$\sigma(K,\phi,A) ~=~ \sigma(K,\phi(A-A^{-1})) \in \{-2,0,2\}~.$$

The symmetric form has determinant

$$\det(\phi(A-A^{-1})) ~=~ \begin{vmatrix} 2c & d-a \\ d-a & -2b \end{vmatrix} ~=~ 4-(a+d)^2 > 0~,$$

By Remark 8.16 (i)

$$\begin{aligned} \sigma(K,\phi,A) ~&=~ \sigma(\mathbb{R}\oplus\mathbb{R},\begin{pmatrix} 2c & d-a \\ d-a & -2b \end{pmatrix}) \\ &=~ 2\,\mathrm{sgn}(c) ~=~ 2\,\mathrm{sgn}(\sin\theta) \in \{-2,2\}~. \end{aligned}$$

The canonical complex structure

$$J ~=~ \frac{A-A^{-1}}{2\sin\theta} ~=~ \frac{1}{2\sin\theta}\begin{pmatrix} a-d & 2b \\ 2c & d-a \end{pmatrix} ~:~ K \to K$$

is compatible with (K,ϕ), and such that $AJ=JA$. For any $x\in K$

$$(A^2-2A\cos\theta+1)(x) ~=~ 0 \in K~,$$

so that

$$\begin{aligned} A(x) ~&=~ \cos(\theta)(x)+((A-A^{-1})/2)(x) \\ &=~ \cos(\theta)(x)+\sin(\theta)(Jx) \\ &=~ \cos(\theta)(x)+\sin(\theta)(Jx) \in K~. \end{aligned}$$

The hermitian automorphism $A:(K,\phi,J)\to(K,\phi,J)$ has the unitary matrix

$$U_A ~=~ (e^{i\theta}) \in U(1)~,$$

so that the η-invariant is

$$\eta(K,\phi,A) ~=~ \eta(\theta) ~=~ 1-2\{\theta/\pi\} \in \mathbb{R}~.$$

Let $\omega=e^{i\psi}\in S^1\backslash\{1\}$. The ω-signature of (K,ϕ,A) is the signature of the hermitian form $(\mathbb{C}\otimes_{\mathbb{R}}K,(\phi,A,\omega))$ over $\mathbb{C}$ with

$$(\phi,A,\omega) ~=~ (1-\omega)(I-A^*)\phi+(1-\overline{\omega})\phi^*(I-A) ~:~ \mathbb{C}\otimes_{\mathbb{R}}K \to \mathbb{C}\otimes_{\mathbb{R}}K^*~,$$

so that

$$\begin{aligned} &\sigma_\omega(K,\phi,A) ~=~ \sigma(\mathbb{C}\otimes_{\mathbb{R}}K,(\phi,A,\omega)) \\ &=~ \sigma(\mathbb{C}\oplus\mathbb{C},(1-\omega)\begin{pmatrix} c & 1-a \\ d-1 & -b \end{pmatrix}+(1-\overline{\omega})\begin{pmatrix} c & d-1 \\ 1-a & -b \end{pmatrix}) \in \{-2,0,2\}~. \end{aligned}$$

The hermitian form has determinant

$$\begin{aligned}\det(\phi,A,\omega) &= \det((1-\omega)\phi(I-A^{-1})(I-\overline{\omega}A))\\ &= (1-\omega)^2\begin{vmatrix} c & 1-a\\ d-1 & -b\end{vmatrix}\begin{vmatrix}1-\overline{\omega}a & -\overline{\omega}b\\ -\overline{\omega}c & 1-\overline{\omega}d\end{vmatrix}\\ &= (1-\omega)^2(2-(a+d))(\overline{\omega}^2-(a+d)\overline{\omega}+1)\\ &= (1-\omega)(1-\overline{\omega})(2-(a+d))((a+d)-(\omega+\overline{\omega}))\in\mathbb{R}\end{aligned}$$

with

$$\operatorname{sgn}(\det(\phi,A,\omega)) = \operatorname{sgn}(a+d-(\omega+\overline{\omega}))\in\{-1,0,1\}\ .$$

By Remark 8.16 (ii)

$$\begin{aligned}\sigma_\omega(K,\phi,A) &= \begin{cases}2\operatorname{sgn}(c) & \text{if } a+d>\omega+\overline{\omega}\\ 0 & \text{if } a+d<\omega+\overline{\omega}\end{cases}\\ &= \begin{cases}2\operatorname{sgn}(\sin\theta) & \text{if } \cos\theta>\cos\psi\\ 0 & \text{if } \cos\theta<\cos\psi\ .\end{cases}\end{aligned}$$

The $L^{(2)}$-signature is

$$\begin{aligned}\rho(K,\phi,A) &= \frac{1}{2\pi}\int_{\psi=0}^{2\pi}\sigma_{e^{\pi i\psi}}(K,\phi,A)d\psi\\ &= \begin{cases}2\operatorname{sgn}(\sin\theta)(2\pi-2\theta)/2\pi & \text{if } 0<\theta<\pi\\ 2\operatorname{sgn}(\sin\theta)(2\pi-2(2\pi-\theta))/2\pi & \text{if } \pi<\theta<2\pi\end{cases}\\ &= 2(1-2\{\theta/2\pi\})\\ &= 2\,\eta(\theta/2)\in\mathbb{R}\ .\end{aligned}$$

Thus

$$\begin{aligned}\rho(K,\phi,A)-\eta(K,\phi,A) &= 2\,\eta(\theta/2)-\eta(\theta)\\ &= \operatorname{sgn}(\sin\theta)\\ &= \sigma(K,\phi,A)/2\in\{-1,1\}\subset\mathbb{R}\ .\end{aligned}$$

□

Proposition 8.18. *The signature, ρ- and η-invariants of a fibred symplectic automorphism (K,ϕ,A) are invariants of the Witt group of fibred symplectic automorphisms, which are related by*

$$\rho(K,\phi,A)-\eta(K,\phi,A) = \sigma(K,\phi,A)/2\in\mathbb{Z}\subset\mathbb{R}\ .$$

Proof. There is no loss of generality in assuming that

$$\begin{aligned}(K,\phi,A) &= \sum_{j=1}^{n}(K_j,\phi_j,A_j)\\ &= \sum_{j=1}^{n}(\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}\cos\alpha_j\theta_j & -\sin\alpha_j\theta_j\\ \sin\alpha_j\theta_j & \cos\alpha_j\theta_j\end{pmatrix})\end{aligned}$$

with

$$\alpha_j = \sigma(K_j,\phi_j(A_j-A_j^{-1}))/2\in\{-1,1\}\ ,\ \theta_j\in(0,\pi)\ .$$

By Example 8.17 applied to each component (K_j,ϕ_j,A_j)

$$\begin{aligned}\sigma(K,\phi,A) &= \sum_{j=1}^{n}\sigma(K_j,\phi_j,A_j) = 2\sum_{j=1}^{n}\alpha_j\ ,\\ \rho(K,\phi,A) &= \sum_{j=1}^{n}\rho(K_j,\phi_j,A_j) = 2\sum_{j=1}^{n}\alpha_j(1-\frac{\theta_j}{\pi})\ ,\\ \eta(K,\phi,A) &= \sum_{j=1}^{n}\eta(K_j,\phi_j,A_j) = \sum_{j=1}^{n}\alpha_j(1-\frac{2\theta_j}{\pi})\ .\end{aligned}$$

□

Proposition 8.19. *Let (K,ϕ,A) be a symplectic automorphism, with* $\dim_{\mathbb{R}}K=2n$.
(i) *If there exists a complex structure $J:K\to K$ which is compatible with (K,ϕ) and such that $AJ=JA:K\to K$ then*

$$(K,\phi,A) = (K_{par},\phi_{par},A_{par})\oplus(K_{ell},\phi_{ell},A_{ell})$$

is a sum of a parabolic and an elliptic symplectic automorphism. The eigenvalues of A are all of the form $e^{\pm i\theta_j}\in S^1$ for $j=1,2,\dots,n$ (meaning a complex conjugate pair $\{e^{i\theta_j},e^{-i\theta_j}\}$ if $e^{i\theta_j}\in S^1\backslash\{1,-1\}$, and $e^{i\theta_j}$ with multiplicity 2 if $e^{i\theta_j}\in\{1,-1\}$). A has a unitary matrix $U_A\in U(n)$ with respect to any ϕJ-orthonormal basis for $((K,J),(\phi,J))$. There exists such a basis consisting of eigenvectors of A, in which case

$$U_A = D(e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_n})\in U(n)$$

is a diagonal matrix, and

$$\eta(K,\phi,A) = \eta(U_A) = \sum_{j=1}^{n}\eta(\theta_j)\in\mathbb{R}\ (\theta_j\in[0,2\pi))\ .$$

(ii) *There exists a complex structure J as in* (i) *if and only if the following conditions are satisfied:*

(a) *A has no hyperbolic eigenvalues, i.e.*

$$\mathrm{ch}_A(z) \;=\; \prod_{j=1}^{n}(z^2 - 2\cos\theta_j\, z + 1)^{r_j}$$

for some $\theta_j \in [0,\pi]$, *so that the eigenvalues* $e^{\pm i\theta_j}$ *are either parabolic* ($\theta_j = 0$ *or* π) *or elliptic* ($0 < \theta_j < \pi$),

(b) *the minimal polynomial of* A *is* $\prod_{j=1}^{n}(z^2 - 2\cos\theta_j\, z + 1)$,

(c) $\ker(A - I : K \to K) = \ker((A-I)^2 : K \to K)$,

(d) $\ker(A + I : K \to K) = \ker((A+I)^2 : K \to K)$.

