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The Jordan Curve Theorem Revisited

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Introduction[1]**.** Among the various theorems which are easy to formulate but difficult to establish, the Jordan curve theorem (abbreviated in the sequel as JCT) undoubtedly occupies a special place. Indeed, there is hardly another theorem which appears as "obvious" as any axiom of elementary geometry, and whose proof is not obvious at all. This probably explains why the Jordan curve theorem remained unnoticed until 1887, when Camille Jordan pointed out and discussed the theorem in his "Cours d'Analyse" [16] ([1]). Needless to say, Jordan's proof was not a proof in the modern sense; yet it aroused the interest of many mathematicians who recognized the significance of the theorem for "analysis situs" as well as complex analysis ([2]). The first rigorous proof of JCT, given by Oswald Veblen in 1905 [32], revealed the complexity of the whole matter. In the subsequent twenty years the theorem was reproved, completed and generalized by several outstanding topologists (to name but a few: Alexander [1], Antoine [3], Brouwer [7], [8], [9], Kerékjártó [17], [18], Schoenflies [27], [28], [29]). Thus, while the topological nature and significance of JCT for the topology of the plane was clearly recognized by the time Kerékjártó published his classic monograph [18], the theorem remained for analysts a troublesome matter. To understand why, it suffices to quote what Professor Salomon Bochner recently wrote on the work of Carathéodory (cf. [6], p. 831): "...Actually Carathéodory was planning the book on complex analysis ahead of the book on the calculus of variations, but what was holding up the book on complex variables was a widespread presumption among analysts in 1920's that any book on complex variables, if to be complete, and deserve the name, has to contain a complete and rigorous proof, without any prerequisites, for the theorem of C. Jordan... The analysts in the 1920's knew full well that this was a "topological" theorem... But, for all of that, the complex analysts of the time were still vying with each other in the quest for producing, for the Theorem of Jordan,

[1]) By ([1]), ([2]), etc., the reader is referred to remarks collected in the concluding Section 3. The numbering of lemmas, propositions and corollaries follows from this example: (2.9) means the ninth statement in Section 2.

a proof to end all such proofs, and Carathéodory was experimenting for many years with a proof of his own...". Indeed, several outstanding analysts of that time did devise new proofs of JCT (cf., e.g., Bieberbach [5], Denjoy [12], Hartogs [15], Pringsheim [24], [25], Schmidt [26]). However, notwithstanding substantial simplifications achieved in the elementary proof of JCT by these and other authors, the theorem has remained and will probably always remain, difficult to establish by purely elementary means. Partly as a result of this situation, it has become customary to omit the proof of JCT from textbooks on complex analysis (among the few exceptions are the books of Dienes [11] and Thron [30]). Moreover, a direct proof of this theorem rarely appears even in introductory texts on topology. On the other hand, one soon finds that despite the abundance of various elementary proofs of JCT in the literature[2]), few of them are really complete and of a truly elementary nature ([3]).

In this note we present an elementary proof of JCT which is based on very intuitive geometric ideas with "combinatorial" elements reduced to a minimum. In order to formalize the proof and thus avoid using geometric illustrations in the arguments, one has to introduce a few auxiliary definitions and related notation. This is done in Section 1, where one can also find those few facts from set topology which are used later in the proof ([4]).

We believe that the main advantage of the proof of JCT presented in Section 2 is that, as an intermediate step, one obtains a very simple proof of a special case of JCT which is sufficient for the needs of complex analysis. Namely, it is shown that JCT holds for the class of *standard curves* (see the definition below), comprising among others all piecewise regular (i.e. piecewise smooth) Jordan curves.

Roughly speaking, a standard curve is any Jordan curve which at (almost) each point looks like the graph of a function of one variable. More precisely, consider a continuous map $\varphi: [0,1] \to \mathbb{R}^2$ which is either a Jordan curve or a Jordan arc[3]). Fix $\bar{t} \in (0,1]$ and suppose that there exist numbers $t_1, t_2, 0 < t_1 < \bar{t} \leq t_2 \leq 1$; a system of (affine) coordinates (x,y) in $\mathbb{R}^2$; and a function f so that the following holds: Let $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ be the parametric expression of φ in coordinates (x,y). Then f is defined on the interval $\varphi_1([t_1,t_2])$ and $\varphi_2 = f \circ \varphi_1$. Hence, φ being Jordan, φ_1 must be one-to-one, which – together with the intermediate value property of continuous functions (cf. ([4])) – shows that φ_1 must be strictly monotonic.

[2]) See the bibliographical entries at the end of the paper marked by an asterisk.

[3]) The meaning of these terms is self-evident. Nevertheless, this terminology is formally introduced (and slightly modified) in Section 1.

We may clearly assume that φ_1 is increasing. Set $a = \varphi_1(t_1)$, $b = \varphi_1(t_2)$. Then φ_1 is a homeomorphism of $[t_1,t_2]$ onto $[a,b]$. Writing $x = \varphi_1(t)$ we see that the set $\varphi([t_1,t_2])$ is actually the graph (in coordinates (x,y)) of the function $y = f(x), a \leqslant x \leqslant b$. If $\bar{t} < t_2(\bar{t} = t_2$, resp.), we shall say that $\varphi(\bar{t})$ is a *standard point* (*left-standard point*, resp.) of φ with respect to f, (x,y). t_1 and t_2 (f, (x,y) and t_1, resp.).

Definition. *Let $\varphi:[0,1] \to \mathbb{R}^2$ be a Jordan curve (Jordan arc, resp.) such that every point $\varphi(t)$, $0 < t \leqslant 1$, is left-standard (with respect to some f, (x,y) and t_1 which all depend on t). Then φ will be called standard.*

Every standard φ contains standard points. Indeed, fix any $\bar{t} \in (0,1]$. Then $\varphi(\bar{t})$ is left-standard with respect to some f, (x,y) and t_1. But then every $\varphi(t)$, $t_1 < t < \bar{t}$, is standard with respect to the same f, (x,y), t_1 and $t_2 = \bar{t}$. Therefore, the set of nonstandard points of a standard φ is nowhere dense in the set $\varphi([0,1])$.

Let $\varphi:[0,1] \to \mathbb{R}^2$ be a continuous map for which there are numbers $0 = t_0 < t_1 < \cdots < t_m = 1$ such that if $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ is the expression of φ (in some fixed system of coordinates (x,y)), then on each interval $[t_{j-1},t_j]$ the functions $\varphi_1(t)$ and $\varphi_2(t)$ are continuously differentiable and $(\varphi_1'(t))^2 + (\varphi_2'(t))^2 \neq 0$. Then φ is called *piecewise regular*. A simple application of the inverse function theorem shows that every piecewise regular Jordan curve (or arc) is standard. Thus, giving a simple proof of JCT for standard curves (see (2.1)–(2.6) below) establishes the theorem for all curves one encounters in classical function theory. This special case is then used to prove JCT in its full generality (see (2.7)–(2.11)).

1. Preliminaries

A. General notation and terminology. If P and Q are points of the plane $\mathbb{R}^2$, then $|P - Q|$ is their Euclidean distance; $\overline{PQ}$ is the straight line segment with endpoints P,Q; and, for $P \neq Q$, $\mathrm{l}(P,Q)$ is the straight line determined by P and Q.

If A and B are subsets of $\mathbb{R}^2$, then $\operatorname{dist}(A,B)$ denotes their distance, i.e. $\operatorname{dist}(A,B) = \inf\{|P - Q| : P \in A, Q \in B\}$; and $\mathring{A}, \bar{A}, \partial A$ and $\complement A$ denote respectively the interior, closure, boundary and complement of A in $\mathbb{R}^2$. We say that a subset S of $\mathbb{R}^2$ does not *disconnect* (or *separate*) $\mathbb{R}^2$, if its complement $\complement S$ is connected. More generally, given points $P,Q \in \complement S$, we say that S does not separate P from Q, provided both P and Q lie in the same component of $\complement S$. We shall adopt the following notation: If A is a connected subset of $\complement S$, then A is contained in a unique component of $\complement S$, which we shall denote $K(A;S)$. If, for instance, S is a bounded subset of $\mathbb{R}^2$, then S

is contained in some disk Δ, hence $C\Delta$ is a connected subset of CS. Thus all the components of CS, except $K(C\Delta;S)$ are contained in Δ, hence are bounded. $K(C\Delta;S)$ is the only unbounded component of CS, and will be denoted $\operatorname{Ext} S$.

B. Curves, paths and arcs. If φ is a mapping of a subset A of $\mathbb{R}^2$ into $\mathbb{R}^2$, $\langle\varphi\rangle$ will denote the range of φ, i.e. $\langle\varphi\rangle = \varphi(A)$. A *curve* φ in the plane $\mathbb{R}^2$ is a continuous mapping of the unit circle $\mathbb{S}^1$ into $\mathbb{R}^2$; if this mapping is one-to-one, φ is called a *Jordan curve*. In particular, by a curve we shall always mean a closed curve. A *path* λ from a point P to a point Q is a continuous mapping of the unit interval $[0,1]$ of the real numbers into $\mathbb{R}^2$ such that $\lambda(0) = P$ and $\lambda(1) = Q$. If $P \neq Q$, the points P, Q will be called endpoints of λ (or of $\langle\lambda\rangle$). We shall then write $\lambda = \lambda\{P,Q\}$ to indicate that λ is a path with endpoints P,Q; and the set $\{P,Q\}$ will be denoted by $\dot{\lambda}$. If λ is a path and I any interval, $I \subset [0,1]$, then λI (rather than the usual symbol $\lambda(I)$) will denote the set $\{\lambda(t): t \in I\}$. Hence the meaning of $\lambda(a,b]$, $\lambda[a,b]$, etc., is clear. Moreover, we shall write $\check{\lambda}$ for $\lambda(0,1)$.

