SMSTC Geometry and Topology 2011–2012

Lecture 4

The fundamental group and covering spaces

Lecturer: Vanya Cheltsov (Edinburgh)

Slides: Andrew Ranicki (Edinburgh)

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The method of algebraic topology

- Algebraic topology uses algebra to distinguish topological spaces from each other, and also to distinguish continuous maps from each other.
- A 'group-valued functor' is a function

  $$\pi \ : \ \{\text{topological spaces}\} \to \{\text{groups}\}$$

  which sends a topological space X to a group $\pi(X)$, and a continuous function $f : X \to Y$ to a group morphism $f_* : \pi(X) \to \pi(Y)$, satisfying the relations

  $$(1 : X \to X)_* \ = \ 1 \ : \ \pi(X) \to \pi(X) \ ,$$

  $$(gf)_* \ = \ g_* f_* \ : \pi(X) \to \pi(Z) \text{ for } f : X \to Y,\ g : Y \to Z \ .$$

- Consequence 1: If $f : X \to Y$ is a homeomorphism of spaces then $f_* : \pi(X) \to \pi(Y)$ is an isomorphism of groups.
- Consequence 2: If X, Y are spaces such that $\pi(X), \pi(Y)$ are not isomorphic, then X, Y are not homeomorphic.

The fundamental group - a first description

- The fundamental group of a space X is a group $\pi_1(X)$.
- The actual definition of $\pi_1(X)$ depends on a choice of base point $x \in X$, and is written $\pi_1(X, x)$. But for path-connected X the choice of x does not matter.
- Ignoring the base point issue, the fundamental group is a functor $\pi_1 : \{\text{topological spaces}\} \to \{\text{groups}\}$.
- $\pi_1(X, x)$ is the geometrically defined group of 'homotopy' classes $[\omega]$ of 'loops at $x \in X$', continuous maps $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. A continuous map $f : X \to Y$ induces a morphism of groups

$$f_* \ : \ \pi_1(X, x) \to \pi_1(Y, f(x)) \ ; \ [\omega] \mapsto [f\omega] \ .$$

- $\pi_1(S^1) = \mathbb{Z}$, an infinite cyclic group.
- In general, $\pi_1(X)$ is not abelian.

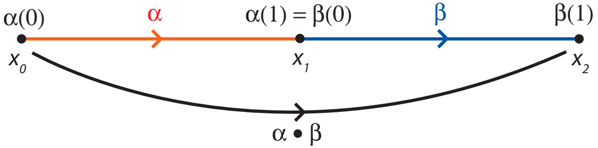
Joined up thinking

- A path in a topological space X is a continuous map $\alpha : I = [0,1] \to X$. Starts at $\alpha(0) \in X$ and ends at $\alpha(1) \in X$.
- **Proposition** The relation on X defined by $x_0 \sim x_1$ if there exists a path $\alpha : I \to X$ with $\alpha(0) = x_0$, $\alpha(1) = x_1$ is an equivalence relation.
- **Proof** (i) Every point $x \in X$ is related to itself by the constant path $\alpha : I \to X; t \mapsto x$.
  (ii) The reverse of a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ is the path $-\alpha; I \to X; t \mapsto \alpha(1-t)$ from $-\alpha(0) = x_1$ to $-\alpha(1) = x_0$.

The concatenation of paths

- (iii) The concatenation of a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ and of a path $\beta : I \to X$ from $\beta(0) = x_1$ to $\beta(1) = x_2$ is the path from x_0 to x_2 given by

$$\alpha \bullet \beta \ : \ I \to X \ ; \ t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \beta(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$



Path components

- The path components of X are the equivalence classes of the path relation on X.
- The path component $[x]$ of $x \in X$ consists of all the points $y \in X$ such that there exists a path in X from x to y.
- The set of path components of X is denoted by $\pi_0(X)$.
- A continuous map $f : X \to Y$ induces a function

$$f_* \ : \ \pi_0(X) \to \pi_0(Y) \ ; \ [x] \mapsto [f(x)] \ .$$

- The function

$$\pi_0 \ : \ \{\text{topological spaces and continuous maps}\} \ ; \\ \to \{\text{sets and functions}\} \ ; \ X \mapsto \pi_0(X) \ , \ f \mapsto f_*$$

is a set-valued functor.