Proof. (i) By construction.
(ii) If $J : K \to K$ is a complex structure which is compatible with (K,ϕ) and such that $AJ = JA : K \to K$ then A is an automorphism of the positive definition hermitian form $((K,J),(\phi,J))$. Such automorphisms can be diagonalized so that

$$(K,\phi,A,J) \;=\; \sum_{j=1}^{n}(K_j,\phi_j,A_j,J_j)$$

with $\dim_{\mathbb{R}}(K_j) = 2$, and the minimal polynomial is $\prod_{j=1}^{n}(z^2 - 2\cos\theta_j\, z + 1)$. There is no loss of generality in only considering the 2-dimensional case, with

$$(K,\phi,J) \;=\; (\mathbb{C},\langle\ ,\ \rangle)\ ,\ A \;=\; (e^{i\theta}) \in U(1)$$

for some $\theta \in [0,2\pi)$. The characteristic polynomial of

$$A \;=\; \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \in U(1) \subset SL_2(\mathbb{R})$$

is

$$\mathrm{ch}_A(z) \;=\; (z - e^{i\theta})(z - e^{-i\theta}) \;=\; z^2 - 2\cos\theta\, z + 1$$

with roots $e^{\pm i\theta} \in S^1$, so (a) and (b) are satisfied. It is immediate from the identities

$$\begin{aligned} A - I \;&=\; e^{i\theta} - 1 \;=\; 2ie^{i\theta/2}\sin\theta/2\ ,\\ A + I \;&=\; e^{i\theta} + 1 \;=\; 2e^{i\theta/2}\cos\theta/2 \end{aligned}$$

that

$$\begin{aligned}\ker(A-I) &= \ker(A-I)^2 = \begin{cases}\mathbb{C} & \text{if } \theta=0\\ \{0\} & \text{otherwise}\end{cases},\\ \ker(A+I) &= \ker(A+I)^2 = \begin{cases}\mathbb{C} & \text{if } \theta=\pi\\ \{0\} & \text{otherwise}\end{cases}\end{aligned}$$

so (c) and (d) are satisfied.

Conversely, suppose (a) and (b) is satisfied, so that by Proposition 8.6

$$(K,\phi,A) = (K_{par},\phi_{par},A_{par})\oplus(K_{ell},\phi_{ell},A_{ell})$$

with

$$\begin{aligned}\mathrm{ch}_{A_{par}}(z) &= (z-1)^{2p}(z+1)^{2q}\ (p,q\geqslant 0)\ ,\\ \mathrm{ch}_{A_{ell}}(z) &= \prod_{j=1}^{n}(z^2+a_jz+1)\ (a_j\in\mathbb{R},\ |a_j|<2)\ .\end{aligned}$$

with $(K_{ell},\phi_{ell},A_{ell})$ a sum of n 2-dimensional symplectic automorphisms. If (c) and (d) are satisfied then $(K_{par},\phi_{par},A_{par})$ is also a sum of 2-dimensional automorphisms. Again, there is no loss in assuming $\dim_{\mathbb{R}}(K) = 2$, with $(K,\phi) = H_-(\mathbb{R})$. If A is parabolic then either $A=I$ or $A=-I$, and the standard complex structure $J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ will do. If A is elliptic then

$$\mathrm{ch}_A(z) = z^2-2\cos\theta\, z+1\in\mathbb{R}[z]$$

with $\theta\in(0,\pi)$, and by the Cayley-Hamilton theorem

$$A^2-2\cos\theta A+I = 0 : K\to K\ .$$

The complex structure

$$J = \frac{(A-A^{-1})}{2\sin\theta} : K\to K$$

is such that $AJ=JA$, and

$$J' = \mathrm{sgn}(\sigma(K,\phi J))J : K\to K$$

is a complex structure which is compatible with (K,ϕ) and such that $AJ'=J'A$. □

Example 8.20. Given

$$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in Sp(21) = SL_2(\mathbb{R})\ (ad-bc=1)$$

let

$$L_1 = \mathbb{R}(1,0)\ ,\ L_2 = A(\mathbb{R}(1,0)) = \mathbb{R}(a,c)\in\Lambda(1)\ .$$

Let $\theta \in [0,\pi]$ be such that

$$\cos\theta \;=\; \frac{a}{\sqrt{a^2+c^2}}\ ,\ \sin\theta \;=\; \frac{c}{\sqrt{a^2+c^2}}\ ,$$

Then

$$U_A \;=\; (e^{i\theta}) \;=\; \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in U(1) \subset Sp(21)$$

is such that $U_A(L_1) = L_2$ and

$$\eta(H_-(\mathbb{R}), J; L_1, L_2) \;=\; \eta(\theta) \;=\; \begin{cases} 1 - \dfrac{2\theta}{\pi} & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \theta = 0 \end{cases} \in \mathbb{R}$$

with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard complex structure on $H_-(\mathbb{R})$. If A is parabolic ($|a+d| = 2$) or elliptic ($|a+d| < 2$) let $\theta' \in [0,\pi]$ be such that

$$\cos\theta' \;=\; (a+d)/2\ ,$$

so that

$$\eta(H_-(\mathbb{R}), A) \;=\; \eta(\theta') \in \mathbb{R}\ .$$

If A is hyperbolic ($|a+d| > 2$) then

$$\eta(H_-(\mathbb{R}), A) \;=\; 0 \in \mathbb{R}\ .$$

If $AJ = JA$ then $A = U_A$ is parabolic or elliptic, $\theta = \theta'$, and

$$\eta(H_-(\mathbb{R}), J; L_1, L_2) \;=\; \eta(H_-(\mathbb{R}), A) \in \mathbb{R}\ .$$

However, in general $AJ \neq JA$ and

$$\eta(H_-(\mathbb{R}), J; L_1, L_2) \;\neq\; \eta(H_-(\mathbb{R}), A) \in \mathbb{R}\ .$$

For example

$$\eta(H_-(\mathbb{R}), J; \mathbb{R}(1,0), \mathbb{R}(1,1)) \;=\; 1/2 \;\neq\; \eta(H_-(\mathbb{R}); \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}) \;=\; 0\ .$$

□

Remark 8.21. The η-invariant $\eta(K,\phi,J;L_1,L_2) \in \mathbb{R}$ depends on the compatible complex structure J. For example, let

$$(K,\phi) \;=\; H_-(\mathbb{R})\ ,\ L_1 \;=\; \mathbb{R}(1,0)\ ,\ L_2 \;=\; \mathbb{R}(\cos\theta,\sin\theta)\ (0 < \theta < \pi)\ .$$

A compatible complex structure on $H_-(\mathbb{R})$

$$J \;=\; \begin{pmatrix} -\nu & -\mu \\ \lambda & \nu \end{pmatrix} \;:\; \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R}$$

is defined by $\lambda,\mu,\nu \in \mathbb{R}$ satisfying

$$\nu^2 \;=\; \lambda\mu - 1 \text{ and } \lambda,\mu > 0\ ,$$

and

$$\phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \ \phi J = \begin{pmatrix} \lambda & \nu \\ \nu & \mu \end{pmatrix} .$$

