Step 11. Since $L = AR$, show that (5) implies $RGR^t = AGA = G$. Equating components in $RG = GR$, noting that $RR^t = I$, show that

$$R = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix},$$

where S is 3×3 orthogonal in the standard basis. Hence R has the same form in the basis e_0, n, v_3, v_4. This concludes the proof outline.

The physical interpretation of the theorem is worth noting: The theorem claims that if L is linear and satisfies (2) and if R is orientation preserving in the sense that $R_{11} = +1$ and $\det S = +1$, then L is the transformation of coordinates from a rocket to a lab frame in which case n points in the direction of motion of the rocket. Finally note that L may be decomposed as $L = \tilde{R}\tilde{A}$. The reader will find that the relationships among $R, A, \tilde{R}, \tilde{A}$ have interesting computational details.

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THE JORDAN CURVE THEOREM VIA THE BROUWER FIXED POINT THEOREM

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A homeomorphic image of a closed interval $[a, b]$ $(a < b)$ is called an arc and a homeomorphic image of a circle is called a Jordan curve. One of the most classical theorems in topology is

Theorem (Jordan Curve Theorem). *The complement in the plane R^2 of a Jordan curve J consists of two components, each of which has J as its boundary.*

Since the first rigorous proof given by Veblen [**4**] in 1905, a variety of elementary (and lengthy) proofs have been provided by many authors. Among them, the one given by Moise [**3**] is intuitive and transparent yet lengthy. The purpose of this note is to provide a short proof by modifying Moise's method. In order to avoid the tedious arguments, we will use the following celebrated theorem of Brouwer (for an elementary proof, for example, see [**1**]).

Theorem (Brouwer Fixed Point Theorem). *Every continuous map from a disk into itself has a fixed point.*

To begin with, we note two simple facts concerning the components of $R^2 - J$, where J is a Jordan curve: (a) $R^2 - J$ has exactly one unbounded component, and (b) each component of $R^2 - J$ is path connected and open. The assertion (a) follows from the boundedness of J, and (b) from the local path-connectedness of R^2 and the closedness of J.

Lemma 1. *If $R^2 - J$ is not connected, then each component has J as its boundary.*

Proof. By assumption, $R^2 - J$ has at least two components. Let U be an arbitrary component. Since any other component W is disjoint from U and open, W contains no point of the closure $\overline{U}$ and hence no point of the boundary $\overline{U} \cap U^c$ of U. Thus $\overline{U} \cap U^c \subset J$. Suppose $\overline{U} \cap U^c \neq J$. Then there exists an arc $A \subset J$ such that

$$(\#) \qquad \overline{U} \cap U^c \subset A.$$

We will show that this leads to a contradiction. By the preceding remark (a), $R^2 - J$ has at least one bounded component. Let o be a point in a bounded component; if U itself is bounded we choose o in U. Let D be a large disk with center o such that its interior contains J. Then the boundary S of D is contained in the unbounded component of $R^2 - J$. Since arc A is homeomorphic to the interval [0,1], the identity map $A \to A$ has a continuous extension $r: D \to A$ by the Tietze Extension Theorem (see, for example, [**2**]). We define a map $q: D \to D - \{o\}$, according as U is bounded or not, by

$$q(z) = \begin{cases} r(z) & \text{for } z \in \overline{U}, \\ z & \text{for } z \in U^c, \end{cases} \quad \text{or } q(z) = \begin{cases} z & \text{for } z \in \overline{U}, \\ r(z) & \text{for } z \in U^c, \end{cases}$$

respectively. By (#), the intersection of the two closed sets $\overline{U}$ and U^c lies in A on which r is the identity map. Thus q is well defined and continuous. Note that $q(z) = z$ if $z \in S$. Let $p: D - \{o\} \to S$ be the natural projection and let $t: S \to S$ be the antipodal map. Then the composition $t \cdot p \cdot q: D \to S \subset D$ has no fixed point. This contradicts the Brouwer fixed point theorem.

