Geometric invariant theory for non-reductive group actions and jet differentials

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(based on joint work with Brent Doran and Gergely Berczi)

Moduli spaces (or stacks) are often constructed as quotients of algebraic varieties by group actions.

Reductive groups $\rightsquigarrow$ we can use Mumford's GIT (+ techniques from symplectic geometry)

Non-reductive groups?

E.g. moduli spaces of hypersurfaces/complete intersections in toric varieties – automorphism group of a toric variety is not in general reductive.

Example: weighted projective plane $\mathbb{P}(1,1,2)$

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$$

$$\text{with } R \cong GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* \cong GL(2) \text{ reductive}$$

$$U \cong (\mathbb{C}^+)^3 \text{ unipotent}$$

where $(x,y,z) \mapsto (x,y,z+\lambda x^2+\mu xy+\nu y^2)$ for $(\lambda,\mu,\nu) \in \mathbb{C}^3$

Mumford's GIT

G complex reductive group
X complex projective variety acted on by G

We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).

$$\begin{array}{ccccl} X & \Rightarrow & A(X) & = & \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X \\ | & & & & = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ | & & \cup| & & \\ \downarrow & & & & \\ X/\!/G & \Leftarrow & A(X)^G & & \text{ring of invariants} \end{array}$$

G reductive implies that $A(X)^G$ is a *finitely generated* graded complex algebra so that $X/\!/G = \mathrm{Proj}(A(X)^G)$ is a projective variety.

The rational map $X - - \rightarrow X//G$ fits into a diagram

$$\boxed{\begin{array}{rcccl} & X & -- \rightarrow & X//G & \text{cx proj variety} \\ & \bigcup & & \| & \\ \text{semistable} & X^{ss} & \stackrel{\text{onto}}{\longrightarrow} & X//G & \\ & \bigcup & & \bigcup & \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/G & \end{array}}$$

where the morphism $X^{ss} \rightarrow X//G$ is G-invariant and surjective.

Topologically $\boxed{X//G = X^{ss}/\sim}$ where

$$x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset.$$

N.B. G reductive $\Leftrightarrow$ G is the complexification $K_{\mathbb{C}}$ of a maximal compact subgroup K (for example $SL(n) = SU(n)_{\mathbb{C}}$), and then

$$X//G = \mu^{-1}(0)/K$$

for a suitable moment map μ for the action of K.

What if G is not reductive?

Problem: We can't define a projective variety

$$X//G = \mathrm{Proj}(A(X)^G)$$

where $A(X) = \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X$ because $A(X)^G$ is not necessarily finitely generated. [In fact G is reductive if and only if $A(X)^G$ is finitely generated for all such X].

Question: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive?

N.B. Any linear algebraic group has a unipotent normal subgroup $U \leqslant G$ (its unipotent radical) such that $R = G/U$ is reductive [for unipotent think strictly upper triangular matrices].

Moreover U has a (canonical) chain of normal subgroups

$$\{1\} = U_0 \leqslant U_1 \leqslant \ldots \leqslant U_s = U$$

such that each $U_j/U_{j-1} \cong \mathbb{C}^+ \times \mathbb{C}^+ \times \cdots \times \mathbb{C}^+$.

Theorem (Doran, K): Let $H = R \ltimes U$ be a linear algebraic group over $\mathbb{C}$ acting linearly on $X \subseteq \mathbb{P}^n$.

Then X has open subsets X^s ('stable points') and X^{ss} ('semistable points') with a geometric quotient $X^s \to X^s/H$ and an 'enveloping quotient' $X^{ss} \to X//H$.
Moreover *if* $A(X)^H$ is finitely generated then

$$X//H = \text{Proj}(A(X)^H).$$

We have a similar diagram to the reductive case

$$\begin{array}{rccccl} & X & - - \to & X//H & \\ & \bigcup & & || & \\ \text{semistable} & X^{ss} & \longrightarrow & X//H & \\ & \bigcup & & \bigcup & \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/H & \end{array}$$

BUT $X//H$ is **not necessarily projective** and $X^{ss} \to X//H$ is **not necessarily onto**.