Note that J is conjugate to the standard compatible complex structure $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with

$$A = \begin{pmatrix} 1 & \nu\lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix} \in GL_2(\mathbb{R})$$

such that

$$J = A^{-1} J_0 A .$$

As in Proposition 7.6 (i) the 2×2 matrix

$$-J_0 J = \begin{pmatrix} \lambda & \nu \\ \nu & \mu \end{pmatrix}$$

is positive definite symmetric and symplectic. The bases $\{b_1, Jb_1\}$, $\{b_1', Jb_1'\}$ for (K, ϕ) given by

$$\begin{aligned} b_1 &= \frac{(1,0)}{\sqrt{\lambda}} \in L \ , \ Jb_1 = \frac{(-\nu,\lambda)}{\sqrt{\lambda}} \in JL \ , \\ b_1' &= \frac{(\cos\theta, \sin\theta)}{\sqrt{\lambda\cos^2\theta + \mu\sin^2\theta + 2\nu\sin\theta\cos\theta}} \in L' \ , \\ Jb_1' &= \frac{(-(\nu\cos\theta + \mu\sin\theta), \lambda\cos\theta + \nu\sin\theta)}{\sqrt{\lambda\cos^2\theta + \mu\sin^2\theta + 2\nu\sin\theta\cos\theta}} \in JL' \end{aligned}$$

are such that

$$\begin{aligned} &(\phi + i\phi J)(b_1)(b_1) = (\phi + i\phi J)(b_1')(b_1') = i \in \mathbb{C} \ , \\ &b_1' = \cos\theta' \, b_1 + \sin\theta' \, Jb_1 \in K \end{aligned}$$

with

$$\tan\theta' = \frac{\tan\theta}{\lambda + \nu\tan\theta} \ ,$$

so that in this case

$$A = (e^{i\theta'}) \in U(1) \ , \ \eta(K, \phi, J; L_1, L_2) = 1 - \frac{2\theta'}{\pi} \in \mathbb{R} \ .$$

□

Proposition 8.22. (i) *If L_1, L_2, L_3 are three lagrangians in (K, ϕ) then*

$$\begin{aligned} \eta(K, \phi, J; L_1, L_2) + \eta(K, \phi, J; L_2, L_3) + \eta(K, \phi, J; L_3, L_1) \\ = \sigma(L_1, L_2, L_3) \in \mathbb{Z} \subset \mathbb{R} \end{aligned}$$

is the triple signature (which is independent of J).
(ii) *Let $(K,\phi) = H_-(\mathbb{R}^n)$ with the standard compatible complex structure $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The η-invariant of a lagrangian $L \in \Lambda(n)$ is such that*

$$\eta(L) \ = \ \eta(H_-(\mathbb{R}^n), J; \mathbb{R}^n \oplus 0, L) \in \mathbb{R} \ .$$

□

Example 8.23. Let

$$(K,\phi) \ = \ H_-(\mathbb{R}^n) \ , \ J \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

A lagrangian $L \subset H_-(\mathbb{R}^n)$ is the image

$$L \ = \ \mathrm{im}(\begin{pmatrix} X \\ Y \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n)$$

with $X, Y \in M_n(\mathbb{R})$ such that

$$X^tY \ = \ Y^tX \in M_n(\mathbb{R}) \ , \ \mathrm{rank} \begin{pmatrix} X \\ Y \end{pmatrix} \ = \ n \ .$$

Since the symmetric form $(\mathbb{R}^n, X^tX + Y^tY)$ is positive definite there exists an orthonormal basis; the matrix $U \in GL_n(\mathbb{R})$ with columns such a basis is such that

$$U^t(X^tX + Y^tY)U \ = \ 1 \in M_n(\mathbb{R})$$

and the unitary matrix

$$A \ = \ (X + iY)U \in U(n)$$

is such that $A(\mathbb{R}^n \oplus 0) = L$, and

$$\eta(K, \phi, J; \mathbb{R}^n \oplus 0, L) \ = \ \eta(A) \in \mathbb{R} \ .$$

□

Example 8.24. For any $\theta \in \mathbb{R}$ the lagrangian in $H_-(\mathbb{R})$

$$L(\theta) \ = \ \mathrm{im}(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}) \subset \mathbb{R} \oplus \mathbb{R}$$

has

$$X \ = \ \cos\theta \ , \ Y \ = \ \sin\theta \ , \ U \ = \ 1 \ , \ A \ = \ (e^{i\theta}) \in U(1)$$

so the η-invariant is

$$\eta(H_-(\mathbb{R}), J; \mathbb{R} \oplus 0, L(\theta)) \ = \ \eta(\theta) \in \mathbb{R} \ .$$

□

Example 8.25. The **boundary** of a symmetric form $(\mathbb{R}^n, Y = Y^t)$ over $\mathbb{R}$ is the lagrangian in $H_-(\mathbb{R}^n)$

$$\partial(\mathbb{R}^n, Y) \;=\; \mathrm{im}(\begin{pmatrix} 1 \\ Y \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n) \;.$$

The eigenvalues of Y are the roots $\tan\theta_j \in \mathbb{R}$ ($1 \leqslant j \leqslant n$, $0 \leqslant \theta_j < \pi$) of the characteristic polynomial

$$\mathrm{ch}_z(Y) \;=\; \det(z - Y : K[z] \to K[z]) \;=\; \prod_{j=1}^{n}(z - \tan\theta_j) \in \mathbb{R}[z] \;.$$

The signature of Y is

$$\sigma(\mathbb{R}^n, Y) \;=\; \sum_{j=1}^{n} \mathrm{sgn}(\tan\theta_j) \in \mathbb{Z} \;.$$

The symmetric form $(\mathbb{R}^n, 1 + Y^tY)$ is positive definite (with $1 + Y^tY = 1 + Y^2$), so there exists $U \in GL_n(\mathbb{R})$ with

$$U^t(1 + Y^tY)U \;=\; 1 \in GL_n(\mathbb{R}) \;.$$

(If $(\mathbb{R}^n, Y)$ is nonsingular let $V \in O_n(\mathbb{R})$ be such that $V^tY^tYV = 1$, and set $U = V/\sqrt{2}$.) The unitary matrix $A = (1 + iY)U \in U(n)$ is such that $A(\mathbb{R}^n \oplus 0) = \partial(\mathbb{R}^n, Y)$, with characteristic polynomial

$$\mathrm{ch}_A(z) \;=\; \det(zI_n - A : \mathbb{C}^n[z] \to \mathbb{C}^n[z]) \;=\; \prod_{j=1}^{n}(z - e^{i\theta_j}) \in \mathbb{C}[z] \;.$$

Thus A has eigenvalues $e^{i\theta_j}$, and the η-invariant is

$$\eta(H_-(\mathbb{R}^n), J; \mathbb{R}^n \oplus 0, \partial(\mathbb{R}^n, Y)) \;=\; \eta(A) \;=\; \sum_{j=1}^{n} \eta(\theta_j) \in \mathbb{R} \;.$$

$\square$

Definition 8.26. (i) The ω**-signature** of $\theta \in \mathbb{R}$ is defined for $\omega \in S^1$ to be

$$\sigma_\omega(\theta) \;=\; \sigma(\mathbb{C}, (1 - \omega)e^{i\theta} + (1 - \overline{\omega})e^{-i\theta}) \in \mathbb{Z} \;.$$

(ii) The **signature** of $\theta \in \mathbb{R}$ is

$$\sigma(\theta) \;=\; \sigma_{-1}(\theta) \;=\; \sigma(\mathbb{C}, e^{i\theta} + e^{-i\theta}) \;=\; \mathrm{sgn}(\cos\theta) \in \{-1, 0, 1\} \;.$$

$\square$

Proposition 8.27. *The ω-signature function $\sigma_\omega : \mathbb{R} \to \{-1, 0, 1\}$ has the following properties:*
(i) *For $\omega = e^{i\psi} \in S^1$ it follows from*

$$1 - \omega \;=\; 1 - \cos\psi - i\sin\psi \;=\; 2\sin(\psi/2)e^{i(\psi - \pi)/2} \in \mathbb{C}$$

that

$$\begin{aligned}\sigma_\omega(\theta) &= \operatorname{sgn}(\operatorname{Re}((1-\omega)e^{i\theta})) \\ &= \operatorname{sgn}(\sin\psi/2\sin(\psi+2\theta)/2) \\ &= \sigma(L(0),L(\psi/2),L(\pi-\theta)) \in \{-1,0,1\}\ .\end{aligned}$$

(ii) $\sigma_\omega(\theta+\pi/2) = \operatorname{sgn}(\sin(\psi+2\theta))\sigma_\omega(\theta)$

$\sigma_\omega(\theta+\pi) = -\sigma_\omega(\theta)\ ,\ \sigma_\omega(-\theta) = \sigma_{\overline{\omega}}(\theta) \in \mathbb{R}\ .$

