CHAPTER II

Dimension Theory and Separation Theorems

§ 1. The Invariance of Dimension

Before 1870 mathematicians only dealt with those subsets X of an $\mathbf{R}^N$ that could (at least locally) be "parametrized" by (usually C^1) injective maps into X of open subsets of some $\mathbf{R}^n$. It was tacitly assumed that the position of a point in $\mathbf{R}^n$ could only be completely determined by a system of n real numbers. The discovery by Cantor in 1877 of a *bijection* of $\mathbf{R}$ onto $\mathbf{R}^n$, for any n, came as a complete surprise and seemed to threaten the bases of analysis. Cantor's map was wildly discontinuous, but the discovery of the Peano curve (1890) showed that there existed *continuous* (although not injective) maps of $\mathbf{R}$ *onto* $\mathbf{R}^n$. The only hope that remained of salvaging the classical notion of dimension was the one expressed by Dedekind as soon as Cantor had communicated his theorem to him: there should not exist *bicontinuous bijections* of $\mathbf{R}^m$ onto $\mathbf{R}^n$ for $m \neq n$. This was elementary for $m = 1, n > 1$, since a point disconnects $\mathbf{R}$ but not $\mathbf{R}^n$; several mathematicians before 1910 were also able to tackle the cases $m = 2$ and $m = 3, n > m$. But the general proof of Dedekind's conjecture was only obtained by Brouwer in the first of the series of papers which he started in 1911 ([89], pp. 430–434).

Brouwer's proof is based on the key lemma showing that if a continuous map f of $[-1, 1]^n$ into $\mathbf{R}^n$ is such that $|f(x) - x| < \frac{1}{2}$ for all x, then $f([-1, 1]^n)$ contains the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ (chap. I, § 3,D). He used that lemma to show that there may not exist an *injective* continuous map g of $[-1, 1]^n$ onto a *rare* subset C of $\mathbf{R}^n$. The proof is by contradiction: Brouwer showed that if such a map existed, it would be possible to define a continuous map h: $C \to [-1, 1]^n$, such that $h(C)$ would be rare and $|h(g(x)) - x| < \frac{1}{2}$ for all $x \in [-1, 1]^n$, in contradiction with the lemma.

To construct h, start from a sufficiently fine triangulation T of a cube $K \supset C$ and consider the union F of the n-simplices of T that meet C. Define $h_0(a)$ for each vertex a of an n-simplex $\sigma \subset F$ as one of the points of $[-1, 1]^n$ such that $g(h_0(a)) \in \sigma$, then extend h_0 to a piecewise affine map h_0 of F into $[-1, 1]^n$. If h is the restriction of h_0 to C, then $h(\sigma \cap C)$ is rare for each n-simplex $\sigma \subset F$, hence h has the required properties provided T has been chosen fine enough.

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From this theorem, Brouwer obtained the invariance of dimension in two steps:

1. Suppose $m > n$; a cube K of $\mathbf{R}^m$ contains a rare image K′ of $[-1,1]^n$ by a continuous injection j. If there existed a continuous injective map f: $\mathrm{K} \to [-1,1]^n$, the map $f \circ j$ would contradict the theorem.
2. If a cube K′ of $\mathbf{R}^n$ contained the image of $[-1,1]^m$ by a homeomorphism g, as there exists a continuous injection h: $\mathrm{K}' \to [-1,1]^m$ such that $h(\mathrm{K}')$ is rare, $h \circ g$ would again contradict the theorem.

These two corollaries imply the nonexistence of a homeomorphism of $\mathbf{R}^m$ onto $\mathbf{R}^n$ for $m \neq n$.

§ 2. The Invariance of Domain

A result closely related to the invariance of dimension was called the "invariance of domain": if A is a compact subset of an $\mathbf{R}^n$ and f: $\mathrm{A} \to \mathbf{R}^n$ is a continuous *injective* map, f sends interior points of A to interior points of $f(\mathrm{A})$ [which implies that it maps the interior of A homeomorphically onto the interior of $f(\mathrm{A})$]. This property implies invariance of dimension: suppose there existed a homeomorphism f of an open set $\mathrm{U} \neq \varnothing$ in $\mathbf{R}^m$ onto an open subset of $\mathbf{R}^n$ with $n < m$; one may consider $\mathbf{R}^n$ as a rare subset of $\mathbf{R}^m$, and for V open nonempty and relatively compact in $\mathbf{R}^m$ and $\bar{\mathrm{V}} \subset \mathrm{U}$, $f(\bar{\mathrm{V}})$, considered as a subset of $\mathbf{R}^m$, would have no interior point.

This is essentially the argument by which Baire, in 1907, wanted to prove the invariance of dimension ([40], [41]). He then endeavored to reduce the invariance of domain to a weak* generalization of the Jordan curve theorem to n dimensions: if f is a homeomorphism of the closed ball $\mathbf{D}_n$: $|x| \leqslant 1$ of $\mathbf{R}^n$ onto a subset of $\mathbf{R}^n$, the complement of $f(\mathbf{S}_{n-1})$ in $\mathbf{R}^n$ has two connected components [traditionally called the "interior" and "exterior" of $f(\mathbf{S}_{n-1})$, the "exterior" being the unbounded one].

In assuming this result, Baire also had to assume that $f(\mathbf{D}_n)$ was not contained in the "exterior" of $f(\mathbf{S}_{n-1})$.† He considered the concentric open balls $\mathrm{B}(\rho)$: $|x| < \rho$ for $0 < \rho \leqslant 1$ [$\mathrm{B}(1) = \mathbf{D}_n$], and their boundaries $\mathrm{S}(\rho)$: $|x| = \rho$. Then $f(\mathrm{B}(1))$ is contained by assumption in the "interior" A of $f(\mathrm{S}(1))$, and by contradiction $f(\mathrm{B}(1)) = \mathrm{A}$. Indeed, if that were not the case, there would be a point $y \in \mathrm{A}$ not in the closed set $f(\overline{\mathrm{B}(1)}) = f(\mathrm{B}(1)) \cup f(\mathrm{S}(1))$ and hence a ball

* That this is not the real generalization of the Jordan theorem is due to the fact that a continuous injection of $\mathbf{S}_{n-1}$ into $\mathbf{R}^n$ cannot in general be extended to a continuous injection of $\mathbf{D}_n$ into $\mathbf{R}^n$.

† If one knows that the order of an "interior" point A with respect to $f(\mathbf{S}_{n-1})$ is ± 1 (see § 3), it implies the fact that $f(\mathbf{D}_n)$ is not contained in the "exterior" of $f(\mathbf{S}_{n-1})$, for as $\mathrm{S}(\rho)$ tends to a point when ρ tends to 0, the order of A with respect to $f(\mathrm{S}(\rho))$ would tend to 0, although it must be constant.

γ of center y and radius r that does not meet $f(\overline{B(1)})$. It is impossible for γ to be contained in the "interior" of $f(S(\rho))$ for all ρ, since the diameter of $f(S(\rho))$ tends to 0 with ρ. Let ρ_0 be the g.l.b. of the $\rho > 0$ such that γ is contained in the "interior" of $f(S(\rho))$. Then, for a sequence (ε_k) tending to 0, y would be at a distance $\geqslant r$ of the "interior" of $f(S(\rho_0 - \varepsilon_k))$ and at a distance $\geqslant r$ of the "exterior" of $f(S(\rho_0 + \varepsilon_k))$, which is impossible by continuity.

Baire, however, could not prove the weak generalization of the Jordan theorem which he needed.* It was again Brouwer who gave two different proofs of the invariance of domain. The first one ([89], p. 485) does not use the Jordan–Brouwer theorem, but what we may call for short the *no separation theorem* (NS), for which Brouwer gave a proof in the same paper (see § 4):

(NS) If U is a connected open subset of $\mathbf{R}^n$, and $F \subset U$ is a homeomorphic image of a compact subset A of S_{n-1}, *distinct* from S_{n-1}, then $U - F$ is connected.

To deduce the invariance of domain from this, Brouwer argued by contradiction: let f be an injective continuous map of $\bar{U}$ into $\mathbf{R}^n$, where U is a nonempty bounded open set in $\mathbf{R}^n$, and suppose there exists a point $P \in U$ such that $f(P)$ is *not* interior to $f(U)$. Let $Q \neq P$ be another point of U; by assumption, there are spheres Σ of center $f(P)$ and arbitrary small radius that are not contained in $f(\bar{U})$; take the radius of such a sphere Σ smaller than the distance of $f(P)$ to $f(Q)$ and such that $F = f^{-1}(\Sigma \cap f(\bar{U}))$ is contained in a closed ball $B \subset U$ of center P that does not contain Q. By (NS), P and Q may be joined by a polygonal line $L \subset U$ that does not meet F; then $f(L) \subset f(U)$ would join $f(P)$ and $f(Q)$ without meeting $\Sigma \cap f(U)$, which is the desired contradiction.

Brouwer's second proof ([89], pp. 509–510) is simpler and only uses properties of the degree [or rather of its localization (chap. I, § 3,D)]. With the same notations, let $P \in U$ and let I be a small open ball of center P such that $\bar{I}$, union of I and its boundary, the sphere K, is contained in U. Let H be the connected component of the open set $\mathbf{R}^n - f(K)$ that contains $f(P)$, hence also $f(I)$ since $f(I) \cap f(K) = \varnothing$; the proof consists in showing that $f(I) = H$. Brouwer's argument, which is only sketched, is clearer if we use the localized degree $d(f, I, p)$; if $f(I) \neq H$ it would imply $d(f, I, H) = 0$ since $\operatorname{Fr}(I) = K$ and $H \cap f(K) = \varnothing$. In his proof of invariance of dimension (§ 1), however, Brouwer had shown that there exists a nonempty open ball $\gamma' \subset f(I)$; then $\gamma = f^{-1}(\gamma') \cap I$ is open in $\mathbf{R}^n$, and the restriction of f to γ is a homeomorphism onto γ'; hence $d(f, \gamma, \gamma') = \pm 1$ (chap. I, § 3,D). If $p \in \gamma'$, then $d(f, I, p) = d(f, I, H) = 0$; but $d(f, I, p) = d(f, \gamma, \gamma')$ [*loc. cit.* formula (14)] and therefore the assumption $f(I) \neq H$ implies a contradiction.

* He complained in a letter to Brouwer that his bad health prevented him from mustering the energy needed to elaborate his ideas.

§ 3. The Jordan–Brouwer Theorem

The full generalization of the Jordan curve theorem (now called the Jordan–Brouwer theorem) was first tackled by Lebesgue and Brouwer in 1911. We can split the problem into three parts. Given a subset J of $\mathbf{R}^n$ homeomorphic to S_{n-1},

(i) The complement $\mathbf{R}^n - J$ has at least two connected components.
(ii) J is the boundary of every connected component of $\mathbf{R}^n - J$.
(iii) $\mathbf{R}^n - J$ has at most two connected components.

A. Lebesgue's Note

Part (i) is independent of the other two and also of (NS). In March 1911 Lebesgue published a sketch of a proof in a *Comptes rendus* note ([294], pp. 173–175). At first Brouwer had doubts that this sketch could be elaborated into a correct proof ([89], p. 452); because of Lebesgue's imprecise language, he thought J was any $(n-1)$-dimensional compact connected manifold in $\mathbf{R}^n$. Later he admitted that (i) could indeed be proved by Lebesgue's method, but [probably owing to his contemporary controversy with Lebesgue on the definition of dimension (see § 5)] he did not wish to write out a complete proof himself. Lebesgue did not write anything on the matter after his *Comptes rendus* note, so no complete proof of (i) was available until 1922.

Lebesgue's method relies on an ingenious interpretation of part (i): for $0 \leqslant k \leqslant n-1$, let L_k be a subset of $\mathbf{R}^n$ homeomorphic to S_k; then there exists a subset L'_{n-k-1} of $\mathbf{R}^n$, homeomorphic to S_{n-k-1}, and such that L_k and L'_{n-k-1} are "enlacées" (i.e., their linking number mod. 2 is $\neq 0$). For $k = n-1$, $S_{n-k-1} = S_0$ consists of two points, and the statement is thus equivalent to (i). For $k = 0$, the theorem is trivial, and Lebesgue's proof is by *induction on k*. He considered a piecewise affine approximation g to a homeomorphism $f: S_k \to L_k$; let A_+, A_-, and L_{k-1} be the images by f of the hemispheres D_+, D_- and of their common boundary S_{k-1}.* By the inductive assumption L_{k-1} is linked by a homeomorphic image L'_{n-k} of S_{n-k}. Replacing L'_{n-k} by an arbitrarily close piecewise affine approximation $h(S_{n-k})$, makes the intersection $g(\mathbf{D}_+) \cap h(S_{n-k})$ finite, and it has an odd number of points (if not, replace D_+ by D_-). If P is one of these points, by a slight change of g, it may be taken to be the intersection of a k-simplex of $g(S_k)$ and an $(n-k)$-simplex σ of $h(S_{n-k})$ and belongs to the interior of these simplices; then the boundary of σ in $h(S_{n-k})$ links $g(S_k)$.