Reductive envelopes

We can choose reductive $G \supseteq H$ and a suitable compactification $\overline{G \times_H X}$ of $G \times_H X$ giving a (non-canonical) compactification $\overline{G \times_H X}/\!/G$ of $X/\!/H$:

$$X^s/H \subseteq X/\!/H \subseteq \overline{G \times_H X}/\!/G$$

However although such a compactification always exists, it is not at all easy in general to decide when a compactification $\overline{G \times_H X}$ of $G \times_H X$ has the properties needed.

Simple example: $\mathbb{C}^+$ acting on $\mathbb{P}^n$
We can choose coordinates in which the generator of $Lie(\mathbb{C}^+)$ has Jordan normal form with blocks of size $k_1+1, \ldots, k_q+1$. The linear $\mathbb{C}^+$ action therefore extends to $G = SL(2)$ with

$$\mathbb{C}^+ = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}\} \leqslant G$$

via $\mathbb{C}^{n+1} \cong \bigoplus_{i=1}^{q} Sym^{k_i}(\mathbb{C}^2)$.

In fact in this case the invariants are finitely generated (Weitzenbock) so we can define

$$\mathbb{P}^n/\!/\mathbb{C}^+ = \mathrm{Proj}((\mathbb{C}[x_0, \ldots, x_n])^{\mathbb{C}^+}).$$

N.B. Via $(g, x) \mapsto (g\mathbb{C}^+, gx)$ we have

$$G \times_{\mathbb{C}^+} \mathbb{P}^n \cong (G/\mathbb{C}^+) \times \mathbb{P}^n \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}^n$$

$$\subseteq \mathbb{C}^2 \times \mathbb{P}^n \subseteq \mathbb{P}^2 \times \mathbb{P}^n$$

and so

$$\boxed{\mathbb{P}^n/\!/\mathbb{C}^+ \cong (\mathbb{P}^2 \times \mathbb{P}^n)/\!/SL(2)}$$

$$
\begin{array}{ccc}
\mathbb{P}^2 \times \mathbb{P}^n & --\rightarrow & \mathbb{P}^2 \times \mathbb{P}^n /\!/ G \\
\cup & & || \\
\mathbb{P}^n = \{[1:0:1]\} \times \mathbb{P}^n & --\rightarrow & \mathbb{P}^n /\!/ \mathbb{C}^+ \\
\cup & & || \\
(\mathbb{P}^n)^{ss} & \stackrel{\text{not nec onto}}{\longrightarrow} & \mathbb{P}^n /\!/ \mathbb{C}^+ \\
\cup & & \cup \\
(\mathbb{P}^n)^s & \longrightarrow & (\mathbb{P}^n)^s / \mathbb{C}^+
\end{array}
$$

Example when $(\mathbb{P}^n)^{ss} \to \mathbb{P}^n /\!/ \mathbb{C}^+$ is *not* onto:

$\mathbb{P}^3 = \mathbb{P}(Sym^3(\mathbb{C}^2)) = \{$ 3 unordered points on $\mathbb{P}^1\}$.

Then $(\mathbb{P}^3)^{ss} = (\mathbb{P}^3)^s$ is
$\{$ 3 unordered points on $\mathbb{P}^1$, at most one at $\infty\}$

and its image in

$$\mathbb{P}^3 /\!/ \mathbb{C}^+ = (\mathbb{P}^3)^s / \mathbb{C}^+ \ \sqcup \ \mathbb{P}^3 /\!/ SL(2)$$

is the open subset $(\mathbb{P}^3)^s / \mathbb{C}^+$ which does not include the ‘boundary’ points coming from

