SMSTC Geometry and Topology 2011–2012

Lecture 5

The Seifert – van Kampen Theorem

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Introduction

- Topology and groups are closely related via the fundamental group construction

$$\pi_1 \ : \ \{\text{spaces}\} \to \{\text{groups}\} \ ; \ X \mapsto \pi_1(X) \ .$$

- The Seifert - van Kampen Theorem expresses the fundamental group of a union $X = X_1 \cup_Y X_2$ of path-connected spaces in terms of the fundamental groups of X_1, X_2, Y.
- The Theorem is used to compute the fundamental group of a space built up using spaces whose fundamental groups are known already.
- The Theorem is used to prove that every group G is the fundamental group $G = \pi_1(X)$ of a space X.
- Lecture follows Section I.1.2 of Hatcher's Algebraic Topology, but not slavishly so.

Revision : the fundamental group

- The fundamental group $\pi_1(X, x)$ of a space X at $x \in X$ is the group of based homotopy classes $[\omega]$ of pointed loops

$$\omega \ : \ (S^1, 1) \to (X, x) \ .$$

- The group law is by the concatenation of loops:

$$[\omega_1][\omega_2] \ = \ [\omega_1 \bullet \omega_2] \ .$$

- The inverse is by the reversal of loops:

$$[\omega]^{-1} \ = \ [\overline{\omega}]$$

  with $\overline{\omega}(t) = \omega(1 - t)$.
- A path $\alpha : I \to X$ induces an isomorphism of groups

$$\alpha_{\#} \ : \ \pi_1(X, \alpha(0)) \to \pi_1(X, \alpha(1)) \ ; \ \omega \mapsto [\overline{\alpha} \bullet \omega \bullet \alpha] \ .$$

- Will mainly consider path-connected spaces X :
  the fundamental group $\pi_1(X, x)$ is independent of the base point $x \in X$, and may be denoted $\pi_1(X)$.

Three ways of computing the fundamental group

I. By geometry

- For an infinite space X there are far too many loops $\omega : S^1 \to X$ in order to compute $\pi_1(X)$ from the definition.
- A space X is **simply-connected**, i.e. X is path connected and the fundamental group is trivial

$$\pi_1(X) \ = \ \{e\} \ .$$

- Sometimes it is possible to prove that X is simply-connected by geometry.
- **Example**: If X is contractible then X is simply-connected.
- **Example**: If $X = S^n$ and $n \geqslant 2$ then X is simply-connected.
- **Example**: Suppose that (X, d) is a metric space such that for any $x, y \in X$ there is unique geodesic ($=$ shortest path) $\alpha_{x,y} : I \to X$ from $\alpha_{x,y}(0) = x$ to $\alpha_{x,y}(1) = y$. If $\alpha_{x,y}$ varies continuously with x, y then X is contractible. Trees. Many examples of such spaces in differentiable geometry.

Three ways of computing the fundamental group

II. From above

- If

$$\begin{array}{c} \widetilde{X} \\ \big\downarrow p \\ X \end{array}$$

  is a covering projection and $\widetilde{X}$ is simply-connected then $\pi_1(X)$ is isomorphic to the group of covering translations

  $\text{Homeo}_p(\widetilde{X}) = \{\text{homeomorphisms } h : \widetilde{X} \to \widetilde{X} \text{ such that } ph = p\}$

  (This will be proved in a later lecture).
- **Example** If

$$p : \widetilde{X} = \mathbb{R} \to X = S^1 \; ; \; x \mapsto e^{2\pi ix}$$

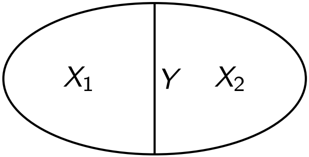
  then

$$\pi_1(S^1) = \text{Homeo}_p(\mathbb{R}) = \mathbb{Z} \; .$$

Three ways of computing the fundamental group

III. From below

- **Seifert-van Kampen Theorem** (preliminary version)



If a path-connected space X is a union $X = X_1 \cup_Y X_2$ with X_1, X_2 and $Y = X_1 \cap X_2$ path-connected then the fundamental group of X is the **free product with amalgamation**

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \; .$$

- $G_1 *_H G_2$ defined for group morphisms $H \to G_1$, $H \to G_2$.
- First proved by van Kampen (1933) in the special case when Y is simply-connected, and then by Seifert (1934) in general.