(iii) *If* $0<\theta<\pi$ *and* $0 \leqslant \psi < 2\pi$

$$\sigma_\omega(\theta) = \begin{cases} +1 & \text{if } 0<\psi<2\pi-2\theta \\ -1 & \text{if } 2\pi-2\theta<\psi<2\pi \\ 0 & \text{if } \psi=0,2\pi-2\theta \end{cases}\ ,\ \sigma_\omega(-\theta) = \begin{cases} -1 & \text{if } 0<\psi<2\theta \\ 1 & \text{if } 2\theta<\psi<2\pi \\ 0 & \text{if } \psi=0,2\theta\ . \end{cases}$$

(iv) *The function*

$$S^1 \to \{-1,0,1\}\ ;\ \omega \mapsto \sigma_\omega(\theta)$$

is locally constant, with jumps

$$\underset{\epsilon}{\underrightarrow{\lim}}(\sigma_{\omega e^{i\epsilon}}(\theta)-\sigma_{\omega e^{-i\epsilon}}(\theta)) = \begin{cases} 2\operatorname{sgn}\cos\theta & \text{if } \omega=1 \\ -2\operatorname{sgn}\cos\theta & \text{if } \omega=e^{(2\pi-2\theta)i} \\ 0 & \text{if } \omega\neq 1,e^{(2\pi-2\theta)i}\ . \end{cases}$$

□

Definition 8.28. The $L^{(2)}$**-signature** of $\theta\in\mathbb{R}$ is defined by

$$\sigma^{(2)}(\theta) = \int_{\omega\in S^1}\sigma_\omega(\theta) = \frac{1}{2\pi}\int_{\psi=0}^{2\pi}\sigma_{e^{i\psi}}(\theta)d\psi \in [-1,1]$$

normalized so that $\sigma^{(2)}(0)=1$.

□

Proposition 8.29. *The* $L^{(2)}$*-signature function* $\sigma^{(2)}:\mathbb{R}\to[-1,1]$ *has the following properties:*

(i) $\sigma^{(2)}(\theta) = \dfrac{1}{2\pi}\int_{\psi=0}^{2\pi}\operatorname{sgn}(\sin\psi/2\sin(\psi/2+\theta))d\psi\ (\omega=e^{i\psi})$

$$= (-)^{[\theta/\pi]}\mu(\theta/\pi) = \begin{cases} (-1)^k\eta(\theta) & \text{if } \pi k<\theta<\pi(k+1) \\ (-1)^k & \text{if } \theta=\pi k\ . \end{cases}.$$

(ii) *For* $\theta\notin\pi\mathbb{Z}$

$$(-)^{[\theta/\pi]}\sigma^{(2)}(\theta) = \eta(\theta) = \mu(\theta/\pi) = 1-2\{\theta/\pi\}\in\mathbb{R}\ .$$

(iii) *The* $L^{(2)}$*-signature function* $\sigma^{(2)}:\mathbb{R}\to\mathbb{R}$ *is such that*

$$\sigma^{(2)}(\theta+\pi) = -\sigma^{(2)}(\theta)\ ,\ \sigma^{(2)}(-\theta) = \sigma^{(2)}(\theta)\ ,\ \rho(0) = 1\ .$$

$\square$

Proposition 8.30. (Bunke [8, Prop. 1.7]) *Let (K,ϕ) be a nonsingular symplectic form over $\mathbb{R}$, with a compatible complex structure $J : K \to K$, and let G be the maximal compact subgroup of the automorphism group (K,ϕ) fixing J. If L_1, L_2 are lagrangians of (K,ϕ) and $A : (K,\phi,J) \to (K,\phi,J)$ is an automorphism such that $A(L_1) = L_2$ then*

$$\eta(A) \;=\; \int_G \sigma(gL, L_1, L_2)dg$$

where $L \subset (K,\phi)$ is an arbitrary lagrangian. The function

$$m \;:\; (L_1, L_2) \mapsto \eta(L_1, L_2) \;=\; \eta(A) \in \mathbb{R}$$

is the unique function on pairs of lagrangians, which is invariant under G with respect to the diagonal action

$$g(L_1L_2) \;=\; (gL_1, gL_2) \;(g \in G)$$

and such that

$$m(L_1, L_2) + m(L_2, L_3) + m(L_3, L_1) \;=\; \sigma(L_1, L_2, L_3) \in \mathbb{Z} \subset \mathbb{R} \;.$$

$\square$

Let K be a real vector space with a complex structure $J : K \to K$. The induced complex linear map $1 \otimes J : \mathbb{C} \otimes_{\mathbb{R}} K \to \mathbb{C} \otimes_{\mathbb{R}} K$ has eigenvalues $\pm i$, and the eigenspaces

$$\begin{aligned}(\mathbb{C} \otimes_{\mathbb{R}} K)_{\pm i} \;&=\; \ker(1 \otimes J \mp i \otimes 1 : \mathbb{C} \otimes_{\mathbb{R}} K \to \mathbb{C} \otimes_{\mathbb{R}} K) \\ &=\; \{z \otimes Jx \pm iz \otimes x \,|\, x \in K, z \in \mathbb{C}\} \subset \mathbb{C} \otimes_{\mathbb{R}} K\end{aligned}$$

are such that

$$\mathbb{C} \otimes_{\mathbb{R}} K \;=\; (\mathbb{C} \otimes_{\mathbb{R}} K)_i \oplus (\mathbb{C} \otimes_{\mathbb{R}} K)_{-i}$$

with isomorphisms

$$\begin{aligned}&K \to (\mathbb{C} \otimes_{\mathbb{R}} K)_i \;;\; x \mapsto i \otimes x + 1 \otimes Jx \;, \\ &K \to (\mathbb{C} \otimes_{\mathbb{R}} K)_{-i} \;;\; x \mapsto -i \otimes x + 1 \otimes Jx\end{aligned}$$

and projections

$$\begin{aligned}&\mathbb{C} \otimes_{\mathbb{R}} K \to (\mathbb{C} \otimes_{\mathbb{R}} K)_i \;;\; 1 \otimes x \mapsto \frac{1}{2i}(i \otimes x + 1 \otimes Jx) \;, \\ &\mathbb{C} \otimes_{\mathbb{R}} K \to (\mathbb{C} \otimes_{\mathbb{R}} K)_{-i} \;;\; 1 \otimes x \mapsto \frac{1}{2i}(i \otimes x - 1 \otimes Jx) \;.\end{aligned}$$

For a nonsingular symmetric form (K,ϕ) over $\mathbb{R}$ the nonsingular hermitian form $(\mathbb{C}\otimes_{\mathbb{R}} K, i\otimes\phi)$ over $\mathbb{C}$ splits as a sum of a positive definite and a negative definite form

$$\begin{aligned}(\mathbb{C}\otimes_{\mathbb{R}} K, i\otimes\phi) \;&=\; ((\mathbb{C}\otimes_{\mathbb{R}} K)_i,\phi_i)\oplus((\mathbb{C}\otimes_{\mathbb{R}} K)_{-i},\phi_{-i})\\ &\cong\; (K,\phi J)\oplus(K,-\phi J)\ .\end{aligned}$$

Given a compatible complex structure J for (K,ϕ) and a lagrangian L the complex linear maps

$$\begin{aligned}&\mathbb{C}\otimes_{\mathbb{R}} L\to K\ ;\ (u+iv)\otimes x\mapsto ux+v(Jx)\ (u,v\in\mathbb{R},\ x\in L)\ ,\\ &\Phi^+(L)\ :\ \mathbb{C}\otimes_{\mathbb{R}} L\to\mathbb{C}\otimes_{\mathbb{R}} K\to(\mathbb{C}\otimes_{\mathbb{R}} K)_i\ ;\ z\otimes x\mapsto\frac{1}{2}(z\otimes x-iz\otimes Jx)\ ,\\ &\Phi^-(L)\ :\ \mathbb{C}\otimes_{\mathbb{R}} L\to\mathbb{C}\otimes_{\mathbb{R}} K\to(\mathbb{C}\otimes_{\mathbb{R}} K)_{-i}\ ;\ z\otimes x\mapsto\frac{1}{2}(z\otimes x+iz\otimes Jx)\end{aligned}$$

are isomorphisms, with $\mathbb{C}\otimes_{\mathbb{R}} L,\mathbb{C}\otimes_{\mathbb{R}} JL\subset\mathbb{C}\otimes_{\mathbb{R}} K$ complementary lagrangians in $(\mathbb{C}\otimes_{\mathbb{R}} K, i\otimes\phi)$. The composite

$$\Phi(L)\;=\;\Phi^-(L)\Phi^+(L)^{-1}\ :\ (\mathbb{C}\otimes_{\mathbb{R}} K)_i\to(\mathbb{C}\otimes_{\mathbb{R}} K)_{-i}$$

is a complex linear isomorphism such that

$$\mathbb{C}\otimes_{\mathbb{R}} L\;=\;\{(x,\Phi(L)(x))\,|\,x\in(\mathbb{C}\otimes_{\mathbb{R}} K)_i\}\subset\mathbb{C}\otimes_{\mathbb{R}} K\ ,$$

as in Kirk and Lesch [23, Definition 8.14]. The isomorphism of nonsingular hermitian forms over $\mathbb{C}$

$$A_L\ :\ H_-(\mathbb{C}\otimes_{\mathbb{R}} L)\to(\mathbb{C}\otimes_{\mathbb{R}} K, i\otimes\phi)\ ;\ (x,y)\mapsto\Phi^+(L)(x)+\Phi^-(JL)((1\otimes\phi_L)^{-1}(y))$$

??