Given a fixed cartesian coordinate system (x,y) in $\mathbb{R}^2$, the natural parametrization $x(t) = \cos 2\pi t$, $y(t) = \sin 2\pi t$, $0 \leqslant t \leqslant 1$, of the circle $\mathbb{S}^1$ associates with each curve φ a unique path $t \mapsto \varphi(x(t), y(t))$. This path will be called the *canonical parametrization* of φ, and will be denoted by the same letter. Hence we shall often speak about a curve $\varphi(t)$, $0 \leqslant t \leqslant 1$, meaning actually the canonical parametrization of φ.

A path $\lambda: [0,1] \to \mathbb{R}^2$ which is one-to-one will be called an *arc*. Hence by an arc we shall mean what is usually called a Jordan arc. In this case $\check{\lambda} = \langle\lambda\rangle \setminus \dot{\lambda}$. If φ is either a Jordan curve or an arc, and λ is an arc such that $\langle\lambda\rangle \subset \langle\varphi\rangle$, λ is said to be a *subarc* of φ. If φ is an arc and λ a subarc of φ with endpoints P, Q, we shall write $\lambda = \varphi\{P,Q\}$ to indicate that λ is "the part of $\langle\varphi\rangle$ from P to Q". If φ is a Jordan curve, $P,Q \in \langle\varphi\rangle$, $P \neq Q$, and A is a point of $\langle\varphi\rangle$, distinct from the points P, Q (A is a subarc of φ, not containing P and Q, resp.), then $\varphi\{P,A,Q\}$ will denote any subarc λ of φ such that $\lambda = \lambda\{P,Q\}$, $A \in \check{\lambda}$ ($A \subset \check{\lambda}$, resp.)[4]).

Finally, given four distinct points A_1, A_2, B_1 and B_2 on a Jordan curve φ, we shall say that the pairs A_1, A_2 and B_1, B_2 are *interlaced* on φ, provided $\langle\varphi\{A_1,B_1,A_2\}\rangle \cup \langle\varphi\{A_1,B_2,A_2\}\rangle = \langle\varphi\rangle$. Being interlaced is clearly a topological property, i.e. invariant under homeomorphisms of $\langle\varphi\rangle$ ([5]). It is easy to see that the role of both pairs is symmetric.

C. Polygonal paths. A path α is called polygonal, provided there exists a homeomorphism (i.e. a "change of parameter") $\tilde{\alpha}: [0,1] \to [0,1]$ and numbers t_j, $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\alpha \circ \tilde{\alpha}$ is linear on each subinterval

[4]) If λ, λ^* are two subarcs with this property, then obviously $\langle\lambda\rangle = \langle\lambda^*\rangle$.

$[t_{j-1}, t_j]$, but is not linear on any larger subinterval of $[0,1]$. It is easy to see that the points $A_i = \alpha(\tilde{\alpha}(t_i))$, called *vertices* of α, actually do not depend on $\tilde{\alpha}$, and clearly $\langle\alpha\rangle = \langle\alpha \circ \tilde{\alpha}\rangle = \bigcup_{i=1}^{k} \overline{A_{i-1}A_i}$. A curve φ is said to be polygonal, if its canonical parametrization is a polygonal path.

A straightforward application of uniform continuity yields the following lemma (cf. [4], Satz 48, p. 72):

(1.1) **Lemma.** *Given a path* $\alpha = \alpha\{P,Q\}$ *and* $\varepsilon > 0$, *there is a polygonal path* $\bar{\alpha} = \bar{\alpha}\{P,Q\}$, *whose vertices lie on* $\langle\alpha\rangle$, *such that* $|\alpha(t) - \bar{\alpha}(t)| < \varepsilon$ *for each* $t \in [0,1]$. *In particular, each of the sets* $\langle\alpha\rangle, \langle\bar{\alpha}\rangle$ *lies in the (closed)* ε*-neighborhood of the other.*

The next lemma is a simple consequence of Lemma (1.1) combined with an obvious reduction of a polygonal path to a polygonal subarc [6]:

(1.2) **Lemma.** *An open subset* $U \subset \mathbb{R}^2$ *is connected if and only if for any two distinct points* P, Q *of* U, *there exists a polygonal arc* $\lambda = \lambda\{P,Q\}$ *such that* $\langle\lambda\rangle \subset U$.

D. The Jordan curve theorem. Outline of the proof. Our aim is to give an elementary proof of the celebrated

(1.3) **Jordan curve theorem.** *Let* φ *be a Jordan curve. Then*
(a) $\langle\varphi\rangle$ *is the boundary of each component of* $\mathrm{C}\langle\varphi\rangle$.
(b) $\mathrm{C}\langle\varphi\rangle$ *has exactly two components, one bounded, denoted* $\mathrm{Int}\langle\varphi\rangle$, *and the other unbounded, namely* $\mathrm{Ext}\langle\varphi\rangle$.

The main steps of the proof of JCT, presented in the next section, are as follows:

(i) JCT is established for standard curves[5] (cf. (2.1)–(2.6)). Then it is shown that

(ii) arcs do not disconnect the plane (cf. (2.9)).

From the last statement one easily obtains part (a) of (1.3). Indeed, we have the following lemma:

(1.4) **Lemma.** *If no subarc of a Jordan curve* φ *separates* $\mathbb{R}^2$, *then* $\langle\varphi\rangle$ *is the boundary of each component of* $\mathrm{C}\langle\varphi\rangle$.

Proof. Let U be a component of $\mathrm{C}\langle\varphi\rangle$. Since a boundary point of U can be in neither U nor any of the components of $\mathrm{C}\langle\varphi\rangle$, we have $\partial U \subset \langle\varphi\rangle$. If ∂U were a proper subset of $\langle\varphi\rangle$, then ∂U would be contained in $\langle\lambda\rangle$ for some subarc λ of φ. Hence $U \cap \mathrm{C}\langle\lambda\rangle = \bar{U} \cap \mathrm{C}\langle\lambda\rangle$, so that U would be

[5]) See the Introduction.

both closed and open in $C\langle\lambda\rangle$, hence U would be a component of $C\langle\lambda\rangle$ (because $U \neq \emptyset$). Since $C\langle\lambda\rangle$ is connected by our hypothesis and U is nonempty, we would then conclude $U = C\langle\lambda\rangle$. This, however, is impossible, because $\langle\varphi\rangle \cap C\langle\lambda\rangle \neq \emptyset$ and $\langle\varphi\rangle \cap U = \emptyset$. Hence $\varphi = \partial U$.

Part (a) of (1.3) is then used to show that

(iii) $\langle\varphi\rangle$ disconnects the plane $\mathbb{R}^2$ (see (2.11)); and finally, that

(iv) there are at most two components of $C\langle\varphi\rangle$ (see (2.11)), which completes the proof of JCT.

We shall also use the fact that for certain Jordan curves statements (a) and (b) of (1.3) are trivial to establish:

(1.5) **Lemma.** *Let φ be a Jordan curve such that $\langle\varphi\rangle$ is either a circle or a (boundary of a) rectangle or a triangle. Then* (a) *and* (b) *of* (1.3) *hold for φ.*

Indeed, in each of the three cases one can give a very simple proof; but one can also prove directly from definitions that (1.3) holds for any starlike curve φ, and (1.5) then follows.

2. Proof of the Jordan Curve Theorem

Two polygonal paths α and β, $\langle\alpha\rangle = \bigcup_{i=1}^{k} \overline{A_{i-1}A_i}$, $\langle\beta\rangle = \bigcup_{j=1}^{m} \overline{B_{j-1}B_j}$, are called *transversal*, if no vertex of one lies on the other. This implies that for each pair (i,j) of integers, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, the set $\overline{A_{i-1}A_i} \cap \overline{B_{j-1}B_j}$ is either empty or consists of one point. The number of pairs (i,j) for which the latter is true is called the intersection number of the paths α and β, and is denoted by $w(\alpha;\beta)$[5a]. The notions of transversality and intersection number can be extended to the case when $\langle\beta\rangle$ is replaced by a straight line and α as above. In either case we have the following:

(2.1) **Lemma.** (i) $w(\alpha;\beta) = w(\beta;\alpha) = \sum_{i=1}^{k} w(\overline{A_{i-1}A_i};\beta)$; (ii) *if $\langle\beta\rangle$ is a straight line or a triangle (i.e. $\langle\beta\rangle = \overline{B_0B_1} \cup \overline{B_1B_2} \cup \overline{B_2B_0}$), then $K(A_1;\langle\beta\rangle) = K(A_k;\langle\beta\rangle)$ if and only if $w(\alpha;\beta)$ is an even integer.*

Proof. Part (i) follows by induction on k. Part (ii): First, assume that $\langle\beta\rangle$ is a nondegenerate triangle, i.e. B_0, B_1, B_2 noncollinear. Let T be the convex hull of the points B_0, B_1, B_2. Let $k = 1$, i.e. $\langle\alpha\rangle = \overline{A_0A_1}$, and let $\langle\alpha\rangle \cap \langle\beta\rangle \neq \emptyset$, i.e. $w(\alpha;\beta) > 0$. Set $\mathfrak{l} = \mathfrak{l}(A_0, A_1)$. By the postulate of Pasch (cf. [1]), $\mathfrak{l} \cap \langle\beta\rangle = \{C_1, C_2\}$; hence by convexity of T, $\overline{C_1C_2} = \mathfrak{l} \cap T$. Considering the mutual position of the points A_0, A_1, C_1, C_2 on the line $\mathfrak{l}$, the proof easily follows. The inductive step is an immediate consequence of the case $k = 1$. For $\langle\beta\rangle$ degenerate, i.e. B_0, B_1, B_2 collinear, the lemma is

[5a]) See page 128.

trivial, because by definition of $w(\alpha;\beta)$, each point of $\langle\alpha\rangle \cap \langle\beta\rangle$ is now counted twice. Finally, if $\langle\beta\rangle$ is a straight line, the lemma is a direct consequence of an equivalent formulation of the postulate of Pasch (cf. ([1])).

