SMSTC Geometry and Topology 2011–2012
Lecture 6
Covering spaces and the Galois equivalence

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17th November, 2011
How to compute the fundamental group?

For a topological space $X$ with a base point $P$, we saw two methods of computing the group $\pi_1(X, P)$.

- The straightforward method using definition of $\pi_1(X, P)$ (contractible spaces, $S^1$, Möbius band, etc),
- Using the Seifert-van Kampen theorem ($S^1 \lor S^1$, complement to a finite subset in $\mathbb{R}^2$, etc),
- Lefschetz hyperplane theorem.

Today we consider another method. This method generalizes our way of computing $\pi_1(S^1) \cong \mathbb{Z}$.

The main objects of this method are the so-called covering spaces. Covering spaces are important on their own.

Let us start with examples and try to generalize them.
Circle: infinite cover

- The continuous map

\[ p: \mathbb{R} \rightarrow S^1 \ ; \ x \mapsto e^{2\pi ix} \]

is a surjection with many wonderful properties!
Circle: fundamental group

Homeo$_p$(\(\mathbb{R}\)) is the group of the homeomorphisms such that

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{h} & \mathbb{R} \\
p \downarrow & & \downarrow p \\
S^1
\end{array}
\]

is a commutative diagram. We have Homeo$_p$(\(\mathbb{R}\)) \(\cong \mathbb{Z}\) given by

\[
\mathbb{Z} \to \text{Homeo}_p(\mathbb{R}) ; n \mapsto (h_n : x \mapsto x + n).
\]

Every loop \(\omega : S^1 \to S^1\) “lifts” to a path \(\alpha : I \to \mathbb{R}\) with

\[
\omega(e^{2\pi it}) = e^{2\pi i\alpha(t)} \in S^1 \ (t \in I).
\]

There is a unique \(h \in \text{Homeo}_p(\mathbb{R})\) with \(h(\alpha(0)) = \alpha(1) \in \mathbb{R}\). We have isomorphisms of groups given by the functions

\[
\text{degree}: \pi_1(S^1) \to \text{Homeo}_p(\mathbb{R}) \cong \mathbb{Z} ; \omega \mapsto \alpha(1) - \alpha(0),
\]

\[
\mathbb{Z} \to \pi_1(S^1) ; n \mapsto (\omega_n : S^1 \to S^1 ; z \mapsto z^n),
\]

where the degree of \(\omega\) is the number of times \(\omega\) winds around 0.
Circle: other covers

Example For every non-zero integer $n$, define a covering

$$p_n : S^1 \rightarrow S^1 ; z \mapsto z^n,$$

and put $\omega = e^{2\pi i/n}$. There is a group isomorphism given by

$$\mathbb{Z}_n \rightarrow \text{Homeo}_{p_n}(S^1) ; r \mapsto (z \mapsto \omega^r z).$$

Circle has many non-connected covering spaces:

- $p : S^1 \times \mathbb{Z} \rightarrow S^1 ; (z, t) \mapsto z,$
- $p : S^1 \times \mathbb{Z}_n \rightarrow S^1 ; (z, t) \mapsto z.$

Recall that every subgroup in $\pi_1(S^1) \cong \mathbb{Z}$ is cyclic. For every subgroup $G \subset \pi_1(S^1)$, we have path-connected cover

$$p : \tilde{X} \rightarrow S^1$$

with a fibre $F$ such that

$$\text{Homeo}_p(\tilde{X}) = G$$

and $|F| = |\mathbb{Z}/G|$ (and $p_*(\pi_1(\tilde{X})) = G$). This is not a coincidence.
Covering spaces: definition

Let $X$ be a (path-connected) topological space.

**Definition** A covering space of $X$ with fibre the discrete space $F$ is

- a space $\tilde{X}$ equipped with
- a covering projection continuous map $p: \tilde{X} \to X$ such that
- for each $x \in X$ there is an open subset $U \subseteq X$ containing $x$
- with a homeomorphism $\phi: F \times U \to p^{-1}(U)$ such that

$$p \circ \phi(a, u) = u \in U \subseteq X \ (a \in F, u \in U).$$

Note that for each $x \in X$, the fibre $p^{-1}(x)$ is homeomorphic to $F$.