$$0 \in \mathbb{C}^2 \subseteq \mathbb{P}^2.$$

The blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at $0 \in \mathbb{C}^2 \subseteq \mathbb{P}^2$ can be identified with $G \times_B \mathbb{P}^1$ where B is the Borel subgroup of $G = SL(2)$ containing $\mathbb{C}^+$ and the standard maximal torus $T \cong \mathbb{C}^*$.
Similarly the blow-up of $\mathbb{P}^2 \times \mathbb{P}^n$ along $\{0\} \times \mathbb{P}^n$ can be identified with $G \times_B (\mathbb{P}^1 \times \mathbb{P}^n)$.
Let $\widetilde{\mathbb{P}^n/\!/\mathbb{C}^+}$ be the blow-up of

$$\mathbb{P}^n/\!/\mathbb{C}^+ = (\mathbb{P}^2 \times \mathbb{P}^n)/\!/G$$

along the subvariety $\mathbb{P}^n/\!/G$ corresponding to $0 \in \mathbb{P}^2$. Then the G-invariant surjection

$$(\mathbb{P}^2 \times \mathbb{P}^n)^{ss,G} \to (\mathbb{P}^2 \times \mathbb{P}^n)/\!/G = \mathbb{P}^n/\!/\mathbb{C}^+$$

induces a B-invariant surjection

$$(\mathbb{P}^1 \times \mathbb{P}^n)^{ss,B} \to \mathbb{P}^n/\!/\mathbb{C}^+$$

from a suitable open subset $(\mathbb{P}^1 \times \mathbb{P}^n)^{ss,B}$ of $\mathbb{P}^1 \times \mathbb{P}^n$, and thus a surjection from an open subset of the GIT quotient

$$\mathcal{X} = (\mathbb{P}^1 \times \mathbb{P}^n)/\!/T$$

to $\mathbb{P}^n/\!/\mathbb{C}^+$.

In constructing the GIT quotient

$$\mathcal{X} = (\mathbb{P}^1 \times \mathbb{P}^n)//T$$

to get a surjection from an open subset $\mathcal{X}^{ss}$ of $\hat{X}$ to $\mathbb{P}^n//\mathbb{C}^+$, the action of $T \cong \mathbb{C}^*$ on $\mathbb{P}^1 \times \mathbb{P}^n$ has to be appropriately linearised; a different choice of linearisation would give

$$(\mathbb{P}^1 \times \mathbb{P}^n)//T = (\mathbb{C}^* \times \mathbb{P}^n)/T = \mathbb{P}^n.$$

Thus the theory of variation of GIT quotients (Thaddeus, Dolgachev-Hu, Ressayre) tells us that $\mathcal{X}$ and $\mathbb{P}^n$ are related by a sequence of explicit blow-ups + blow-downs (flips in the sense of Thaddeus).

VGIT $\rightsquigarrow$ $\boxed{\mathcal{X} = (\mathbb{P}^1 \times \mathbb{P}^n)//T \overset{\text{flips}}{\longleftarrow - \longrightarrow} X = \mathbb{P}^n}$

$$\begin{array}{ccccc}
\mathbb{P}^n & & \overset{\text{flips}}{\longleftarrow -- \longrightarrow} & & \mathcal{X} \\
\bigcup & & & & \bigcup \\
(\mathbb{P}^n)^{ss} & \longrightarrow & \mathbb{P}^n//\mathbb{C}^+ & \overset{\text{onto}}{\longleftarrow} & \mathcal{X}^{ss} \\
\bigcup & & \bigcup & & \bigcup \\
(\mathbb{P}^n)^s & \longrightarrow & (\mathbb{P}^n)^s/\mathbb{C}^+ & \longleftarrow & (\mathbb{P}^n)^s
\end{array}$$

Defn: Call a unipotent linear algebraic group U *graded unipotent* if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all strictly positive. Then let

$$\hat{U} = U \rtimes \mathbb{C}^* = \{(u,t) : u \in U, t \in \mathbb{C}^*\}$$

with multiplication $(u,t)\cdot(u',t') = (u(\lambda(t)(u')), tt')$.