(2.2) **Lemma** ([7]). *Let φ be as in (1.5) and X_1, Y_1 and X_2, Y_2 two interlaced pairs of points on φ. Let $\sigma_i = \sigma_i\{X_i, Y_i\}$, $i = 1, 2$, be given paths such that $(\check{\sigma}_1 \cup \check{\sigma}_2) \cap \langle\varphi\rangle = \emptyset$ and $K(\check{\sigma}_1;\langle\varphi\rangle) = K(\check{\sigma}_2;\langle\varphi\rangle)$. Then the paths σ_1 and σ_2 intersect, i.e. $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle \neq \emptyset$.*

Proof. I. Let σ_1 and σ_2 be polygonal.

Case Ia: $K(\check{\sigma}_1;\langle\varphi\rangle) = K(\check{\sigma}_2;\langle\varphi\rangle) = \mathrm{Int}\langle\varphi\rangle$ (cf. (1.5)). Obviously we may assume that σ_1 and σ_2 are transversal, since otherwise $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle \neq \emptyset$ as desired. It suffices to show that

$$w(\sigma_1,\sigma_2) \text{ is an odd integer.} \tag{$*$}$$

Let $\langle\sigma_i\rangle = \bigcup_{j=1}^{m_i} \overline{A^i_{j-1}A^i_j}$, $A^i_0 = X_i$, $A^i_{m_i} = X_i (i = 1,2)$. We shall prove $(*)$ by induction on m_2. If $m_2 = 1$, then by the convexity of the region $\overline{\mathrm{Int}\langle\varphi\rangle}$, we have $\overline{\mathrm{Int}\langle\varphi\rangle} \cap \mathrm{I}(X_2, Y_2) = \overline{X_2Y_2}$ ([8]), hence $\overline{A^1_{j-1}A^1_j} \cap \mathrm{I}(X_2,Y_2) = \overline{A^1_{j-1}A^1_j} \cap \overline{X_2Y_2}$. Applying (2.1) to $\alpha = \sigma_1$ and $\beta = \mathrm{I}(X_2,Y_2)$ we conclude that $w(\sigma_1;\sigma_2)$ is odd[6]). Assume now that $(*)$ holds for $m_2 \leqslant n-1$, $n > 1$, and any m_1. We shall prove that $(*)$ holds for $m_2 = n$ and m_1 arbitrary. Fix $Z \in \langle\varphi\{X_1,X_2,Y_1\}\rangle$ so that $A^1_j \notin \langle\sigma_Z\rangle = \overline{ZA^2_1} \cup \overline{A^2_1A^2_2} \cup \overline{A^2_2Z}$ for $j = 1,\ldots,m_1$ and $|Z - X_2| < \min_j \delta_j$, where δ_j is defined as follows:

$$\delta_j = \mathrm{dist}(X_2, \mathrm{I}(A^2_1, A^1_{j-1}) \cup \mathrm{I}(A^1_{j-1}, A^1_j) \cup \mathrm{I}(A^1_j, A^2_1))$$

if the segments $\overline{A^1_{j-1}A^1_j}, \overline{X_2A^2_1}$ intersect; and $\delta_j = \mathrm{dist}(\overline{A^1_{j-1}A^1_j}, \overline{X_2A^2_1})$ otherwise. By our choice of Z,

$$w(\sigma_1;\overline{ZA^2_1}) = w(\sigma_1;\overline{X_2A^2_1});\ \sigma_1 \text{ and } \sigma_Z \text{ are transversal.} \tag{1}$$

Let σ^i_2 be such that $\langle\sigma^i_2\rangle = \overline{ZA^2_i} \cup \bigcup_{j>i} \overline{A^2_{j-1}A^2_j}$, $i = 1,2$. The paths σ_1, σ^2_2 being transversal (cf. (1)), $w(\sigma_1;\sigma^2_2)$ is an odd integer by the induction hypothesis. Since by (1), $w(\sigma_1;\sigma^1_2) = w(\sigma_1;\sigma_2)$, it suffices to show that the integers $w(\sigma_1;\sigma^2_2)$ and $w(\sigma_1;\sigma^1_2)$ have the same odd-even parity. By (2.1) (i) this amounts to showing that the integers $w(\sigma_1;\overline{ZA^2_2})$ and $w(\sigma_1;\overline{ZA^2_1}) + w(\sigma_1;\overline{A^2_1A^2_2})$ have the same parity; but this follows immediately, since by (2.1) (i) their sum equals $w(\sigma_1;\sigma_Z)$, which by (2.1) (ii) is an even number[6a]).

Case Ib: $K(\check{\sigma}_1;\langle\varphi\rangle) = K(\check{\sigma}_2;\langle\varphi\rangle) = \mathrm{Ext}\langle\varphi\rangle$. Choose a point P and a disk Δ centered at P so that $\Delta \subset \mathrm{Int}\langle\varphi\rangle$. If T is any point of φ, set $T' =$

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[6]) That $K(X_1;\beta) \neq K(Y_1;\beta)$ follows easily from the special choice of φ, but can also be established by a simple convexity argument for any convex φ.

[6a]) See page 128.

$\overline{PT} \cap \partial\Delta$[7]). The mapping $T \mapsto T'$ being a homeomorphism of $\langle\varphi\rangle$ onto $\partial\Delta$, the pairs X'_1, Y'_1 and X'_2, Y'_2 are interlaced on $\partial\Delta$. Let σ_i^* be a path such that $\langle\sigma_i^*\rangle = \overline{X'_iX_i} \cup \langle\sigma_i\rangle \cup \overline{Y_iY'_i}$, $i = 1,2$. Applying the reflection r with respect to the circle $\partial\Delta$ and then Case I a to the paths $r \circ \sigma_1^*, r \circ \sigma_2^*$ (cf. [6a])), we obtain $r(\langle\sigma_1^*\rangle) \cap r(\langle\sigma_2^*\rangle) \neq \emptyset$, hence also $\emptyset \neq \langle\sigma_1^*\rangle \cap \langle\sigma_2^*\rangle = \langle\sigma_1\rangle \cap \langle\sigma_2\rangle$ which completes the proof.

II. σ_1,σ_2 arbitrary. Assume that there exists a pair of paths σ_1,σ_2 satisfying the hypotheses of the lemma but for which $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \emptyset$. Hence there exists $\varepsilon > 0$ such that $\operatorname{dist}(\langle\sigma_1\rangle,\langle\sigma_2\rangle) > \varepsilon$ (cf. ([4])). By (1.1) it is easy to find polygonal paths $\tilde{\sigma}_1, \tilde{\sigma}_2$ which satisfy the assumptions of the lemma and lie in the $\varepsilon/2$-neighborhood of σ_1,σ_2, resp. Hence $\langle\tilde{\sigma}_1\rangle \cap \langle\tilde{\sigma}_2\rangle = \emptyset$, a contradiction.

(2.3) **Lemma.** *Consider a continuous function* $f:[a,b] \to (c,d)$. *If* $\gamma(t) = (t,f(t)), a \leqslant t \leqslant b$, *is the graph of* f, *then the set* $W = ((a,b) \times (c,d))\backslash\langle\gamma\rangle$ *has exactly two components.*

Proof. Set $G_i = \{(x,y) \in W : (-1)^i y > (-1)^i f(x)\}$ $(i = 0,1)$. Then each G_i is a connected set. Indeed, let $\varepsilon > 0$ be such that $f([a,b]) \subset (c + \varepsilon, d - \varepsilon)$. If $P_j = (x_j,y_j)$ $(i = 1,2)$ are two distinct points of G_0, then assuming $x_1 < x_2$ and setting $P'_j = (x_j, d - \varepsilon)$ $(i = 1,2)$, consider any path σ such that $\langle\sigma\rangle = \overline{P_1P'_1} \cup \overline{P'_1P'_2} \cup \overline{P'_2P_2}$. Then $\sigma = \sigma\{P_1,P_2\}$ and $\langle\sigma\rangle \subset G_0$. If $x_1 = x_2$, then it suffices to take σ such that $\langle\sigma\rangle = \overline{P_1P_2}$. Similarly one shows that G_1 is connected. Since G_0, G_1 are clearly open disjoint sets and $G_0 \cup G_1 = W$, they are also closed in W. Hence each of them is a component of W.

(2.4) **Lemma**[8]). *Let* $\varphi:[0,1] \to \mathbb{R}^2$ *be either a Jordan curve or an arc. Let* $\varphi(\bar{t})$ *be a standard point of* φ *with respect to some* $f, (x,y), t_1, t_2$. *Set* $(a,f(a)) = \varphi(t_1)$, $(b,f(b)) = \varphi(t_2)$. *Then, for any disk* Δ *centered at* $\varphi(\bar{t}) = (\bar{x},\bar{y})$, *there exists a rectangle* $R = [\bar{x} - \varepsilon_1, \bar{x} + \varepsilon_1] \times [\bar{y} - \varepsilon_2, \bar{y} + \varepsilon_2]$ *with the following property:*

(S) $R \subset \Delta, [\bar{x} - \varepsilon_1, \bar{x} + \varepsilon_1] \subset (a,b)$ *and there are numbers* t_- *and* $t_+, 0 < t_- < \bar{t} < t_+ < 1$, *such that* $\varphi(t_\pm) = (\bar{x} \pm \varepsilon_1, f(\bar{x} \pm \varepsilon_1))$, $|f(x) - \bar{y}| < \varepsilon_2$ *if* $|\bar{x} - \bar{x}| \leqslant \varepsilon_1, \langle\varphi\rangle \cap \partial R = \{\varphi(t_-), \varphi(t_+)\}, \langle\varphi\rangle \cap \mathring{R} = \varphi(t_-,t_+)$.