Path-connected spaces

- A space X is path-connected if $\pi_0(X)$ consists of just one element. Equivalently, there is only one path component, i.e. if for every $x_0, x_1 \in X$ there exists a path $\alpha : I \to X$ starting at $\alpha(0) = x_0$ and ending at $\alpha(1) = x_1$.
- Example Any connected open subset $U \subseteq \mathbb{R}^n$ is path-connected. This result is often used in analysis, e.g. in checking that the contour integral in the Cauchy formula

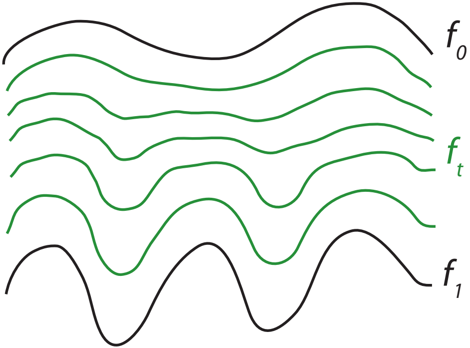
$$\frac{1}{2\pi i} \oint_{\omega} \frac{f(z)dz}{z - z_0}$$

  is well-defined, i.e. independent of the loop $\omega \subset \mathbb{C}$ around $z_0 \in \mathbb{C}$, with $U = \mathbb{C} \backslash \{z_0\} \subset \mathbb{C} = \mathbb{R}^2$.
- Exercise Every path-connected space is connected.
- Exercise Construct a connected space which is not path-connected.

Homotopy I.

- **Definition** A homotopy of continuous maps $f_0 : X \to Y$, $f_1 : X \to Y$ is a continuous map $f : X \times I \to Y$ such that for all $x \in X$

$$f(x,0) \;=\; f_0(x)\;,\;\; f(x,1) \;=\; f_1(x) \in Y\;.$$



Homotopy II.

- A homotopy $f : X \times I \to Y$ consists of continuous maps

$$f_t \ : \ X \to Y \ ; \ x \mapsto f_t(x) \ = \ f(x,t)$$

which varies continuously with 'time' $t \in I$. Starts at f_0 and ending at f_1, like the first and last shot of a take in a film.

- For each $x \in X$ there is defined a path

$$\alpha_x \ : \ I \to Y \ ; \ t \mapsto \alpha_x(t) \ = \ f_t(x)$$

starting at $\alpha_x(0) = f_0(x)$ and ending at $\alpha_x(1) = f_1(x)$. The path α_x varies continuously with $x \in X$.

- **Example** The constant map $f_0 : \mathbb{R}^n \to \mathbb{R}^n; x \mapsto 0$ is homotopic to the identity map $f_1 : \mathbb{R}^n \to \mathbb{R}^n; x \mapsto x$ by the homotopy

$$h \ : \ \mathbb{R}^n \times I \to \mathbb{R}^n \ ; \ (x,t) \mapsto tx \ .$$

Homotopy equivalence I.

- **Definition** Two spaces X, Y are homotopy equivalent if there exist continuous maps $f : X \to Y$, $g : Y \to X$ and homotopies

$$h \ : \ gf \simeq 1_X \ : \ X \to X \ , \ k \ : \ fg \simeq 1_Y \ : \ Y \to Y \ .$$

- A continuous map $f : X \to Y$ is a homotopy equivalence if there exist such g, h, k. The continuous maps f, g are inverse homotopy equivalences.
- **Example** The inclusion $f : S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is a homotopy equivalence, with homotopy inverse

$$g \ : \ \mathbb{R}^{n+1} \backslash \{0\} \to S^n \ ; \ x \mapsto \frac{x}{\|x\|} \ .$$

Homotopy equivalence II.

- The relation defined on the set of topological spaces by

$$X \simeq Y \text{ if } X \text{ is homotopy equivalent to } Y$$

  is an equivalence relation.
- **Slogan 1.** Algebraic topology views homotopy equivalent spaces as being isomorphic.
- **Slogan 2.** Use topology to construct homotopy equivalences, and algebra to prove that homotopy equivalences cannot exist.
- **Exercise** Prove that a homotopy equivalence $f : X \to Y$ induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$. Thus X is path-connected if and only if Y is path-connected.