B. Brouwer's First Paper on the Jordan–Brouwer Theorem

Brouwer published two papers on the Jordan–Brouwer theorem. The first one ([89], pp. 489–494), exclusively deals with part (iii) of the problem. Part

* This seems to be the first occurrence of this splitting of the sphere, which will be used again and again later in many contexts.

(ii) is dismissed with the remark that it follows from the (NS) theorem (§ 2), for which he had written a proof in an earlier paper (see § 4), without giving any detail. For any point $x_0 \in \mathrm{J}$, it is enough to delete from J the interior of an arbitrarily small $(n-1)$-simplex σ of a sufficiently fine triangulation of J containing x_0. If G_1 and G_2 are two connected components of $\mathbf{R}^n - \mathrm{J}$, $y_1 \in \mathrm{G}_1$ and $y_2 \in \mathrm{G}_2$, there is, by the (NS) theorem, a polygonal arc joining y_1 and y_2 in $\mathbf{R}^n - (\mathrm{J} - \sigma)$; on that arc there are points of G_1 and points of G_2 at a distance from x_0 smaller than the diameter of σ; this proves (ii).

The proof of (iii) occupies four pages; it is quite involved and, in spite of its length, full of cryptic statements that make it very hard to follow in detail. What follows is my own interpretation and simplification of what I think are the main points of Brouwer's arguments. He repeatedly uses a lemma first stated in the paper on the (NS) theorem ([89], p. 478):

(L) The boundary F of a pseudomanifold-with-boundary P (chap. I, § 3,A), of dimension n, is a disjoint union of closed $(n-1)$-dimensional pseudomanifolds F_j.

Simple examples show that, if taken literally, this is not correct, for an $(n-2)$-simplex of F may be contained in more than two $(n-1)$-simplices of F. Brouwer acknowledges this but dismisses the matter by saying that p-simplices of F, for $p \leqslant n-2$, that appear to contradict the fact that the F_j are pseudomanifolds and are pairwise disjoint, should be "demultiplied" ("als verschieden zu betrachten sind") so to speak. It would have been clearer if he had bothered to give a proof, and said that one can do away with those occurrences by slightly moving the vertices of F!

The proof of (iii) is essentially based on the idea of linking number, which Brouwer only defined in a general way six months later; here it is used in the particular case of a polygonal Jordan curve L and the frontier j of an $(n-1)$-simplex σ of a (curvilinear) triangulation T of J; his arguments can be simplified by using the definition of linking numbers as degrees of mappings (chap. I, § 3,C). Let E be the unbounded component of $\mathbf{R}^n - \mathrm{J}$, G another (bounded) component, P a point of G; the bulk of Brouwer's proof consists in constructing a polygonal Jordan curve L containing P and such that $\mathrm{lk}(\mathrm{L}, j) = \pm 1$.

He first constructed in $\mathbf{R}^n$ an infinite locally finite $(n-1)$-dimensional simplicial complex $g \subset \mathrm{G}$ whose closure in $\mathbf{R}^n$ is $g \cup j$. Starting with the cubical subdivisions A_ν of $\mathbf{R}^n$ whose vertices are the points of $2^{-\nu}\mathbf{Z}^n$, for each integer τ, let μ_τ be the union of the closed cubes of A_ν that meet the interior of σ and have a distance at least $\sqrt{n}/2^{\tau-1}$ from $\mathrm{J}'' = \overline{\mathrm{J} - \sigma}$; if τ is taken large enough, P does not belong to any $\mu_{\tau+k}$ for $k \geqslant 0$. The union V_τ of the $\mu_{\tau+k}$ for $k \geqslant 0$ is a kind of "thickening" of σ in $\mathbf{R}^n$ with a "decent" boundary; $\mathrm{V}_\tau \cup \mathrm{J}''$ is closed and connected, and $\overline{\mathrm{V}}_\tau \cap \mathrm{J}'' = j$. Define I_τ as the intersection of G and of the open component in $\mathbf{R}^n$ of the complement of $\mathrm{V}_\tau \cup \mathrm{J}''$ that contains P; g is the part of the boundary of I_τ contained in V_τ, the union of the g_ν, where g_ν is a finite rectilinear cell complex, the cells of which are cells in the frontiers of some of the cubes whose union is $\mu_{\tau+k}$ for $\tau + k \leqslant v$. After subdividing of the cubes into simplices and using lemma (L), one sees that g_v is the disjoint union of finitely many $(n - 1)$-dimensional pseudomanifolds.

To construct L, one first joins P to a point R on one of the rectilinear simplices of g, by a polygonal arc L_1 contained* in I_τ. On the other hand, one can join P, by a polygonal arc L_2 contained in I_τ, to a point B′ arbitrarily close to a point of $J - \sigma$ [using (ii)]; then [again using (ii)], one can join B′ to a point B″ in E by a line segment of arbitrarily small length s_2. Similarly, one can join R to a point A′ of V_τ arbitrarily close to a point in the interior of σ, by a polygonal arc L_3 in V_τ; then, again using (ii), a line segment s_1 of arbitrarily small length joins A′ to a point A″ in E; finally, one may join A″ and B″ by a polygonal arc L_4 contained in E. The polygonal Jordan curve L is the union of L_1, L_3, s_1, L_4, s_2, and L_2.

If g were a closed pseudomanifold with boundary j, one would have $\mathrm{lk}(L, j) = \pm 1$, since L meets g in the single point R. But the argument by which Brouwer proved the equivalence of the definition of the linking number as a degree and its definition by counting intersection points does not apply to "open" complexes such as g. To circumvent this difficulty, Brouwer apparently considered the connected component γ_v of g_v containing R, which is a rectilinear $(n - 1)$-dimensional pseudomanifold with boundary η_v, and he takes for granted that η_v tends to j when v tends to $+\infty$, but gives no proof for this statement. Taking v large enough and a sufficiently fine triangulation of γ_v, a simplicial mapping φ of η_v into j can be defined, homotopic to the identity,† so that $\mathrm{lk}(L, \eta_v)$ is equal to the degree of the map $(x, y) \mapsto (\varphi(x) - y)/|\varphi(x) - y|$ of $\eta_v \times L$ into S_{n-1}; by the multiplicative property of the degree, this implies that

$$\mathrm{lk}(L, \eta_v) = \deg(\varphi) \,.\, \mathrm{lk}(L, j); \tag{1}$$

but for γ_v and η_v, the equivalence of the two definitions of the linking number applies, so that $\mathrm{lk}(L, \eta_v) = \pm 1$, and from (1) it follows that $\mathrm{lk}(L, j) = \pm 1$.

Now assume there exists a third (bounded) component G′ of $\mathbf{R}^n - J$, and construct the corresponding intersection I'_τ of G′ and an open component in $\mathbf{R}^n$ of the complement of $V_\tau \cap J''$. If g' is the part of the boundary of I'_τ contained in V_τ, the construction of the polygonal Jordan curve L shows that $L \cap g' = \varnothing$, if s_1 and s_2 are small enough. But then $\mathrm{lk}(L, j) = 0$ by the argument made above where g is replaced by g'; this brings the required contradiction.

C. Brouwer's Second Paper on the Jordan–Brouwer Theorem

This paper immediately follows the first one in *Mathematische Annalen* ([89], pp. 498–505). In it Brouwer capitalized on the hard work he did in the first

* To simplify the language, we say that a polygonal arc is "contained" in an open set I_τ if the complement of its extremities is a subset of I_τ.

† Note that φ need not be injective.

paper to obtain additional properties of the "Jordan hypersurfaces" in $\mathbf{R}^n$, generalizing results Schoenflies had proved for Jordan curves in $\mathbf{R}^2$.

(I) J is *accessible* from both components I and E (the "interior" and "exterior" of J) of $\mathbf{R}^n - \mathrm{J}$. This means that for any point A of J, there is a Jordan arc having A as one extremity and contained in I (resp. in E). The idea is to consider a sequence (T_k) of triangulations of J obtained by repeated subdivisions of T, and a decreasing sequence (σ_k) of $(n-1)$-simplices of the triangulation T_k, whose diameter tends to 0, such that A is interior to each σ_k. For each k, Brouwer constructed a "thickening" $\mathrm{V}_{\tau_k}^{(k)}$ of σ_k as in the first paper, for a sufficiently large τ_k, in such a way that $\mathrm{V}_{\tau_{k+1}}^{(k+1)}$ is contained in the interior of $\mathrm{V}_{\tau_k}^{(k)}$. Then, starting from a point $\mathrm{P}_0 \in \mathrm{I}$ not in $\mathrm{V}_{\tau_1}^{(1)}$, he constructed a sequence of polygonal arcs L_k, joining a point $\mathrm{P}_k \in \mathrm{V}_{\tau_k}^{(k)} - \mathrm{V}_{\tau_{k+1}}^{(k+1)}$ to a point $\mathrm{P}_{k+1} \in \mathrm{V}_{\tau_{k+1}}^{(k+1)} - \mathrm{V}_{\tau_{k+2}}^{(k+2)}$ and contained in $\mathrm{V}_{\tau_k}^{(k)} \cap \mathrm{I}$. The union of the L_k and of the point A is the required Jordan arc. The same argument applies for a point in E.

(II) A similar argument proves the property called "Unbewaltheit" by Schoenflies: if Q and Q′ are two points of J, and $m(\mathrm{Q},\mathrm{Q}')$ is the infimum of the *diameters* of Jordan arcs joining Q and Q′ in I (resp. E), then $m(\mathrm{Q},\mathrm{Q}')$ tends to 0 with the distance $d(\mathrm{Q},\mathrm{Q}')$ of the two points in $\mathbf{R}^n$. This time one considers a sequence $(\mathrm{Q}_k, \mathrm{Q}'_k)$ of pairs of points of J with $d(\mathrm{Q}_k, \mathrm{Q}'_k)$ tending to 0, and a sequence (σ_k) of $(n-1)$-simplices of triangulations of J, whose diameter tends to 0, and are such that both Q_k and Q'_k are in the interior of σ_k. The construction in I) shows that $m(\mathrm{Q}_k, \mathrm{Q}'_k)$ is at most the diameter of a "thickening" $\mathrm{V}_{\tau_k}^{(k)}$, which obviously tends to 0 when the diameter of σ_k tends to 0 and τ_k tends to $+\infty$.

(III) Finally, Brouwer sketched a proof that the *order* of a point $\mathrm{P} \in \mathrm{I}$ with respect to J (chap. I, § 3,B) is ± 1 (that order is of course constant in I). With the notations of the first paper, he took for granted that there exists an $(n-1)$-simplex σ of the triangulation T of J, and a half-line D of origin a suitable point P of I, such that $\mathrm{D} \cap \mathrm{J}'' = \varnothing$. To show this, take the first point of intersection Q of J and of an oriented line D_0 that meets I and does not meet the $(n-2)$-simplices of T; then if σ is the $(n-1)$-simplex of T containing Q, the distance of Q to $\mathrm{J}'' = \overline{\mathrm{J} - \sigma}$ is > 0. There is therefore a point $\mathrm{P} \in \mathrm{D}_0 \cap \mathrm{I}$ close enough to Q that the half-line D of origin P and containing Q satisfies the requirement.

Next he took a subdivision T_1 of T, and considered the piecewise affine map h of J into $\mathbf{R}^n$ coinciding with the identity on the vertices of T_1; h is homotopic with the identity by a homotopy F whose image does not contain P if T_1 is fine enough. Hence the order of P with respect to J is, up to sign, the sum of the intersection numbers of D and of the rectilinear complex $\mathrm{J}_1 = h(\mathrm{J})$ [one may always suppose that D does not meet the $(n-2)$-simplices of T_1]. Brouwer stated without proof that this number m is ± 1. It is possible to supply a simple argument justifying this claim by using the construction of the first paper: first take a polygonal arc L′ joining P to a point P′ of $\mathrm{D} \cap \mathrm{E}$, which does not meet $\mathrm{J}''_1 = h(\mathrm{J}'')$, and next a polygonal arc L″ joining P′ to P and which does not meet $\sigma_1 = h(\sigma)$. If $\mathrm{L} = \mathrm{L}' \cup \mathrm{L}''$, the construction gives

$\text{lk}(L, j_1) = \pm 1$, where $j_1 = h(j)$. Now, if L_0 is the segment of D joining P and P′, then by definition $\text{lk}(L_0 \cup L'', j_1) = \pm m$. If L_0' is the loop $L_0 \cup L'$, it is only necessary to show that $\text{lk}(L_0', j_1) = 0$, and as L_0' does not meet J_1'', and j_1 is homotopic to a point *in the complex* J_1'', this is obvious.

