SMSTC Geometry and Topology 2011–2012
Lecture 6
Covering spaces and the Galois equivalence

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17th November, 2011

How to compute the fundamental group?

For a topological space X with a base point P,
we saw two methods of computing the group $\pi_1(X, P)$.

- The straightforward method using definition of $\pi_1(X, P)$ (contractible spaces, S^1, Möbius band, etc),
- Using the Seifert-van Kampen theorem ($S^1 \vee S^1$, complement to a finite subset in $\mathbb{R}^2$, etc),
- Lefschetz hyperplane theorem.

Today we consider another method.
This method generalizes our way of computing $\pi_1(S^1) \cong \mathbb{Z}$.

The main objects of this method are the so-called covering spaces.
Covering spaces are important on their own.

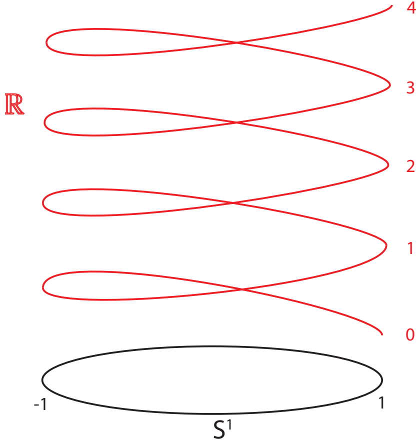
Let us start with examples and try to generalize them.

Circle: infinite cover

- The continuous map

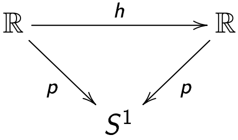
$$p\colon \mathbb{R} \to S^1 \;;\; x \mapsto e^{2\pi i x}$$

is a surjection with many wonderful properties!



Circle: fundamental group

$\text{Homeo}_p(\mathbb{R})$ is the group of the homeomorphisms such that



is a commutative diagram. We have $\text{Homeo}_p(\mathbb{R}) \cong \mathbb{Z}$ given by

$$\mathbb{Z} \to \text{Homeo}_p(\mathbb{R}) \ ; \ n \mapsto (h_n \colon x \mapsto x + n).$$

Every loop $\omega \colon S^1 \to S^1$ "lifts" to a path $\alpha \colon I \to \mathbb{R}$ with

$$\omega(e^{2\pi i t}) = e^{2\pi i \alpha(t)} \in S^1 \ (t \in I).$$

There is a unique $h \in \text{Homeo}_p(\mathbb{R})$ with $h(\alpha(0)) = \alpha(1) \in \mathbb{R}$. We have isomorphisms of groups given by the functions

$$\text{degree} \colon \pi_1(S^1) \to \text{Homeo}_p(\mathbb{R}) \cong \mathbb{Z} \ ; \ \omega \mapsto \alpha(1) - \alpha(0),$$
$$\mathbb{Z} \to \pi_1(S^1) \ ; \ n \mapsto (\omega_n : S^1 \to S^1 ; z \mapsto z^n),$$

where the degree of ω is the number of times ω winds around 0.

Circle: other covers

Example For every non-zero integer n, define a covering

$$p_n \colon S^1 \to S^1; z \mapsto z^n,$$

and put $\omega = e^{2\pi i/n}$. There is a group isomorphism given by

$$\mathbb{Z}_n \to \mathrm{Homeo}_{p_n}(S^1) \ ; \ r \mapsto (z \mapsto \omega^r z).$$

Circle has many non-connected covering spaces:

- $p \colon S^1 \times \mathbb{Z} \to S^1; (z, t) \mapsto z,$
- $p \colon S^1 \times \mathbb{Z}_n \to S^1; (z, t) \mapsto z.$

Recall that every subgroup in $\pi_1(S^1) \cong \mathbb{Z}$ is cyclic.
For every subgroup $G \subset \pi_1(S^1)$, we have path-connected cover

$$p \colon \tilde{X} \to S^1$$

with a fibre F such that

$$\mathrm{Homeo}_p(\tilde{X}) = G$$

and $|F| = |\mathbb{Z}/G|$ (and $p_*(\pi_1(\tilde{X})) = G$). This is not a coincidence.

Covering spaces: definition

Let X be a (path-connected) topological space.