If the point $\varphi(\bar{t})$ *is only left-standard, then* $\bar{t} = t_2, \bar{x} = b$, *and* (S) *is replaced by*

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[7]) The existence of the intersection T' easily follows if we define Δ as the disk (centered at P) with radius $\varrho < \operatorname{dist}(P,\langle\varphi\rangle)$. Then $\langle\varphi\rangle$ lies in $C\Delta$.

[8]) In (2.4), (2.5) and (2.6) we use the terminology and notation discussed in the Introduction.

(S′) *$R \subset \Delta, \bar{x} - \varepsilon_1 > a$, and there exists a number $t_- \in (0,\bar{t})$ such that $\varphi(t_-) = (\bar{x} - \varepsilon_1, f(\bar{x} - \varepsilon_1)), \varphi[0,\bar{t}] \cap R = \varphi[t_-,\bar{t}]$ and $\varphi[0,\bar{t}] \cap \partial R = \{\varphi(t_-)\}$.*

Proof. Let $\varphi(\bar{t})$ be a standard point. For any $X_0 = (x_0,y_0) \in \mathbb{R}^2$ set $\|X_0\| = \max\{|x_0|,|y_0|\}$. Then there exists $\varepsilon_0 > 0$ so that $\|X - \varphi(\bar{t})\| > \varepsilon_0$ for any X in the compact set $\langle\varphi\rangle \setminus \varphi(t_1,t_2)$. Let $\varepsilon \in (0,\varepsilon_0)$ be so small that $R_1 = [\bar{x} - \varepsilon, \bar{x} + \varepsilon] \times [\bar{y} - \varepsilon, \bar{y} + \varepsilon] \subset \Delta, [\bar{x} - \varepsilon, \bar{x} + \varepsilon] \subset (a,b)$. If $I = (\bar{x} - \varepsilon', \bar{x} + \varepsilon'')$ is the largest open interval containing $\bar{x}$ for which $I \subset [\bar{x} - \varepsilon, \bar{x} + \varepsilon]$ and $|f(x) - \bar{y}| < \varepsilon$ for all $x \in I$, set $\varepsilon_1 = (1/2)\min(\varepsilon',\varepsilon''), \varepsilon_2 = \varepsilon$. Then the rectangle $R = [\bar{x} - \varepsilon_1, \bar{x} + \varepsilon_1] \times [\bar{y} - \varepsilon_2, \bar{y} + \varepsilon_2]$ and the numbers $t_\pm$ defined by $\varphi(t_\pm) = (\bar{x} \pm \varepsilon_1, f(\bar{x} \pm \varepsilon_1))$ have all the required properties. (One has to use the fact that $\varphi[t_1,t_2]$ is the graph of f.) The proof of (S′) is analogous but simpler.

(2.5) **Proposition.** *Standard arcs do not disconnect the plane.*

Proof. Let $\lambda:[0,1] \to \mathbb{R}^2$ be a standard arc. It suffices to show that given any pair of points $P,Q \in \mathrm{C}\langle\lambda\rangle, P \neq Q$, λ does not separate them. Set $U = \{t \in [0,1] : \lambda[0,t]$ does not separate $P,Q\}$. Clearly, $0 \in U$. Hence U is a nonempty interval which does not reduce to $\{0\}$, because U is open in $[0,1]$. Indeed, if $t \in U, t < 1$, then by continuity of λ, for some $\varepsilon > 0, t + \varepsilon \in U$, whence $[0,t + \varepsilon] \subset U$, because U is an interval. Thus, if we show that $t_0 = \sup U \in U$, then necessarily $t_0 = 1$, and the proposition will follow. Applying (2.4) (S′) to $\varphi = \lambda, \bar{t} = t_0$ and any Δ such that $P,Q \notin \Delta$, we obtain the rectangle R and the number $t_- < t$. Fix an arbitrary $t' \in (t_-,t_0)$. U being an interval, $t' \in U$, hence there is an arc $\alpha = \alpha\{P,Q\}$ such that $\langle\alpha\rangle \cap \lambda[0,t'] = \emptyset$. Let α_1,α_2 be maximal subarcs[9] of the form $\alpha_1 = \alpha\{P,P'\}$, $\alpha_2 = \alpha\{Q',Q\}$ for which $\breve{\alpha}_i \cap R = \emptyset$, $i = 1,2$. (Here we are assuming that $\langle\alpha\rangle \cap R \neq \emptyset$, since otherwise $\langle\alpha\rangle \cap \lambda[0,t_0] = \emptyset$, as desired.) Hence $P',Q' \in \partial R$. Let $\varrho = \varrho\{P',Q'\}$ be any arc such that $\langle\varrho\rangle \subset \partial R$ and $\lambda(t_-) \notin \langle\varrho\rangle$. Take any path such that $\alpha' = \alpha'\{P,Q\}, \langle\alpha'\rangle = \langle\alpha_1\rangle \cup \langle\varrho\rangle \cup \langle\alpha_2\rangle$. Then by (2.4) (S′) we see that

$$\langle\alpha'\rangle \cap \lambda[0,t_0] = \langle\alpha'\rangle \cap \partial R \cap \lambda[0,t_0] \subset \langle\alpha'\rangle \cap \{\lambda(t_-)\} = \emptyset\,.$$

Therefore, $t_0 \in U$.

(2.6) **Proposition.** *If a Jordan curve φ contains at least one standard point, then $\langle\varphi\rangle$ disconnects the plane $\mathbb{R}^2$. Moreover, if φ is a standard curve, then the Jordan curve theorem holds for φ.*

Proof. Let $\varphi(\bar{t})$ be a standard point of φ for some function f, a system of coordinates (x,y) in $\mathbb{R}^2$, etc. Let $\varphi(\bar{t}) = (\bar{x},\bar{y})$. Then there exists a rectangle

[9]) maximal in the sense of inclusion of their images.

R and numbers t_-, t_+ as in (2.4) (S) (for a fixed Δ). Set $A_\pm = \varphi(t_\pm)$ and $B_\pm = (\bar{x}, \bar{y} \pm \varepsilon_2)$. We claim that $K(B_-;\langle\varphi\rangle) \neq K(B_+;\langle\varphi\rangle)$. Assume the contrary. Then for some path $\alpha = \alpha\{B_-, B_+\}, \langle\alpha\rangle \cap \langle\varphi\rangle = \emptyset$. Let Δ' be a disk centered at $\varphi(\bar{t})$ such that $\Delta' \subset R$ and $\Delta' \cap \langle\alpha\rangle = \emptyset$. Applying (2.4) (S) to $\varphi(\bar{t})$ and Δ', we obtain the rectangle $R' = [\bar{x} - \varepsilon_1', \bar{x} + \varepsilon_1'] \times [\bar{y} - \varepsilon_2', \bar{y} + \varepsilon_2']$ and the numbers t'_-, t'_+. Set $A'_\pm = \varphi(t'_\pm)$, $B'_\pm = (\bar{x}, \bar{y} \pm \varepsilon_2')$. Let α' be any path such that $\alpha' = \alpha'\{B'_-, B'_+\}$ and $\langle\alpha'\rangle = \langle\alpha\rangle \cup \overline{B_+B'_+} \cup \overline{B_-B'_-}$, and let φ' be an arc such that $\langle\varphi'\rangle = \langle\varphi\rangle \setminus \varphi(t'_-, t'_+)$. Since $R' \subset \Delta' \subset R$ and $\langle\alpha\rangle \cap \langle\varphi\rangle = \emptyset$, we have $\langle\alpha'\rangle \cap \langle\varphi'\rangle = \emptyset$. Let ψ be a Jordan curve such that $\langle\psi\rangle = \partial R'$. By (2.4) (S), $(\breve{\alpha}' \cup \breve{\varphi}') \cap \langle\psi\rangle = \emptyset$ and the points A'_-, A'_+ lie in opposite "vertical" sides of $\langle\psi\rangle$. Since B'_-, B'_+ lie in opposite "horizontal" sides of $\langle\psi\rangle$, the pairs $\{A'_-, A'_+\}$ and $\{B'_-, B'_+\}$ are interlaced on ψ. This, together with the obvious inclusion $\breve{\alpha}' \cup \breve{\varphi}' \subset \mathrm{Ext}\langle\psi\rangle$, implies by (2.2) that $\langle\alpha'\rangle \cap \langle\varphi'\rangle \neq \emptyset$ which is impossible, since clearly $\langle\alpha'\rangle \cap \langle\varphi'\rangle = \langle\alpha\rangle \cap \langle\varphi\rangle = \emptyset$.

Now let φ be a standard curve. By (1.4) and (2.5), φ satisfies part (a) of (1.3). We have just shown that $\langle\varphi\rangle$ disconnects $\mathbb{R}^2$. Let R be the rectangle corresponding by (2.4) (S) to any chosen standard point on φ. Then part (a) of (1.3) combined with (2.3) show that $\mathrm{C}\langle\varphi\rangle$ has at most two components.