Seifert and van Kampen

Herbert Seifert
(1907-1996)

Egbert van Kampen
(1908-1942)

The free product of groups

- Let G_1, G_2 be groups with units $e_1 \in G_1$, $e_2 \in G_2$.
- A **reduced word** $g_1 g_2 \dots g_m$ is a finite sequence of length $m \geqslant 1$ with
  - $g_i \in G_1 \setminus \{e_1\}$ or $g_i \in G_2 \setminus \{e_2\}$,
  - g_i, g_{i+1} not in the same G_j.
- The **free product** of G_1 and G_2 is the group

$$G_1 * G_2 \ = \ \{e\} \cup \{\text{reduced words}\}$$

  with multiplication by concatenation and reduction.
- The unit e = empty word of length 0.
- See p.42 of Hatcher for detailed proof that $G_1 * G_2$ is a group.
- **Exercise** Prove that

$$\{e\} * G \cong G \ , \ G_1 * G_2 \cong G_2 * G_1 \ , \ (G_1 * G_2) * G_3 \cong G_1 * (G_2 * G_3)$$


Free Product &
Technical Advice

The free group F_g

- For a set S let

$$\langle S \rangle \;=\; \text{free group generated by } S \;=\; \mathop{\star}_{s \in S} \mathbb{Z} \; .$$

- Let $g \geqslant 1$. The **free group on g generators** is the free product of g copies of $\mathbb{Z}$

$$F_g \;=\; \langle a_1, a_2, \dots, a_g \rangle \;=\; \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} \; .$$

- Every element $x \in F_g$ has an expression as a word

$$x \;=\; (a_1)^{m_{11}}(a_2)^{m_{12}} \dots (a_g)^{m_{1g}} a_1^{m_{21}} \dots$$

  with (m_{ij}) an $N \times g$ matrix (N large), $m_{ij} \in \mathbb{Z}$.
- $F_1 = \mathbb{Z}$.
- For $g \geqslant 2$ F_g is nonabelian.
- $F_g * F_h = F_{g+h}$.

The infinite dihedral group

- **Warning** The free product need not be free.
- **Example** Let

$$G_1 = \mathbb{Z}_2 = \{e_1, a\} \ , \ \ G_2 = \mathbb{Z}_2 = \{e_2, b\}$$

be cyclic groups of order 2, with generators a, b such that

$$a^2 = e_1 \ , \ \ b^2 = e_2 \ .$$

- The **infinite dihedral group** is the free product

$$D_\infty \ = \ \mathbb{Z}_2 * \mathbb{Z}_2 \ = \ \{e, a, b, ab, ba, aba, bab, abab, \dots\} \ ,$$

with

$$a^2 = b^2 = e \ .$$

This is an infinite group with torsion, so not free.

The subgroups generated by a subset

- Needed for statement and proof the Seifert - van Kampen Theorem.
- Let G be a group. The **subgroup generated** by a subset $S \subseteq G$
$$\langle S \rangle \subseteq G$$
is the smallest subgroup of G containing S.
- $\langle G \rangle$ consists of finite length words in elements of S and their inverses.
- Let S^G be the subset of G consisting of the conjugates of S
$$S^G = \{gsg^{-1} \,|\, s \in S,\, g \in G\}$$
- The **normal subgroup generated** by a subset $S \subseteq G$ $\langle S^G \rangle \subseteq G$ is the smallest normal subgroup of G containing S.