Thm: *Let U be graded unipotent acting linearly on a projective variety X, and suppose that the action extends to $\hat{U} = U \rtimes \mathbb{C}^*$. Then*
(i) the ring $A(X)^U$ of U-invariants is finitely generated, so that $X//U = \text{Proj}(A(X)^U)$;
(ii) there is a projective variety $\mathcal{X}$ which is related to X via VGIT and a surjection $\mathcal{X}^{ss} \to X//U$ to $X//U$ from an open subset $\mathcal{X}^{ss}$ of $\mathcal{X}$.

$$\begin{array}{ccccc}
X & & \overset{\text{flips}}{\longleftarrow - - \longrightarrow} & & \mathcal{X} \\
\cup & & & & \cup \\
X^{ss} & \longrightarrow & X//U = \text{Proj}(A(X)^U) & \overset{\text{onto}}{\longleftarrow} & \mathcal{X}^{ss} \\
\cup & & \cup & & \cup \\
X^s & \longrightarrow & X^s/U & \longleftarrow & X^s
\end{array}$$

Example: The automorphism group of the weighted projective plane $\mathbb{P}(1,1,2)$ is

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$$

with $R \cong GL(2)$ reductive and $U \cong (\mathbb{C}^+)^3$ unipotent .... $(\lambda, \mu, \nu) \in (\mathbb{C}^+)^3$ acts as $(x, y, z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2)$.

The central one-parameter subgroup $\mathbb{C}^*$ of $R \cong GL(2)$ acts on $Lie(U)$ with all positive weights, and the associated extension $\hat{U} = U \rtimes \mathbb{C}^*$ can be identified with a subgroup of $\mathrm{Aut}(\mathbb{P}(1,1,2))$.

Corollary When $H = \mathrm{Aut}(\mathbb{P}(1,1,2))$ acts linearly on a projective variety X, the ring of invariants $A(X)^H$ is finitely generated as a complex algebra, so that

$$X//H = \mathrm{Proj}(A(X)^H),$$

and moreover there is a projective variety $\mathcal{X}$ which is related to X via VGIT and a surjection $\mathcal{X}^{ss} \to X//H$ to $X//H$ from an open subset $\mathcal{X}^{ss}$ of $\mathcal{X}$.

Application to jet differentials (following Demailly 1995)

X complex manifold, $\dim X = n$
$J_k \to X$ bundle of k-jets of holomorphic curves $f : (\mathbb{C}, 0) \to X$
[f *and* g *have the same* k*-jet if their Taylor expansions at 0 coincide up to order* k]. More generally $J_{k,p} \to X$ is the bundle of k-jets of holomorphic maps $f : \mathbb{C}^p \to X$.

Under composition modulo t^{k+1} we have a group $\mathbb{G}_k$ given by
$\{k$-jets of germs of biholomorphisms of $(\mathbb{C}, 0)\}$
$t \mapsto \phi(t) = a_1 t + a_2 t^2 + \ldots + a_k t^k, \quad a_j \in \mathbb{C}, a_1 \neq 0$

$\mathbb{G}_k$ acts on J_k fibrewise by reparametrising k-jets. Similarly we have $\mathbb{G}_{k,p}$ acting fibrewise on $J_{k,p}$.

$$\mathbb{G}_k \cong \{ \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ 0 & a_1^2 & \dots & \\ & & \dots & \\ 0 & 0 & \dots & a_1^k \end{pmatrix} : a_1 \in \mathbb{C}^*, a_2, \dots a_k \in \mathbb{C}\}$$

$\mathbb{G}_k$ has a subgroup $\mathbb{C}^*$ (represented by $\phi(t) = a_1 t$) and a unipotent subgroup $\mathbb{U}_k$ (represented by $\phi(t) = t + a_2 t^2 + \dots + a_k t^k$) such that

$$\mathbb{G}_k \cong \mathbb{U}_k \rtimes \mathbb{C}^*.$$

Similarly

$$\mathbb{G}_{k,p} \cong \mathbb{U}_{k,p} \rtimes GL(p)$$

where $\mathbb{U}_{k,p}$ is the unipotent radical of $\mathbb{G}_{k,p}$, and the central one-parameter subgroup $\mathbb{C}^*$ of $GL(p)$ acts on $Lie(\mathbb{U}_{k,p})$ with all weights strictly positive. Thus

linear actions of $\mathbb{G}_{k,p}$ have finitely generated invariants.