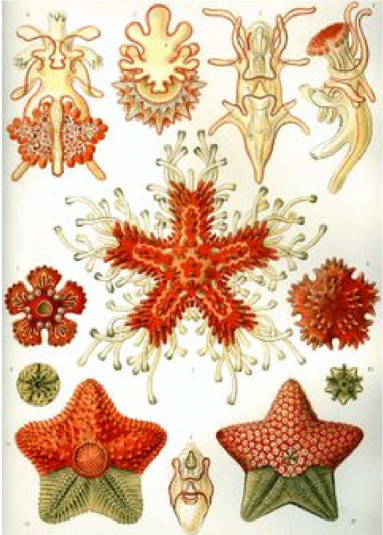
Contractible spaces

- A space X is contractible if it is homotopy equivalent to the space $\{\text{pt.}\}$ consisting of a single point.
- **Exercise** A subset $X \subseteq \mathbb{R}^n$ is star-shaped at $x \in X$ if for every $y \in X$ the line segment joining x to y

$$[x, y] \;=\; \{(1-t)x + ty \,|,\; 0 \leqslant t \leqslant 1\}$$

  is contained in X. Prove that X is contractible.
- **Example** The n-dimensional Euclidean space $\mathbb{R}^n$ is contractible.
- **Example** The unit n-ball $D^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$ is contractible.
- By contrast, the n-dimensional sphere S^n is not contractible, although this is not easy to prove. In fact, it can be shown that S^n is homotopy equivalent to S^m if and only if $m = n$. As S^n is the one-point compactification of $\mathbb{R}^n$, it follows that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

Every starfish is contractible



"Asteroidea" from Ernst Haeckel's Kunstformen der Natur, 1904 (Wikipedia)

Based spaces

- **Definition** A based space (X, x) is a space with a base point $x \in X$.
- **Definition** A based continuous map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y \in Y$.
- **Definition** A based homotopy $h : f \simeq g : (X, x) \to (Y, y)$ is a homotopy $h : f \simeq g : X \to Y$ such that

$$h(x, t) = y \in Y \quad (t \in I) \ .$$

- For any based spaces (X, x), (Y, y) based homotopy is an equivalence relation on the set of based continuous maps $f : (X, x) \to (Y, y)$.

Loops = closed paths

- A path $\alpha : I \to X$ is closed if $\alpha(0) = \alpha(1) \in X$.
- Identify S^1 with the unit circle $\{z \in \mathbb{C} \,|\, |z| = 1\}$ in the complex plane $\mathbb{C}$.
- A based loop is a based continuous map $\omega : (S^1, 1) \to (X, x)$.
- In view of the homeomorphism

$$I/\{0 \sim 1\} \to S^1 \; ; \; [t] \mapsto e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t$$

there is essentially no difference between based loops $\omega : (S^1, 1) \to (X, x)$ and closed paths $\alpha : I \to X$ at $x \in X$, with

$$\alpha(t) \; = \; \omega(e^{2\pi i t}) \in X \quad (t \in I)$$

such that

$$\alpha(0) \; = \; \omega(1) \; = \; \alpha(1) \in X \; .$$

Homotopy relative to a subspace

- Let X be a space, $A \subseteq X$ a subspace. If $f, g : X \to Y$ are continuous maps such that $f(a) = g(a) \in Y$ for all $a \in A$ then a homotopy rel A (or relative to A) is a homotopy $h : f \simeq g : X \to Y$ such that

$$h(a,t) \;=\; f(a) \;=\; g(a) \in Y \quad (a \in A, t \in I) \ .$$

- **Exercise** If a space X is path-connected prove that any two paths $\alpha, \beta : I \to X$ are homotopic.
- **Exercise** Let $e_x : I \to X; t \mapsto x$ be the constant closed path at $x \in X$. Prove that for any closed path $\alpha : I \to X$ at $\alpha(0) = \alpha(1) = x \in X$ there exists a homotopy rel $\{0, 1\}$

$$\alpha \bullet -\alpha \;\simeq\; e_x \;:\; I \to X$$

with $\alpha \bullet -\alpha$ the concatenation of α and its reverse $-\alpha$.