Group presentations

- Given a set S and a subset $R \subseteq \langle S \rangle$ define the group

$$\begin{aligned}\langle S|R\rangle &= \langle S\rangle/\langle R^{\langle S\rangle}\rangle \\ &= \langle S\rangle/\text{normal subgroup generated by } R\ .\end{aligned}$$

  with **generating set** S and **relations** R.
- **Example** Let $m \geqslant 1$. The function

$$\langle a \,|\, a^m\rangle \ = \ \{e, a, a^2, \dots, a^{m-1}\} \to \mathbb{Z}_m \ ; \ a^n \mapsto n$$

  is an isomorphism of groups, with $\mathbb{Z}_m$ the finite cyclic group of order m.
- R can be empty, with

$$\langle S|\emptyset\rangle \ = \ \langle S\rangle \ = \ \mathop{\star}_{S} \mathbb{Z}$$

  the free group generated by S.
- The free product of $G_1 = \langle S_1 | \, R_1\rangle$ and $G_2 = \langle S_2 \,|\, R_2\rangle$ is

$$G_1 * G_2 = \langle S_1 \cup S_2 \,|\, R_1 \cup R_2\rangle\ .$$

Every group has a presentation

- Every group G has a group presentation, i.e. is isomorphic to $\langle S|R\rangle$ for some sets S, R.
- **Proof** Let $S = \langle G\rangle = \underset{G}{\star}\,\mathbb{Z}$ and let $R = \ker(\Phi)$ be the kernel of the surjection of groups

$$\Phi \ : \ S \to G \ ; \ (g_1^{n_1}, g_2^{n_2}, \dots) \mapsto (g_1)^{n_1}(g_2)^{n_2}\dots \ .$$

  Then

$$\langle S|R\rangle \to G \ ; \ [x] \mapsto \Phi(x)$$

  is an isomorphism of groups.
- It is a nontrivial theorem that R is a free subgroup of the free group S. But we are only interested in S and R as sets here. This presentation is too large to be of use in practice! But the principle has been established.
- While presentations are good for specifying groups, it is not always easy to work out what the group actually is. Word problem: when is $\langle S|R\rangle \cong \langle S'|R'\rangle$? Undecidable in general.

How to make a group abelian

- The **commutator** of $g, h \in G$ is
$$[g,h] \ = \ ghg^{-1}h^{-1} \in G \ .$$
Let $F = \{[g,h]\} \subseteq G$. G is abelian if and only if $F = \{e\}$.
- The **abelianization** of a group G is the abelian group
$$G^{ab} \ = \ G/\langle F^G \rangle \ ,$$
with $\langle F^G \rangle \subseteq G$ the normal subgroup generated by F.
- If $G = \langle S | R \rangle$ then $G^{ab} = \langle S | R \cup F \rangle$.
- **Universal property** G^{ab} is the largest abelian quotient group of G, in the sense that for any group morphism $f : G \to A$ to an abelian group A there is a unique group morphism $f^{ab} : G^{ab} \to A$ such that
$$f \ : \ G \longrightarrow G^{ab} \xrightarrow{f^{ab}} A \ .$$
- $\pi_1(X)^{ab} = H_1(X)$ is the **first homology group** of a space X – about which more in a later lecture.

Free abelian at last

- **Example** Isomorphism of groups

$$(F_2)^{ab} \;=\; \langle a, b \,|\, aba^{-1}b^{-1}\rangle \to \mathbb{Z}\oplus\mathbb{Z} \;;$$
$$a^{m_1} b^{n_1} a^{m_2} b^{n_2} \ldots \mapsto (m_1 + m_2 + \ldots, n_1 + n_2 + \ldots)$$

  with $\mathbb{Z}\oplus\mathbb{Z}$ the free abelian group on 2 generators.
- More generally, the abelianization of the free group on g generators is the free abelian group on g generators

$$(F_g)^{ab} \;=\; \bigoplus_g \mathbb{Z} \text{ for any } g \geqslant 1 \; .$$

- It is clear from linear algebra (Gaussian elimination) that

$$\bigoplus_g \mathbb{Z} \text{ is isomorphic to } \bigoplus_h \mathbb{Z} \text{ if and only if } g = h \; .$$

- It follows that

$$F_g \text{ is isomorphic to } F_h \text{ if and only if } g = h \; .$$

Amalgamated free products

- The **amalgamated free product** of group morphisms

$$i_1 \ : \ H \to G_1 \ , \ i_2 \ : \ H \to G_2$$

is the group

$$G_1 *_H G_2 \ = \ (G_1 * G_2)/N$$

with $N \subseteq G_1 * G_2$ the normal subgroup generated by the elements

$$i_1(h)i_2(h)^{-1} \ (h \in H) \ .$$

- For any $h \in H$

$$i_1(h) \ = \ i_2(h) \in G_1 *_H G_2 \ .$$

- In general, the natural morphisms of groups

$$j_1 : G_1 \to G_1 *_H G_2 \ , \ j_2 : G_2 \to G_1 *_H G_2 \ , \ j_1 i_1 = j_2 i_2 : H \to G_1 *_H G_2$$

are not injective.