Green-Griffiths (1979): For $x \in X$ consider
$$(J_k)_x \cong \bigoplus_{j=1}^k Sym^j(\mathbb{C}^n)$$
and
$(E^{GG}_{k,m})_x = \{\mathbb{C}$-valued polynomials on $(J_k)_x$ of weighted degree m wrt $\mathbb{C}^* \leqslant \mathbb{G}_k\}$.

Demailly-Semple jet differentials:

$$(E_{k,m})_x = (E^{GG}_{k,m})_x^{\mathbb{U}_k} = \mathcal{O}((J_k)_x)^{\mathbb{G}_k}$$

is the fibre at x of the bundle $E_{k,m}$ of invariant jet differentials of order k and degree m over X. (N.B. The action of $\mathbb{G}_k$ on $\mathcal{O}((J_k)_x)$ is twisted by the character $\mathbb{G}_k \to \mathbb{C}^*$ with kernel $\mathbb{U}_k$).

Merker (2008) gave algorithm to generate all invariants, and showed invariants are finitely generated for small n and k (in the case $p = 1$).

Kobayashi hyperbolicity
X compact complex manifold

$$X \text{ is Kobayashi hyperbolic} \iff \nexists \text{ nonconstant entire holo curve in } X$$

Idea: global holo sections of $E_{k,m}$ vanishing on a fixed divisor $\rightsquigarrow$ global algebraic differential equations satisfied by every entire holo curve $f : \mathbb{C} \to X$.

Conjecture (Kobayashi 1970)
$X \subseteq \mathbb{P}^{n+1}$ generic hypersurface of degree $d \gg n$ $\Rightarrow X$ hyperbolic.

Siu (2004): method of proof but no effective lower bound for d.

Conjecture (Green-Griffiths)
$X \subseteq \mathbb{P}^{n+1}$ generic hypersurface of degree

$$d \geq d(n) \gg n$$

$\Rightarrow \exists$ proper algebraic subvariety $Y \subset X$ such that every nonconstant entire holo curve $f : \mathbb{C} \to X$ is contained in Y.

Diverio-Merker-Rousseau (2009) prove this with

$$d(n) \sim n^{(n+1)^{n+5}}$$

(for $n \geq 2$).

Berczi-K use non-reductive GIT to obtain

$$d(n) \sim n^{3/2}.$$

Method goes back to Demailly

$$J_k^{reg} = \{f \in J_k \ : \ f'(0) \neq 0\}$$

Thm (Demailly 1995) $J_k^{reg}/\mathbb{G}_k$ is a locally trivial bundle over X with a compactification

$$\pi : X_k \to X$$

and a line bundle $\mathcal{O}_{X_k}(1)$ satisfying $\pi_*(\mathcal{O}_{X_k}(m)) = E_{k,m}$.

If $\exists$ an ample line bundle $L \to X$ such that

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi^* L^{-1}) \cong H^0(X, E_{k,m} \otimes L^{-1}) \neq 0$$

with basis $\sigma_1, \ldots, \sigma_N$ and base locus Z, then every entire holo curve $f : \mathbb{C} \to X$ is contained to kth order in Z.

The bound $n^{(n+1)^{n+5}}$ comes from the relatively complicated nature of the compactification X_k (an iterated projective bundle). The better bound $n^{3/2}$ comes from using the compactification of $J_k^{reg}/\mathbb{G}_k$ obtained from non-reductive GIT.