As in Bunke [8, p.403] define the automorphism

$$\sigma_L\ :\ K\;=\;L\oplus JL\to K\;=\;L\oplus JL\ ;\ (x,y)\mapsto(-x,y)$$

with eigenvalues $-1,+1$, such that

$$\begin{aligned}&(\sigma_L)^2\;=\;1\ ,\ \sigma_L(L)\;=\;L\ ,\ \sigma_L(JL)\;=\;JL\ ,\\ &\sigma_L J\;=\;-J\sigma_L\ ,\ (1-\sigma_L)(K)\;=\;L\ ,\ (1+\sigma_L)(K)\;=\;JL\ .\end{aligned}$$

If L' is another lagrangian of (K,ϕ) then

$$B\;=\;-\sigma_{L'}\sigma_L\ :\ (K,(\phi,J))\to(K,(\phi,J))$$

is the square $B=A^2$ of an automorphism such that $A_L=L'$.

9. Asymmetric forms

Definition 9.1. (i) An **asymmetric form** (K,λ) is a bilinear pairing

$$\lambda ~:~ K\times K\to R~;~(x,y)\mapsto \lambda(x,y)$$

on a finite-dimensional real vector space K. The **adjoint** of λ is the linear map

$$\lambda ~:~ K\to K^*~;~x\mapsto (y\mapsto \lambda(x,y))~.$$

There is virtually no difference between the pairing and the adjoint.
(ii) The **dual** of an asymmetric form (K,λ) is the asymmetric form (K,λ^*) with

$$\lambda^* ~:~ K\times K\to R~;~(x,y)\mapsto \lambda^*(x,y)=\lambda(y,x)~.$$

The adjoint $\lambda^*:K\to K^*$ is the dual of the adjoint $\lambda:K\to K^*$.
(iii) An asymmetric form (K,λ) is **nonsingular** if $\lambda:K\to K^*$ is an isomorphism, or equivalently if $\lambda^*:K\to K^*$ is an isomorphism.
(iv) The **monodromy** of a nonsingular asymmetric form (K,λ) is the automorphism

$$A ~=~ \lambda^{-1}\lambda^* ~:~ K\to K$$

such that for $\epsilon=\pm 1$

$$I+\epsilon A ~=~ \lambda^{-1}(\lambda+\epsilon\lambda^*) ~:~ K\to K~.$$

(v) A nonsingular asymmetric form (K,λ) is **fibred** if A is fibred, i.e. if $I-A, I+A : K\to K$ are isomorphisms, or equivalently if the symmetric form $(K,\lambda+\lambda^*)$ and the symplectic form $(K,\lambda-\lambda^*)$ are nonsingular.

□

Proposition 9.2. *The monodromy defines a one-one correspondence of isomorphism classes*

$$\begin{aligned}&\{\textit{fibred nonsingular asymmetric forms}\}\\ &\to\{\textit{fibred symplectic automorphisms}\}~;\\ &\qquad\qquad (K,\lambda)\mapsto (K,\lambda-\lambda^*,A)~,~A~=~\lambda^{-1}\lambda^*\end{aligned}$$

with inverse

$$\begin{aligned}&\{\textbf{fibred symplectic automorphisms}\}\\ &\to\{\textbf{fibred nonsingular asymmetric forms}\}~;\\ &\qquad\qquad (K,\phi,A)\mapsto (K,\lambda)~,~\lambda~=~\phi(I-A)^{-1}~.\end{aligned}$$

For any such (K,ϕ,A), (K,λ) there is defined an isomorphism of non-singular symmetric forms

$$I-A \;:\; (K,\phi(A-A^{-1})) \to (K,\lambda+\lambda^*)\ .$$

$\square$

Definition 9.3. Let (K,λ) and (K,ϕ,A) be related as in Proposition 9.2

$$A \;=\; \lambda^{-1}\lambda^*\ ,\ \phi \;=\; \lambda-\lambda^*\ ,\ \lambda \;=\; \phi(1-A)^{-1}\ .$$

(i) The **Alexander polynomial** of (K,λ) is the Alexander polynomial of the monodromy $A=\lambda^{-1}\lambda^*:K\to K$

$$\begin{aligned}\Delta_{(K,\lambda)}(z) &= \Delta_A(z)\\ &= \det((\lambda-\lambda^*)^{-1}(z\lambda-\lambda^*):K[z]\to K[z])\\ &= \prod_{j=1}^{n}\frac{(z^2-2z\cos\theta_j+1)}{4\sin^2\theta_j/2}\in\mathbb{R}[z]\end{aligned}$$

with $0<\theta_1<\theta_2<\dots<\theta_n<\pi$. Set

$$\begin{aligned}\theta_0 &= 0\ ,\ \theta_{2n+1} \;=\; 2\pi\ ,\\ \theta_j &= 2\pi-\theta_{2n-j+1}\in(\pi,2\pi)\ (j=n+1,n+2,\dots,2n)\ .\end{aligned}$$

The eigenspaces of the monodromy A

$$K_j \;=\; \{x\in K\,|\,(A^2-2A\cos\theta_j+1)(x)=0\}$$

give decompositions

$$(K,\phi,A) \;=\; \sum_{j=1}^{n}(K_j,\phi_j,A_j)\ ,\ (K,\lambda) \;=\; \sum_{j=1}^{n}(K_j,\lambda_j)$$

with

$$\sigma(K_j,\phi_j(A_j-A_j^{-1})) \;=\; \sigma(K_j,\lambda_j+\lambda_j^*)\in 2\mathbb{Z}\ .$$

Let

$$\alpha_j \;=\; \begin{cases}\sigma(K_j,\phi_j(A_j-A_j^{-1}))/2 & \text{if } 1\leqslant j\leqslant n\\ -\alpha_{2n-j+1} & \text{if } n+1\leqslant j\leqslant 2n\end{cases}\in\mathbb{Z}\ .$$

(ii) The **ω-signature** of (K,λ) is given for $\omega=e^{i\psi}\in S^1$ by

$$\begin{aligned}\sigma_\omega(K,\lambda) &= \sigma_\omega(K,\phi,A) \;=\; \sigma(\mathbb{C}\otimes_{\mathbb{R}}K,(1-\omega)\lambda+(1-\overline{\omega})\lambda^*)\\ &= \begin{cases}2\sum\limits_{j=1}^{m}\alpha_j & \text{if } \theta_m<\psi<\theta_{m+1}\\ 2(\sum\limits_{j=1}^{m-1}\alpha_j)+\alpha_m & \text{if } \psi=\theta_m\end{cases}\end{aligned}$$

for $m = 0, 1, 2, \ldots, 2n-1$. The L^2**-signature**, the η**-invariant** and the **signature** of (K, λ) are given by

$$\begin{aligned}
\rho(K,\lambda) \;&=\; \rho(K,\phi,A) \;=\; \int_\omega \sigma_\omega(K,\lambda) \\
&=\; \sum_{j=1}^{2n} \sigma_{(\theta_j+\theta_{j+1})/2}(K,\lambda)(\theta_{j+1}-\theta_j)/\pi \;=\; 2\sum_{j=1}^{n} \alpha_j \eta(\theta_j/2)\ , \\
\eta(K,\lambda) \;&=\; \eta(K,\phi,A) \;=\; \sum_{j=1}^{n} \alpha_j \eta(\theta_j)\ , \\
\sigma(K,\lambda) \;&=\; \sigma(K,\phi,A) \;=\; \sigma_{-1}(K,\lambda) \;=\; 2\sum_{j=1}^{n} \alpha_j \in \mathbb{Z}\ .
\end{aligned}$$