Some examples of amalgamated free products

- **Example** $G *_G G = G$.
- **Example** $\{e\} *_H \{e\} = \{e\}$.
- **Example** For $H = \{e\}$ the amalgamated free product is just the free product

$$G_1 *_{\{e\}} G_2 \;=\; G_1 * G_2 \;.$$

- **Example** For any group morphism $i : H \to G$

$$G *_H \{e\} \;=\; G/N$$

with $N = \langle i(H)^G \rangle \subseteq G$ the normal subgroup generated by the subgroup $i(H) \subseteq G$.

Injective amalgamated free products

- If $i_1 : H \to G_1$, $i_2 : H \to G_2$ are injective then G_1, G_2, H are subgroups of $G_1 *_H G_2$ with $\langle G_1 \cup G_2 \rangle = G$, $G_1 \cap G_2 = H$.
- Conversely, suppose that G is a group and that $G_1, G_2 \subseteq G$ are subgroups such that

$$\langle G_1 \cup G_2 \rangle \ = \ G \ .$$

- This condition is equivalent to the group morphism

$$\Phi \ : \ G_1 * G_2 \to G \ ; \ g_k \mapsto g_k \ (g_k \in G_k)$$

  being surjective.
- Then $G = G_1 *_H G_2$ with

$$H \ = \ G_1 \cap G_2 \subseteq G$$

  and $i_1 : H \to G_1$, $i_2 : H \to G_2$ the inclusions.

The Seifert - van Kampen Theorem

- Let $X = X_1 \cup_Y X_2$ with X_1, X_2 and $Y = X_1 \cap X_2$ open in X and path-connected. Let
$$i_1 : \pi_1(Y) \to \pi_1(X_1)\ ,\ i_2 : \pi_1(Y) \to \pi_1(X_2)$$
be the group morphisms induced by $Y \subseteq X_1$, $Y \subseteq X_2$, and let
$$j_1 : \pi_1(X_1) \to \pi_1(X)\ ,\ j_2 : \pi_1(X_2) \to \pi_1(X)$$
be the group morphisms induced by $X_1 \subseteq X$, $X_2 \subseteq X$. Then
$$\Phi \ :\ \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X)\ ;\ x_k \mapsto j_k(x_k)$$
$(x_k \in \pi_1(X_k), k = 1$ or $2)$ is a surjective group morphism with
$$\begin{aligned}\ker \Phi \ =\ N \ =\ & \text{the normal subgroup of } \pi_1(X_1) * \pi_1(X_2) \\ & \text{generated by } i_1(y) i_2(y)^{-1}\ (y \in \pi_1(Y))\ .\end{aligned}$$
- **Theorem** Φ induces an isomorphism of groups
$$\Phi \ :\ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ \cong\ \pi_1(X)\ .$$

Φ is surjective I.

- Will only prove the easy part, that Φ is surjective.
- For the hard part, that Φ is injective, see pp.45-46 of Hatcher.
- Choose the base point $x \in Y \subseteq X$. Regard a loop $\omega : (S^1, 1) \to (X, x)$ as a closed path

$$f \ : \ I = [0,1] \to X = X_1 \cup_X X_2$$

such that $f(0) = f(1) = x \in X$, with $\omega(e^{2\pi is}) = f(s) \in X$.
- By the compactness of I there exist

$$0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$$

such that $f[s_i, s_{i+1}] \subseteq X_1$ or X_2, written $f[s_i, s_{i+1}] \subseteq X_i$.
- Then

$$f = f_1 \bullet f_2 \bullet \cdots \bullet f_m : I \to X$$

is the concatenation of paths $f_i : I \to X_i$ with

$$f_i(1) \ = \ f_{i+1}(0) \ = \ f(s_i) \in Y \ (1 \leqslant i \leqslant m) \ .$$

Φ is surjective II.