There is also in this second paper a curious section in which Brouwer claimed to have proved (by a fairly intricate construction) the orientability of J. Did he forget that by definition J is homeomorphic to S_{n-1}, and that S_{n-1} is orientable as a "manifold" in his sense, for any triangulation, according to his own definition of orientability ([89], p. 458)?

§ 4. The No Separation Theorem

In *Mathematische Annalen*, this paper precedes the one on the Jordan–Brouwer theorem, and is entitled "Proof of the invariance of domain," although invariance of domain is only mentioned in the last section; the bulk of the paper (six pages) consists in the proof of what we have called in § 2 the "no separation theorem." It is certainly the most intricate proof of all Brouwer's theorems and the most difficult to follow; the details are so sketchy that I find it impossible to give more than a summary of the main arguments as I understand them.

A preliminary result is a generalization of a theorem of Janiszewski on sets of the plane [267]: let P, Q be two points of S_{n-1}, X and Y be two disjoint relatively closed subsets of the open ball $\mathbf{B}_n$: $|x| < 1$. Suppose P and Q are *not separated* by X nor by Y in $\mathbf{B}_n$, a statement which Brouwer interpreted as meaning that there are Jordan arcs L, M, joining P and Q in $\mathbf{B}_n$ such that $L \cap X = M \cap Y = \varnothing$; then P and Q are not separated by $X \cup Y$, i.e., there is a Jordan arc N joining P and Q in $\mathbf{B}_n$ such that $N \cap (X \cup Y) = \varnothing$. Brouwer's proof consists in approximating X and Y by neighborhoods that are subcomplexes of a sufficiently fine triangulation T of $\bar{\mathbf{B}}_n$, and showing that the theorem may be proved when X and Y are replaced by these neighborhoods. In that simpler case Brouwer used, in addition to lemma (L) of § 3, the following unproved assertion:

(L′) A subcomplex K of T separates P and Q if and only if any polygonal arc joining P and Q in $\mathbf{B}_n$, and which does not meet any $(n-2)$-simplex of T, meets K in an *odd* number of points.

He then simply observed that if a polygonal arc joining P and Q in $\mathbf{B}_n$ and having empty intersections with the $(n-2)$-simplices of T meets each of the subcomplexes X, Y in an even number of points, it also meets $X \cup Y$ in an even number of points.

Brouwer then used this theorem to show that if the points P, Q in S_{n-1} are separated in $\mathbf{B}_n$ by a relatively closed subset X of $\mathbf{B}_n$, then they also are separated by a suitably chosen *connected component* of X.

The proof of (NS) proper is by contradiction, and can be divided into three steps.

First step. Let J be a "Jordan hypersurface" in $\mathbf{R}^n$, M be a closed subset of J *distinct from* J. By arguments that are not at all clear, Brouwer claimed that the assumption that $\mathbf{R}^n - \mathrm{M}$ has more than one connected component leads to the following situation: P is a *frontier point* of M in J, D is an open ball of $\mathbf{R}^n$ with center at P, H is the $(n-1)$-dimensional sphere, boundary of D in $\mathbf{R}^n$, A, B are two points of H separated by $\mathrm{M} \cap \mathrm{D}$ in D. From the preliminary result he deduced that there is a connected set t contained in $\mathrm{M} \cap \mathrm{D}$, relatively closed in D, containing P and separating A and B in D. Let u be the intersection $\bar{t} \cap \mathrm{H}$, contained in $\mathrm{H} \cap \mathrm{M}$, and G the connected component of $\mathrm{M} - u$ in J containing t. The first step in Brouwer's argument was to show that $t \neq \mathrm{G}$; otherwise P would be an interior point of $t = \mathrm{G}$, contrary to the assumption that P is a frontier point of M in J. For a sufficiently fine triangulation T of J there is therefore an $(n-1)$-simplex of T contained in $\mathrm{G} - t$.

Second step. For the second and third step Brouwer found it easier to transform $\mathbf{R}^n - \{\mathrm{B}\}$ by an inversion of pole B, bringing about the following situation (where we use the *same* notation for elements of the former situation and for their transforms by inversion): D is now an open half space of $\mathbf{R}^n$, having a hyperplane H as its frontier, and one has $\mathrm{A} \in \mathrm{H}$; u is a closed subset of H that does not contain A; $\mathrm{G} \subset \mathrm{D}$ is a homeomorphic image of a subset of J, open in J; $u = \bar{\mathrm{G}} \cap \mathrm{H}$; finally t is a subset of G, relatively closed in G, $u = \bar{t} \cap \mathrm{H}$, and $\mathrm{G} - t$ contains an $(n-1)$-simplex σ of a triangulation T of J. If $\pi: \mathrm{H} - \{\mathrm{A}\} \to \mathrm{S}$ is the projection from A of $\mathrm{H} - \{\mathrm{A}\}$ onto an $(n-2)$-dimensional sphere $\mathrm{S} \subset \mathrm{H}$ of center A, then, as $\mathrm{A} \notin u$, the restriction $p: u \to \mathrm{S}$ of π to u is defined. The second step of the proof consists in *extending* p to a continuous map $\bar{p}: t \cup u \to \mathrm{S}$. As nothing is known of the connected set t, p is in fact extended to a continuous map $p_0: (\mathrm{G} - \sigma) \cup u \to \mathrm{S}$, and then $\bar{p}$ is the restriction of p_0 to $t \cup u$.

Begin by triangulating the open subset G of J by the usual method, taking a sequence (T_ν) of successive subdivisions of T, whose mesh tends to 0. G_ν is the union of the simplices of T_ν contained in G, and $\mathrm{G}_\nu \subset \mathrm{G}_{\nu+1}$; $g_\nu = \mathrm{G}_{\nu+1} - \mathrm{G}_\nu$ converges uniformly to u, and G is the union of the g_ν. Next define p_0 in two steps:

First take a sufficiently large number r, and define p_0 on the union G'_r of all the g_ν for $\nu \geqslant r$ by projecting each vertex of all g_ν for $\nu \geqslant r$ on H by the orthogonal projection $f: \mathbf{R}^n \to \mathrm{H}$; p_0 is then defined on the vertices of G'_r as the map $\pi \circ f$. Extend it to a piecewise affine map $\mathrm{G}'_r \to \mathrm{S}$ (using barycentric coordinates in the simplices of each T_ν and in simplices of S). This defines p_0 on the *frontier* E_r *in* G of the union G''_r of the G_ν for $\nu < r$.

Next define p_0 on $\mathrm{G} - \sigma$ by extending it "backward," so to speak, from E_r; for each $(n-1)$-simplex σ_r in G''_r with one of its faces τ_r in E_r, assign an arbitrary value in S to the only vertex of σ_r not in τ_r; then p_0 can be extended from τ_r to the whole of σ_r as a piecewise affine map (again using barycentric coordinates in curvilinear simplices). Then p_0 is defined on the union of these simplices σ_r, hence it is known on the frontier E_{r-1} of the union of the remaining simplices of G_r'', and the procedure can be repeated. This would not work if one wanted to define p_0 in the *whole* set G (there would be an "obstruction" in σ); but it does work for $G - \sigma$.

Third step. Let N be the connected component of the open complement of t in the half space D, such that $A \in \bar{N}$. Let T′ be a triangulation of the half space $\bar{D}$, such that the n-simplices of T′ that meet H have as an intersection with H a p-simplex of their frontier ($p \leqslant n - 1$); these intersections form a triangulation T″ of H. Then construct a triangulation in the usual way for the set $N \cup (H - u)$ (open in $\bar{D}$) by taking successive subdivisions T'_ν of T′ with mesh tending to 0, and defining N_ν as the union of the simplices of T'_ν contained in $N \cup (H - u)$; A may always be supposed interior to an $(n - 1)$-simplex σ_0 of that triangulation. By lemma (L), the frontier of N_ν in $\mathbf{R}^n$ is the union of $N_\nu \cap H$ (which contains A) and a union L_ν of pseudomanifolds-with-boundary, and $F_\nu = L_\nu \cap H$ is a union of closed $(n - 2)$-dimensional pseudomanifolds.

For each vertex C of L not in H let C_1 be a point of t at a distance $d(C, t)$ of C, and let $q(C) = \bar{p}(C_1) \in S$; if $C \in F_\nu$, let $q(C) = \pi(C)$. Then extend q to L_ν as a piecewise affine map in S (using barycentric coordinates as above); q is then a continuous map of L_ν into S.

The contradiction needed to end the proof consists in computing, for sufficiently large values of ν, the degree of the restriction $q|F_\nu$ (as a mapping into S) in two different ways, using the fact that F_ν is *both* the intersection $L_\nu \cap H$ and the frontier of $\bar{N}_\nu \cap H$ in H. For the first computation take ν large enough; for any $(n - 1)$-simplex σ_1 in L_ν, $q(\sigma_1)$ is then contained in a half sphere of S (depending on σ_1). The degree of the restriction of q to the boundary of σ_1 is then 0. By the additivity of the degree and the fact that any $(n - 2)$-simplex of L_ν is the face of two $(n - 1)$-simplices except those in F_ν the degree of $q|F_\nu$ is 0.

For the second computation, consider the $(n - 1)$-simplices of $\bar{N}_\nu \cap H$; it may be assumed that they are so small that, with the exception of σ_0 (which contains A), their images by π each belong to a half sphere of S; the degree of the restriction of π to the boundary of each such simplex is therefore 0. The additivity of the degree then shows that the degree of $q|F_\nu = \pi|F_\nu$ is the same as the degree of the restriction of π to the boundary of σ_0, and it is clear that the latter is ± 1.

§ 5. The Notion of Dimension for Separable Metric Spaces

The theorem on the invariance of dimension (§ 1) did not give a definition of the word "dimension" as a number attached to a topological space and invariant under homeomorphisms except for spaces locally homeomorphic to $\mathbf{R}^n$ ("pure" C^0 manifolds), and even for these spaces the introduction of the auxiliary space $\mathbf{R}^n$ was not satisfactory for a notion that should have been an intrinsic one. This incongruity was stressed by Poincaré in 1903 [371] and again in 1912, the last year of his life [372], in articles written for a nonmathematical public. He pointed out that, just as in classical geometry, one thought of a surface as "limiting" a solid, a curve as "limiting" a surface, and a point as "limiting" a curve, it should be possible to define the "dimension" of a connected space by an *inductive process*: the dimension should be *one* if the space may be disconnected by points, *two* if it may be disconnected by sets of dimension 1, *three* if it may be disconnected by sets of dimenion 2, "and so on."

Meanwhile, in October 1910, Lebesgue, who had heard from Blumenthal of Brouwer's proof of the invariance of dimension (then in the process of being published in *Mathematische Annalen*, of which Blumenthal was one of the editors) sketched, in a letter to Blumenthal (which the latter published immediately after Brouwer's proof) another proof, based on a completely new and remarkable idea ([293], pp. 170–171). Observing that for a covering of a plane domain by sufficiently small closed "bricks" there always are points of the domain belonging to at least three bricks, he stated as a theorem that for any finite covering (E_j) of an open bounded connected set D in $\mathbf{R}^n$ by sufficiently small closed sets there always are points in D belonging to at least $n + 1$ sets. He added that for a cube D it is always possible to find a finite covering by arbitrary small parallelotopes for which no point of D belongs to more than $n + 1$ sets of the covering (both statements of course together imply the invariance of dimension).

This last part was easy enough to show by a simple arrangement of "bricks" in the cube D; but although Lebesgue's sketch of a proof for the first statement was later seen to be capable of yielding a correct argument, the way in which he tried to apply it led to incorrect statements, as Brouwer almost immediately observed. The proof is easily reduced to the case in which D is the cube $[0, 1]^n$, and the E_j are unions of closed cubes of side $1/2^\nu$, having as vertices points of $2^{-\nu}\mathbf{Z}^n$ for sufficiently large ν; it is only necessary to suppose that no E_j meets *both* opposite faces C_i, C_i' of D (defined, respectively, by $x_i = 0$ and $x_i = 1$) for $1 \leqslant i \leqslant n$. Lebesgue's idea was to inductively construct nonempty closed sets $K_1 \supset K_2 \supset \cdots \supset K_n$, for which it could be proved that each K_h contains points belonging to at least $h + 1$ sets E_j (cf. [261], p. 43). He thought he could define the K_h by taking the union G_1 of those E_j that meet C_1, and letting K_1 be a connected component of the frontier of G_1 in $\mathbf{R}^n$ contained in D, different from C_1 and meeting *both* C_i and C_i' for $2 \leqslant i \leqslant n$. He could then take the union G_2 of those E_j not contained in G_1 and meeting both K_2 and C_2, and let K_2 be a connected component of the frontier of $G_2 \cap K_1$, not contained in C_2 and meeting *both* C_i and C_i' for $3 \leqslant i \leqslant n$. Lebesgue claimed he could proceed inductively in this way (without giving any detail) to define the K_h; however Brouwer found a counterexample (for $n = 3$) where Lebesgue's procedure does not yield any set K_3 having the properties he claimed ([89], p. 545). It was only in 1921 that Lebesgue published a correct proof of his theorem ([295], pp. 177–206).