Definition A covering space of X with fibre the discrete space F is

- a space $\tilde{X}$ equipped with
- a covering projection continuous map $p\colon \tilde{X} \to X$ such that
- for each $x \in X$ there is an open subset $U \subseteq X$ containing x
- with a homeomorphism $\phi\colon F \times U \to p^{-1}(U)$ such that

$$p \circ \phi(a, u) = u \in U \subseteq X \quad (a \in F, u \in U).$$

Note that for each $x \in X$, the fibre $p^{-1}(x)$ is homeomorphic to F.

The covering projection $p\colon \tilde{X} \to X$ is a "local homeomorphism":

- for each $\tilde{x} \in \tilde{X}$ there is an open subset $U \subseteq \tilde{X}$ containing x
- such that $U \to p(U); u \mapsto p(u)$ is a homeomorphism
- with $p(U) \subseteq X$ an open subset.

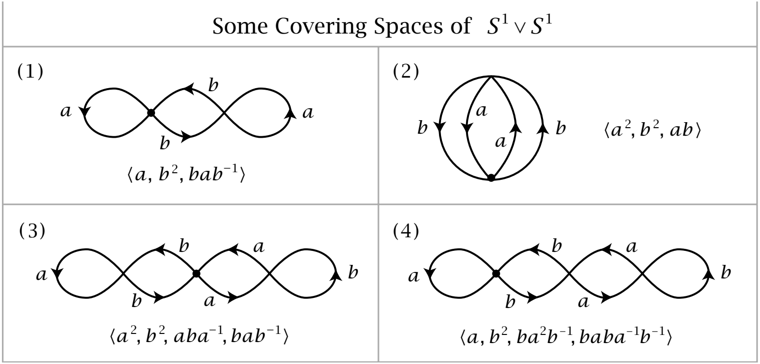
We will see that $p\colon \tilde{X} \to X$ gives a geometric method for computing $\pi_1(X)$ if $\tilde{X}$ is simply-connected.

Covering spaces: examples

We already know many examples of covering spaces:

- any space X is a covering space of itself with identity map as covering projection,
- the maps $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, $\mathbb{R} \times S^1 \to S^1 \times S^1$, and $S^1 \times \mathbb{R} \to S^1 \times S^1$,
- the standard map $S^2 \to \mathbb{RP}^2$.

How to construct coverings of $S^1 \vee S^1$?

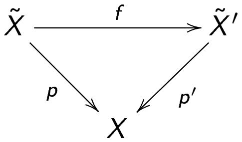


Isomorphisms of covering spaces

Let $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ be covering spaces.

When $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ are the "same"?

An isomorphism between $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ is a homeomorphism $f\colon \tilde{X} \to \tilde{X}'$ such that the diagram



commutes.
If $\tilde{X} = \tilde{X}'$ and $p = p'$, then f is not necessary the identity map.

The isomorphism of covering spaces is an equivalence relation.

All automorphisms of a given covering space $p\colon \tilde{X} \to X$ form a group.

Pullback covers

Let $p \colon \tilde{X} \to X$ be a covering space with a fibre F.
Let $f : V \to X$ be a continuous map. Put

$$\tilde{V} = \Big\{ (\tilde{x}, v) \in \tilde{X} \times V \text{ such that } p(\tilde{x}) = f(v) \Big\},$$

let $q \colon \tilde{V} \to V$ be the map induced by the projection $\tilde{X} \times V \to V$.
The map $q \colon \tilde{V} \to V$ is a covering space with a fibre F.
Moreover, there exists a commutative diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow[f]{} & X \end{array}$$

for the continuous map $\tilde{f} \colon \tilde{V} \to \tilde{X}; (\tilde{x}, v) \mapsto \tilde{x}$.
The covering space $q \colon \tilde{V} \to V$ is sometimes denoted by $f^*(\tilde{X})$.
It is known as the "pullback" of the covering space $p \colon \tilde{X} \to X$ via f.