□

Example 9.4. The fibred symplectic automorphism defined for $\alpha \in \{-1, 1\}$ and $\theta \in (0, \pi)$ by

$$(K,\phi,A) \;=\; (\ \mathbb{R}\oplus\mathbb{R}\ ,\ \alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ,\ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\ )$$

corresponds to the fibred nonsingular asymmetric form

$$(K,\lambda) \;=\; (\ \mathbb{R}\oplus\mathbb{R}\ ,\ \frac{\alpha}{2}\begin{pmatrix} \cot\theta/2 & 1 \\ -1 & \cot\theta/2 \end{pmatrix}\ )$$

with

$$\begin{aligned}
\Delta_{(K,\lambda)}(z) \;&=\; \frac{(z^2 - 2z\cos\theta + 1)}{4\sin^2\theta/2} \in \mathbb{R}[z]\ , \\
\sigma_\omega(K,\lambda) \;&=\; \begin{cases} 2\alpha & \text{if } \theta < \psi < 2\pi - \theta \\ 0 & \text{if } 0 < \psi < \theta \text{ or } 2\pi - \theta < \psi < 2\pi\ , \end{cases} \\
\sigma(K,\lambda) \;&=\; 2\alpha \in 2\mathbb{Z}\ ,\ \rho(K,\lambda) \;=\; 2\alpha\eta(\theta/2)\ ,\ \eta(K,\lambda) \;=\; \alpha\eta(\theta) \in \mathbb{R}\ , \\
\rho(K,\lambda) - \eta(K,\lambda) \;&=\; \alpha \;=\; \sigma(K,\lambda)/2 \in \mathbb{Z} \subset \mathbb{R}\ .
\end{aligned}$$

□

Proposition 9.5. *The L^2-signature, the η-invariant and the signature are Witt group invariants of a fibred nonsingular asymmetric form (K, λ) over $\mathbb{R}$, and are related by*

$$\rho(K,\lambda) - \eta(K,\lambda) \;=\; \sigma(K,\lambda)/2 \in \mathbb{Z} \subset \mathbb{R}\ .$$

□

10. Knots

KEY RESULT: if $k : S^1 \subset S^3$ is a knot with Seifert surface $F \subset S^3$ and Seifert form $(H_1(F), \lambda)$ the relative cobordism

$$(D^4; X_F; F \times I; F \cup_\partial F)$$

is a fundamental domain for the infinite cyclic cover $(\overline{Y}, \overline{X})$ of the 4-manifold with boundary

$$(Y, X) \ = \ (\mathrm{cl.}(D^4 \backslash (F \times D^2)), \mathrm{cl.}(S^3 \backslash (k(S^1) \times D^2))) \ .$$

The fundamental domain has real signature

$$\sigma_{\mathbb{R}}(D^4; X_F; F \times I; F \cup_\partial F) \ = \ \sigma(D^4) + \eta(A) \ = \ \eta(A) \in \mathbb{R} \ .$$

The double cover of (D^4, S^3) branched over $(F, k(S^1))$ is a relative cobordism

$$(D^4; X_F; F \times I; F \cup_\partial F) \cup (D^4; F \times I, X_F; F \cup_\partial F)) \ = \ (D^4 \cup_{F \times I} D^4; X_F, X_F; F \cup_\partial F)$$

with intersection form $(H_1(F), \lambda + \lambda^*)$, and real signature

$$\begin{aligned} & \sigma_{\mathbb{R}}(D^4 \cup_{F \times I} D^4; X_F, X_F; F \cup_\partial F) \\ & = \ \sigma_{\mathbb{R}}(D^4; X_F; F \times I; F \cup_\partial F) + \sigma_{\mathbb{R}}(D^4; F \times I, X_F; F \cup_\partial F) \\ & = \ \sigma(H_1(F), \lambda + \lambda^*) + \eta(A^2) \ = \ 2\eta(A) \in \mathbb{R} \end{aligned}$$

so that the signature of k is

$$\sigma(k) \ = \ \sigma(H_1(F), \lambda + \lambda^*) \ = \ 2\eta(A) - \eta(A^2) \ .$$

$\square$

The exterior of a knot $k : S^1 \subset S^3$

$$X \ = \ S^3 \backslash k(S^1)$$

is a homology circle, with

$$H_*(X) \ = \ H_*(S^1) \ , \ H^*(X) \ = \ H^*(S^1) \ .$$

The generator $1 \in H^1(X) = \mathbb{Z}$ is represented by a map $p : X \to S^1$ such that making p transverse regular at $* \in S^1$ gives a Seifert surface for k, a codimension 1 submanifold

$$F \ = \ p^{-1}(*) \subset S^3$$

with

$$\partial F \ = \ k(S^1) \subset S^3 \ ,$$

and with a neighbourhood $F \times I \subset S^3$. The 3-dimensional relative cobordism

$$(X_F; F_0, F_1; \partial F) \ = \ (\mathrm{cl.}(X \backslash F \times I); F \times \{0\}, F \times \{1\}; \partial F \times \{1/2\})$$

is a fundamental domain for the canonical infinite cyclic cover

$$\overline{X} \;=\; p^*\mathbb{R} \;=\; \bigcup_{k=-\infty}^{\infty} \zeta^k(X_F; F_0, F_1; \partial F)$$

with $\zeta : \overline{X} \to \overline{X}$ a generating covering translation.

Proposition 10.1. (Milnor [34]) *The $\mathbb{R}$-coefficient cohomology of $\overline{X}$ is such that*

$$\dim_{\mathbb{R}} H_2(\overline{X};\mathbb{R}) \;=\; 1\ ,\ \dim_{\mathbb{R}} H^1(\overline{X};\mathbb{R}) < \infty\ .$$

The characteristic polynomial of the monodromy automorphism

$$A \;=\; \zeta^* \;:\; K \;=\; H^1(\overline{X};\mathbb{R}) \to K$$

is the Alexander polynomial of k

$$\Delta(z) \;=\; \det(z - A : K[z,z^{-1}] \to K[z,z^{-1}]) \in \mathbb{R}[z,z^{-1}]$$

with $\Delta(1) = 1 \in \mathbb{R}$. The monodromy is an elliptic automorphism of the nonsingular symplectic form over $\mathbb{R}$

$$\phi \;:\; K \times K \to \mathbb{R}\ ;\ (x,y) \mapsto \langle (A(x) \cup y) + (x \cup A(y), [\overline{X}] \rangle\ .$$

so that (K,ϕ,A) is an elliptic symplectic automorphism, with eigenvalues the roots of $\Delta(z)$. The linear isomorphism

$$\lambda \;=\; \phi(1-A)^{-1} \;:\; K \to K^*$$

is such that

$$\lambda - \lambda^* \;=\; \phi \;:\; K \to K^*\ .$$

□

Proposition 10.2. *For any Seifert surface $F \subset S^3$ the linear isomorphism*

$$\lambda \;:\; H_1(F;\mathbb{R}) \to H_1(S^3\backslash F;\mathbb{R}) \;=\; H^1(F;\mathbb{R}) \;=\; H_1(F;\mathbb{R})^*$$

defines a Seifert (asymmetric) form for k, with

$$\lambda - \lambda^* \;:\; H_1(F;\mathbb{R}) \to H_1(F;\mathbb{R})^*$$

the nonsingular symplectic form over $\mathbb{R}$, and

$$\begin{aligned}
&K \;=\; \operatorname{coker}(\lambda - z\lambda^* : H_1(F;\mathbb{R})[z,z^{-1}] \to H_1(F;\mathbb{R})^*[z,z^{-1}])\ ,\\
&\phi \;:\; K \times K \to \mathbb{R}\ ;\\
&(x,y) \mapsto \big(\textit{coefficient of } z^{-1} \textit{ in } (\lambda - z\lambda^*)^{-1}(x)(y) \in \mathbb{R}(z)\big)\\
&A \;=\; \lambda^{-1}\lambda^* \;=\; \zeta^* \;=\; z \;:\; K \to K
\end{aligned}$$

□

Definition 10.3. (i) The ω**-signature** of a knot $k : S^1 \subset S^3$ is defined for any $\omega \in S^1$ by

$$\sigma_\omega(k) \;=\; \sigma_\omega(K,\lambda) \;=\; \sigma(K,(1-\omega)\lambda+(1-\overline{\omega})\lambda^*)) \in \mathbb{Z}\ .$$

(ii) The **signature** of k is

$$\sigma(k) \;=\; \sigma_{-1}(k) \;=\; \sigma(K,\lambda+\lambda^*) \in \mathbb{Z}\ .$$

□

The 0-framed surgery on $k : S^1 \subset S^3$ is the closed 3-dimensional manifold

$$M_k \;=\; X \cup_{S^1\times S^1} D^2 \times S^1$$

cobordant to S^3, with $H_1(M_k) = \mathbb{Z}$.