- Since Y is path-connected there exists paths $g_i : I \to Y$ from $g_i(0) = x$ to $g_i(1) = f(s_i) \in X$. The loop

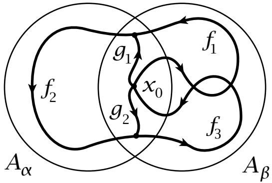
  $$(f_1 \bullet \overline{g}_1) \bullet (g_1 \bullet f_2 \bullet \overline{g}_2) \bullet \cdots \bullet (g_{m-2} \bullet f_{m-1} \bullet \overline{g}_{m-1}) \bullet (g_{m-1} \bullet f_m) : I \to X$$

  is homotopic to f rel $\{0, 1\}$, with

  $$[g_i \bullet f_i \bullet \overline{g}_{i+1}] \in \text{im}(\pi_1(X_i)) \subseteq \pi_1(X) ,$$

  so that

  $$[f] = [f_1 \bullet \overline{g}_1][g_1 \bullet f_2 \bullet \overline{g}_2] \ldots [g_{m-2} \bullet f_{m-1} \bullet \overline{g}_{m-1}][g_{m-1} \bullet f_m] \in \text{im}(\Phi) \subseteq \pi_1(X) .$$



Hatcher diagram: $A_\alpha = X_1$, $A_\beta = X_2$

The universal property I.

- An amalgamated free product $G_1 *_H G_2$ defines a a commutative square of groups and morphisms

$$\begin{array}{ccc} H & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ G_2 & \xrightarrow{j_2} & G_1 *_H G_2 \end{array}$$

with the universal property that for any commutative square

$$\begin{array}{ccc} H & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ G_2 & \xrightarrow{k_2} & G \end{array}$$

there is a unique group morphism $\Phi : G_1 *_H G_2 \to G$ such that $k_1 = \Phi j_1 : G_1 \to G$ and $k_2 = \Phi j_2 : G_2 \to G$.

The universal property II.

- By the Seifert - van Kampen Theorem for $X = X_1 \cup_Y X_2$ the commutative square

$$\begin{array}{ccc} \pi_1(Y) & \xrightarrow{i_1} & \pi_1(X_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \pi_1(X_2) & \xrightarrow{j_2} & \pi_1(X) \end{array}$$

has the universal property of an amalgamated free product, with an isomorphism

$$\Phi \ : \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ \cong \ \pi_1(X) \ .$$

The one-point union

- Let X_1, X_2 be spaces with base points $x_1 \in X_1$, $x_2 \in X_2$. The **one-point union** is

$$X_1 \vee X_2 \ = \ X_1 \times \{x_2\} \cup \{x_1\} \times X_2 \subseteq X_1 \times X_2$$

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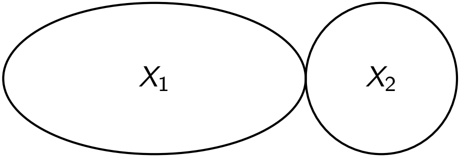


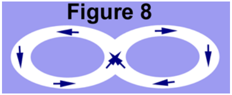
- **The Seifert - van Kampen Theorem for** $X_1 \vee X_2$
If X_1, X_2 are path connected then so is $X_1 \vee X_2$, with fundamental group the free product

$$\pi_1(X_1 \vee X_2) \ = \ \pi_1(X_1) * \pi_1(X_2) \ .$$

The fundamental group of the figure eight

- The figure eight is the one-point union of two circles

$$X = S^1 \vee S^1$$



- The fundamental group is the free **nonabelian** group on two generators:

$$\pi_1(X) = \pi_1(S^1) * \pi_1(S^1) = \langle a, b \rangle = \mathbb{Z} * \mathbb{Z} .$$

- An element

$$a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots \in \pi_1(X)$$

can be regarded as the loop traced out by an iceskater who traces out a figure 8, going round the first circle m_1 times, then round the second circles n_1 times, then round the first circle m_2 times, then round the second circle n_2 times, ...