In the meantime Brouwer had taken up Poincaré's idea in 1913, and had given it mathematical content ([89], pp. 540–546). He first observed that

Poincaré's tentative definition had to be slightly modified to really conform to intuition*: if one deletes the vertex of a cone with two sheets in $\mathbf{R}^3$, the cone is disconnected although no one would consider its dimension to be 1! For a space E,† Brouwer said that two disjoint closed sets F, F′ are *separated* by a set C if any connected subset of E that meets F and F′ also meets C‡; he then defined a space of *dimension* 0 as one containing no connected set with more than one point, and a space E of *dimension* $n > 0$ by the property that n is the smallest integer > 0 such that *any two* disjoint closed subsets of E are separated by a subset of dimension $\leqslant n - 1$. That definition can immediately be localized: a space E has *dimension n at a point* P if P has a fundamental system of neighborhoods of dimension n.

The bulk of Brouwer's paper is devoted to proving that, with his definition of dimension, $\mathbf{R}^n$ has dimension n at every point. By induction on n, it is easy to show that this dimension is $\leqslant n$. To prove that it is $\geqslant n$, an argument similar to Lebesgue's reduced the proof to a simplicial version of Lebesgue's theorem:

(S) Let $\sigma = A_1 A_2 \cdots A_{n+1}$ be an n-simplex in $\mathbf{R}^n$, and consider a triangulation T of σ in rectilinear simplices, none of which meets *both* $A_1 A_2 \cdots A_\nu$ and $A_{\nu+1} A_{\nu+2} \cdots A_{n+1}$, for any $\nu \leqslant n$. Define σ_j inductively for $1 \leqslant j \leqslant n$ by letting γ be the subcomplex of T, union of all the n-simplices of T having A_1 as one of their vertices; lemma (L) of §3 shows that γ is a pseudomanifold-with-boundary; the part σ_1 of that boundary, the union of the $(n-1)$-simplices that does not contain A_1, is a union of closed pseudomanifolds, and $A_1 \notin \sigma_1$. In general, σ_ν is defined by induction on $\nu \leqslant n$: let γ_ν be the subcomplex of σ_ν, union of the $(n-\nu)$-simplices of σ_ν that meet $A_1 A_2 \cdots A_{\nu+1}$, but do not meet $A_1 A_2 \cdots A_\nu A_{\nu+2} \cdots A_{n+1}$; this is again a pseudomanifold-with-boundary; the part $\sigma_{\nu+1}$ of that boundary which is the union of the $(n-\nu-1)$-simplices of σ_ν that do not meet $A_1 A_2 \cdots A_\nu$ is a union of closed pseudomanifolds. Then the σ_ν, which form a decreasing sequence of sets, are all *nonempty*.

The proof uses the properties of the degree of a map, and, as usual, is very sketchy and has to be interpreted to make sense. Let π_ν be the projection of $\sigma - (A_1 A_2 \cdots A_\nu)$ onto $A_{\nu+1} A_{\nu+2} \cdots A_{n+1}$ [$\pi_\nu(M)$ being the intersection of $A_{\nu+1} \cdots A_{n+1}$ with the ν-dimensional linear affine variety generated by M and $A_1 A_2 \cdots A_\nu$]. Let p_ν be the restriction of π_ν to σ_ν.

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* A similar observation had already been made by Riesz [396].

† This paper is the only one of the period 1911–1913 in which Brouwer considers general topological spaces. He says his spaces must be "Normalmenge in Fréchetschen Sinne" (?) but does not use any property beyond the definition of a topological space.

‡ In the paper as it was first published, he had written "closed connected subset" instead of "connected subset"; after Urysohn had pointed out to him that this definition was incompatible with the proof of the main theorem of Brouwer's paper, the latter published in 1923 a corrected version ([89], p. 547), which he elaborated in a 1924 paper ([89], pp. 554–557).

The induction starts with the obvious remark that the degree of p_1 is equal to 1.* The main point of the proof is to show that if the degree of p_ν is 1, so is the degree of $p_{\nu+1}$; this of course implies that $\sigma_\nu \neq \varnothing$ for all ν.

The passage from ν to $\nu + 1$ is done by considering each $(n - \nu - 1)$-simplex of $\sigma_\nu \cap (A_1 \cdots A_\nu A_{\nu+2} \cdots A_{n+1})$, which is the face of a unique $(n - \nu)$-simplex of σ_ν; it follows easily, by a continuity argument, that the restriction of $p_{\nu+1}$ to $\sigma_\nu \cap (A_1 \cdots A_\nu A_{\nu+2} \cdots A_{n+1})$, considered as a mapping into $A_{\nu+2} \cdots A_{n+1}$, has degree 1. On the other hand, the restriction of $p_{\nu+1}$ to the frontier of each $(n - \nu)$-simplex meeting $A_1 \cdots A_\nu A_{\nu+2} \cdots A_{n+1}$ has degree 0. By additivity of the degree, it follows that, deleting all these simplices from σ_ν, which by definition gives as remnant $\gamma_{\nu+1}$, the restriction of $p_{\nu+1}$ to $\sigma_{\nu+1}$ has degree 1.

§ 6. Later Developments

The first complete proofs of the "no separation" (§ 4) and Jordan–Brouwer (§ 3) theorems entirely devoid of the obscurities linked to the fantastic complexity of Brouwer's constructions were given by Alexander in 1922. They constitute the first and second steps, respectively, in the proof of his duality theorem (Part 1, chap. II, § 6). As we have seen, these proofs, based on convenient splittings of a cube or a sphere, are reminiscent of the (later) Mayer–Vietoris theorems. Indeed the use of the general Mayer–Vietoris exact sequence in cohomology (Part 1, chap. IV, § 6) very easily determines the whole de Rham cohomology $H^{\cdot}(\mathbf{R}^n - X)$ (Part 1, chap. III, § 3) when X is homeomorphic to a cube or to a sphere, and the "no separation" and Jordan–Brouwer theorems are just consequences of the computation of $H^0(\mathbf{R}^n - X)$.

Another way of obtaining these theorems was used by Leray [324] who proved a general result containing both as special cases†: if K and K′ are two homeomorphic compact subsets of $\mathbf{R}^n$, then $\mathbf{R}^n - K$ and $\mathbf{R}^n - K'$ have the same cardinal number (finite or infinite) of connected components. This follows from the multiplicative property of the localized degree [chap. I, § 3, formula (13)] and the purely algebraic property of invariance of (linear) dimension of a vector space over **Q**.

Although Brouwer gave a definition of the notion of dimension applying to arbitrary spaces, he was obviously chiefly interested in proving that for $\mathbf{R}^n$ that definition gives the number n. This is probably the reason why his paper was considered merely another way of proving the invariance of dimension, and the fact that he had given a general definition of dimension was neglected. At any rate, when in 1922 Urysohn and Menger proposed (independently of

* As a simplex is not a "manifold" in Brouwer's sense, it is in fact the localized degree $d(p_1 . I, M)$ which is equal to 1, where I is the interior of the simplex $A_2 A_3 \cdots A_{n+1}$ and M is a point of I. Similarly for the p_ν, $\nu \geqslant 2$.

† It also contains the "invariance of closed curves" that Brouwer had attempted to prove ([89], pp. 523–526).

each other) a definition that is equivalent to Brouwer's for locally connected or compact separable metric spaces, they were at first unaware of Brouwer's priority.

The Urysohn–Menger definition applies to *all separable metric spaces*. For them the empty set has dimension -1, and the dimension of a nonempty space is the least integer $n \geqslant 0$ for which every point has a fundamental system of neighborhoods whose boundaries have dimension $< n$ (the dimension is taken to be $+\infty$ if there is no such integer n).*

This definition's consequences were studied in the period, extending to about 1940, during which dimension theory became a very active branch of mathematics. But apart from the Brouwer theorem on the dimension of $\mathbf{R}^n$ the methods of proof in that theory belonged to general (also called "set-theoretic") topology and made no use of triangulations or homology.† We will therefore not describe all the results of that theory, but refer the reader to [261]. Some of results, however have interesting connections with algebraic topology.

First, Lebesgue's theorem furnishes (for separable metric spaces) an alternative definition of dimension. The *order* of a finite open covering $\mathfrak{R}$ of a space E is the largest integer p such that there exists $p+1$ distinct sets of $\mathfrak{R}$ with nonempty intersection. If $m(\mathfrak{R})$ is the g.l.b. of the orders of the finite open coverings of E *finer* than $\mathfrak{R}$, Lebesgue's theorem says that for $\mathbf{R}^n$ the l.u.b. of the $m(\mathfrak{R})$ for all finite open coverings $\mathfrak{R}$ of $\mathbf{R}^n$ is equal to n. For a general space E this l.u.b. is the dimension of E as defined by Urysohn and Menger.

From this it follows at once that for a separable metric space E of dimension n, the *Čech homology groups* $\check{\mathrm{H}}_p(\mathrm{E};\mathrm{G})$ based on *finite* open coverings (Part 1, chap. IV, § 2) are all 0 for $p > n$. Surprisingly enough this is not true for *singular* homology groups: there exist compact metric spaces of finite dimension for which *infinitely many* singular homology groups are $\neq 0$ [44]. On the other hand, there are obviously contractible compact spaces of any finite dimension, so that there are no very strong links between dimension and homology of a space. In Part 3, chap. II, we shall see that homotopy theory is much closer to the notion of dimension.

With the arrival of sheaf cohomology (Part 1, chap. IV, § 7,C), another notion of "dimension" of a space could be defined. A space X, on which is given a family Φ of supports (Part 1, chap. IV, § 7,C), has *finite* Φ-*dimension* if there is an integer $n \geqslant 0$ such that

$$\mathrm{H}^i_\Phi(\mathrm{X};\mathscr{F}) = 0 \quad \text{for every } i > n \text{ and } \textit{every sheaf } \mathscr{F} \text{ over X;} \tag{2}$$

the smallest integer n having that property is called the Φ-*dimension* of X and

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* Brouwer's definition differs from that of Urysohn–Menger because he takes totally disconnected spaces to have dimension 0, whereas for the Urysohn–Menger definition, there are totally disconnected spaces of arbitrary finite dimension ([261], p. 23).

† Brouwer's proof was later replaced by a purely combinatorial lemma of Sperner ([30], p. 376).

written $\dim_\Phi X$; when Φ is the family of all closed sets in X, n is called the *cohomological dimension* of X (or simply *dimension*) if no confusion arises. If $\Phi_1 \supset \Phi_2$ are two families of supports on X, and if X has Φ_1-dimension $\leqslant n$, then it has Φ_2-dimension $\leqslant n$. If Y is a subset of X that is *locally closed*, and if X has Φ-dimension $\leqslant n$, then Y has Φ'-dimension $\leqslant n$, where $\Phi' = \Phi \cap Y$. When X is *metrizable* and has cohomological dimension $\leqslant n$, the same is true for *every subset* of X. For a *paracompact* space X to have cohomological dimension $\leqslant n$ it is necessary and sufficient that each point of X have a neighborhood of cohomological dimension $\leqslant n$. If a compact metric space has dimension $\leqslant n$ in the sense of Urysohn–Menger, it also has a cohomological dimension $\leqslant n$ ([66], [87]).