(2.7) Remark. If φ is a Jordan curve, let $I(\varphi)$ be the union of all bounded components of $\mathrm{C}\langle\varphi\rangle$. Since $I(\varphi)$ and $\mathrm{Ext}\langle\varphi\rangle$ are disjoint open sets, the decomposition $I(\varphi) \cup \langle\varphi\rangle \cup \mathrm{Ext}\langle\varphi\rangle = \mathbb{R}^2$ implies $\partial I(\varphi) \subset \langle\varphi\rangle$, hence $\overline{I(\varphi)} \setminus \langle\varphi\rangle = I(\varphi)$. (If (1.3) (a) holds for φ, then $\langle\varphi\rangle \subset \partial I(\varphi)$, i.e. $\overline{I(\varphi)} = I(\varphi) \cup \langle\varphi\rangle$.) With this notation we have the following:

(2.8) **Lemma.** (i) *Let φ (ψ, resp.) be a Jordan curve satisfying part* (a) *(part* (b), *resp.) of* (1.3) *and let $\langle\psi\rangle \subset \overline{I(\varphi)}$. Then* $\mathrm{Int}\langle\psi\rangle \subset I(\varphi)$.
(ii) *Let φ be a Jordan curve and $\alpha_1, \ldots, \alpha_n$ arcs such that $\langle\alpha_i\rangle \cap \langle\varphi\rangle = \dot{\alpha}_i$ for all i and $\breve{\alpha}_l \cap \breve{\alpha}_k = \emptyset$ whenever $k \neq \ell$. Moreover, assume that the Jordan curve theorem holds for any Jordan curve χ satisfying $\langle\chi\rangle \subset \langle\varphi\rangle \cup \langle\alpha_1\rangle \cup \cdots \cup \langle\alpha_n\rangle$. Then, for any arc β such that $\langle\beta\rangle \subset \langle\varphi\rangle \setminus (\dot{\alpha}_1 \cup \cdots \cup \dot{\alpha}_n)$, there exists a Jordan curve ψ such that $\langle\beta\rangle \subset \langle\psi\rangle \subset \langle\varphi\rangle \cup \bigcup_{i=1}^{n} \langle\alpha_i\rangle$ and* $\mathrm{Int}\langle\psi\rangle \subset (\mathrm{Int}\langle\varphi\rangle) \setminus \bigcup_{j=1}^{n} \langle\alpha_j\rangle$ [10]).

Proof. (i) Let Δ be a disk such that $\overline{I(\varphi)} \subset \mathring{\Delta}$. Fix $Z \in \partial\Delta$. Then $Z \in \mathrm{Ext}\langle\varphi\rangle \cap \mathrm{Ext}\langle\psi\rangle$ (cf. paragraph A of Section 1). Notice that by (1.3) (a) $\mathrm{C}I(\varphi) = \langle\varphi\rangle \cup \mathrm{Ext}\langle\varphi\rangle = \overline{\mathrm{Ext}\langle\varphi\rangle}$. Suppose that there exists a point $X \in \mathrm{Int}\langle\psi\rangle \cap \mathrm{C}I(\varphi) = \mathrm{Int}\langle\psi\rangle \cap \overline{\mathrm{Ext}\langle\varphi\rangle}$. We may clearly assume that $X \in \mathrm{Int}\langle\psi\rangle$

[10]) Notice that by our hypothesis JCT holds for both φ and ψ.

$\cap \operatorname{Ext}\langle\varphi\rangle$. By (1.2) there exists an arc $\lambda = \lambda\{X,Z\}$ such that $\langle\varphi\rangle \cap \langle\lambda\rangle = \emptyset$. Since $Z \in \operatorname{Ext}\langle\psi\rangle$ and $X \in \operatorname{Int}\langle\psi\rangle$, $\langle\psi\rangle \cap \langle\lambda\rangle \neq \emptyset$. Hence, if Y is any point of $\langle\psi\rangle \cap \langle\lambda\rangle$, then by (2.7), $Y \in \overline{I(\varphi)}\setminus\langle\varphi\rangle = I(\varphi)$. Therefore, $\operatorname{Ext}\langle\varphi\rangle = K(Z;\langle\varphi\rangle) = K(\langle\lambda\rangle;\langle\varphi\rangle) = K(Y;\langle\varphi\rangle) \subset I(\varphi)$, which is impossible.

(ii) It follows from our hypotheses that

$$\check{\alpha}_i \cap \langle\varphi\rangle = \emptyset \text{ for all } i;\ \check{\alpha}_i \cap \langle\alpha_j\rangle = \emptyset \text{ for all } i \neq j. \tag{2}$$

We shall proceed by induction on n. Let $n = 1, \alpha_1 = \alpha_1\{X,Y\}$. If $\check{\alpha}_1 \subset \operatorname{Ext}\langle\varphi\rangle$, set $\psi = \varphi$. If $\check{\alpha}_1 \subset \operatorname{Int}\varphi$, let ψ be any Jordan curve such that $\langle\psi\rangle = \langle\varphi\{X,\beta,Y\}\rangle \cup \langle\alpha_1\rangle$ and use (i). Assuming that (ii) holds whenever the number of α_i's is less than $n (n > 1)$, suppose we are given n arcs α_i as in the lemma. Applying the case $n = 1$ to φ and α_n, we obtain a Jordan curve ψ' such that $\langle\beta\rangle \subset \langle\psi'\rangle \subset \langle\varphi\rangle \cup \langle\alpha_n\rangle$ and $\operatorname{Int}\langle\psi'\rangle \subset (\operatorname{Int}\langle\varphi\rangle)\setminus\langle\alpha_n\rangle$. Renumbering $\alpha_1,\dots,\alpha_{n-1}$, if necessary, we may assume that $\langle\alpha_i\rangle \cap \operatorname{Int}\langle\psi'\rangle \neq \emptyset$ precisely for all $i \leqslant k$ for some $k < n$ (if $k = 0$, there is nothing to prove, hence we assume $k \geqslant 1$). Since $\langle\psi'\rangle \subset \langle\varphi\rangle \cup \langle\alpha_n\rangle$, it follows from (2) that $\check{\alpha}_i \cap \langle\psi'\rangle = \emptyset$ for all $i \leqslant n - 1$. Hence, for each $i \leqslant k, \check{\alpha}_i \subset \operatorname{Int}\langle\psi'\rangle$, and thus also $\dot{\alpha}_i \subset \overline{\operatorname{Int}\langle\psi'\rangle} \cap \langle\varphi\rangle = (\langle\psi'\rangle \cup \operatorname{Int}\langle\psi'\rangle) \cap \langle\varphi\rangle \subset \langle\psi'\rangle$, whence $\langle\alpha_i\rangle \cap \langle\psi'\rangle = \dot{\alpha}_i$ for $1 \leqslant i \leqslant k$. Applying the induction hypothesis to ψ' and $\alpha_1,\dots,\alpha_k$, we obtain a Jordan curve ψ such that $\langle\beta\rangle \subset \langle\psi\rangle \subset \langle\psi'\rangle \cup \langle\alpha_1\rangle \cup \cdots \cup \langle\alpha_k\rangle \subset \langle\varphi\rangle \cup \langle\alpha_1\rangle \cup \cdots \cup \langle\alpha_n\rangle$ and $\operatorname{Int}\langle\psi\rangle \subset (\operatorname{Int}\langle\psi'\rangle)\setminus(\langle\alpha_1\rangle \cup \cdots \cup \langle\alpha_k\rangle) \subset (\operatorname{Int}\langle\varphi\rangle)\setminus(\langle\alpha_1\rangle \cup \cdots \cup \langle\alpha_n\rangle)$, the last inclusion following from the definition of the number k.

(2.9) **Proposition.** *Arcs do not disconnect the plane.*

Proof. We shall use the notation of the proof of (2.5). As before, it suffices to show that $t_0 = \sup U \in U$. Let Δ_1 be a closed disk centered at $\lambda(t_0)$, so small that the points $\lambda(0), P, Q$ lie outside Δ_1. Set $t_1 = \inf\{t : \lambda[t,t_0] \subset \mathring{\Delta}_1\}$; then $\lambda(t_1) \in \partial\Delta_1$ and $\lambda(t_1,t_0] \subset \mathring{\Delta}_1$. Let Δ_2 be a concentric disk so small that $\Delta_2 \cap \lambda[0,t_1] = \emptyset$. Hence, $\Delta_2 \subset \mathring{\Delta}_1$, and if we set $t_2 = \inf\{t : \lambda[t,t_0] \subset \mathring{\Delta}_2\}$, then $t_1 < t_2 < t_0$. Therefore, $t_2 \in U$[11]) and so by (1.1) there exists a polygonal arc $\mu = \mu\{P,Q\}$ such that $\langle\mu\rangle \cap \lambda[0,t_2] = \emptyset$. If $\langle\mu\rangle \cap \mathring{\Delta}_2 = \emptyset$, then also $\langle\mu\rangle \cap \lambda[0,t_0] = \emptyset$, i.e. $t_0 \in U$. We may therefore assume that $\langle\mu\rangle \cap \mathring{\Delta}_2 \neq \emptyset$, which yields $\langle\mu\rangle \cap \partial\Delta_1 \neq \emptyset$[12]). Since μ is a polygonal arc and $\partial\Delta_1$ a circle, the set $\langle\mu\rangle \cap \partial\Delta_1$ is finite and thus defines a decomposition of μ into subarcs $\mu_1,\dots,\mu_m$ such that for each $i = 1,\dots,m$, we have: $\dot{\mu}_i \subset \{P,Q\} \cup (\langle\mu\rangle \cap \partial\Delta_1)$; either $\check{\mu}_i \subset \mathring{\Delta}_1$ or $\check{\mu}_i \cap \Delta_1 = \emptyset$; and $\check{\mu}_i \cap \check{\mu}_j = \emptyset$

[11]) Indeed, U is an interval containing $[0,t_0)$.

[12]) Here we are tacitly applying (1.5) to $\partial\Delta_1$.

whenever $i \neq j$. Assume that $\mu_1 = \mu_1\{P_1,Q_1\}$ is such that $\check{\mu}_1 \cap \mathring{\Delta}_2 \neq \emptyset$. Hence $\check{\mu}_1 \subset \mathring{\Delta}_1$ and $P_1, Q_1 \in \partial\Delta_1$. We shall define a new arc $\mu'_1 = \mu'_1\{P_1,Q_1\}$ such that $\langle\mu'_1\rangle \cap \lambda[0,t_0] = \emptyset$. Repeating this construction for every index i for which $\check{\mu}_i \cap \mathring{\Delta}_2 \neq \emptyset$, and replacing the subarc μ_i of μ by μ'_i we finally obtain a path $\mu' = \mu'\{P,Q\}$ such that $\langle\mu'\rangle \cap \lambda[0,t_0] = \emptyset$, which shows that $t_0 \in U$.