The fundamental group (official definition)

- The fundamental group $\pi_1(X, x)$ is the set of based homotopy classes of loops $\omega : (S^1, 1) \to (X, x)$, or equivalently the rel $\{0, 1\}$ homotopy classes $[\alpha]$ of closed paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$.
- The group law is by the concatenation of closed paths

$$\pi_1(X, x) \times \pi_1(X, x) \to \pi_1(X, x) \; ; \; ([\alpha], [\beta]) \mapsto [\alpha \bullet \beta]$$

- Inverses are by the reversing of paths

$$\pi_1(X, x) \to \pi_1(X, x) \; ; \; [\alpha] \mapsto [\alpha]^{-1} = [-\alpha] \; .$$

- The constant closed path e_x is the identity element

$$[\alpha \bullet e_x] \; = \; [e_x \bullet \alpha] \; = \; [\alpha] \in \pi_1(X, x) \; .$$

- See Theorem 4.2.15 of the notes for a detailed proof that $\pi_1(X, x)$ is a group.

Fundamental group morphisms

- **Proposition** A continuous map $f : X \to Y$ induces a group morphism

$$f_* \;:\; \pi_1(X,x) \to \pi_1(Y,f(x)) \;;\; [\omega] \mapsto [f\omega] \;.$$

  with the following properties:
  (i) The identity $1 : X \to X$ induces the identity, $1_* = 1 : \pi_1(X,x) \to \pi_1(X,x)$.
  (ii) The composite of $f : X \to Y$ and $g : Y \to Z$ induces the composite, $(gf)_* = g_* f_* : \pi_1(X,x) \to \pi_1(Z,gf(x))$.
  (iii) If $f,g : X \to Y$ are homotopic rel $\{x\}$ then $f_* = g_* : \pi_1(X,x) \to \pi_1(Y,f(x))$.
  (iv) If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ is an isomorphism.
  (v) A path $\alpha : I \to X$ induces an isomorphism

$$\alpha_\# \;:\; \pi_1(X,\alpha(0)) \to \pi_1(X,\alpha(1)) \;;\; \omega \mapsto (-\alpha) \bullet \omega \bullet \alpha \;.$$

- In view of (v) we can write $\pi_1(X,x)$ as $\pi_1(X)$ for a path-connected space.

Simply-connected spaces

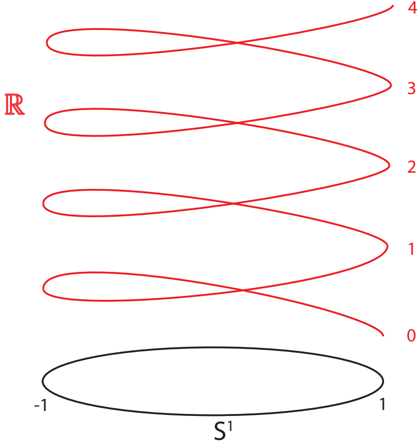
- **Definition** A space X is simply-connected if it is path-connected and $\pi_1(X) = \{1\}$. In words: every loop in X can be lassoed down to a point!
- **Example** A contractible space is simply-connected.
- **Exercise** A space X is simply-connected if and only if for any points $x_0, x_1 \in X$ there is a unique rel $\{0,1\}$ homotopy class of paths $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$.
- **Exercise** If $n \geqslant 2$ then the n-sphere S^n is simply-connected: easy to prove if it can be assumed that every loop $\omega : S^1 \to S^n$ is homotopic to one which is not onto (which is true).
- **Remark** The circle S^1 is path-connected, but not simply-connected.

The universal cover of the circle by the real line

- The continuous map

$$p : \mathbb{R} \to S^1 \; ; \; x \mapsto e^{2\pi i x}$$

is a surjection with many wonderful properties!