The condition (2) may be restricted by considering only sheaves $\mathscr{F}$ of modules over a fixed Dedekind ring Λ; if (2) holds for all such sheaves $\mathscr{F}$ and *all* paracompactifying families Φ, one says the dimension of X *over* Λ is $\leqslant n$, and the smallest integer n having that property is the *dimension of* X *over* Λ, written $\dim_\Lambda X$; it is also the smallest integer n for which the cohomology with compact supports $H_c^{n+1}(U; \Lambda) = 0$ for all *open* subsets U of X [208]. When $\dim_\Lambda X \leqslant n$, the *Borel–Moore homology* (Part 1, chap. IV, § 7,F) satisfies

$$H_q^\Phi(X; \Lambda) = 0 \quad \text{for } q \geqslant n + 1 \text{ and any family } \Phi \text{ of supports;} \tag{3}$$

$$H_q^x(X; \Lambda) = 0 \quad \text{for } q \geqslant n + 1 \text{ and all } x \in X. \tag{4}$$

If $\mathscr{F}$ is the constant sheaf Λ, or if Φ is paracompactifying, there is a canonical isomorphism

$$H_n^\Phi(X; \mathscr{F}) \simeq \Gamma_\Phi(\mathscr{H}_n(X; \Lambda) \otimes \mathscr{F}) \quad \text{([66], pp. 151–152).} \tag{5}$$

CHAPTER III

Fixed Points

§ 1. The Theorems of Brouwer

Brouwer had been considering continuous maps of the sphere $\mathbf{S}_2$ into itself as early as 1909; he first studied the particular case of a *bijection** f (which is therefore bicontinuous) preserving orientation, and he gave a proof that in that case there exists at least one *fixed point* x for f, i.e., such that $f(x) = x$; the proof is very long (nine pages) and involved, using intricate arguments on deformations of curves on $\mathbf{S}_2$ ([89], pp. 195–205). In 1910 he gave another proof of the same result as a corollary to the existence of at least one singular point for a continuous vector field on $\mathbf{S}_2$ (§ 3) by another intricate argument ([89], pp. 303–318).

It was only in 1911, in the paper in which he gave the definition of the degree of a map (chap. I, § 2), that he realized that this notion could be used to prove that a continuous map f of $\mathbf{S}_n$ into itself, satisfying *the only condition that* $\deg(f) \neq (-1)^{n+1}$, has at least one fixed point. Equivalently, he showed that if f has *no* fixed point, then $\deg(f) = (-1)^{n+1}$; but his first proof is far from simple, and uses the computation (done earlier in that paper) of the sum of the indices of a continuous vector field on $\mathbf{S}_n$ having only isolated singularities (see § 3). Fixing a point O on $\mathbf{S}_n$, he considered, for every point P $\neq$ O for which $f(\mathrm{P}) \neq \mathrm{O}$, the unit vector tangent at P to the arc of the circle through O, P and $f(\mathrm{P})$ having extremities at P and $f(\mathrm{P})$ and not containing O.† To apply his theorem on vector fields, he had to define the vector field in the neighborhood of O and of the points of $f^{-1}(\mathrm{O})$ where the previous definition is meaningless.

Finally, in the next paper he published in 1911 ([89], pp. 454–472), Brouwer arrived at a very simple proof without using vector fields: if f has no fixed point, the consideration of the great circle joining x and $f(x)$ at once provides a *homotopy* of f to the antipodal map $x \mapsto -x$ for which the degree is obviously $(-1)^{n+1}$.

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* Brouwer only assumed that f is injective, but by degree theory (which he had not invented at that time) it follows that f is necessarily bijective.

† He had already used that device in 1910 for $n = 2$ ([89], p. 315).

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Being linked to the as yet unfamiliar notion of degree, this result did not attract much attention from the mathematicians of that time. Things were quite different for the corollary Brouwer added concerning a continuous map g of a *cube* I^n into itself. He showed that such a map *always* has at least one fixed point. His argument consisted in replacing I^n by the homeomorphic upper hemisphere D_+ of the sphere S_n and extending g to a continuous map $f: \mathrm{S}_n \to \mathrm{D}_+$ by taking $f(x) = g(s(x))$ in the lower hemisphere D_-, s being the symmetry with respect to the equator; then $\deg(f) = 0$ since f is not surjective, and a fixed point of f must of necessity be a fixed point of g. The interest aroused by this result was due to its unexpected generality, which made possible its application to *existence* proofs in analysis, using much weaker assumptions than had been customary in earlier existence theorems; later it was realized that Brouwer's fixed point theorem could even be used in infinite-dimensional spaces, under assumptions allowing suitable approximations by finite dimensional compact sets (see chap. VII).

§ 2. The Lefschetz Formula

It is clear that for a continuous map f of a compact space X into itself the existence of fixed points will in general depend not only on the space X, but on f itself (the Brouwer case $\mathrm{X} = \mathrm{I}^n$ being an exception). This fact was given precise expression in a remarkable formula discovered by Lefschetz in 1926 [300].

Lefschetz limited himself to a *combinatorial manifold* X (Part 1, chap. II, § 4), but considerably enlarged the concept of "fixed point." He first observed that it was a special case of "coincidences" for two continuous maps f, g of X into itself, namely, the points $x \in \mathrm{X}$ such that $f(x) = g(x)$. As he was at that time working on the topology of product spaces, he translated that notion in terms of the *graphs* $\Gamma(f)$ and $\Gamma(g)$ of f and g in the product space $\mathrm{X} \times \mathrm{X}$ which is also a combinatorial manifold: a "coincidence" is the first projection in X of a *common point* of $\Gamma(f)$ and $\Gamma(g)$. Lefschetz was thus led back to a problem of *intersection*, a question on which we have seen he was also working (Part 1, chap. II, §§ 4 and 5).

It is quite obvious that he was strongly influenced by the similar problems in algebraic geometry, and in particular by the theory of *correspondences*, studied since the middle of the nineteenth century by Chasles and the school of "enumerative geometry" (de Jonquières, Zeuthen, Schubert), then by Hurwitz in the theory of Riemann surfaces, and which had been thoroughly investigated by Severi in the first years of the twentieth century; this influence explains the rather unusual frame within which Lefschetz developed his theory.

Let X be a compact, connected, orientable combinatorial manifold without boundary, of dimension n, Lefschetz studied that he calls a "transformation" T in X, by which he means an *n-cycle* Γ_{T} *in the product space* $\mathrm{X} \times \mathrm{X}$. If T′ is a second "transformation" in X, the algebraic intersection number* $(\Gamma_T . \Gamma_{T'})$ is defined (Part 1, chap. IV, §4). Once homology bases (distinct or not), (γ_p^i), (δ_p^j), are known for $H_.(X; Q)$, as well as their multiplication table in the "intersection ring" $H_.(X \times X; Q)$, the $\gamma_p^i \times \delta_q^j$ for $p + q = r$ form a base of $H_r(X \times X; Q)$ by Künneth's theorem, and the intersection products of these elements in $H_.(X \times X; Q)$ are given by formula (30) of Part 1, chap. II, § 5. The number $(\Gamma_T . \Gamma_{T'})$ could therefore be computed at once from the expressions

$$\Gamma_T = \sum_{0 \leqslant p \leqslant n} \varepsilon_p^{ij}(\gamma_p^i \times \delta_{n-p}^j), \qquad \Gamma_{T'} = \sum_{0 \leqslant p \leqslant n} \varepsilon_p'^{ij}(\gamma_p^i \times \delta_{n-p}^j). \tag{1}$$

But Lefschetz's original idea was to look for another computation of that number by introducing *actions* of T and T′ on the homology groups $H_p(X; Q)$. Even before singular homology had been defined, it was possible to associate to every continuous map $f: X \to Y$ of finite cell complexes, a homomorphism

$$f_*: H_.(X; Q) \to H_.(Y; Q)$$

of graded vector spaces, by simplicial approximation (Part 1, chap. II, §3). Lefschetz [probably inspired by similar processes in algebraic geometry, the images of divisors by correspondences (see [299])], extended this idea to his "transformations." He considered a homology class $\alpha_p \in H_p(X; Q)$ and its product $\alpha_p \times [X]$ by the fundamental class of X (chap. I, §3,A) in $H_{p+n}(X \times X; Q)$; its intersection $\Gamma_T . (\alpha_p \times [X])$ with Γ_T is a class in $H_p(X \times X; Q)$, and the image of that class by the homomorphism $(\mathrm{pr}_2)_*$ in $H_p(X; Q)$ is, by definition, the image $T_*(\alpha_p)$ by the action of T on $H_p(X; Q)$.

From his intersection theory (Part 1, chap. II, §4), Lefschetz deduced the fundamental result

$$(\Gamma_T . (\gamma_p^i \times \delta_{n-p}^j)) = (-1)^p (T_*(\gamma_p^i) . \delta_{n-p}^j) \tag{2}$$

which gave him the expressions of the ε_p^{ij} as linear forms in the coefficients of the matrix (α_p^{ij}) of the homomorphism $(T_*)_p$, the restriction of T_* to $H_p(X; Q)$. From these expressions he derived the expression of $(\Gamma_T . \Gamma_{T'})$ as function of the matrices of the $(T_*)_p$ and $(T'_*)_q$. He did not at first express this formula in terms of traces of matrices, but in a second paper [301] he obtained such an expression, and in particular when T′ is the identity (so that $\Gamma_{T'}$ is the *diagonal* Δ of $X \times X$, which is an n-cycle), he arrived at the famous *Lefschetz formula*

$$(\Gamma_T . \Delta) = \sum_{0 \leqslant p \leqslant n} (-1)^p \operatorname{Tr}((T_*)_p). \tag{3}$$

When the cycle Γ_T and the diagonal Δ intersect "transversally" in a finite number of points, the left-hand side of (3) could be interpreted as the "algebraic number of fixed points" of the "transformation" T.

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* We abuse language by writing an intersection number for *cycles* instead of writing it for their *homology classes*.

In 1928 Hopf returned to the initial problem of the existence of fixed points for an arbitrary continuous *map* $f: X \to X$, but this time he considered not merely a combinatorial manifold X, but an *arbitrary* finite euclidean simplicial complex of dimension n. He associated to such a map, according to (3), what came to be called the *Lefschetz number* of f

$$\Lambda(f) = \sum_{0 \leqslant p \leqslant n} (-1)^p \operatorname{Tr}((f_*)_p) \tag{4}$$

and he proved first that if f has *no* fixed point, then $\Lambda(f) = 0$.

As X is compact, the assumption implies that $|f(x) - x| \geqslant \delta > 0$ for all $x \in X$. There is therefore a subdivision K of the triangulation of X and a simplicial approximation g of f for that triangulation, homotopic to f and such that $|g(x) - x| \geqslant \delta/2 > 0$ for $x \in X$; since $g_* = f_*$, it is enough to prove the theorem for g instead of f. If $(\sigma_j)_{1 \leqslant j \leqslant \alpha_p}$ is the canonical basis of the **Q**-vector space $C_p(K)$ of the p-chains of K, and if the diameters of the simplices of K are small enough, the endomorphism $\tilde{g}_p$ of $C_p(T)$ corresponding to g (Part 1, chap. II, § 3) is such that

$$\tilde{g}_p(\sigma_j) = \pm\sigma_k \quad \textit{for an index } k \neq j \text{ if } g(\sigma_j) \text{ is not degenerate,}$$

$$\tilde{g}_p(\sigma_j) = 0 \qquad \text{otherwise;}$$

this implies that $\operatorname{Tr}(\tilde{g}_p) = 0$. From this Hopf concluded that all he had to do was prove the formula that he rightly considered the natural generalization of the Euler–Poincaré formula [Part 1, chap. I, § 3, formula (4)]:

$$\sum_{p=0}^{n} (-1)^p \operatorname{Tr}(\tilde{g}_p) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr}((g_*)_p) \tag{5}$$

for every simplicial map $g: X \to X$; it reduces to the Euler–Poincaré formula when g is the identity. The proof is similar, using the fact that $\tilde{g}_p(Z_p) \subset Z_p$, $\tilde{g}_p(B_p) \subset B_p$ for cycles and boundaries and that

$$\operatorname{Tr}(\tilde{g}_p) = \operatorname{Tr}(\tilde{g}_p | Z_p) + \operatorname{Tr}(\tilde{g}_{p-1} | B_{p-1}),$$

$$\operatorname{Tr}((g_*)_p) = \operatorname{Tr}(\tilde{g}_p | Z_p) - \operatorname{Tr}(\tilde{g}_p | B_p).$$

When $\Lambda(f) \neq 0$ and X is again a *combinatorial manifold*, so that (3) is applicable for $T = f$, Hopf gave an interpretation of the left-hand member when f has only a finite number of fixed points, by defining for each fixed point a of f an *index* j_a, the definition of which is meaningful for any C^0 manifold, triangulable or not. Consider a homeomorphism h of an open neighborhood of a in X [with $h(a) = 0$], onto an open neighborhood of 0 in $\mathbf{R}^n$; then, for sufficiently small $\rho > 0$, $g = hfh^{-1}$ is defined in the ball $\mathbf{B}$: $|x| \leqslant \rho$ and is a continuous map $\mathbf{B} \to \mathbf{R}^n$, with only one fixed point 0. The map $x \mapsto g(x)/|g(x)|$ is defined on S: $|x| = \rho$ and maps S into $\mathbf{S}_{n-1}$, so that its *degree* is defined; it is independent of ρ and of the choice of the homeomorphism h, and its value is by definition the index j_a. Hopf's interpretation of (3) for $T = f$ is then

$$\sum_{a \in \mathrm{Fix}(f)} j_a = \Lambda(f), \tag{6}$$

$\mathrm{Fix}(f)$ being the finite set of fixed points of f.