Note that the preceding proof implicitly contains a proof that no arc separates R^2, which is often a lemma to the Jordan curve theorem.

We need another lemma for our purpose. Let $E(a, b; c, d)$ denote the rectangular set $\{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$ in the plane R^2, where $a < b$ and $c < d$.

LEMMA 2. *Let $h(t) = (h_1(t), h_2(t))$ and $v(t) = (v_1(t), v_2(t))$ $(-1 \leqslant t \leqslant 1)$ be continuous paths in $E(a, b; c, d)$ satisfying*

$$(\#\#) \qquad h_1(-1) = a, \quad h_1(1) = b, \quad v_2(-1) = c, \quad v_2(1) = d.$$

Then the two paths meet, i.e., $h(s) = v(t)$ for some s, t in $[-1, 1]$.

Proof. Suppose $h(s) \neq v(t)$ for all s, t. Let $N(s, t)$ denote the maximum-norm of $h(s) - v(t)$, i.e.,

$$N(s,t) = \operatorname{Max}\{\, |h_1(s) - v_1(t)|, \quad |h_2(s) - v_2(t)| \,\}.$$

Then $N(s, t) \neq 0$ for all s, t. We define a continuous map F from $E(-1,1; -1,1)$ into itself by

$$F(s,t) = \left(\frac{v_1(t) - h_1(s)}{N(s,t)}, \frac{h_2(s) - v_2(t)}{N(s,t)} \right).$$

Note that the image of F is in the boundary of $E(-1,1; -1,1)$. To see that F has no fixed point, assume $F(s_0, t_0) = (s_0, t_0)$. By the above remark, we have $|s_0| = 1$ or $|t_0| = 1$. Suppose, for example, $s_0 = -1$. Then by (##), the first coordinate of $F(-1, t_0)$, $(v_1(t_0) - h_1(-1))/N(-1, t_0)$, is nonnegative and hence cannot equal $s_0 (= -1)$. Similarly, the other possibilities of $|s_0| = 1$ or $|t_0| = 1$ cannot occur. This contradicts the Brouwer fixed point theorem since $E(-1,1; -1,1)$ is homeomorphic to a disk.

We are now ready to prove the Jordan curve theorem. By Lemma 1, we need only show that $R^2 - J$ has one and only one bounded component. The proof will consist of the following three steps: Establishing the notation and defining a point z_0 in $R^2 - J$; proving that the component U containing z_0 is bounded; and proving that there is no bounded component other than U.

Since J is compact, there exist points a, b in J such that the distance $\|a - b\|$ is the largest. We may assume that $a = (-1, 0)$ and $b = (1, 0)$. Then the rectangular set $E(-1,1; -2,2)$ contains J, and its boundary Γ meets J at exactly two points a and b. Let n be the middle point of the top side of $E(-1,1; -2,2)$, and s the middle point of the bottom side; i.e., $n = (0,2)$ and $s = (0, -2)$. The segment $\overline{ns}$ meets J by Lemma 2. Let l be the y-maximal point (that means the point $(0, y)$ with maximal y) in $J \cap \overline{ns}$. Points a and b divide J into two arcs; we denote the one containing l by J_n and the other by J_s. Let m be the y-minimal point in $J_n \cap \overline{ns}$ (possibly, $l = m$). Then the segment $\overline{ms}$ meets J_s; otherwise, the path $\overline{nl} + \widehat{lm} + \overline{ms}$ (where $\widehat{lm}$ denotes the subarc of J_n with end points l and m) could not meet J_s, contradicting Lemma 2. Let p and q

denote the y-maximal point and the y-minimal point in $J_s \cap \overline{ms}$, respectively. Finally, let z_0 be the middle point of the segment $\overline{mp}$. (see Fig. 1).