The covering projection $p: \tilde{X} \to X$ is a “local homeomorphism”:

- for each $\tilde{x} \in \tilde{X}$ there is an open subset $U \subseteq \tilde{X}$ containing $x$
- such that $U \to p(U); u \mapsto p(u)$ is a homeomorphism
- with $p(U) \subseteq X$ an open subset.

We will see that $p: \tilde{X} \to X$ gives a geometric method for computing $\pi_1(X)$ if $\tilde{X}$ is simply-connected.
Covering spaces: examples

We already know many examples of covering spaces:

- any space $X$ is a covering space of itself with identity map as covering projection,
- the maps $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, $\mathbb{R} \times S^1 \to S^1 \times S^1$, and $S^1 \times \mathbb{R} \to S^1 \times S^1$,
- the standard map $S^2 \to \mathbb{RP}^2$.

How to construct coverings of $S^1 \vee S^1$?

<table>
<thead>
<tr>
<th>Some Covering Spaces of $S^1 \vee S^1$</th>
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<tr>
<td>(1)</td>
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<tr>
<td>$\langle a, b^2, bab^{-1} \rangle$</td>
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<td>(2)</td>
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<tr>
<td>$\langle a^2, b^2, ab \rangle$</td>
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<td>(3)</td>
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<tr>
<td>$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$</td>
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<td>(4)</td>
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<tr>
<td>$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$</td>
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Isomorphisms of covering spaces

Let \( p : \tilde{X} \rightarrow X \) and \( p' : \tilde{X}' \rightarrow X \) be covering spaces.

When \( p : \tilde{X} \rightarrow X \) and \( p' : \tilde{X}' \rightarrow X \) are the “same”?

An isomorphism between \( p : \tilde{X} \rightarrow X \) and \( p' : \tilde{X}' \rightarrow X \) is a homeomorphism \( f : \tilde{X} \rightarrow \tilde{X}' \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{X}' \\
\downarrow{p} & & \downarrow{p'} \\
X & & X
\end{array}
\]

commutes.

If \( \tilde{X} = \tilde{X}' \) and \( p = p' \), then \( f \) is not necessary the identity map.

The isomorphism of covering spaces is an equivalence relation.

All automorphisms of a given covering space \( p : \tilde{X} \rightarrow X \) form a group.
Pullback covers

Let $p: \tilde{X} \to X$ be a covering space with a fibre $F$. Let $f: V \to X$ be a continuous map. Put

$$
\tilde{V} = \left\{ (\tilde{x}, v) \in \tilde{X} \times V \text{ such that } p(\tilde{x}) = f(v) \right\},
$$

let $q: \tilde{V} \to V$ be the map induced by the projection $\tilde{X} \times V \to V$. The map $q: \tilde{V} \to V$ is a covering space with a fibre $F$. Moreover, there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{f}} & \tilde{X} \\
q \downarrow & & \downarrow p \\
V & \xrightarrow{f} & X
\end{array}
\]

for the continuous map $\tilde{f}: \tilde{V} \to \tilde{X}; (\tilde{x}, v) \mapsto \tilde{x}$.

The covering space $q: \tilde{V} \to V$ is sometimes denoted by $f^*(\tilde{X})$.

It is known as the "pullback" of the covering space $p: \tilde{X} \to X$ via $f$. 
Examples, remarks, comments, and questions
Covering translations

For any space $X$ let $\text{Homeo}(X)$ be the group of all homeomorphisms $h: X \to X$, with composition as group law. Let $p: \tilde{X} \to X$ be a covering space. 

**Definition** Let $\text{Homeo}_p(\tilde{X})$ be the subgroup of $\text{Homeo}(\tilde{X})$ consisting of all homeomorphisms $h: \tilde{X} \to \tilde{X}$ such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{h} & \tilde{X} \\
\downarrow p & & \downarrow p \\
X & \xleftarrow{p} & X \\
\end{array}
\]

commutes, i.e. $p \circ h = p: \tilde{X} \to X$. The group $\text{Homeo}_p(\tilde{X})$ is called

- either the group of **covering translations**,  
- or the group of **deck transformations**.

**Example** For every non-zero integer $n$, define a covering

$$p_n: S^1 \to S^1; z \mapsto z^n,$$

and put $\omega = e^{2\pi i/n}$. There is a group isomorphism given by

$$\mathbb{Z}_n \to \text{Homeo}_{p_n}(S^1); r \mapsto (z \mapsto \omega^r z).$$
Trivial covering

Let \( p : \tilde{X} \to X \) be a covering space with fibre \( F \).