The fundamental group of the circle

- Define $\mathrm{Homeo}_p(\mathbb{R})$ to be the group of the homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ such that $ph = p : \mathbb{R} \to S^1$. The group is infinite cyclic, with an isomorphism of groups

$$\mathbb{Z} \to \mathrm{Homeo}_p(\mathbb{R}) \; ; \; n \mapsto (h_n : x \mapsto x + n) \; .$$

- Every loop $\omega : S^1 \to S^1$ 'lifts' to a path $\alpha : I \to \mathbb{R}$ with

$$\omega(e^{2\pi it}) \; = \; e^{2\pi i\alpha(t)} \in S^1 \quad (t \in I) \; .$$

  There is a unique $h \in \mathrm{Homeo}_p(\mathbb{R})$ with $h(\alpha(0)) = \alpha(1) \in \mathbb{R}$.
- The functions

$$\mathrm{degree} \; : \; \pi_1(S^1) \to \mathrm{Homeo}_p(\mathbb{R}) = \mathbb{Z} \; ; \; \omega \mapsto \alpha(1) - \alpha(0) \; ,$$

$$\mathbb{Z} \to \pi_1(S^1) \; ; \; n \mapsto (\omega_n : S^1 \to S^1; z \mapsto z^n)$$

  are inverse isomorphisms of groups. The degree of ω is the number of times ω winds around 0, and equals $\frac{1}{2\pi i} \oint_\omega \frac{dz}{z}$.

Covering spaces

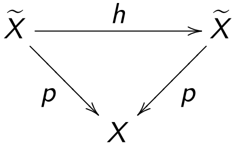
- Covering spaces give a geometric method for computing the fundamental groups of path-connected spaces X with a 'covering projection' $p : \widetilde{X} \to X$ such that $\widetilde{X}$ is simply-connected.
- **Definition** A covering space of a space X with fibre the discrete space F is a space $\widetilde{X}$ with a covering projection continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi : F \times U \to p^{-1}(U)$ such that

$$p\phi(a, u) \ = \ u \in U \subseteq X \ \ (a \in F, u \in U) \ .$$

- For each $x \in X$ $p^{-1}(x)$ is homeomorphic to F.
- The covering projection $p : \widetilde{X} \to X$ is a 'local homeomorphism': for each $\widetilde{x} \in \widetilde{X}$ there exists an open subset $U \subseteq \widetilde{X}$ such that $\widetilde{x} \in U$ and $U \to p(U); u \mapsto p(u)$ is a homeomorphism, with $p(U) \subseteq X$ an open subset.

The group of covering translations

- For any space X let $\text{Homeo}(X)$ be the group of all homeomorphisms $h : X \to X$, with composition as group law.
- **Definition** Given a covering projection $p : \widetilde{X} \to X$ let $\text{Homeo}_p(\widetilde{X})$ be the subgroup of $\text{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, called covering translations, with commutative diagram



- **Example** For each $n \neq 0 \in \mathbb{Z}$ complex n-fold multiplication defines a covering $p_n : S^1 \to S^1; z \mapsto z^n$ with fibre $F = \{1, 2, \ldots, |n|\}$. Let $\omega = e^{2\pi i/n}$. The function

$$\mathbb{Z}_n \to \text{Homeo}_{p_n}(S^1) \; ; \; j \mapsto (z \mapsto \omega^j z)$$

is an isomorphism of groups.

The trivial covering

- **Definition** A covering projection $p : \widetilde{X} \to X$ with fibre F is trivial if there exists a homeomorphism $\phi : F \times X \to \widetilde{X}$ such that
$$p\phi(a,x) \ = \ x \in X \ \ (a \in F,\ x \in X)\ .$$
A particular choice of ϕ is a trivialisation of p.
- **Example** For any space X and discrete space F the covering projection
$$p \ : \ \widetilde{X} \ = \ F \times X \to X \ ; \ (a,x) \mapsto x$$
is trivial, with the identity trivialization $\phi = 1 : F \times X \to \widetilde{X}$. For path-connected X $\text{Homeo}_p(\widetilde{X})$ is isomorphic to the group of permutations of F.

A non-trivial covering

- **Example** The universal covering

$$p \ : \ \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $\mathbb{Z}$, and $\mathrm{Homeo}_p(\mathbb{R}) = \mathbb{Z}$.

- Note that p is not trivial, since $\mathbb{R}$ is not homeomorphic to $\mathbb{Z} \times S^1$.
- Warning The bijection

$$\phi \ : \ \mathbb{Z} \times S^1 \to \mathbb{R} \ ; \ (n, e^{2\pi i t}) \mapsto n + t \quad (0 \leqslant t < 1)$$

is such that $p\phi = \text{projection} : \mathbb{Z} \times S^1 \to \mathbb{R}$, but ϕ is not continuous.