Construction of μ'_1: Let $\beta = \beta\{P_1,Q_1\}$ be an arc such that $\langle\beta\rangle \subset \partial\Delta_1$, $\lambda(t_1) \notin \langle\beta\rangle$ and let φ be any Jordan curve for which $\langle\varphi\rangle = \langle\beta\rangle \cup \langle\mu_1\rangle$. The intersection of the circle $\partial\Delta_2$ and the polygonal line $\langle\mu_1\rangle$ is a finite set. Hence there are arcs $\alpha_1,\ldots,\alpha_n$ with the following properties: $\langle\alpha_j\rangle \subset \partial\Delta_2$ (for all j); $\dot{\alpha}_1 \cup \cdots \cup \dot{\alpha}_n = \langle\mu_1\rangle \cap \partial\Delta_2 = \langle\varphi\rangle \cap \partial\Delta_2$; and $\check{\alpha}_i \cap \check{\alpha}_j = \emptyset$ if $i \neq j$. By (2.6) we can apply (2.8) (ii) ([9]) and obtain a curve ψ, for which JCT holds and such that $\langle\beta\rangle \subset \langle\psi\rangle \subset \langle\varphi\rangle \cup \partial\Delta_2$ and $\mathrm{Int}\langle\psi\rangle \subset (\mathrm{Int}\langle\varphi\rangle)\backslash\partial\Delta_2$. By (2.8) (i) we see that $\mathrm{Int}\langle\varphi\rangle \subset \mathring{\Delta}_1$, and since $\lambda(t_1) \notin \langle\varphi\rangle$, we conclude that $\lambda(t_1) \in \mathrm{Ext}\langle\varphi\rangle$. This, together with $\lambda(t_1,t_2] \cap \langle\varphi\rangle \subset \lambda(t_1,t_2] \cap \langle\mu\rangle = \emptyset$, implies $\lambda[t_1,t_2] \subset \mathrm{Ext}\langle\varphi\rangle \subset \mathrm{Ext}\langle\psi\rangle$, hence also $\lambda[0,t_2] \cap \langle\psi\rangle = \emptyset$. If we show that $\lambda(t_2,t_0] \cap \langle\psi\rangle = \emptyset$, we can clearly take any μ'_1 such that $\langle\mu'_1\rangle = \langle\psi\rangle\backslash\check{\beta}$. It suffices to show that $\mathring{\Delta}_2 \cap \langle\psi\rangle = \emptyset$. Since $\mathrm{Int}\langle\psi\rangle \subset (\mathrm{Int}\langle\varphi\rangle)\backslash\partial\Delta_2$, the connected set $\mathrm{Int}\langle\psi\rangle$ is the disjoint union of the open sets $\mathrm{Int}\langle\psi\rangle \cap \mathring{\Delta}_2$ and $\mathrm{Int}\langle\psi\rangle \cap \mathrm{C}\Delta_2$, hence one of these sets must be empty. If we had $\mathrm{Int}\langle\psi\rangle \cap \mathrm{C}\Delta_2 = \emptyset$, then $\mathrm{Int}\langle\psi\rangle \subset \mathring{\Delta}_2$; in particular, $\langle\beta\rangle \subset \Delta_2$, which is impossible. Therefore, $(\mathrm{Int}\langle\psi\rangle) \cap \mathring{\Delta}_2 = \emptyset$, and the proof is complete.

(2.10) **Corollary.** *The Jordan curve theorem holds for any Jordan curve containing at least one standard point.*

The proof is exactly the same as the last paragraph of the proof of (2.6), provided we replace there (2.5) by (2.9).

(2.11) *Proof of JCT for an arbitrary Jordan curve* φ.

Part (a) of (1.3) follows by (1.4) from (2.9).
Part (b): First we shall prove that $\langle\varphi\rangle$ disconnects $\mathbb{R}^2$.

Let $F', G' \in \langle\varphi\rangle, F' \neq G'$. If $\overline{F'G'} \subset \langle\varphi\rangle$, we are done by (2.6) (or by (2.10)). Hence we may assume that there is a point $A \in \overline{F'G'}\backslash\langle\varphi\rangle$. Let $\varkappa$ be an arc such that $\langle\varkappa\rangle = \overline{FG}$ is a maximal segment for which $A \in \langle\varkappa\rangle \subset \overline{F'G'}$ and $\check{\varkappa} \cap \langle\varphi\rangle = \emptyset$. Let φ_1, φ_2 be subarcs of φ such that $\dot{\varphi}_1 = \dot{\varphi}_2 = \{F,G\}$, $\langle\varphi_1\rangle \cup \langle\varphi_2\rangle = \langle\varphi\rangle, \check{\varphi}_1 \cap \check{\varphi}_2 = \emptyset$. Let Φ_i $(i = 0,1)$ be Jordan curves such that $\langle\Phi_i\rangle = \langle\varphi_i\rangle \cup \langle\varkappa\rangle$. By (2.10), JCT holds for Φ_0, Φ_1. Choose a disk Δ centered at a point of $\check{\varphi}_1$ with $\Delta \cap \langle\Phi_2\rangle = \emptyset$. By part (a) of the theorem we can find points $A', B' \in \Delta\backslash\langle\Phi_1\rangle$ for which $K(A';\langle\Phi_1\rangle) \neq K(B';\langle\Phi_1\rangle)$.

We claim that $K(A';\langle\varphi\rangle) \neq K(B';\langle\varphi\rangle)$, which will show that $\langle\varphi\rangle$ disconnects $\mathbb{R}^2$.

Assume the contrary. Then there is an arc $\gamma = \gamma\{A',B'\}$ such that $\langle\gamma\rangle \cap \langle\varphi\rangle = \emptyset$. Let $\langle\alpha\rangle$, $\langle\beta\rangle$ be the maximal subarcs of γ such that $\breve{\alpha} \cap \langle\varkappa\rangle = \emptyset = \breve{\beta} \cap \langle\varkappa\rangle$, $A' \in \langle\alpha\rangle$, $B' \in \langle\beta\rangle$. By the choice of A', B' we have $\langle\gamma\rangle \cap \langle\varkappa\rangle = \langle\gamma\rangle \cap \langle\Phi_1\rangle \neq \emptyset$. Hence by definition of α,β, $\langle\alpha\rangle \cap \breve{\varkappa} \neq \emptyset \neq \langle\beta\rangle \cap \breve{\varkappa}$. Assuming without loss of generality that the system of coordinates is chosen so that $\langle\varkappa\rangle$ is an interval of the x-axis, we have $\langle\alpha\rangle \cap \breve{\varkappa} = A = (a,0)$ and $\langle\beta\rangle \cap \breve{\varkappa} = B = (b,0)$. Interchanging A', B', if necessary, we may assume that $a \leqslant b$. Set $\varepsilon = (1/2)\mathrm{dist}(\overline{AB},\langle\varphi\rangle)$ and let R be the open rectangle $\{(x,y): a - \varepsilon < x < b + \varepsilon, |y| < \varepsilon\}$. By (2.3) the set $R\backslash\langle\varkappa\rangle$ has two components R_+,R_-. Since $R \cap \langle\varphi\rangle = \emptyset$, we obtain $R_\pm \cap \langle\Phi_i\rangle = \emptyset$ $(i = 1,2)$. Hence by part (a) of the theorem, applied to any point of $R \cap \breve{\varkappa}$, we see that

$$K(R_+;\langle\Phi_i\rangle) \neq K(R_-;\langle\Phi_i\rangle) \quad (i = 1,2). \tag{3}$$

Since A', B' lie in a connected subset Δ of $\complement\langle\Phi_2\rangle$ and $(\breve{\alpha} \cup \breve{\beta}) \cap \langle\Phi_2\rangle = \emptyset$, we conclude that

$$K(\breve{\alpha};\langle\Phi_2\rangle) = K(A';\langle\Phi_2\rangle) = K(B';\langle\Phi_2\rangle) = K(\breve{\beta};\langle\Phi_2\rangle). \tag{4}$$

Since R is a neighborhood of both A and B, we have $\breve{\alpha} \cap (R\backslash\langle\varkappa\rangle) \neq \emptyset \neq \breve{\beta} \cap (R\backslash\langle\varkappa\rangle)$, hence either (i) $\breve{\alpha} \cap R_+ \neq \emptyset$; or (ii) $\breve{\alpha} \cap R_- \neq \emptyset$. We claim that (i) implies $\breve{\beta} \cap R_+ \neq \emptyset$, which by (3) is clearly equivalent to $\breve{\beta} \cap R_- = \emptyset$. Suppose not, i.e. $\breve{\beta} \cap R_- \neq \emptyset$. Then by (4) and (i), $K(R_-;\langle\Phi_2\rangle) = K(\breve{\beta};\langle\Phi_2\rangle) = K(\breve{\alpha};\langle\Phi_2\rangle) = K(R_+;\langle\Phi_2\rangle)$ which contradicts (3). Thus $\breve{\alpha} \cap R_+ \neq \emptyset$ implies $\breve{\beta} \cap R_+ \neq \emptyset$, hence the set $E = \{A',B'\} \cup \breve{\alpha} \cup R_+ \cup \breve{\beta}$ is a connected subset of $\complement\Phi_1$, i.e. $K(A';\langle\Phi_1\rangle) = K(E;\langle\Phi_1\rangle) = K(B';\langle\Phi_1\rangle)$. This, however, contradicts the choice of A', B'. Similarly, one arrives at a contradiction in the remaining case (ii).