**Definition** A covering projection \( p : \tilde{X} \to X \) is trivial if there exists a homeomorphism \( \phi : F \times X \to \tilde{X} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & F \times X \\
p \downarrow & & pr_X \downarrow \\
X & = & X
\end{array}
\]

commutes, where \( pr_X : F \times X \) is a projection to \( X \).

A particular choice of \( \phi \) is a trivialisation of \( p \).

**Example** If \( \tilde{X} = F \times X \) and \( p \) is a projection to \( X \), then

- \( p : \tilde{X} \to X \) is trivial covering (by definition),
- if we assume that the space \( X \) is path-connected, then Homeo_\( p \)(\( \tilde{X} \)) is the group of permutations of \( F \).
Non-trivial coverings

**Example** The universal covering

\[ p: \mathbb{R} \to S^1; \ x \mapsto e^{2\pi i x} \]

is a covering projection with fibre \( \mathbb{Z} \). Moreover, we saw that

\[ \text{Homeo}_p(\mathbb{R}) \cong \mathbb{Z}, \]

and \( p \) is not trivial (\( \mathbb{R} \) is not homeomorphic to \( \mathbb{Z} \times S^1 \)).

**Warning** There is a bijection

\[ \phi: \mathbb{Z} \times S^1 \to \mathbb{R}; \ (n, e^{2\pi i t}) \mapsto n + t \ (0 \leq t < 1) \]

such that \( p: \phi = \text{projection}: \mathbb{Z} \times S^1 \to \mathbb{R} \), but \( \phi \) is not continuous.

**Example** Let \( p: S^2 \to \mathbb{R}\mathbb{P}^2 \) be a standard covering. Then

\[ \text{Homeo}_p(S^2) \cong \mathbb{Z}_2, \]

and \( p \) is not trivial (\( S^2 \) is not homeomorphic to \( \mathbb{R}\mathbb{P}^2 \times \mathbb{Z}_2 \)).

Let \( f: S^1 \to \mathbb{R}\mathbb{P}^2 \) be a loop. Then the pullback of \( p \) to \( S^1 \) via \( f \) is

- trivial if \( [f(S^1)] = 0 \) in \( \pi_1(\mathbb{R}\mathbb{P}^2) \),
- non-trivial if \( [f(S^1)] \neq 0 \) in \( \pi_1(\mathbb{R}\mathbb{P}^2) \cong \mathbb{Z}_2 \).
Lifts

Let \( p: \tilde{X} \to X \) be a covering projection with fibre \( F \), and let \( f: Y \to X \) be a continuous map.

**Definition** A lift of \( f \) is a continuous map \( \tilde{f}: Y \to \tilde{X} \) such that

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow{f} & & \downarrow{p} \\
Y & & X
\end{array}
\]

is a commutative diagram, i.e. \( p(\tilde{f}(y)) = f(y) \in X \) for any \( y \in Y \).

**Example** If \( \tilde{X} = F \times X \) and \( p \) is a projection to \( X \), then we can easily define a lift of \( f \) by setting

\[
\tilde{f}_a: Y \to \tilde{X} = F \times X ; \ y \mapsto (a, f(y)),
\]

where \( a \) is some fixed chosen point in \( F \).

If \( Y \) is path-connected \( Y \), then \( a \mapsto \tilde{f}_a \) gives a bijection between \( F \) and the lifts of \( f \).
Path lifting

Let $p: \tilde{X} \to X$ be a covering projection with fibre $F$.
Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.
Let $\alpha: I \to X$ be any path with $\alpha(0) = x_0 \in X$, where $I = [0, 1]$.

**Path lifting property** There is a unique lift of $\alpha$ to a path $\tilde{\alpha}: I \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0 \in \tilde{X}$.

Let $\beta: I \to X$ be another path with $\beta(0) = x_0 \in X$.
Let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of $\alpha$ and $\beta$, respectively, such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0 \in \tilde{X}$.