Definition 10.4. The ρ**-invariant** or **reduced** $\sigma^{(2)}$**-signature** of k is

$$\rho(k) \;=\; \sigma^{(2)}(W) - \sigma(W) \in \mathbb{R}$$

for any 4-dimensional manifold W with $\partial W = M_k$ and an extension of $H_1(M_k) = \mathbb{Z}$ to a morphism $H_1(W) \to \mathbb{Z}$, where $\sigma^{(2)}(W) \in \mathbb{R}$ is the $L^{(2)}$-signature and $\sigma(W) \in \mathbb{Z}$ is the ordinary signature.

□

Proposition 10.5. *The reduced $\sigma^{(2)}$-signature of a knot $k : S^1 \subset S^3$ is*

$$\rho(k) \;=\; \int\limits_{\omega\in S^1} \sigma_\omega(H_1(F),\lambda) \in \mathbb{R}$$

for any Seifert surface $F^2 \subset S^3$, with Seifert form $(H_1(F),\lambda)$.

Proof. The 4-dimensional manifold with boundary

$$(W^4,\partial W) \;=\; (\mathrm{cl.}(D^4\backslash F \times D^2), X \cup F \times S^1)$$

is such that $\pi_1(W) = \mathbb{Z}$ and the intersection form on the infinite cyclic cover $(\overline{W},\overline{\partial W})$ is

$$(H_2(\overline{W};\mathbb{R}(z)),\text{intersection form}) \;=\; (K(z),(1-z)\lambda+(1-z^{-1})\lambda^*)$$

so that

$$\sigma(W) \;=\; \sigma(K,0) \;=\; 0 \in \mathbb{Z}\ ,\ \sigma^{(2)}(W) \;=\; \int\limits_{\omega\in S^1} \sigma_\omega(K,\lambda) \in \mathbb{R}$$

The reduced $\sigma^{(2)}$-signature of k is thus

$$\rho(k) \;=\; \sigma^{(2)}(W) - \sigma(W) \;=\; \sigma^{(2)}(W) \;=\; \int\limits_{\omega\in S^1} \sigma_\omega(K,\lambda) \in \mathbb{R}\ .$$

□

Definition 10.6. The **Alexander polynomial** of $k : S^1 \subset S^3$ is the Alexander polynomial of A

$$\Delta_k(z) \ = \ \Delta_A(z) \ = \ \prod_{j=1}^{n} \frac{z^2 - 2z\cos\theta_j + 1}{4\sin^2\theta_j/2} \in \mathbb{R}[z]$$

with roots $e^{\pm i\theta_j}$ $(\theta_j \in (0,\pi))$.

□

The eigenspaces of A

$$K_j \ = \ \{x \in K \,|\, (A^2 - 2A\cos\theta_j + 1)(x) = 0\}$$

give decompositions

$$(K,\phi,A) \ = \ \sum_{j=1}^{n}(K_j,\phi_j,A_j) \ , \ (K,\lambda) \ = \ \sum_{j=1}^{n}(K_j,\lambda_j) \ .$$

As before, let

$$\theta_0 \ = \ 0 \ , \ \theta_{2n+1} \ = \ 2\pi \ , \ \theta_j \ = \ 2\pi - \theta_{2n-j+1} \in (\pi,2\pi) \ (j = n+1, n+2, \dots, 2n)$$

and

$$\alpha_j \ = \ \begin{cases} \sigma(K_j,\phi_j(A_j - A_j^{-1}))/2 & \text{if } 1 \leqslant j \leqslant n \\ -\alpha_{2n-j+1} & \text{if } n+1 \leqslant j \leqslant 2n \end{cases} \in \mathbb{Z} \ .$$

The signatures

$$2\alpha_j \ = \ \sigma(K_j,\phi_j(A_j - A_j^{-1})) \ = \ \sigma(K_j,\lambda_j + \lambda_j^*) \in 2\mathbb{Z} \ (1 \leqslant j \leqslant n)$$

are the knot signatures of Milnor [34]. The ω-signature of k (Levine [27], Tristram [55]) is given for $\omega = e^{i\psi} \in S^1$ by

$$\begin{aligned} \sigma_\omega(k) \ &= \ \sigma_\omega(K,\phi,A) \ = \ 2\sum_{j=1}^{n} \alpha_j \sigma_\omega(-\theta_j/2) \\ &= \ 2\sum_{j=1}^{n} \alpha_j(\mathrm{sgn}(\sin(\psi/2)\sin((\psi - \theta_j)/2)) \\ &= \ \begin{cases} 2\sum\limits_{j=1}^{m} \alpha_j & \text{if } \theta_m < \psi < \theta_{m+1} \\ 2(\sum\limits_{j=1}^{m-1} \alpha_j) + \alpha_m & \text{if } \psi = \theta_m \ . \end{cases} \end{aligned}$$

Definition 10.7. The **η-invariant** of a knot $k : S^1 \subset S^3$ is

$$\eta(k) \ = \ \eta(M_k, F \cup_\partial F, J) \in \mathbb{R}$$

with

$$F \cup_\partial F \ \subset M_k \ = (F \times I \cup D^2 \times S^1) \cap_{F\cup_\partial F} (F \times I \cup X) \ .$$

□

Proposition 10.8. *The ρ-invariant, η-invariant and signature of k are given by*

$$\begin{array}{rcl} \rho(k) & = & \rho(K,\lambda) \ = \ 2\sum\limits_{j=1}^{n} \alpha_j \eta(\theta_j/2) \in \mathbb{R} \ , \\ \eta(k) & = & \eta(K,\lambda) \ = \ \sum\limits_{j=1}^{n} \alpha_j \eta(\theta_j) \in \mathbb{R} \ , \\ \sigma(k) & = & \sigma(K,\lambda) \ = \ 2\sum\limits_{j=1}^{n} \alpha_j \in \mathbb{Z} \end{array}$$

are related by

$$\rho(k) - \eta(k) \ = \ \sigma(k)/2 \in \mathbb{Z} \subset \mathbb{R} \ .$$

□

Let $F^2 \subset S^3$ be a Seifert surface for k, with $\partial F = k(S^3)$, and let

$$A \ : \ (K,\phi) \ = \ (H_1(F;\mathbb{R}), \text{intersection form}) \to (K,\phi)$$

be the monodromy automorphism, which is elliptic, corresponding to the Seifert form

$$\lambda \ = \ \phi(1-A)^{-1} \ : \ K \to K^* \ .$$

The 4-dimensional manifold with boundary

$$(W^4,\partial W) \ = \ (\text{cl.}(D^4\backslash F \times D^2), T(A\cup 1 : F\cup_\partial F \to F \cup_\partial F))$$

is such that $\eta(k) = \eta(\partial W) \in \mathbb{R}$. The infinite cyclic cover $(\overline{W},\overline{\partial W})$ classified by a map $(W,\partial W) \to S^1$ representing a generator $1 \in H^1(W,\partial W) = \mathbb{Z}$ is such that

$$(H_2(\overline{M};\mathbb{R}(z)), \text{intersection form}) \ = \ (K(z), (1-z)\lambda + (1-z^{-1})\lambda^*) \ .$$

The isomorphism

$$L\text{Asy}^0(\mathbb{R}) \to L^4(\mathbb{R}(z)) \, ; \, (K,\lambda) \mapsto \sigma^*(W) = (K(z), (1-z)\lambda + (1-z^{-1})\lambda^*)$$

sends the Witt class of the Seifert form to the multisignature

$$\begin{array}{rcl} \sigma^*(W) & = & (K(z), (1-z)\lambda + (1-z^{-1})\lambda^*) \\ & = & (0, \sum\limits_{j=1}^{n} \alpha_j\theta_j) \in L^4(\mathbb{R}(z)) \ = \ \mathbb{Z} \oplus \sum\limits_{0<\theta<\pi} \mathbb{Z} \end{array}$$

The L^2-signature map of Cochran, Orr and Teichner [11, §5]

$$\sigma^{(2)} \ : \ L^4(\mathbb{R}(z)) \to L^4(\mathcal{U}\mathbb{Z}) \ = \ \mathbb{R}$$

sends $\sigma^*(W)$ to

$$\begin{aligned}\sigma^{(2)}(W) &= \int\limits_{\omega\in S^1} \sigma_\omega(k) \\ &= 2\sum_{j=1}^{n}\alpha_j\eta(\theta_j/2) \;=\; \sum_{j=1}^{n}\alpha_j\eta(\theta)+\sum_{j=1}^{n}\alpha_j \\ &= \eta(k)+\sigma(k)/2\in\mathbb{R}\ .\end{aligned}$$