Examples, remarks, comments, and questions

Covering translations

For any space X let $\text{Homeo}(X)$ be the group of all homeomorphisms $h\colon X \to X$, with composition as group law.
Let $p\colon \tilde{X} \to X$ be a covering space.
Definition Let $\text{Homeo}_p(\tilde{X})$ be the subgroup of $\text{Homeo}(\tilde{X})$ consisting of all homeomorphisms $h\colon \tilde{X} \to \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\quad h \quad} & \tilde{X} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

commutes, i.e. $p \circ h = p\colon \tilde{X} \to X$. The group $\text{Homeo}_p(\tilde{X})$ is called

- either the group of covering translations,
- or the group of deck transformations.

Example For every non-zero integer n, define a covering

$$p_n\colon S^1 \to S^1; z \mapsto z^n,$$

and put $\omega = e^{2\pi i/n}$. There is a group isomorphism given by

$$\mathbb{Z}_n \to \text{Homeo}_{p_n}(S^1)\ ;\ r \mapsto (z \mapsto \omega^r z).$$

Trivial covering

Let $p\colon \tilde{X} \to X$ be a covering space with fibre F.
Definition A covering projection $p\colon \tilde{X} \to X$ is trivial if there exists a homeomorphism $\phi\colon F \times X \to \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \phi\ } & F \times X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr_X} \\ X & = & X \end{array}$$

commutes, where $pr_X\colon F \times X$ is a projection to X.

A particular choice of ϕ is a trivialisation of p.

Example If $\tilde{X} = F \times X$ and p is a projection to X, then

- $p\colon \tilde{X} \to X$ is trivial covering (by definition),
- if we assume that the space X is path-connected, then $\mathrm{Homeo}_p(\tilde{X})$ is the group of permutations of F.

Non-trivial coverings

Example The universal covering

$$p\colon \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $\mathbb{Z}$. Moreover, we saw that

$$\mathrm{Homeo}_p(\mathbb{R}) \cong \mathbb{Z},$$

and p is not trivial ($\mathbb{R}$ is not homeomorphic to $\mathbb{Z} \times S^1$).
Warning There is a bijection

$$\phi\colon \mathbb{Z} \times S^1 \to \mathbb{R} \ ; \ (n, e^{2\pi i t}) \mapsto n + t \quad (0 \leqslant t < 1)$$

such that $p\colon \phi =$ projection : $\mathbb{Z} \times S^1 \to \mathbb{R}$, but ϕ is not continuous.

Example Let $p\colon S^2 \to \mathbb{RP}^2$ be a standard covering. Then

$$\mathrm{Homeo}_p(S^2) \cong \mathbb{Z}_2,$$

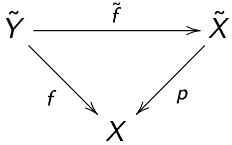
and p is not trivial (S^2 is not homeomorphic to $\mathbb{RP}^2 \times \mathbb{Z}_2$).
Let $f : S^1 \to \mathbb{RP}^2$ be a loop. Then the pullback of p to S^1 via f is

- trivial if $[f(S^1)] = 0$ in $\pi_1(\mathbb{RP}^2)$,
- non-trivial if $[f(S^1)] \neq 0$ in $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.

Lifts

Let $p\colon \tilde{X} \to X$ be a covering projection with fibre F,
and let $f\colon Y \to X$ be a continuous map.

Definition A lift of f is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that



is a commutative diagram, i.e. $p(\tilde{f}(y)) = f(y) \in X$ for any $y \in Y$.

Example If $\tilde{X} = F \times X$ and p is a projection to X,
then we can easily define a lift of f by setting

$$\tilde{f}_a\colon Y \to \tilde{X} = F \times X \;;\; y \mapsto (a, f(y)),$$

where a is some fixed chosen point in F.
If Y is path-connected Y, then $a \mapsto \tilde{f}_a$ gives a bijection
between F and the lifts of f.

Path lifting

Let $p\colon \tilde{X} \to X$ be a covering projection with fibre F.
Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.
Let $\alpha\colon I \to X$ be any path with $\alpha(0) = x_0 \in X$, where $I = [0,1]$.
Path lifting property There is a unique lift of α to a path

$$\tilde{\alpha}\colon I \to \tilde{X}$$

such that $\tilde{\alpha}(0) = \tilde{x}_0 \in \tilde{X}$.