**Homotopy lifting property** Every rel $\{0, 1\}$ homotopy $h: \alpha \simeq \beta: I \to X$ has a unique lift to a rel $\{0, 1\}$ homotopy $\tilde{h}: \tilde{\alpha} \simeq \tilde{\beta}: I \to \tilde{X}$ and in particular $\tilde{\alpha}(1) = \tilde{h}(1, t) = \tilde{\beta}(1) \in \tilde{X}$ for every $t \in I$. 
Homotopy lifting property

Let $p : \tilde{X} \to X$ be a covering projection with fibre $F$. Homotopy lifting property for paths can be generalized. Let $f_0 : Y \to X$ be a continuous map, and let $\tilde{f}_0 : Y \to \tilde{X}$ be its lift, i.e. the diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow{f_0} & & \downarrow{p} \\
X & & X
\end{array}
$$

commutes. Here we assume that the lift $\tilde{f}_0$ exists.

Let $f_t : Y \to X$ be a homotopy of $f_0$, where $t \in I = [0, 1]$. Homotopy lifting property There exists a unique homotopy

$$
\tilde{f}_t : Y \to \tilde{X}
$$

of the map $\tilde{f}_0$ that is a lift of the homotopy $f_t$. 
Examples, remarks, comments, and questions
Fundamental group action: fibers

Let $p : \tilde{X} \to X$ be a covering projection with fibre $F$.
Take a point $x_0 \in X$. Put $F_{x_0} = p^{-1}(x_0)$.
For every path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0 \in X$,
and for every point $y \in F_{x_0}$, there is a lift $\tilde{\alpha} : I \to \tilde{X}$ such that
$\tilde{\alpha}(0) = y$ by the path lifting property.
We have $\tilde{\alpha}(1) \in F_{x_0}$, since $\tilde{\alpha}$ is a lift of $\alpha$.
Taking another path $\alpha' \in [\alpha] \in \pi_1(X, x_0)$ and its lift $\tilde{\alpha}'$, we have

$$\tilde{\alpha}'(1) = \tilde{\alpha}(1) \in F_{x_0}$$

by the homotopy lifting property.
So $\pi_1(X, x_0)$ acts on $F_{x_0}$ (what a surprise).
For every $x \in X$, put $F_x = p^{-1}(x)$. Then $\pi_1(X, x)$ acts on $F_x$.
We can write $\sigma(\tilde{x})$ for $\sigma \in \pi_1(X, x)$ and $\tilde{x} \in F_x$.
Can we extend this actions to the action of $\pi_1(X, x_0)$ on $\tilde{X}$?
This would be a nice thing to do.
Especially if $X$ is path-connected and we can drop $x_0$ in $\pi_1(X, x_0)$. 
Fundamental group action: space

Let \( p: \tilde{X} \to X \) be a covering projection with fibre \( F \).
Suppose that \( \tilde{X} \) and \( X \) are path-connected.
For every \( x \in X \), put \( F_x = p^{-1}(x) \). Then \( \pi_1(X, x) \) acts on \( F_x \).
Take points \( x_0 \in X \) and \( \tilde{x}_0 \in F_{x_0} \).
For every \( \tilde{x} \in \tilde{X} \), let \( \gamma_{\tilde{x}}: I \to \tilde{X} \) be a path from \( \tilde{x}_0 \) to \( \tilde{x} \).
For every \( \sigma \in \pi_1(X, x_0) \), put
\[
\sigma(\tilde{x}) = \left( p \circ \gamma_{\tilde{x}} \right) \cdot \sigma \cdot \left( p \circ \gamma_{\tilde{x}}^{-1} \right)(\tilde{x}).
\]
Note that \( \sigma(\tilde{x}) \) is well-defined.
But \( \sigma(\tilde{x}) \) may depend on the choice of \( \gamma_{\tilde{x}} \), which is no good.

**Theorem** \( \sigma(\tilde{x}) \) does not dependent on the choice of \( \gamma_{\tilde{x}} \) if
\[
p_*(\pi_1(\tilde{X})) \triangleleft \pi_1(X),
\]
i.e. if \( p_*(\pi_1(\tilde{X})) \) is a normal subgroup in \( \pi_1(X) \).
Recall that a subgroup \( H \subseteq G \) is **normal** if
\[
gH = Hg
\]
for all \( g \in G \), in which case the quotient group \( G/H \) is defined.
Examples, remarks, comments, and questions
Regular covers

Let \( \tilde{p}: \tilde{X} \rightarrow X \) be a covering projection such that \( \tilde{X} \) and \( X \) are path-connected. Take points \( x_0 \in X \) and \( \tilde{x}_0 \in \tilde{X} \) with \( \tilde{p}(\tilde{x}_0) = x_0 \).