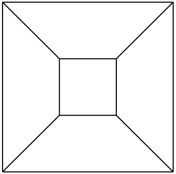
A famous Edinburgh iceskater

The fundamental group of a graph

- Let X be the path-connected space defined by a connected finite graph with V vertices and E edges, and let

$$g = 1 - V + E .$$

- **Exercise** Prove that X is homotopy equivalent to the one-point union $S^1 \vee S^1 \vee \cdots \vee S^1$ of g circles, and hence that $\pi_1(X) = F_g$, the free group on g generators. Prove that X is a tree if and only if it is contractible, if and only if $g = 0$.
- Example I.1.22 of Hatcher is a special case with $V = 8$, $E = 12$, $g = 5$.



Knots

- A **knot** is an embedding

$$K \;:\; S^1 \subset S^3 \;.$$

- The **complement** of K is

$$X_K \;=\; S^3 \backslash K(S^1) \subset S^3 \;.$$

- Two knots $K_1, K_2 : S^1 \subset S^3$ are **equivalent** if there exists a homeomorphism of their complements

$$h \;:\; X_{K_1} \;\cong\; X_{K_2} \;.$$

- The fundamental group $\pi_1(X_K)$ of the complement of a knot K is an invariant of the equivalence class of K. Fact:

$$\pi_1(X_K)^{ab} \;=\; H_1(X_K) \;=\; \mathbb{Z} \;.$$

- The Seifert - van Kampen Theorem can be applied to obtain the Wirtinger presentation of $\pi_1(X_K)$ from a knot projection – Exercise I.1.2.22 of Hatcher.

The unknot

- The **unknot**

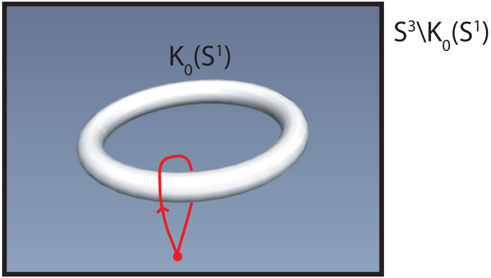
$$K_0 : S^1 \subset S^3; z \mapsto (z, 0, 0)$$

has complement

$$X_{K_0} \ = \ S^1 \times \mathbb{R}^2$$

with group

$$\pi_1(X_{K_0}) \ = \ \mathbb{Z}$$



Knotting and unknotting

- A knot $K : S^1 \subset S^3$ is **unknotted** if it is equivalent to K_0.
- A knot $K : S^1 \subset S^3$ is **knotted** if it is not equivalent to K_0.
- It is easy to prove that if the surjection

$$\pi_1(X_K) \to \pi_1(X_K)^{ab} \;=\; \mathbb{Z}$$

  is not an isomorphism then K is knotted, since

$$\pi_1(X_K) \ncong \pi_1(X_{K_0}) = \mathbb{Z} \ .$$

- It is hard to prove (but true) that if the surjection

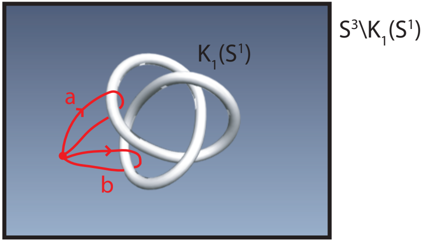
$$\pi_1(X_K) \to \pi_1(X_K)^{ab} \;=\; \mathbb{Z}$$

  is an isomorphism then K is unknotted. This is Dehn's Lemma, originally stated in 1911, and only finally proved by Papakyriakopoulos in 1957.
- Musical interlude: Get knotted! (Fascinating Aida)

The trefoil knot

- The trefoil knot $K_1 : S^1 \subset S^3$ has group

$$\pi_1(X_{K_1}) = \{a, b \mid aba = bab\} .$$



- The groups of K_0, K_1 are not isomorphic (since one is abelian and the other one is not abelian), so that K_0, K_1 are not equivalent: the algebra shows that the trefoil knot is knotted.

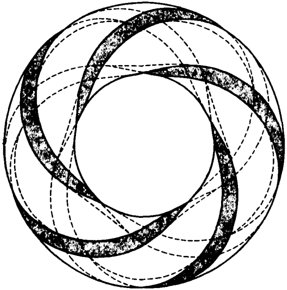
The torus knots I.

