Applications of algebra to a problem in topology
Joint work with

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and

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Pontryagin (1930’s)
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cobordism group of stably framed k-manifolds

\[ \pi_{n+k} S^n, n \gg 0 \]
Pontryagin (1930’s)

\[ \pi_0 S^0 = \mathbb{Z} \]

\[ \pi_1 S^0 = \mathbb{Z}/2 \]
Pontryagin (1930s)  \( k=2 \)
Pontryagin (1930s)  $k=2$
Pontryagin (1930s) \( k = 2 \)
Pontryagin (1930s) This defines a function

\[ \varphi : H_1(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \]

If genus \( M > 0 \), dimension \( \dim H_1(M) > 1 \) and so

\[ \ker \varphi \neq 0 \]

You can always lower the genus with surgery
Pontryagin (?)

φ is not linear

it's quadratic and refines the intersection pairing
Pontryagin (?)

\[ \Phi(\Sigma) = Ar f(\varphi) \]

\[ \pi_2 S^0 = \mathbb{Z}/2 \]
Kervaire (1960)

\[ M = M^{4k+2} \] (framed)

defined \( \varphi : H^{2k+1}(M; \mathbb{Z}/2) \to \mathbb{Z}/2 \)

quadratic refinement of the intersection pairing

\[ \Phi(M) = Arf(\varphi) \]

showed \( \Phi(M^{10}) = 0 \)
Kervaire (1960)

produced a piecewise linear $N^{10}$

with $\Phi(N^{10}) \neq 0$

hence $N^{10}$ has no smooth structure
Browder (1969)

\[ n \neq 2^j - 1 \quad \Phi(M^{2n}) = 0 \]

\[ n = 2^j - 1 \quad \Phi(M^{2n}) \neq 0 \]

\[ \iff \quad \text{there exists} \quad \theta_j \in \pi_{2^j+2-2}S^0 \]

\[ \text{represented by} \quad h_j^2 \in \text{Ext}_A(\mathbb{Z}/2, \mathbb{Z}/2) \]
The elements $\theta_j$ exist for $j = 1, 2, 3, 4, 5$ dimensions $2, 6, 14, 30, 62$.

so the first open dimension is 126
The Kervaire invariant problem

In which dimensions can \( \Phi(M) \) be non-zero?
Doomsday Theorem (Hill, H., Ravenel)

If $\Phi(M^n) \neq 0$ then $n = 2, 6, 14, 30, 62$ or $126$

In other words $\theta_j$ does not exist for $j \geq 7$
Adams Spectral Sequence

Adams-Novikov Spectral Sequence

$h^2_j$

$\theta_j$

Something easier to compute

$K$-theory

Tuesday, April 21, 2009
Adams Spectral Sequence

Adams-Novikov Spectral Sequence

Something easier to compute

$K$-theory
\[ H^s(\mathbb{Z}/2; K_t) \implies KO_{t-s} \]
for $j \geq 2$, $\theta_j$ supports a non-zero differential
Adams Spectral Sequence

Adams-Novikov Spectral Sequence

$c + \theta_j$

$h^2_j$

Something easier to compute

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periodicity Theorem

Rochlin’s Theorem
K-theory and reality (Atiyah, 1966)

\[ X \] space with a \( \mathbb{Z}/2 \) action

\[ KR(X) \] vector bundles with compatible conjugate-linear action

\[ KR(X) \cong KR(X \wedge S^{n,n}) \]

\[ S^{n,n} = \mathbb{C}^n \]
Assemble K-theory
from the equivariant
chains on $S^{n,n}$
slice filtration
periodicity

$\theta_1 \quad \theta_2 \quad \theta_3$
level 5 topological modular forms

Like $KR$ with $\mathbb{Z}/4$ instead of $\mathbb{Z}/2$
$j \geq 4$

2 below the period

$\theta_{j}$

the period
Assemble tmf(5) from the equivariant chains on \( S^m \rho_4 \)

\( \rho_4 \) the 4 dimensional real regular representation of \( \mathbb{Z}/4 \)
2 below the period

the period
gap + periodicity

differentials on the $\theta_j$
The actual proof

Step 1: Use $\mathbb{Z}/8$ and an appropriate cohomology theory

Step 2: Show that all the choices of $\theta_j$ are distinguished

Step 3: Prove a gap theorem (easy)

Step 4: Prove a periodicity theorem (of period 256)
Relation to Geometry/Physics?

4 dimensional field theory?

generalization of Clifford algebras with periodicity of $2^8 \cdot 3^3 \cdot 5 = 34,560$

(maybe twice that)
Given a real manifold $M^{2d}$ whose fixed point space $N$ bounds an unoriented manifold, find a cobordism invariant of $M$ which, when $N = \emptyset$ is

$$\int_{M/(\mathbb{Z}/2)} \omega_1^{2d}$$