**Claim** The homomorphism \( \tilde{p}_*: \pi_1(\tilde{X}) \rightarrow \pi_1(X) \) is injective.

Indeed, if \( \omega: S^1 \rightarrow \tilde{X} \) is a loop at \( \tilde{x}_0 \) with a homotopy

\[
h: p \circ \omega \simeq e_{x_0}: S^1 \rightarrow X \text{ rel } 1,
\]

then \( h \) can be lifted to a homotopy

\[
\tilde{h}: \omega \simeq e_{\tilde{x}_0}: S^1 \rightarrow \tilde{X} \text{ rel } 1
\]

by the homotopy lifting property.

The subgroup \( \tilde{p}_*(\pi_1(\tilde{X}, x_0)) \) consists of the homotopy classes of loops in \( X \) based at \( x_0 \) whose lifts to \( \tilde{X} \) starting at \( \tilde{x}_0 \) are still loops.

**Definition** A covering \( p \) is regular or normal if

\[
p_*(\pi_1(\tilde{X})) \triangleleft \pi_1(X).
\]

**Remark** If \( \tilde{X} \) is simply-connected, then \( p \) is regular (why?).

**Example** \( p: \mathbb{R} \rightarrow S^1 \) and \( p_n: S^1 \rightarrow S^1 \) are regular.

**Example** \( p: S^2 \rightarrow \mathbb{R}P^2 \) is regular.
Constructing regular coverings

Let $Y$ be a space, let $G$ be a subgroup in $\text{Homeo}(Y)$. Define an equivalence relation $\sim$ on $Y$ by

$$y_1 \sim y_2 \text{ if there exists } g \in G \text{ such that } y_2 = g(y_1).$$

Put $X = Y/\sim$. Let $p: Y \to X$ be a quotient map.

Is $X$ a topological space? Yes (quotient topology).

Is $p: Y \to X$ a covering projection? Usually No. Why?

Suppose that for any $y \in Y$ there is an open subset $U \subseteq Y$ such that $U$ contains $y$ and (this is more important)

$$g(U) \cap U = \emptyset \text{ for every } g \in G \text{ such that } g \neq 1,$n

where $1$ denotes the identity element in $G$.

Such an action of $G$ on $Y$ is called free and properly discontinuous.

**Theorem** $p: Y \to X$ is a regular covering projection with fibre $G$.

If $Y$ is path-connected then

- $X$ is path-connected,
- $\text{Homeo}_p(Y) \cong G$. 
Deck theorem

Let $p: \tilde{X} \to X$ be a covering projection with a fibre $F$. Suppose that $\tilde{X}$ and $X$ are path-connected. For every point $x \in X$, put $F_x = p^{-1}(x)$ for every point $x \in X$.

**Theorem** Put $H = p_\ast(\pi_1(\tilde{X})) \subset \pi_1(X)$. Then

- there is a bijection $F \cong$ the set of left cosets of $H$ in $\pi_1(X)$,
- if $p$ is regular, then $\text{Homeo}_p(\tilde{X}) \cong \pi_1(X)/H$,
- if $p$ is not necessarily regular, then

$$\text{Homeo}_p(\tilde{X}) \cong N(H)/H,$$

where $N(H)$ is a normalizer of $H$ in $\pi_1(X)$,

- $p$ is regular iff $\text{Homeo}_p(\tilde{X})$ acts transitively on $F_x$ for any $x \in X$,
- if $\tilde{X}$ is simply connected, then $\text{Homeo}_p(\tilde{X}) \cong \pi_1(X)$.

If $|F| < +\infty$, then $|F|$ is the index of $H$ in $\pi_1(X)$.

$\text{Homeo}_p(\tilde{X})$ play the role of the Galois group in topology.
Decker theorem: sketch of the proof

Let us give a sketch of \( \text{Homeo}_p(\tilde{X}) \cong \pi_1(X)/H \).

If \( p \) is regular, then the action of \( \pi_1(X) \) on \( \tilde{X} \) induces a surjection

\[
\pi_1(X) \to \text{Homeo}_p(\tilde{X})
\]

whose kernel is \( H \). This is basically the idea.

Take points \( x_0 \in X \) and \( \tilde{x}_0 \in \tilde{X} \) such that \( p(\tilde{x}_0) = x_0 \).