- For coprime $m, n \geqslant 1$ use the embedding

$$S^1 \subset S^1 \times S^1 \ ; \ z \mapsto (z^m, z^n)$$

  to define the **torus knots**

$$T_{m,n} : S^1 \subset S^1 \times S^1 \subset S^3 \ .$$



- Every torus knot $S^1 \subset S^1 \times S^1 \subset S^3$ is equivalent to one of $T_{m,n}$ (Brauner, 1928).

The torus knots II.

- See Example I.1.24 of Hatcher for the application of the Seifert - van Kampen Theorem to compute

$$\pi_1(X_{T_{m,n}}) \ = \ \langle x, y \,|\, x^m = y^n \rangle \ .$$

- The centre (= the elements which commute with all others)

$$C \ = \ \langle x^m \rangle \ = \ \langle y^n \rangle \subseteq \pi_1(T_{m,n})$$

  is an infinite cyclic normal subgroup such that

$$\pi_1(X_{T_{m,n}})/C \ = \ \mathbb{Z}_m * \mathbb{Z}_n \ .$$

- Two torus knots $T_{m,n}$, $T_{m',n'}$ are equivalent if and only if $(m,n) = (m',n')$ or (n',m').
- The trefoil knot is $K_1 = T_{2,3}$.

Links

- Let $\mu \geqslant 1$. A μ-**component link** is an embedding

$$L \ : \ \bigcup_{\mu} S^1 \ = \ S^1 \cup S^1 \cup \cdots \cup S^1 \subset S^3 \ .$$

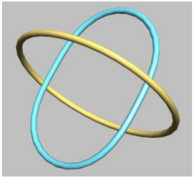
- A knot is a 1-component link.
- The **complement** of L is

$$X_L \ = \ S^3 \backslash L(\bigcup_{\mu} S^1) \subset S^3 \ .$$

- The fundamental group $\pi_1(X_L)$ detects the linking of the circles among each other. (And much else besides).

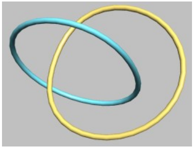
The unlink and the Hopf link

- The unlink $L_0 : S^1 \cup S^1 \subset S^3$



has $X_{L_0} \simeq S^1 \vee S^1 \vee S^2$, $\pi_1(X_{L_0}) = F_2 = \mathbb{Z} * \mathbb{Z}$.

- The Hopf link $L_1 : S^1 \cup S^1 \subset S^3$



has $X_{L_1} \simeq S^1 \times S^1$, $\pi_1(X_{L_1}) = \mathbb{Z} \oplus \mathbb{Z}$.

- Hatcher Example I.1.2, noting difference between S^3 and $\mathbb{R}^3$.

Cell attachment

- Let $n \geqslant 0$. Given a space W and a map $f : S^{n-1} \to W$ let

$$X = W \cup_f D^n$$

be the space obtained from W by **attaching an n-cell**.

- X is the quotient of the disjoint union $W \cup D^n$ by the equivalence relation generated by

$$(x \in S^{n-1}) \sim (f(x) \in W) .$$

- An **n-dimensional cell complex** is a space obtained from $\emptyset$ by successively attaching k-cells, with $k = 0, 1, 2, \ldots, n$
- **Example** A graph is a 1-dimensional cell complex.
- **Example** S^n is the n-dimensional cell complex obtained from $\emptyset$ by attaching a 0-cell and an n-cell

$$S^n = D^0 \cup_f D^n$$

with $f : S^{n-1} \to D^0$ the unique map.

The effect on π_1 of a cell attachment I.

- Let $n \geqslant 1$. If W is path-connected then so is $X = W \cup_f D^n$.
- What is $\pi_1(X)$?
- If $n = 1$ then X is homotopy equivalent to $W \vee S^1$, so that

$$\pi_1(X) \ = \ \pi_1(W \vee S^1) \ = \ \pi_1(W) * \mathbb{Z} \ .$$

- For $n \geqslant 2$ apply the Seifert - van Kampen Theorem to the decomposition

$$X \ = \ X_1 \cup_Y X_2$$

with

$$X_1 \ = \ W \cup_f \{x \in D^n \mid \|x\| \geqslant 1/2\} \ ,$$
$$X_2 \ = \ \{x \in D^n \mid \|x\| \leqslant 1/2\} \ ,$$
$$Y \ = \ X_1 \cap X_2 \ = \ \{x \in D^n \mid \|x\| = 1/2\} \ = \ S^{n-1}$$

The effect on π_1 of a cell attachment II.