To complete the proof, it suffices to show that, together with any pair of distinct points A_0, A_1, the set $I(\varphi)$ ($\equiv$ union of all bounded components of $\complement\langle\varphi\rangle$) contains a connected subset G such that $A_0, A_1 \in G$[13]). Given A_0, A_1, let $\mathfrak{l}_0, \mathfrak{l}_1$ be any two parallel lines passing through A_0, A_1, respectively. Let $\varkappa_i$ $(i = 0,1)$ be an arc such that $\langle\varkappa_i\rangle = \overline{C_iC^i}$ is the maximal segment on $\mathfrak{l}_i$ for which $A_i \in \breve{\varkappa}_i \subset \complement\langle\varphi\rangle$. Then $\breve{\varkappa}_0 \cup \breve{\varkappa}_1 \subset I(\varphi)$ and $C_i, C^i \in \langle\varphi\rangle (i = 0,1)$. Let φ_1, φ^1 be arcs such that $\varphi_1 = \varphi\{C_0,C_1,C^0\}$, $\varphi^1 = \varphi\{C_0,C^1,C^0\}$. Assume first that the pairs $\{C_0,C^0\}, \{C_1,C^1\}$ are interlaced on φ. Then (cf. ([5])), $\breve{\varphi}_1 \cap \breve{\varphi}^1 = \emptyset$. Hence we can find a Jordan curve ψ such that $\langle\psi\rangle = \overline{C_0C^0} \cup \varphi^1\{C^0,C^1\} \cup \overline{C^1C_1} \cup \varphi_1\{C_1,C_0\}$. Then

[13]) The rest of the proof follows an idea taken from [33] (cf. also [14]).

$\langle\psi\rangle \subset I(\varphi) \cup \langle\varphi\rangle$. However by (2.7) the latter set equals $\overline{I(\varphi)}$, because by the beginning of this proof, φ satisfies part (a) of the theorem, hence $\langle\varphi\rangle \subset \overline{I(\varphi)}$. Hence $\langle\psi\rangle \subset \overline{I(\varphi)}$. By (2.10) applied to ψ, we may use (2.8) (i), hence $\mathrm{Int}\langle\psi\rangle \subset I(\varphi)$. Set $G = \{A_0, A_1\} \cup \mathrm{Int}\langle\psi\rangle$. Then clearly $G \subset I(\varphi)$ and G is connected, because $\mathrm{Int}\langle\psi\rangle \subset G \subset \overline{\mathrm{Int}\langle\psi\rangle}$ ([4]). If the pairs $\{C_0, C^0\}, \{C_1, C^1\}$ are not interlaced on φ, then $\langle\varphi_1\rangle = \langle\varphi^1\rangle$, and interchanging C_1 with C^1, if necessary, we may assume that $\langle\varphi_1\{C^0, C^1\}\rangle \cap \langle\varphi_1\{C_1, C_0\}\rangle = \emptyset$ (cf. ([5])). Then ψ can be defined as above and the rest of the proof is the same as in the previous case. This completes the proof of JCT.

3. Remarks

([1]) It is interesting to observe that there is a similar history for the postulate of Pasch (cf. [19, 21, 23]): (P) Given three noncollinear points B_0, B_1, B_2, consider the triangle $\beta = \overline{B_0B_1} \cup \overline{B_1B_2} \cup \overline{B_2B_0}$ and any straight line $\mathfrak{l}$ in the plane of β, which intersects the segment $\overline{B_0B_1}$, but does not pass through either B_0 or B_1. Then $\mathfrak{l}$ intersects either $\overline{B_0B_2}$ or $\overline{B_1B_2}$. An equivalent formulation of (P) could be called "the Jordan curve theorem for straight lines". Indeed, it reads as follows: (J) The complement (in the plane) of any straight line consists of two disjoint convex sets. (For the proof of (P) $\Leftrightarrow$ (J), cf. [21], Thm. 1 on p. 62 and Exercise 15 on p. 63). The importance of this postulate for foundations of Euclidean geometry was explicitly recognized only in 1882 by the well-known German geometer Moritz Pasch [23]. However, the postulate itself can be traced back to the Arab mathematician Nasîr-Eddîn at Tusî (1201 – 1274), who used it in his "proof" of Euclid's fifth postulate on parallels. His work remained in manuscript until 1594, when it was published in Rome [22]. Later this work was discussed by J. Wallis, G. Saccheri and N. I. Lobačevskij, always in connection with Euclid's fifth postulate. It is not without interest to note that Lobačevskij used form (J) of the postulate (cf. [19]). We use (P) and (J) to establish Lemma (2.1).

([2]) Jordan states only that part of the theorem which says that the graph $\langle\varphi\rangle$ of a simple closed planar curve φ decomposes the plane into two regions, $\mathrm{Ext}\langle\varphi\rangle$ and $\mathrm{Int}\langle\varphi\rangle$. (For the notation and terminology, see Sec. 1.) Thus he omits the fact that $\langle\varphi\rangle$ is the common boundary of both $\mathrm{Ext}\langle\varphi\rangle$ and $\mathrm{Int}\langle\varphi\rangle$. Moreover, Jordan assumes the theorem to be valid for polygons. His proof is based on constructing two sequences of simple polygonal curves $\varphi_n, \Phi_n (n = 1, 2, \ldots)$, whose "distance" to φ is positive but converges to zero for $n \to \infty$, and $\langle\varphi\rangle \subset (\mathrm{Ext}\langle\varphi_n\rangle) \cap (\mathrm{Int}\langle\Phi_n\rangle)$, $\langle\varphi_n\rangle \subset \mathrm{Int}\langle\varphi_{n+1}\rangle$, $\langle\Phi_n\rangle \subset \mathrm{Ext}\langle\Phi_{n+1}\rangle$. Jordan's proof, as well as another early proof due to de la Vallée-Poussin [31], was critically analyzed and completed by Schoenflies [29]. How some geometers around the turn of the century felt about Jordan's theorem can be seen from the beginning of Veblen's paper [32]: "Jordan's explicit formulation of the fundamental theorem that a simple closed curve lying wholly in a plane decomposes the plane into an inside and an outside region is justly regarded as a most important step in the direction of a perfectly rigorous mathematics. This may be confidently asserted whether we believe that perfect rigor is attainable or not. His proof however, is unsatisfactory to many mathematicians..."

([3]) Some authors assume – often without an explicit mention – the validity of JCT and its corollaries for special classes of curves (e.g. for polygonal curves). Others omit proofs of certain steps which are "geometrically obvious". (But what is more obvious than JCT itself?) Finally, certain proofs can hardly be considered rigorous in the modern sense.

([4]) On the other hand, we do not state in Section 1 several elementary facts such as: (i) two disjoint compact sets (in $\mathbb{R}^2$) have a positive distance and thus possess disjoint neighborhoods; (ii) the intermediate value property of continuous functions of one variable; (iii) definition and basic properties of a connected set; of components of a set (in $\mathbb{R}^2$), etc. They can all be found in any introductory text on general topology.

([5]) In particular, applying the homeomorphism $\varphi^{-1}:\langle\varphi\rangle \to \mathbb{S}^1$, we see that all questions about interlaced pairs of points on φ can be reduced to the corresponding questions about the canonical parametrization of the circle $\mathbb{S}^1$. Thus, for instance, it is easy to establish the following statement: Let A_1, A_2, B_1 and B_2 be four distinct points on $\langle\varphi\rangle$. Let $\varphi_1 = \varphi\{A_1,B_1,A_2\}$ $\varphi_2 = \varphi\{A_1,B_2,A_2\}$. Then either the pairs $\{A_1,A_2\}$, $\{B_1,B_2\}$ are interlaced on φ, which is equivalent to $\check{\varphi}_1 \cap \check{\varphi}_2 = \emptyset$; or these pairs are not interlaced on φ; then $\langle\varphi_1\rangle = \langle\varphi_2\rangle$ and interchanging B_1 with B_2, if necessary, we may achieve that the points A_i, B_j follow in this order on $\varphi_1 : A_1, B_1, B_2, A_2$; in particular, $\langle\varphi_1\{A_1,B_1\}\rangle \cap \langle\varphi_1\{B_2,A_2\}\rangle = \emptyset$.

([6]) By this we mean the following: Let $\lambda = \lambda\{P,Q\}$ be a polygonal path, $P \neq Q$. Then there exists a polygonal arc $\mu = \mu\{P,Q\}, \langle\mu\rangle \subset \langle\lambda\rangle$. (Sketch of the proof: We may assume that $\lambda:[0,1] \to \mathbb{R}^2$ is piecewise linear. The set $M_0 = \{\lambda(t):\lambda(t) = \lambda(t') \text{ for some } t' \neq t\}$ is a finite collection of segments and isolated points and the number m_0 of components of the closed set $N_\lambda, N_\lambda = \lambda^{-1}(M_0) \subset [0,1]$, is finite. Let $t_0 = \min N_\lambda, t_1 = \max\{t:\lambda(t) = \lambda(t_0)\}$. By removing the "loop" $\lambda(t_0,t_1)$ from $\langle\lambda\rangle$, i.e. by pasting together the restriction of λ to $[0,t_0)$ (or to $\{0\}$ if $t_0 = 0$) with the restriction of λ to $[t_1,1]$ we obtain a new mapping such that if m_1 is the number of components of the analogous set N_{λ_1}, then $m_1 < m_0$. After finitely many steps we obtain a λ_k for which $m_k = 0$. The desired arc μ is then λ_k parametrized on $[0,1]$.)