Let $\beta\colon I \to X$ be another path with $\beta(0) = x_0 \in X$.
Let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β, respectively, such that

$$\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0 \in \tilde{X}.$$

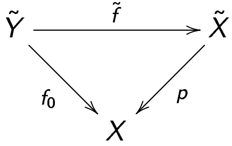
Homotopy lifting property Every rel $\{0,1\}$ homotopy $h\colon \alpha \simeq \beta\colon I \to X$ has a unique lift to a rel $\{0,1\}$ homotopy

$$\tilde{h}\colon \tilde{\alpha} \simeq \tilde{\beta}\colon I \to \tilde{X}$$

and in particular $\tilde{\alpha}(1) = \tilde{h}(1,t) = \tilde{\beta}(1) \in \tilde{X}$ for every $t \in I$.

Homotopy lifting property

Let $p \colon \tilde{X} \to X$ be a covering projection with fibre F.
Homotopy lifting property for paths can be generalized.
Let $f_0 \colon Y \to X$ be a continuous map, and let $\tilde{f}_0 \colon Y \to \tilde{X}$ be its lift, i.e. the diagram



commutes. Here we assume that the lift $\tilde{f}_0$ exists.

Let $f_t \colon Y \to X$ be a homotopy of f_0, where $t \in I = [0, 1]$.
Homotopy lifting property There exists a unique homotopy

$$\tilde{f}_t \colon Y \to \tilde{X}$$

of the map $\tilde{f}_0$ that is a lift of the homotopy f_t.

Examples, remarks, comments, and questions

Fundamental group action: fibers

Let $p \colon \tilde{X} \to X$ be a covering projection with fibre F.
Take a point $x_0 \in X$. Put $F_{x_0} = p^{-1}(x_0)$.
For every path $\alpha \colon I \to X$ with $\alpha(0) = \alpha(1) = x_0 \in X$,
and for every point $y \in F_{x_0}$, there is a lift $\tilde{\alpha} \colon I \to \tilde{X}$ such that $\tilde{\alpha}(0) = y$ by the path lifting property.
We have $\tilde{\alpha}(1) \in F_{x_0}$, since $\tilde{\alpha}$ is a lift of α.
Taking another path $\alpha' \in [\alpha] \in \pi_1(X, x_0)$ and its lift $\tilde{\alpha}'$, we have

$$\tilde{\alpha}'(1) = \tilde{\alpha}(1) \in F_{x_0}$$

by the homotopy lifting property.
So $\pi_1(X, x_0)$ acts on F_{x_0} (what a surprise).
For every $x \in X$, put $F_x = p^{-1}(x)$. Then $\pi_1(X, x)$ acts on F_x.
We can write $\sigma(\tilde{x})$ for $\sigma \in \pi_1(X, x)$ and $\tilde{x} \in F_x$.
Can we extend this actions to the action of $\pi_1(X, x_0)$ on $\tilde{X}$?
This would be a nice thing to do.
Especially if X is path-connected and we can drop x_0 in $\pi_1(X, x_0)$.

Fundamental group action: space

Let $p \colon \tilde{X} \to X$ be a covering projection with fibre F.
Suppose that $\tilde{X}$ and X are path-connected.
For every $x \in X$, put $F_x = p^{-1}(x)$. Then $\pi_1(X, x)$ acts on F_x.
Take points $x_0 \in X$ and $\tilde{x}_0 \in F_{x_0}$.
For every $\tilde{x} \in \tilde{X}$, let $\gamma_{\tilde{x}} \colon I \to \tilde{X}$ be a path from $\tilde{x}_0$ to $\tilde{x}$.
For every $\sigma \in \pi_1(X, x_0)$, put

$$\sigma(\tilde{x}) = \Big(p \circ \gamma_{\tilde{x}}\Big) \cdot \sigma \cdot \Big(p \circ \gamma_{\tilde{x}}^{-1}\Big)(\tilde{x}).$$

Note that $\sigma(\tilde{x})$ is well-defined.
But $\sigma(\tilde{x})$ may depend on the choice of $\gamma_{\tilde{x}}$, which is no good.