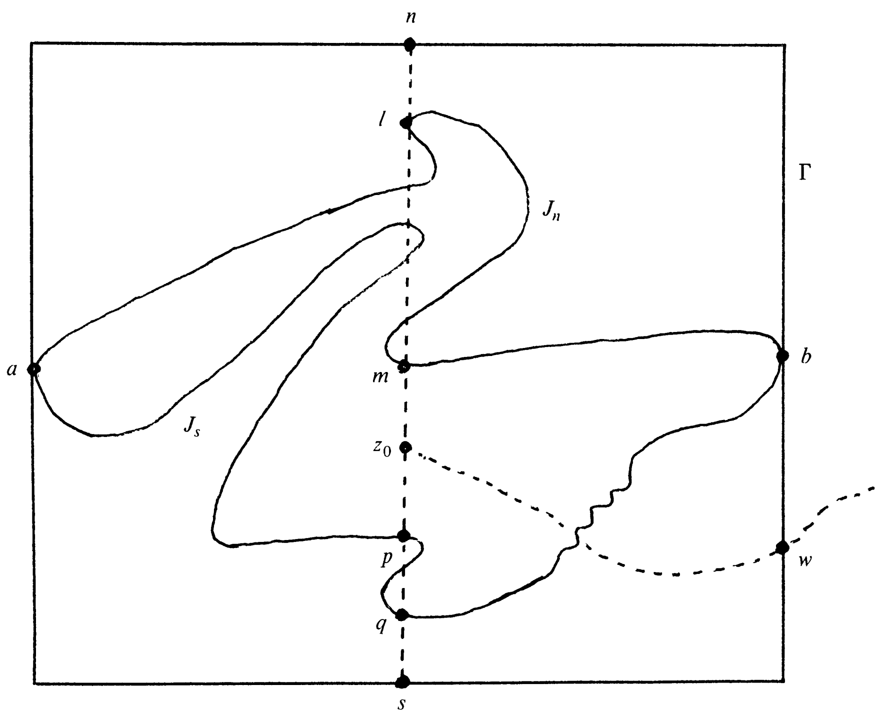

FIG. 1

Now we show that U, the component of $R^2 - J$ which contains z_0, is bounded. Suppose that U is unbounded. Since U is path connected, there exists a path α in U from z_0 to a point outside $E(-1,1;\ -2,2)$. Let w be the first point at which α meets the boundary Γ of $E(-1,1;\ -2,2)$. Denote by α_w the part of α from z_0 to w. If w is on the lower half of Γ, we can find a path $\widehat{ws}$ in Γ from w to s which contains neither a nor b. Now consider the path $\overline{nl} + \widehat{lm} + \overline{mz_0} + \alpha_w + \widehat{ws}$. This path does not meet J_s, contradicting Lemma 2. Similarly, if w is on the upper half of Γ, the path $\overline{sz_0} + \alpha_w + \widehat{wn}$ fails to meet J_n, where $\widehat{wn}$ is the shortest path in Γ from w to n. The contradiction shows that U is a bounded component.

Finally suppose that there exists another bounded component $W(\neq U)$ of $R^2 - J$. Clearly $W \subset E(-1,1;\ -2,2)$. We denote by β the path $\overline{nl} + \widehat{lm} + \overline{mp} + \widehat{pq} + \overline{qs}$, where $\widehat{pq}$ is the subarc of J_s from p to q. As seen easily, β has no point of W. Since a and b are not on β, there are circular neighborhoods V_a, V_b of a, b, respectively, such that each of them contains no point of β. By Lemma 1, a and b are in the closure $\overline{W}$. Hence, there exist $a_1 \in W \cap V_a$ and $b_1 \in W \cap V_b$. Let $\widehat{a_1b_1}$ be a path in W from a_1 to b_1. Then the path $\overline{aa_1} + \widehat{a_1b_1} + \overline{b_1b}$ fails to meet β. This contradicts Lemma 2 and completes our proof.

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