([7]) The proof of this lemma is taken from [13] where (2.2) was established for $\langle\varphi\rangle$ being the boundary of a square, and X_i, Y_i lying inside the opposite sides of $\langle\varphi\rangle$. To find an elementary proof of this statement was posed as a problem in the problem section of Čas. Pěst. Mat. (vol. 81, 1956, p. 470) by Jan Mařík. – For the case of an arbitrary polygonal Jordan curve the lemma appears in Denjoy's proof of the Jordan curve theorem (cf. [12]). However, both the proof and application of this lemma in [12] are different from ours: Denjoy assumes the validity of the Jordan curve theorem for polygonal Jordan curves, which yields an immediate proof of his version of (2.2); he then uses the lemma to establish directly part (b) of (1.3), thus omitting the fact that simple arcs do not disconnect the plane. – Lemma (2.2) clearly does not hold on surfaces of positive genus (e.g. on the torus).

([8]) Indeed, set $K = \overline{\mathrm{Int}\langle\varphi\rangle}$, $\overline{ST} = K \cap \mathrm{l}(X_2, Y_2)$. By the convexity of K, $\overline{X_2 Y_2} \subset K$. Since $\overline{ST} \not\subset \langle\varphi\rangle$ (because in view of $\check{\sigma}_2 \cap \langle\varphi\rangle = \emptyset$ we have $\overline{X_2 Y_2} = \langle\sigma_2\rangle \not\subset \langle\varphi\rangle$), the special shape of K (or just the convexity of K) implies that $\overline{ST} \cap \langle\varphi\rangle = \overline{ST} \cap \partial K = \{S, T\}$, but $X_2, Y_2 \in \overline{ST} \cap \langle\varphi\rangle$, hence $\overline{X_2 Y_2} = \overline{ST}$.

([9]) Indeed, let χ be a Jordan curve such that $\langle\chi\rangle \subset \langle\varphi\rangle \cup \partial\Delta_2$. Since $\langle\varphi\rangle \cap \partial\Delta_2$ is a finite set, there are (straight line or circular) arcs γ_i, such that $\check{\gamma}_i \cap \check{\gamma}_j = \emptyset$ for $i \neq j$ and $\langle\varphi\rangle \cup \partial\Delta_2 = \bigcup_{i=1}^{m} \langle\gamma_i\rangle$. Hence if $\langle\chi\rangle \cap \langle\gamma_i\rangle \neq \emptyset$ for some i, then $\langle\gamma_i\rangle \subset \langle\chi\rangle$. Therefore, $\langle\chi\rangle$ is a union of a subfamily of $\{\langle\gamma_1\rangle, \ldots, \langle\gamma_m\rangle\}$ so that after a change of parameter, if necessary, χ is a piecewise regular, hence also standard, curve. By (2.6) JCT holds for χ.

References[14])

[1]* Alexander, J. W.: A proof of Jordan's theorem about a simple closed curve. Ann. of Math. **21** (1920) 180–184

[2] Alexandroff, P. S.: Combinatorial Topology, Vol. 1 of the English transl., Rochester, N.Y., 1956 (see Chap. II, pp. 39–64; presents the proof of [26])

[3] Antoine, L.: Sur l'homéomorphie de deux figures et de leurs voisinages. J. Math. pures appl. **4** (1921) 221–325

[4] Behnke, H.; Sommer, F.: Theorie der analytischen Funktionen einer komplexen Veränderlichen. 3rd ed. Berlin-Heidelberg-New York 1963

[14]) By an asterisk we denote those papers which contain an elementary proof of JCT. We do not list any of the numerous articles where the theorem is established for a special class of Jordan curves (e.g. polygonal curves, curves of finite length, etc.). References to some of these can be found in the survey article [37].

[5]* Bieberbach, L.: Über den Jordanschen Kurvensatz, die Schoenflischen Sätze von Erreichbarkeit und Unbewalltheit und den Satz von der Invarianz des ebenen Gebietes. Jber. d. Dt. Math.-Verein. **22** (1913) 144–153

[6] Bochner, S.: Mathematical reflections. Amer. Math. Month. **81** (1974) 827–852

[7]* Brouwer, L. E. J.: Beweis des Jordanschen Kurvensatz. Math. Ann. **69** (1910) 169–175

[8]* Brouwer, L. E. J.: Beweis der Invarianz der geschlossenen Kurve. Math. Ann. **72** (1912) 442–425 (contains implicitly another proof of the Jordan curve theorem)

[9] Brouwer, L. E. J.: Beweis des Jordanschen Satzes für den *n*-dimensionalen Raum. Math. Ann. **71** (1911) 314–319

[10]* Cairns, S. S.: An elementary proof of the Jordan-Schoenflies theorem. Proc. of the A.M.S. **2** (1951) 860–867

[11] Dienes, P.: The Taylor Series. Oxford 1931 (presents Kerékjártó's proof [17, 18])

[12]* Denjoy, A.: Nouvelle démonstration du theorème de Jordan sur les courbes planes, Proc. Kon. Ak. Wetensch. Amsterdam 21 (1918); repr. in: Denjoy, A.: Un demisiècle de Notes communiquées aux Académies, vol. II. Paris 1957, 474–480

[13] Dostal, M.: Solution of a problem of Jan Mařík. Čas. Pěst. Mat. **83** (1958) 240–241

[14]* Filippov, A. F.: An elementary proof of the Jordan theorem. Uspekhi Mat. Nauk **5** (39) (1950) 173–176

[15]* Hartogs, F.: Beweis des Jordanschen Kurvensatzes. Math. Z. **22** (1925) 62–74

[16]* Jordan, C.: Cours d'Analyse, vol. III. Paris 1887 (see pp. 587–594); vol. I, 3rd ed. Paris 1893 (see §§ 96–103)

[17]* Kerékjártó, B. von: Beweis des Jordanschen Kurvensatzes, Math.-naturwiss. Ber. d. Ungar. Akad. **38** (1919) 194–198

[18] Kerékjártó, B. von: Vorlesungen über Topologie I. Berlin 1923 (see pp. 59–65; the proof is that of [17]; the same proof is presented in English in [11])

[19] Kagan, V. F.: Foundations of Geometry I. Moscow-Leningrad 1949

[20]* Lennes, N. J.: Curves in non-metrical analysis situs with an application in the calculus of variations. Amer. J. of Math. **33** (1911) 287–326 (in particular, see pp. 314–318)

[21] Moise, E.: Elementary Geometry from an Advanced Standpoint. 2nd ed. Reading, Mass.-Palo Alto-London 1974

[22] Nasîr-Eddîn at Tûsî: Euclidis Elementorum geometricorum libri tredecim. Ex traditione doctissimi Nasiridini Tusini nunc primum Arabice impressi. Rome 1594

[23] Pasch, M.: Vorlesungen über neuere Geometrie. Leipzig 1882

[24]* Pringsheim, A.: Über die äußere Berandung eines im Endlichen gelegenen Gebietes und den Jordanschen Kurvensatz. Sitz.-Ber. d. Bayer. Akad. Wiss., Math.-Phys. Kl. **2** (1922) 187–212

[25] Pringsheim, A.: Vorlesungen über Funktionentheorie, vol. II, part I. Leipzig 1925 (repr. New York 1968)

[26] Schmidt, E.: Über den Jordanschen Kurvensatz. Sitz.-Ber. d. Preuß. Akad. Wiss. **28** (1923) 318–329 (cf. also [2])

[27]* Schoenflies, A.: Bemerkungen zu dem vorstehenden Aufsatz des Herrn L. E. J. Brouwer. Math. Ann. **68** (1910) 435–414

[28] Schoenflies, A.: Über das eineindeutige und stetige Abbildung des Kreises (Jordan-kurve). Jber. d. Dt. Math.-Verein. **33** (1924) 147–157

[29] Schoenflies, A.: Zusatz (Bemerkungen) zu den Beweisen von C. Jordan und Ch. J. de la Vallée Poussin. Jber. d. Dt. Math.-Verein. **33** (1924) 157–160 (critical comments and completion of the proofs in [16], [31])

[30] Thron, W. J.: The Theory of Functions of a Complex Variable. New York 1953

[31] Vallée-Poussin, Ch. J. de la: Cours d'Analyse infinitésimale. vol. I, Louvain-Paris 1903 (1st. ed., pp. 308–310; more details in the 3rd ed., 1914, pp. 374–379)

[32]* Veblen, O.: Theory on plane curves in non-metrical analysis situs. Trans. of the A.M.S. **6** (1905) 83–98

[33]* Vol'pert, A. I.: An elementary proof of the Jordan theorem. Uspekhi Mat. Nauk **5** (39) (1950) 168–172

[34]* Winternitz, A.: Über den Jordanschen Kurvensatz und verwandte Sätze der Analysis Situs. Math. Z. **1** (1918) 329–337

[35]* Whyburn, G. T.: Analytic Topology, A.M.S. Colloquium Publ. vol. 28, Providence 1942

[36]* Young, W. H.; Young, G. Ch.: The Theory of Sets of Points. Cambridge 1906 (repr. Bronx, N.Y. 1972)

[37] Zoretti, L.; Rosenthal, A.: Die Punktmengen. Leipzig 1923–1927. = Encyklopädie der Math. Wissenschaften, II C 9a (see vol. II, Part 3, 2nd half, pp. 916–923)

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[5a]) Actually, $w(\alpha;\beta)$ depends only on $\langle\alpha\rangle$ and $\langle\beta\rangle$; thus if $\langle\alpha\rangle$ is a segment $\overline{AB}$ we shall write $w(\overline{AB};\beta)$ instead of $w(\alpha;\beta)$.

[6a]) Before proceeding to Case I b the reader should read Part II of this proof (cf. p. 118) applied to arbitrary (i.e. non-polygonal) σ_1, σ_2 for which $K(\check{\sigma}_i;\langle\varphi\rangle) = \operatorname{Int}\varphi$.