Theorem $\sigma(\tilde{x})$ does not dependent on the choice of $\gamma_{\tilde{x}}$ if

$$p_*\Big(\pi_1(\tilde{X})\Big) \lhd \pi_1(X),$$

i.e. if $p_*(\pi_1(\tilde{X}))$ is a normal subgroup in $\pi_1(X)$.
Recall that a subgroup $H \subseteq G$ is normal if

$$gH = Hg$$

for all $g \in G$, in which case the quotient group G/H is defined

Examples, remarks, comments, and questions

Regular covers

Let $p\colon \tilde{X} \to X$ be a covering projection such that $\tilde{X}$ and X are path-connected. Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$.

Claim The homomorphism $p_*\colon \pi_1(\tilde{X}) \to \pi_1(X)$ is injective.
Indeed, if $\omega\colon S^1 \to \tilde{X}$ is a loop at $\tilde{x}_0$ with a homotopy

$$h\colon p \circ \omega \simeq e_{x_0}\colon S^1 \to X \text{ rel } 1,$$

then h can be lifted to a homotopy

$$\tilde{h}\colon \omega \simeq e_{\tilde{x}_0}\colon S^1 \to \tilde{X} \text{ rel } 1$$

by the homotopy lifting property.
The subgroup $p_*(\pi_1(\tilde{X}, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are still loops.
Definition A covering p is regular or normal if

$$p_*\Big(\pi_1(\tilde{X})\Big) \lhd \pi_1(X).$$

Remark If $\tilde{X}$ is simply-connected, then p is regular (why?).
Example $p\colon \mathbb{R} \to S^1$ and $p_n\colon \mathbb{S}^1 \to S^1$ are regular.
Example $p\colon \mathbb{S}^2 \to \mathbb{RP}^2$ is regular.

Constructing regular coverings

Let Y be a space, let G be a subgroup in Homeo(Y).
Define an equivalence relation $\sim$ on Y by

$$y_1 \sim y_2 \text{ if there exists } g \in G \text{ such that } y_2 = g(y_1).$$

Put $X = Y/\sim$. Let $p\colon Y \to X$ be a quotient map.
Is X a topological space? Yes (quotient topology).
Is $p\colon Y \to X$ a covering projection? Usually No. Why?
Suppose that for any $y \in Y$ there is an open subset $U \subseteq Y$ such that U contains y and (this is more important)

$$g(U) \cap U = \emptyset \text{ for every } g \in G \text{ such that } g \neq 1,$$

where 1 denotes the identity element in G.
Such an action of G on Y is called free and properly discontinuous.
Theorem $p\colon Y \to X$ is a regular covering projection with fibre G.
If Y is path-connected then

- X is path-connected,
- $\mathrm{Homeo}_p(Y) \cong G$.

Deck theorem

Let $p\colon \tilde{X} \to X$ be a covering projection with a fibre F.
Suppose that $\tilde{X}$ and X are path-connected.
For every point $x \in X$, put $F_x = p^{-1}(x)$ for every point $x \in X$.

Theorem Put $H = p_*(\pi_1(\tilde{X}) \subset \pi_1(X)$. Then

- there is a bijection $F \leftrightarrows$ the set of left cosets of H in $\pi_1(X)$,
- if p is regular, then $\mathrm{Homeo}_p(\tilde{X}) \cong \pi_1(X)/H$,
- if p is not necessarily regular, then

$$\mathrm{Homeo}_p(\tilde{X}) \cong N(H)/H,$$

  where $N(H)$ is a normalizer of H in $\pi_1(X)$,
- p is regular iff $\mathrm{Homeo}_p(\tilde{X})$ acts transitively on F_x for any $x \in X$,
- if $\tilde{X}$ is simply connected, then $\mathrm{Homeo}_p(\tilde{X}) \cong \pi_1(X)$.

If $|F| < +\infty$, then $|F|$ is the index of H in $\pi_1(X)$.

$\mathrm{Homeo}_p(\tilde{X})$ play the role of the Galois group in topology.