Hopf's first proof of (6) ([241a], p. 153) is particularly interesting. In the neighborhood of a fixed point a, he modified both the cell complex X and the map f. One may assume that a is contained in an n-simplex σ, of frontier τ, and (with the preceding notation) the homeomorphism h maps $\bar{\sigma}$ onto $\mathbf{B}$ and τ onto S; a homotopy can modify f in a neighborhood of a in such a way that $f(\tau)$ does not meet $\bar{\sigma}$. Then Hopf added a new n-simplex σ' to X by gluing it to σ along τ in such a way that $\bar{\sigma} \cup \sigma'$ becomes homeomorphic to $\mathbf{S}_n$, τ being mapped on the "equator" $\mathbf{S}_{n-1}$. Transferring to $\bar{\sigma} \cup \sigma'$ the symmetry with respect to the equator gives an automorphism s of $\bar{\sigma} \cup \sigma'$, exchanging σ and σ' and leaving the points of τ invariant. Next Hopf changed f in $\bar{\sigma}$, replacing it by $\bar{f} = s \circ f$, and defined $\bar{f}$ in σ' equal to $f \circ s$.

Doing this for every fixed point of f yields a cell complex X′ and a continuous map $\bar{f}$ of X′ into itself with *no fixed point*; $\Lambda(\bar{f}) = 0$; but the construction gives the relation

$$\Lambda(\bar{f}) = \Lambda(f) - \sum_{a \in \mathrm{Fix}(f)} j_a$$

hence the result. This is one of the first examples of the use of *attachment* of new cells to a cell complex that later became an important tool (see chap. V, § 3).

Hopf's second proof [241b] starts from a triangulation T of X such that all the fixed points of f belong to n-simplices. He refined T to a sufficiently iterated subdivision T′, for which he constructed a simplicial approximation g homotopic to f, such that there are *no fixed r-simplices* of T′ for g when $r < n$; then $\mathrm{Tr}(\tilde{g}_r) = 0$ for $r < n$ and $\mathrm{Tr}(\tilde{g}_n) = \sum_{a \in \mathrm{Fix}(f)} j_a$, so that formula (6) becomes a consequence of (5).

Lefschetz endeavored to generalize his formula to compact metric spaces using Vietoris homology, but Hopf provided him with an example of a compact subset X of $\mathbf{R}^2$ and a continuous map without fixed point for which $\Lambda(f) \neq 0$ both for singular and Vietoris homology: X is the union of two concentric circles and a spiral curve winding between both and asymptotic to each of them, whereas f is just a rotation of a fixed angle ω for points on each circle and on the spiral ([304], p. 347). Later Lefschetz realized that the validity of the formula could be recovered by making assumptions on the "local connectedness in the sense of homology" on X (cf. chap. IV) [46].

§ 3. The Index Formula

We have already mentioned (chap. I, § 2) that in 1881 Poincaré, in his work on global theory of differential equations, had introduced the notion of *index* for a vector field on the sphere $\mathbf{S}_2$. He was studying in $\mathbf{R}^2$ the integral curves of a differential equation

$$\frac{dx}{X} = \frac{dy}{Y}$$

where X and Y are *polynomials*. He took a point O in $\mathbf{R}^3$ not in the plane, and projected from O the vector field (X, Y) on a sphere S of center O, extending it by continuity on the "equator" of S (section by the plane parallel to $\mathbf{R}^2$). This gave him a vector field on S, symmetrical with respect to O. He showed that there were always at least *two* (*symmetrical*) *singular points* of the field (i.e., points where the field vanishes). Then he restricted himself to "general" such fields in the following sense: (1) X and Y have the same degree m; (2) if X_m, Y_m are their homogeneous parts of degree m, $xY_m - yX_m$ is not identically 0; (3) the curves $X = 0$ and $Y = 0$ intersect transversally in points not on the equator; (4) the roots of the homogeneous equation $xY_m - yX_m = 0$ are simple.

Next Poincaré introduced the notion of *index* of *any* closed curve on an hemisphere of S containing no singular point: if h (resp. k) is the number of points where Y/X passed from $-\infty$ to $+\infty$ (resp. from $+\infty$ to $-\infty$) when moving on the (positively oriented) curve, the index is defined as $i = (h - k)/2$. He showed that $i = \pm 1$ for a small enough curve around a singular point, and took that value as the *index of the singular point*; he then proved the remarkable result that the *sum* of the indices of the singular points is *equal to* 2 ([365], p. 29).

In 1909 Brouwer, who at that time probably was not aware of Poincaré's paper, considered a vector field on $\mathbf{S}_2$ that he only supposed *continuous* (whereas in Poincaré's case, the field is *analytic* at nonsingular points); he wanted to prove that there exists *at least one* singular point. He argued by contradiction, using the detailed study of the trajectories of the vector field (he could not use local uniqueness since the field is not supposed to be C^1) ([89], p. 279).

In his 1911 paper on the definition of the degree ([89], pp. 454–472) Brouwer considered, for any n, a vector field on $\mathbf{S}_n$ that he merely supposed continuous, with at most finitely many singular points; he proceeded to prove that the sum of the indices of the singular points is 2 for even n, 0 for odd n. To apply his definition of the degree to that problem he used a very complicated and obscure process, starting from a simplicial triangulation T of $\mathbf{S}_n$ obtained by intersections of $\mathbf{S}_n$ with hyperplanes, among which is the equator; T is supposed symmetrical with respect to the equator and the singular points of the vector field are all contained in the interior of n-simplices of T. If T is fine enough, the sum of the indices of the singular points of the vector field is given by a sum of degrees of maps, written $c_{1\alpha}$ and $c_{2\alpha}$. To define $c_{1\alpha}$, project each n-simplex $s_{1\alpha}$ of T in the northern hemisphere stereographically on the tangent hyperplane H_1 at the north pole, consider the map of the frontier of the projected simplex in $\mathbf{S}_{n-1}$ given by the (stereographically projected) vector field in H_1, and take its degree $c_{1\alpha}$; do the same for the southern hemisphere, stereographically projected on the tangent hyperplane H_2 at the south pole, to get the degrees $c_{2\alpha}$. Brouwer showed that, owing to the symmetry of T with respect to the equator, the sum of the degrees $c_{1\alpha}$ and $c_{2\alpha}$ (for all n-simplices of T) reduces to the sum of the degrees of two maps of the equator $\mathbf{S}_{n-1}$ into itself. He then claimed that the computation of that sum could be reduced to the case of a *constant* vector field on $\mathbf{S}_{n-1}$, but his description of what he does to reach that result is so sketchy and intricate that it is hard to decide if his procedure really constitutes a proof.

In 1925 ([238], p. 2) Hopf announced that Brouwer's theorem for vector fields on $\mathbf{S}_n$ would generalize to arbitrary compact "manifolds" X: for a continuous vector field on X, with finitely many singular points, the sum of the indices of these points is equal to the *Euler–Poincaré characteristic*. Hopf indicated that this result could be derived from the theory of fixed points of continuous maps. Alexandroff and Hopf showed in their book ([30], p. 549) how this can be done very simply for a C^1 *manifold* X and a C^1 *vector field* Z on X by considering the *flow* $(x, t) \mapsto F_Z(x, t)$ of Z. Recall that this is defined for all $x \in X$ and all $t \in \mathbf{R}$; if $v(t) = F_Z(x, t)$, $t \mapsto v(t)$ is the integral curve of the field Z starting from $x = v(0)$, i.e., $v'(t) = Z(v(t))$. A compactness argument shows that there is an interval $|t| \leqslant \varepsilon$ such that the fixed points of the map $x \mapsto F_Z(x, t)$ are exactly the singular points of Z for any t in that interval, with the same indices. Since that map is also obviously homotopic to the identity, the result follows from formula (6). It can be generalized to a vector field Z on a C^1 manifold X that is merely supposed continuous, for such a field is homotopic to a C^1 vector field with the same singular points.

The notion of vector field on X is not clearly defined for a combinatorial manifold X, since there may be several distinct differential structures on X (or none at all) compatible with the topology. In 1928 [240] Hopf considered vectors attached to each point of X and satisfying conditions depending not only on the topology of X but on its triangulation, and he proved that they still satisfy the index formula.

CHAPTER IV

Local Homological Properties

§ 1. Local Invariants

Local properties of topological spaces were considered at the beginning of the twentieth century, chiefly by Schoenflies, who was a pioneer in that matter. They were mainly studied for subsets of $\mathbf{R}^2$, and without any intervention of homological notions. Examples of these properties are accessibility and "Unbewaltheit," which we saw developed by Brouwer using simplicial methods but still no homology (chap. II, § 3,C). After 1910 the concept of *local connectedness** was also the theme of many papers in "point-set" (or "analytic") topology (see [517] and [518], chap. I).

The fact that all contractible spaces have the same homology showed that homology is a very coarse notion to use for the description of properties of a space invariant under homeomorphism. At the end of the 1920s the idea emerged that, just as global connectedness of a space is a property that gives very little information, and "localizing" it gives much more, so one could perhaps "localize" *homology groups* of any dimension in order to make a deeper study of the topology of a space.

In this chapter, it shall always be understood that "homology group" means reduced homology group (Part 1, chap. IV, §6,E).

The first instance of such ideas probably occurs in print in a footnote of a 1928 paper by Alexandroff ([27], p. 181, note 63), in which he introduces the notion of "r-local connectedness" for any $r \geqslant 0$; we shall examine it in § 2; he mentions that Alexander had considered the same definition but did not publish it.

A. Local Homology Groups and Local Betti Numbers

It was only in 1934 that Alexandroff [28], Čech [122], and Seifert and Threlfall in their book ([421], chap. VIII) independently gave definitions of "local" homology groups or Betti numbers.

* For the many uncertainties and even priority claims to which the notion of connectedness gave rise in the early 1900s, see [89], p. 486.

J. Dieudonné, *A History of Algebraic and Differential Topology, 1900–1960*,
Modern Birkhäuser Classics, DOI 10.1007/978-0-8176-4907-4_10,

Alexandroff only considered compact subspaces of $\mathbf{R}^n$ and Vietoris homology (Part 1, chap. IV, § 2) with rational coefficients; Seifert and Threlfall limited themselves to locally finite simplicial complexes and simplicial homology; Čech gave definitions for arbitrary topological spaces and used Čech homology based on finite open coverings (Part 1, chap. IV, § 2) with coefficients in $\mathbf{Q}$ or in a finite field.

Both Alexandroff and Čech referred to Lefschetz's "relative homology" (Part 1, chap. II, § 6), whereas Seifert and Threlfall gave direct definitions and only mentioned relative homology in a footnote. The natural procedure stemming from relative homology would be to take the relative homology groups $H_p(X, X - \{x\}; G)$ as local invariants at a point $x \in X$ for *some* homology theory (Part 1, chap. IV, § 6,B), and if that theory satisfies the excision axiom (*loc.cit.*) these groups may be replaced by $H_p(V, V - \{x\}; G)$ where V is an arbitrary open neighborhood of x; however, this is not the way the authors mentioned above proceeded.

They attached to any point $x \in X$ an "r-dimensional Betti number $p_r(x)$ at x" for every $r \geqslant 0$, in the following way (reformulated for convenience in the present language, and for *any* homology theory with coefficients in a field). Consider two open neighborhoods $U \supset V$ of x, and the natural map

$$H_r(X, X - U) \to H_r(X, X - V);$$

write $p_{r,U,V}$ the rank of that homomorphism [dimension of the image of $H_r(X, X - U)$], which decreases when V decreases and hence has a limit $p_{r,U}$ (finite integer or $+\infty$) for the directed set $\mathfrak{U}(x)$ of open neighborhoods of x. Furthermore, when U decreases $p_{r,U}$ increases and hence has a limit $p_r(x)$ for the directed set $\mathfrak{U}(x)$. Observe that instead of the dimension $p_{r,U,V}$, the homology groups $H_r(X, X - U)$ themselves may be considered, and one can take direct limits over the directed set $\mathfrak{U}(x)$. The group obtained in that manner is not necessarily isomorphic to $H_r(X, X - \{x\})$.