Every closed path \( \alpha: I \to X \) with \( \alpha(0) = \alpha(1) = x_0 \) has a unique lift to a path \( \tilde{\alpha}: I \to \tilde{X} \) such that \( \tilde{\alpha}(0) = \tilde{x}_0 \). The function

\[
\pi_1(X, x_0)/p_\star \pi_1(\tilde{X}, \tilde{x}_0) \to p^{-1}(x_0) ; \alpha \mapsto \tilde{\alpha}(1)
\]

is a bijection. For each \( \tilde{x} \in F_{x_0} \) there is a unique covering translation \( h_{\tilde{y}} \in \text{Homeo}_p(\tilde{X}) \) such that \( h_{\tilde{x}}(\tilde{x}_0) = \tilde{x} \). The function

\[
p^{-1}(x_0) \to \text{Homeo}_p(\tilde{X}); \tilde{x} \mapsto h_{\tilde{x}}
\]

is a bijection, with inverse \( h \mapsto h(\tilde{x}_0) \). The composite bijection

\[
\pi_1(X, x_0)/H \to p^{-1}(x_0) \to \text{Homeo}_p(\tilde{X})
\]

is an isomorphism of groups.
Examples, remarks, comments, and questions
Universal covers

Let \( p : \tilde{X} \to X \) be a covering projection with a fibre \( F \).

Suppose that \( \tilde{X} \) and \( X \) are path-connected.

If \( \pi_1(\tilde{X}) = 1 \), then \( \pi_1(X) \cong \text{Homeo}_p(\tilde{X}) \) and \( \pi_1(X) \not\cong F \).

**Definition** A covering \( p : \tilde{X} \to X \) is universal if \( \pi_1(\tilde{X}) = 1 \).

**Example** \( p : \mathbb{R} \to S^1 \) is universal.

**Example** \( p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \) is universal. So

\[
\pi_1(S^1 \times S^1) \cong \text{Homeo}_{p \times p}(\mathbb{R} \times \mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}.
\]

**Remark** Universal covering \( p : \tilde{X} \to X \) exists if

(A) \( X \) is path-connected,

(B) and \( X \) is locally path-connected

( each point has arbitrarily small open neighborhood like this ),

(C) and \( X \) is semi-locally simply-connected

( each point \( P \in X \) has a neighborhood \( U \) such that

the inclusion-induced map \( \pi_1(U, P) \to \pi_1(X, P) \) is trivial ).

Every reasonable \( X \) (e.g. connected manifold) fits (A), (B), (C).

Universal cover is unique (this justifies the word “universal”).
Existence of covers

Let $p : \tilde{X} \to X$ be a covering space. Suppose that $X$ satisfies $(A)$, $(B)$, $(C)$, and $\tilde{X}$ is path-connected. Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. We have a function (or a correspondence)

covering space $p : \tilde{X} \to X \implies$ subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$.

Is this function surjective?
For a trivial subgroup in $\pi_1(X)$, the answer is Yes. For any subgroup $G \subset \pi_1(X)$, we can put

$$\tilde{X} = \mathcal{X}/G,$$

where $\mathcal{X}$ is the universal cover of $X$. So the answer to the above question is Yes in general. Is this function “injective”?
Galois correspondence

Let \( p : \tilde{X} \rightarrow X \) and \( p' : \tilde{X}' \rightarrow X \) be covering spaces.
Suppose that \( X \) satisfies \((A), (B), (C)\).
Suppose that \( \tilde{X} \) and \( \tilde{X}' \) are path-connected.
Take points \( x_0 \in X \), \( \tilde{x}_0 \in \tilde{X} \), \( \tilde{x}_0' \in \tilde{X}' \), with \( p(\tilde{x}_0) = p(\tilde{x}_0') = x_0 \).

Claim The following two conditions are equivalent:

- there is a homeomorphism \( f : \tilde{X} \rightarrow \tilde{X}' \) such that
  \[
  \begin{array}{ccc}
  \tilde{X} & \xrightarrow{f} & \tilde{X}' \\
  p & \swarrow & p' \\
  X & \searrow & \\
  \end{array}
  \]
  is commutative diagram and \( f(\tilde{x}_0) = \tilde{x}_0' \),
- \( p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p'_*(\pi_1(\tilde{X}', \tilde{x}_0')) \).

Theorem There exists a natural bijection between

- isomorphism classes of path-connected covering spaces \( p : \tilde{X} \rightarrow X \),
- subgroups in \( \pi_1(X) \) up to conjugation.