Deck theorem: sketch of the proof

Let us give a sketch of $\text{Homeo}_p(\tilde{X}) \cong \pi_1(X)/H$.
If p is regular, then the action of $\pi_1(X)$ on $\tilde{X}$ induces a surjection

$$\pi_1(X) \to \text{Homeo}_p(\tilde{X})$$

whose kernel is H. This is basically the idea.
Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.
Every closed path $\alpha \colon I \to X$ with $\alpha(0) = \alpha(1) = x_0$ has a unique lift to a path $\tilde{\alpha} \colon I \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}_0$. The function

$$\pi_1(X, x_0)/p_* \pi_1(\tilde{X}, \tilde{x}_0) \to p^{-1}(x_0) \ ; \ \alpha \mapsto \tilde{\alpha}(1)$$

is a bijection. For each $\tilde{x} \in F_{x_0}$ there is a unique covering translation $h_{\tilde{y}} \in \text{Homeo}_p(\tilde{X})$ such that $h_{\tilde{x}}(\tilde{x}_0) = \tilde{x}$. The function

$$p^{-1}(x_0) \to \text{Homeo}_p(\tilde{X}); \tilde{x} \mapsto h_{\tilde{x}}$$

is a bijection, with inverse $h \mapsto h(\tilde{x}_0)$. The composite bijection

$$\pi_1(X, x_0)/H \to p^{-1}(x_0) \to \text{Homeo}_p(\tilde{X})$$

is an isomorphism of groups.

Examples, remarks, comments, and questions

Universal covers

Let $p\colon \tilde{X} \to X$ be a covering projection with a fibre F.
Suppose that $\tilde{X}$ and X are path-connected.
If $\pi_1(\tilde{X}) = 1$, then $\pi_1(X) \cong \mathrm{Homeo}_p(\tilde{X})$ and $\pi_1(X) \rightleftarrows F$.
Definition A covering $p\colon \tilde{X} \to X$ is universal if $\pi_1(\tilde{X}) = 1$.
Example $p\colon \mathbb{R} \to S^1$ is universal.
Example $p \times p\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is universal. So

$$\pi_1(S^1 \times S^1) \cong \mathrm{Homeo}_{p\times p}(\mathbb{R} \times \mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Remark Universal covering $p\colon \tilde{X} \to X$ exists if

(A) X is path-connected,

(B) and X is locally path-connected
(each point has arbitrarily small open neighborhood like this),

(C) and X is semi-locally simply-connected
(each point $P \in X$ has a neighborhood U such that
the inclusion-induced map $\pi_1(U, P) \to \pi_1(X, P)$ is trivial).

Every reasonable X (e.g. connected manifold) fits (A), (B), (C).
Universal cover is unique (this justifies the word “universal”).

Existence of covers

Let $p \colon \tilde{X} \to X$ be a covering space.
Suppose that X satisfies (A), (B), (C), and $\tilde{X}$ is path-connected.
Take points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$.
We have a function (or a correspondence)

$$\text{covering space } p \colon \tilde{X} \to X \implies \text{ subgroup } p_*\Big(\pi_1(\tilde{X}, \tilde{x}_0)\Big) \subset \pi_1(X, x_0).$$

Is this function surjective?
For a trivial subgroup in $\pi_1(X)$, the answer is Yes.
For any subgroup $G \subset \pi_1(X)$, we can put

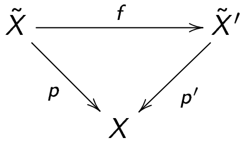
$$\tilde{X} = \mathcal{X}/G,$$

where $\mathcal{X}$ is the universal cover of X.
So the answer to the above question is Yes in general.
Is this function "injective"?

Galois correspondence

Let $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ be covering spaces.
Suppose that X satisfies (A), (B), (C).
Suppose that $\tilde{X}$ and $\tilde{X}'$ are path-connected.
Take points $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$, $\tilde{x}_0' \in \tilde{X}'$, with $p(\tilde{x}_0) = p(\tilde{x}_0') = x_0$.
Claim The following two conditions are equivalent:

- there is a homeomorphism $f\colon \tilde{X} \to \tilde{X}'$ such that



  is commutative diagram and $f(\tilde{x}_0) = \tilde{x}_0'$,
- $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p'_*(\pi_1(\tilde{X}', \tilde{x}_0'))$.

Theorem There exists a natural bijection between

- isomorphism classes of path-connected covering spaces $p\colon \tilde{X} \to X$,
- subgroups in $\pi_1(X)$ up to conjugation.