Suppose x has a fundamental decreasing system of neighborhoods (U_m), such that $X - U_m$ is a strong deformation retract (Part 1, chap. IV, § 6,B) of $X - \{x\}$ and of $X - U_n$ for $n > m$. It then follows from the exact sequence of relative homology [Part 1, chap. IV, § 6,B, formula (94)] that the maps

$$H_r(X, X - U_m) \to H_r(X, X - U_n) \to H_r(X, X - \{x\})$$

are all *bijective*; hence the groups obtained by the preceding limit processes actually are the $H_r(X, X - \{x\})$, which then deserve to be called the *local homology groups* at x.

This is particularly the case when X is a C^0-manifold or a locally finite simplicial complex. In the first case, if the dimension of X at the point x is $n > 0$,

$$\begin{aligned} &H_j(X, X - \{x\}) = 0 && \text{for } j \neq n \\ &H_n(X, X - \{x\}; \Lambda) \simeq \Lambda && \text{for any ring } \Lambda. \end{aligned} \tag{1}$$

In the second case (the only one considered by Seifert and Threlfall), if $\mathrm{St}(x)$ is the star of x for a triangulation of X, $X - \mathrm{St}(x)$ is a strong deformation retract of $X - \{x\}$. As $H_p(\overline{St(x)}) = 0$ for all $p \geqslant 0$, since $\overline{St(x)}$ is contractible,

$$H_p(X, X - \{x\}) \simeq H_{p-1}(K_x) \tag{2}$$

where K_x is the subcomplex $\overline{St(x)} - St(x)$ of the triangulation of X. This is actually the *definition* given by Seifert and Threlfall for the local homology groups, and of course they had to prove it independent of the triangulation of X ([421], pp. 120–125). They used these groups to show that some properties, defined *a priori* with respect to some triangulation, are in fact independent of the choice of that triangulation; for instance, this is the case for the union of the j-simplices that are not on the frontier of a $(j+1)$-simplex $[0 \leqslant j \leqslant \dim(X)]$.

B. Application to the Local Degree

Let u be a C^∞ map of $\mathbf{R}^n$ into $\mathbf{R}^n$ such that $u(0) = 0$, $u(\mathbf{R}^n - \{0\}) \subset \mathbf{R}^n - \{0\}$, so that u defines an endomorphism u^* of $H^{n-1}(\mathbf{R}^n - \{0\}; \mathbf{Z})$, which is isomorphic to $\mathbf{Z}$. Then $u^*(\zeta) = c\zeta$ for any cohomology class ζ, and $c \in \mathbf{Z}$; the integer c is called the *local degree* of u at 0, and written $\deg_0(u)$. If the jacobian J of u at 0 is $\neq 0$, then $\deg_0(u) = 1$ if $J > 0$ and $\deg_0(u) = -1$ if $J < 0$.

Now consider two smooth manifolds X, Y, both oriented and having the same dimension $n \geqslant 2$, and let $f: X \to Y$ be a C^∞ map. A point $a \in X$ is *isolated* for f if there is an open neighborhood U of a such that $f(x) \neq f(a)$ for $x \in \bar{U} - \{a\}$. One may assume that U is the domain of a chart $\varphi: U \to \mathbf{R}^n$ of X such that $\varphi(U) = \mathbf{R}^n$ and $\varphi(a) = 0$ and there is a chart $\psi: V \to \mathbf{R}^n$ of Y such that $f(U) \subset V$, $\psi(V) = \mathbf{R}^n$ and $\psi(f(a)) = 0$. Then define the *local degree* $\deg_a f$ at the point a as $\deg_a f = \deg_0(\psi \circ f \circ \varphi^{-1})$; it does not depend on the choices of U, V, φ, and ψ. If the tangent mapping $T_a(f)$ is a bijection of $T_a(X)$ onto $T_{f(a)}(Y)$, then $\deg_a f = 1$ if $T_a(f)$ preserves orientations, $\deg_a f = -1$ if not.

Let Z be another smooth oriented manifold of dimension n, and $g: Y \to Z$ be a C^∞ map such that $f(a)$ is isolated for g; then a is isolated for $g \circ f$, and

$$\deg_a(g \circ f) = \deg_{f(a)} g \cdot \deg_a f.$$

Finally, suppose X and Y are compact and connected and that there is a point $y_0 \in Y$ such that $f^{-1}(y_0) = \{x_1, x_2, \ldots, x_r\}$, a finite set. The x_j are isolated for f, and

$$\deg f = \sum_{j=1}^{r} \deg_{x_j} f.$$

C. Later Developments

The papers of Alexandroff and Čech defined Betti numbers $p_r(x)$ but not groups attached to a point x. Alexandroff proposed definitions of other groups at a point x, the dimension of which may be different from $p_r(x)$. One of his definitions is similar to one that is better understood within the context of Borel–Moore homology: the definitions and notations of Part 1, chap. IV,

§7,G give the *homology graded sheaf* $\mathscr{H}_{\cdot}(\mathscr{C}_{\mathrm{H}\cdot}(\mathrm{X};\mathrm{L}))$ for the generalized chain complex of sheaves $\mathscr{C}_{\mathrm{H}\cdot}(\mathrm{X};\mathrm{L})$, written $\mathscr{H}_{\cdot}(\mathrm{X};\mathrm{L})$. The *stalk* $(\mathscr{H}_j(\mathrm{X};\mathrm{L}))_x$ at a point x can be called the j-th *local homology group at* x; the exact sequence of relative homology shows that it is isomorphic to the Borel–Moore relative homology group $\mathrm{H}_j(\mathrm{X}, \mathrm{X} - \{x\}; \mathrm{L})$.

The work of Alexandroff and Čech was considerably enlarged and diversified by Wilder between 1935 and 1955. He conclusively showed how all the results (mainly relative to plane sets) obtained by the "point-set topologists" of the Polish and American schools who shunned simplicial methods could be enormously generalized and put in their proper perspective by the use of homological notions [518]. He not only used Čech homology, but also Čech cohomology with compact supports and coefficients in a field (which did not yet exist when Alexandroff and Čech wrote their papers): for two open neighborhoods $\mathrm{U} \supset \mathrm{V}$ of x in a locally compact space X, there is a natural homomorphism $\mathrm{H}_c^r(\mathrm{V}) \to \mathrm{H}_c^r(\mathrm{U})$ (Part 1, chap. IV, §7,G). If $p_{\mathrm{U,V}}^r$ is the dimension of the image of that homomorphism, the numbers $p_{\mathrm{U,V}}^r$ behave exactly as the numbers $p_{r,\mathrm{U,V}}$ of Alexandroff; hence, by the same limit processes a number $p^r(x)$ can be attached to each $x \in \mathrm{X}$, called the *local co-Betti number* at x, which is an integer or $+\infty$; Wilder showed that in fact $p_r(x) = p^r(x)$ for all $x \in X$ ([518], p. 191).

Wilder's book contains a large number of local properties linked to homology and cohomology. Since it was written when modern algebraic techniques (Part 1, chap. IV, §5) had not yet been introduced into algebraic topology, it would be worthwhile rewriting it with the help of these techniques, which very likely would make it shorter and more perspicuous.

In the remainder of this chapter, we shall restrict our description to the notions and results of [518] that have proved most striking and useful in other directions in algebraic topology (see [385]).

D. Phragmén–Brouwer Theorems and Unicoherence

As an illustration of Wilder's ideas, I think it worthwhile to insert as a small digression an example of topological properties that are put into a better light when they are connected with notions of algebraic topology.

In 1885 Phragmén published a short note on topology of plane sets [361] in which he proved the following property: if A is a compact connected subset of $\mathbf{R}^2$, and U is the unbounded connected component of the open set $\mathbf{R}^2 - \mathrm{A}$, then the frontier of U is connected. His method consisted in decomposing $\mathbf{R}^2$ into squares with sides of length 2^{-m}, considering the union of those squares that met the frontier of U, and letting m tend to infinity.

In one of his first topological papers, in which he gave a new proof of the Jordan theorem for plane curves, Brouwer extended Phragmén's result by showing that the frontier of *any* connected component of $\mathbf{R}^2 - \mathrm{A}$ is connected ([89], p. 378). Later it was discovered that this property is linked to several others, and "point-set topologists" were able to extend them when $\mathbf{R}^2$ is replaced by much more general spaces X. But apparently it was only in the

Alexandroff–Hopf ([30], p. 292) book that these properties were shown to depend on the fact that $H_1(X; \mathbf{Z}) = 0$. The key property is:

If X is a Hausdorff arcwise connected space, *such that* $H_1(X; \mathbf{Z}) = 0$, and if A, B are two nonempty *disjoint* closed sets such that $X - A$ and $X - B$ are arcwise connected (neither A nor B "cuts" the space), then $X - (A \cup B)$ also is arcwise connected ($A \cup B$ does not "cut" the space). This is an immediate consequence of the Mayer—Vietoris homology exact sequence.

Elementary arguments of "point-set topology" easily produce from that property the following so-called "Phragmén–Brouwer theorems," under the additional assumption that X is *locally arcwise connected.*

(i) If A, B are two closed nonempty sets in X such that $A \cap B = \varnothing$, and if x, y belong both to the same connected component of $X - A$ and to the same connected component of $X - B$, then they also belong to the same connected component of $X - (A \cup B)$.
(ii) If A is a closed, connected, nonempty subset of X, each connected component of $X - A$ has a connected frontier.
(iii) If A, B are two closed connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected (a property that was much studied under the name of "unicoherence").
(iv) If A is a closed subset of X, and C_1, C_2 two nonempty connected components of $X - A$ having the same frontier B, then B is connected.

§ 2. Homological and Cohomological Local Connectedness

In a *locally connected* space X each $x \in X$ has a fundamental system of open neighborhoods that are connected. It follows from the definitions (Part 1, chap. IV, § 3) that for Alexander–Spanier cohomology, 0-cocycles are just locally constant functions; hence for a connected space X the reduced cohomology $\tilde{H}^0(X) = 0$. Conversely, if a compact space K is the union of two nonempty open and closed sets U_1, U_2, then a function constant in U_1 and constant in U_2 but with different values is locally constant; hence $\tilde{H}^0(K) \neq 0$. From this it follows at once that for locally compact spaces X, saying that X is locally connected is equivalent to saying that $p^0(x) = 0$ for all $x \in X$.

This leads to the generalization of local connectedness formulated by Alexandroff in 1929 and mentioned in § 1. He said that X is *homologically locally connected in dimension* $q \geqslant 0$ (later abbreviated into $q - \mathrm{lc}$) at a point x, if for every open neighborhood U of x there is an open neighborhood $V \subset U$ of x such that every q-cycle in V bounds in U. There is, however, no direct relation between that property and the fact that $p_q(x) = 0$, as Alexandroff himself showed by examples ([28], p. 9). What $p_q(x) = 0$ [or equivalently $p^q(x) = 0$] means is the corresponding notion for Čech–Alexander cohomology with coefficients in a field: X is *cohomologically locally connected in dimension q* (abbreviated to $q-$ clc) at the point x if for any open neighborhood U of x there is an open neighborhood $V \subset U$ of x such that the image of the homomorphism $H^q(U) \to H^q(V)$ is 0.

Examples show that at a point x of a locally compact space X, X may be $q-$ lc for all integers q in an arbitrary finite set, but *not* $q-$ lc for the other values of q ([304], p. 92). In 1935 Lefschetz [307] and Wilder defined the property of being lc^n at a point as meaning that the space is $q-$ lc at that point for *all* values $q \leqslant n$. They needed this for their definition of generalized manifolds (see § 3); the notion was studied in detail by Begle for compact spaces [46]. There is a corresponding notion (clc^n) for cohomology.

Results concerning these notions are now best expressed in the context of Borel–Moore homology. In their notation (L being a Dedekind ring) the locally compact space X is homologically (resp. cohomologically) locally connected in dimension q [abbreviated to $q - \mathrm{hlc}_L$ (resp. $q - \mathrm{clc}_L$)] at the point x if, for any neighborhood U of x, there is a neighborhood $V \subset U$ of x such that the image of the homomorphism

$$H_q^c(V;L) \to H_q^c(U;L) \quad [\text{resp. } H^q(U;L) \to H^q(V;L)] \tag{3}$$

is 0. The space is $q - \mathrm{hlc}_L$ (resp. $q - \mathrm{clc}_L$) if it has that property at every point, and hlc_L^r (resp. clc_L^r) if it is $q - \mathrm{hlc}_L$ (resp. $q - \mathrm{clc}_L$) for all integers $q \leqslant r$. Finally, X is hlc_L (resp. clc_L) if for any neighborhood U of any point x it is possible to choose the neighborhood $V \subset U$ *independently of q* such that the image of the map (3) is 0 for every q.