Let ∂ be the boundary map in the localization exact sequence

$$0 \longrightarrow L^4(\mathbb{R}[z,z^{-1}]) \longrightarrow L^4(\mathbb{R}(z)) \xrightarrow{\partial} LAut^2(\mathbb{R}) \longrightarrow 0\ .$$

The isomorphism

$$\sigma\oplus\partial\ :\ L^4(\mathbb{R}(z)) \to L^4(\mathbb{R}[z,z^{-1}])\oplus LAut^2(\mathbb{R}) \;=\; \mathbb{Z}\oplus\sum_{0<\theta<\pi}\mathbb{Z}$$

sends the symmetric signature to

$$\begin{aligned}(\sigma(W),\partial\sigma^*(W)) &= (0,(K,\phi,A)\oplus(K,-\phi,1)) \\ &= (0,\sum_{j=1}^{n}\alpha_j\theta_j)\in L^4(\mathbb{R}(z)) \;=\; \mathbb{Z}\oplus\sum_{0<\theta<\pi}\mathbb{Z}\end{aligned}$$

Example 10.9. The right-handed trefoil knot $k: S^1\subset S^3$ has Seifert form

$$(K,\lambda) \;=\; (\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix})$$

and elliptic monodromy

$$A \;=\; \lambda^{-1}\lambda^* \;=\; \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\ :\ (K,\phi=\lambda-\lambda^*) \;=\; (\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}) \to (K,\phi)$$

The Alexander polynomial, symmetric form, ρ-invariant, η-invariant and signature of k are

$$\Delta_k(z) \;=\; z^2-z+1 \;=\; (z-e^{\pi i/3})(z-e^{-\pi i/3})\ ,\ \theta_1 \;=\; \pi/3\ ,$$

$$(K,\phi(A-A^{-1})) \;=\; (\mathbb{R}\oplus\mathbb{R},\begin{pmatrix}-2 & 1\\ 1 & -2\end{pmatrix})\ ,\ \alpha_1 \;=\; -1\ ,$$

$$\rho(k) \;=\; 2\alpha_1\eta(\theta_1/2) \;=\; -4/3\ ,$$

$$\eta(k) \;=\; \alpha_1\eta(\theta_1) \;=\; -1/3\ ,$$

$$\sigma(k) \;=\; 2\alpha_1 \;=\; -2\in\mathbb{Z}\ .$$

$\square$

Example 10.10. (Kervaire [21, III.12], Milnor [34]) For $m \neq 0 \in \mathbb{Z}$ let K_m be the m-twist knot, with Seifert matrix

$$(K,\lambda) \;=\; (\mathbb{R}\oplus\mathbb{R}, \begin{pmatrix} m & 0 \\ -1 & 1 \end{pmatrix}) \;.$$

The Alexander polynomial is

$$\Delta_{K_m}(z) \;=\; \frac{\det(\lambda - z\lambda^*)}{\det(\lambda-\lambda^*)} \;=\; mz^2-(2m-1)z+m \in \mathbb{R}[z]$$

with roots

$$z_1,z_2 \;=\; \frac{(2m-1)\pm\sqrt{1-4m}}{2m} \in \mathbb{C}$$

such that $z_1z_2=1$. The monodromy is

$$A \;=\; \lambda^{-1}\lambda^* \;=\; \begin{pmatrix} 1 & -\dfrac{1}{m} \\ 1 & \dfrac{m-1}{m} \end{pmatrix} \;:\; (K,\phi)\to(K,\phi)$$

with

$$\phi \;=\; \lambda-\lambda^* \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;:\; K\to K^* \;.$$

For $m \leqslant -1$ the roots of A are real and K_m is algebraically slice, with

$$\sigma_\omega(K_m) \;=\; 0 \; (\omega\in S^1) \;,\; \sigma(K_m) \;=\; \rho(K_m) \;=\; \eta(K_m) \;=\; 0 \;.$$

K_m is slice if and only if $-m=n(n+1)$ for some $n\geqslant 1$ by Casson and Gordon [9].

For $m\geqslant 1$ A is elliptic, with $\{z_1,z_2\}=\{e^{i\theta_m},e^{-i\theta_m}\}$ for $\theta_m\in(0,\pi/2)$ such that

$$\cos\theta_m \;=\; \frac{2m-1}{2m} \;,\; \sin\theta_m \;=\; \frac{\sqrt{4m-1}}{2m} \;,$$

with

$$(K,\phi,A) \;=\; (\mathbb{R}\oplus\mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{pmatrix}) \;.$$

The ρ-invariant, η-invariant and signature of K_m

$$\begin{aligned} \rho(K_m) \;&=\; 2\eta(\theta_m/2) \;=\; 2-\frac{2\theta_m}{\pi} \in \mathbb{R} \;, \\ \eta(K_m) \;&=\; \eta(\theta_m) \;=\; 1-\frac{2\theta_m}{\pi} \in \mathbb{R} \;, \\ \sigma(K_m) \;&=\; 2 \in \mathbb{Z} \end{aligned}$$

are related by

$$\rho(K_m)-\eta(K_m) \;=\; \sigma(K_m)/2 \;=\; 1 \in \mathbb{Z}\subset\mathbb{R} \;.$$

(The computation of $\rho(K_{2m})$ first appeared in Cochran, Orr and Teichner [12, 5.7]). $\square$

Example 10.11. Let $T(p,q)$ be the (p,q)-torus knot, with $p,q \geqslant 2$ coprime integers. Let $n=(p-1)(q-1)/2$, and let $1=r_1<r_2<\cdots<r_n$ be such that

$$\{r \,|\, 1 \leqslant r < pq/2,\ p \not| r,\ q \not| r\} ~=~ \{r_1, r_2, \dots, r_n\}~,$$

with

$$r_j ~=~ a_j q + b_j p ~(a_j \neq 0,~ 1 \leqslant b_j < q)~.$$

The Alexander polynomial of $T(p,q)$ is

$$\Delta_{T(p,q)}(z) ~=~ \frac{(z^{pq}-1)(z-1)}{(z^p-1)(z^q-1)} \in \mathbb{R}[z]$$

with roots $\{e^{\pm i\theta_j} \,|\, 1 \leqslant j \leqslant n\}$ given by

$$\theta_j ~=~ \frac{2\pi r_j}{pq} ~=~ 2\pi(\frac{a_j}{p}+\frac{b_j}{q})~.$$

The elliptic symplectic automorphism (K,ϕ,A) over $\mathbb{R}$ is given by

$$\begin{aligned}(K,\phi,A) ~&=~ \sum_{j=1}^{n}(K_j,\phi_j,A_j)\\ &=~ \sum_{j=1}^{n}(\mathbb{R}\oplus\mathbb{R}, \alpha_j\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}\cos\theta_j & -\sin\theta_j\\ \sin\theta_j & \cos\theta_j\end{pmatrix})\end{aligned}$$

with

$$\alpha_j ~=~ \mathrm{sgn}(a_j) \in \{-1,1\}~.$$

The ρ-invariant, η-invariant and signature of $T(p,q)$

$$\begin{aligned}\rho(T(p,q)) ~&=~ 2\sum_{j=1}^{n}\alpha_j\eta(\theta_j/2) ~=~ 2\sum_{j=1}^{n}\alpha_j\eta(\frac{\pi r_j}{pq}) ~=~ 2\sum_{j=1}^{n}\alpha_j(1-\frac{2r_j}{pq}) \in \mathbb{R}~,\\ \eta(T(p,q)) ~&=~ \sum_{j=1}^{n}\alpha_j\eta(\theta_j) ~=~ \sum_{j=1}^{n}\alpha_j\eta(\frac{2\pi r_j}{pq}) ~=~ \sum_{j=1}^{n}\alpha_j(1-\frac{4r_j}{pq}) \in \mathbb{R}~,\\ \sigma(T(p,q)) ~&=~ 2\sum_{j=1}^{n}\alpha_j \in \mathbb{Z}\end{aligned}$$

are related by

$$\rho(T(p,q))-\eta(T(p,q)) ~=~ \sigma(T(p,q))/2 \in \mathbb{Z} \subset \mathbb{R}~.$$

Kirby and Melvin [22, 3.8], Borodzik [6] and Collins [13] have obtained a formula for the ρ-invariant

$$\rho(T(p,q)) ~=~ -\frac{(p-1)(p+1)(q-1)(q+1)}{3pq} \in \mathbb{R}$$

Borodzik and Oleszkiewicz [7] have recently obtained a formula for the signature for coprime p, q odd

$$\sigma(T(p,q)) \;=\; -\frac{pq}{2}+\frac{2p}{3q}+\frac{2q}{3p}+\frac{1}{6pq}-4(s(2p,q)+s(2q,p))-1\in\mathbb{Z}\ .$$

□

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