- The inclusion $W \subset X_1$ is a homotopy equivalence, and $X_2 \cong D^n$ is simply-connected, so that

$$\begin{aligned}\pi_1(X) &= \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \\ &= \pi_1(W) *_{\pi_1(S^{n-1})} \pi_1(D^n) \ = \ \pi_1(W) *_{\pi_1(S^{n-1})} \{e\} \ .\end{aligned}$$

- If $n \geqslant 3$ then $\pi_1(S^{n-1}) = \{e\}$, so that

$$\pi_1(X) \ = \ \pi_1(W) *_{\{e\}} \{e\} \ = \ \pi_1(W) \ .$$

- If $n = 2$ then

$$\pi_1(X) \ = \ \pi_1(W) *_{\mathbb{Z}} \{e\} \ = \ \pi_1(W)/N$$

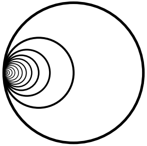
  the quotient of $\pi_1(W)$ by the normal subgroup $N \subseteq \pi_1(W)$ generated by the homotopy class $[f] \in \pi_1(W)$ of $f : S^1 \to W$.
- See Hatcher's Proposition I.1.26 for detailed exposition.
- If $X = \bigvee_S S^1 \cup \bigcup_R D^2 \cup \bigcup_{n \geqslant 3} \bigcup D^n$ is a cell complex with a single 0-cell, S 1-cells and R 2-cells then $\pi_1(X) = \langle S | R \rangle$.

Every group is a fundamental group

- Let $G = \langle S | R \rangle$ be a group with a presentation.
- Realize the generators S by the 1-dimensional cell complex

$$W = \bigvee_S S^1$$

with $\pi_1(W) = \langle S \rangle$ the free group generated by S.



- Realize each relation $r \in R \subseteq \pi_1(W)$ by a map $r : S^1 \to W$.
- Attach a 2-cell to W for each relation, to obtain a 2-dimensional cell complex $X = W \cup \bigcup_R D^2$ such that

$$\pi_1(X) = \langle S | R \rangle = G .$$

Realizing the cyclic groups topologically

- **Example** Let $m \geqslant 1$. The cyclic group $\mathbb{Z}_m = \langle a \mid a^m \rangle$ of order m is the fundamental group

$$\pi_1(X_m) = \mathbb{Z}_m$$

of the 2-dimensional cell complex

$$X_m = S^1 \cup_m D^2 ,$$

with the 2-cell attached to $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by

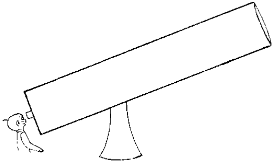
$$m : S^1 \to S^1 ; z \mapsto z^m .$$

- $X_1 = D^2$ is contractible, with $\pi_1(X_1) = \{e\}$
- X_2 is homeomorphic to the real projective plane

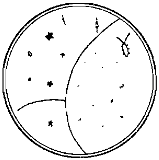
$$\begin{aligned} \mathbb{RP}^2 &= S^2/\{v \sim -v \mid v \in S^2\} \\ &= D^2/\{w \sim -w \mid w \in S^1\} \end{aligned}$$

with $\pi_1(X_2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$.

If the Universe were a projective space
(from 'Relativity for the Layman' by J. Coleman, Pelican (1959))



(a) He looks ...



(b) and sees ...

We can conclude that according to the General Theory of relativity, the universe was considered to be finite and unbounded. Whether it is or not may never actually be determined experimentally. However, it is amusing to predict what may take place many years from now. An astronomer might some day build a super-duper telescope, and we can imagine what will happen when he looks through it. He may see a shiny luminous object which looks like the moon, but with a very peculiar-looking curved tree growing out of it. Only after many hours of quiet and careful scrutiny will it dawn on him that he is looking at his own gleaming bald pate, the light from which has gone completely around the universe and returned!