For a hlc_L^r space X and an L-module B, there is for every $q \leqslant r$ a split exact sequence

$$0 \to \mathrm{Ext}(H_{q-1}^c(X;L), B) \to H^q(X;B) \to \mathrm{Hom}(H_q^c(X;L), B) \to 0 \tag{4}$$

corresponding to the exact sequence for $H_q(X;B)$ applicable to all locally compact spaces [Part 1, chap. IV, § 7,G), formula (184)].

Property hlc_L^r implies clc_L^r, but clc_L^r only implies hlc_L^{r-1}. When L is a field, however, hlc_L^r and clc_L^r are equivalent, and hlc_L^r and clc_L^r are always equivalent.

If X is *compact* and hlc_L^r, then the L-modules $H_q(X;L)$ and $H^q(X;L)$ are finitely generated for $q \leqslant r$; $\mathrm{Ext}(H^{q+1}(X;L), L)$ is then the torsion submodule of $H_q(X;L)$ and $\mathrm{Ext}(H_{q-1}(X;L), L)$ the torsion submodule of $H^q(X;L)$.

We conclude this section with the remark that *singular homology* can be used for the definition of local properties instead of Čech homology or Borel–Moore homology. This was done in 1935 by Lefschetz,* who defined properties $q-$ HLC, HLC^r, and HLC by replacing Čech homology by singular homology in the definitions of $q-$ lc, lc^r, and hlc. The important property

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* Do not confuse these notions with other concepts of "local connectedness" based on homotopy rather than on homology, which we shall consider in Part 3, chap. II, § 2,B. They were also introduced by Lefschetz, who used the symbol LC (with indices or exponents) to designate them (the H in HLC stands for "homology").

of HLC spaces is that for them Alexander–Spanier cohomology is naturally *isomorphic* to singular cohomology.

§ 3. Duality in Manifolds and Generalized Manifolds

A. Fundamental Classes and Duality

Local properties of a C^∞ manifold M are used to extend the concept of "fundamental class" in the homology of a compact manifold (chap. I, § 3,A) to "relative fundamental classes" for a noncompact one.

Suppose M is an *oriented* smooth n-dimensional manifold (connected or not). Choose an orientation on $\mathbf{R}^n$ and on S_{n-1} and let γ_n be the generator of the group $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z}) \simeq \mathbf{Z}$ that is mapped on $[S_{n-1}]$ by the isomorphism $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z}) \xrightarrow{\partial} H_{n-1}(S_{n-1}; \mathbf{Z})$. For any chart φ: $V \to \mathbf{R}^n$ preserving orientation, and $x \in V$ such that $\varphi(x) = 0$, there is an isomorphism φ_*: $H_\bullet(V, V - \{x\}; \mathbf{Z}) \simeq H_\bullet(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z})$. Thus $H_p(V, V - \{x\}; \mathbf{Z}) = 0$ for $p \neq n$ and $H_n(V, V - \{x\}; \mathbf{Z})$ is isomorphic to $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z})$. By excision, this gives a composite isomorphism

$$H_n(M, M - \{x\}; \mathbf{Z}) \simeq H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z})$$

which is independent of the chart φ; let μ_x be the element of $H_n(M, M - \{x\}; \mathbf{Z})$ mapped onto γ_n by that isomorphism. Now let $K \subset M$ be any compact subset. Then there exists a unique class $\mu_{M,K} \in H_n(M, M - K; \mathbf{Z})$, called the *fundamental class* relative to K, such that for any $x \in K$ the image of $\mu_{M,K}$ by the homomorphism

$$j_*: H_n(M, M - K; \mathbf{Z}) \to H_n(M, M - \{x\}; \mathbf{Z})$$

deduced from the natural injection is the class μ_x. The proof uses a technique similar to the one in H. Cartan's paper of 1945 [106]. Consider first the case $M = \mathbf{R}^n$ and then the case in which K is small enough, then apply the Mayer–Vietoris exact sequence to treat the union of finitely many such compact sets by induction on their number. Poincaré duality for homology and cohomology of M with *integer* coefficients can then be obtained by considering M as union of an increasing sequence (K_m) such that each K_m is a compact neighborhood of K_{m-1}. Let z_m be a relative n-cycle whose homology class is $\mu_{M,K} \in H_n(M, M - K_m; \mathbf{Z})$. Then, for each p-cocycle f on M with compact support, the class of the cap product $z_m \frown f$ is the same for all sufficiently large m and only depends on the class c of f in $H_c^p(M; \mathbf{Z})$. Call $D_M c$ that class in $H_{n-p}(M; \mathbf{Z})$; then the homomorphism

$$D_M: H_c^p(M; \mathbf{Z}) \to H_{n-p}(M; \mathbf{Z})$$

is bijective (*Poincaré duality*).

In a similar way for a closed subset A of M, there is an isomorphism

$$D_{M,A}: \bar{H}_c^p(A; \mathbf{Z}) \to H_{n-p}(M, M - A; \mathbf{Z})$$

for Alexander–Spanier cohomology with compact supports and singular homology (*Alexander duality*).

There are analogous results for nonorientable manifolds and coefficients in F_2.

B. Duality in Generalized Manifolds

Until 1930 Poincaré and Alexander duality theorems for integer coefficients had only been proved for orientable compact *triangulable* C^0-manifolds. This was soon felt to be an unsatisfactory situation, since the notion of triangulation depends on auxiliary subspaces $\mathbf{R}^n$, whereas the duality theorems only deal with homology and cohomology; even an extension to all C^0-manifolds (for which triangulability was unknown) would have suffered from the same defect. Starting with Čech [121] and Lefschetz [306] in 1933 topologists endeavored to define classes of spaces by *purely homological conditions* which would include both combinatorial manifolds and C^0-manifolds, and for which the duality theorems would hold.

The general idea was to impose homological properties known to hold for C^0-manifolds on these spaces, particularly *local* homological conditions (§ 2). Several definitions were proposed in succession by Wilder, Alexandroff and Pontrjagin [31], P. Smith [437] and Begle [46]. Here again the introduction of Borel–Moore homology, with substantial improvements by Bredon [87], brought a more satisfactory state of the theory.

If L is a Dedekind ring, a locally compact space X is a *homology n-manifold* over L (abbreviated $n - \mathrm{hm}_L$) if:

1. The cohomological dimension $\dim_L X$ of X over L (chap. II, § 6) is finite.
2. The relative Borel–Moore homology

$$H_q(X, X - \{x\}; L) = \begin{cases} L & \text{for } q = n \\ 0 & \text{for } q \neq n \end{cases} \tag{5}$$

for any $x \in X$.

These conditions imply that the cohomological dimension $\dim_L X \leqslant n + 1$ and that the sheaves $\mathscr{H}_q(X; L)$ are 0 for $q \neq n$. Bredon has also proved that $\mathscr{O} = \mathscr{H}_n(X; L)$ is *locally isomorphic to the constant sheaf* L. One says $\mathscr{O}$ is the *orientation sheaf* and X is *orientable over* L if $\mathscr{O}$ is isomorphic to L; an isomorphism of $\mathscr{O}$ onto L is called an *orientation* of X over L.

We have seen that in 1945 H. Cartan had already started to drop assumptions of differentiability or triangulability in the theory of "manifolds" (Part 1, chap. IV, § 5,A). In 1947 he realized that sheaf theory (which he still used at that time in Leray's formulation) provided a way to "localize" the concept of orientation. In his 1950–1951 Seminar he defined a generalized cochain complex (with indices $\leqslant 0$) of sheaves of singular chains and introduced an orientation sheaf in that context, with the help of which he could prove Poincaré and Alexander duality theorems for C^0-manifolds.

In the context of Borel–Moore homology the duality theorems are derived from a spectral sequence applicable to all locally compact spaces X with *finite* cohomological dimension. Suppose $\dim_L X \leqslant n$, and let $\mathscr{B}^{\bullet}$ be the generalized cochain complex of sheaves defined by

$$\mathscr{B}^q = \mathscr{C}_{H,n-q}(X; L) \tag{6}$$

so that $\mathscr{H}^q(\mathscr{B}^{\bullet}) = 0$ for $q < 0$, and $\mathscr{H}^0(\mathscr{B}^{\bullet}) = \mathscr{H}_n(X; L)$. Then ([66], p. 152) for any *paracompactifying* family of supports Φ there is a spectral sequence having as E_2 terms

$$E_2^{pq} = H_{\Phi}^p(X; \mathscr{H}^q(\mathscr{B}^{\bullet})) \tag{7}$$

and $H^0(\Gamma_{\Phi}(\mathscr{B}^{\bullet}))$ for abutment with a suitable filtration.

If X is now a *homology n-manifold* over L and $\dim_{\Phi} X < +\infty$, there is a natural isomorphism

$$H_{\Phi}^p(X; \mathscr{O} \otimes L) \xrightarrow{\sim} H_{n-p}^{\Phi}(X; L) \tag{8}$$

("Poincaré duality"). In addition, if A is a closed set in X, and $\dim_{\Phi|A} X < +\infty$, there are natural isomorphisms

$$H_{\Phi}^p(X, X - A; \mathscr{O} \otimes L) \xrightarrow{\sim} H_{n-p}^{\Phi|A}(A; L) \tag{9}$$

$$H_{\Phi \cap (X-A)}^p(X - A; \mathscr{O} \otimes L) \xrightarrow{\sim} H_{n-p}^{\Phi}(X, A; L) \tag{10}$$

("Alexander duality").

In the Borel–Moore theory a *generalized n-manifold* X over L (abbreviated $n - \mathrm{gm}_L$), also called *cohomology n-manifold* ($n - \mathrm{cm}_L$), is an $n - \mathrm{hm}_L$ which is also clc_L (§ 2), and $\dim_L X \leqslant n$. If L is a field, a metric $n - \mathrm{hm}_L$ space is also a $n - \mathrm{cm}_L$.

Using excision and the Künneth theorem, it is easy to see that combinatorial manifolds of dimension n in the sense of Alexander (Part 1, chap. II, § 4) are generalized n-manifolds over **Z**.

C^0-manifolds are trivially generalized manifolds, but generalized manifolds are genuine generalizations of C^0-manifolds. There are generalized manifolds of dimension 4 in which for some points x there is an open neighborhood U of x such that for *no* open neighborhood $V \subset U$ of x is $V - \{x\}$ simply connected ([421], p. 241).

The main interest of generalized manifolds is that they are much easier to work with than C^0-manifolds. For instance, if a product $A \times B$ of locally compact spaces is a generalized manifold, both A and B are generalized manifolds. In the theory of transformation groups, fixed point sets and "slices" in a generalized manifold are generalized manifolds.

Wilder's general program was to find conditions under which the Schoenflies results for $\mathbf{R}^2$ could be extended to generalized manifolds. A whole chapter of his book ([518], chap. 12) is devoted to the notion of *accessibility*. He generalized Schoenflies' "Unbewaltheit" (chap. II, §) to the notion of *uniform local q-connectedness*: in a compact space X, an open subset D is *uniformly locally* *connected in dimension q* (abbreviated to q – ulc) if for every finite open covering $\mathfrak{U} = (U_\alpha)$ of X there exists a finite open covering $\mathfrak{V} = (V_\beta)$ of X finer than U and such that, for any V_β, there exists a $U_\alpha \supset V_\beta$ for which the image of the map $H_q(V_\beta \cap D) \to H_q(U_\alpha \cap D)$ is 0; D is ulc^r if it is q – ulc for $0 \leqslant q \leqslant r$.

We only mention here a few of the numerous properties proved by Wilder.

1. If X is an orientable n – gm which is a homology sphere [$H_q(X) = 0$ for $q \neq n$] and M is a compact $(n - 1)$ – gm contained in X, then the components of X – M are ulc^{n-1}.
2. If X is as in 1 and $M \subset X$ is the common frontier in X of at least two connected open sets, one of which is ulc^{n-2}, then M is an orientable $(n - 1)$ – gm.
3. If X is an orientable n – gm such that $H_1(X) = 0$ and $U \subset X$ is an open connected set which is ulc^{n-2} and has a connected frontier B in X, then B is an orientable $(n - 1)$ – gm.
4. Finally, if X is an orientable generalized manifold and $f: X \to Y$ a surjective continuous map of X onto a Hausdorff space Y, such that the reduced homology of each fiber $f^{-1}(y)$ is 0, then Y is an *orientable generalized manifold* and $f_*: H_\bullet(X) \to H_\bullet(Y)$ is an isomorphism [a remarkable refinement of the Vietoris-Begle theorem (Part 1, chap. IV, §§ 7,B and 7,E)].

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