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Exotic homology manifolds - Oberwolfach 2003

EDITED BY FRANK QUINN AND ANDREW RANICKI

Abstract

AMS Classification

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Preface

This volume is the proceedings of the Mini-Workshop *Exotic Homology manifolds* held at Oberwolfach 29th June - 5th July, 2003.

Homology manifolds were developed in the first half of the 20th century to give a precise setting for Poincaré’s ideas on duality. Major results in the second half of the century came from two different areas. Methods from the point-set tradition were used to study homology manifolds obtained by dividing genuine manifolds by families of contractible subsets. “Exotic” homology manifolds are ones that cannot be obtained in this way, and these have been investigated using algebraic and geometric methods.

The Mini-Workshop brought together experts from the point-set and algebraic traditions, along with new Ph.D.s and people in related areas. There were 17 participants, 14 formal lectures and a problem session. There was a particular focus on the proof of the existence of exotic homology manifolds. This gave experts in each area an opportunity to learn more about details coming from other areas. There had also been concerns about the stability (“shrinking”) theorem that in retrospect is a crucial step in the proof but had not been worked out when the theorem was originally announced. This was discussed in detail. One of the high points of the conference was the discovery of a short and very general new proof of this result by Pedersen and Yamasaki (published in these proceedings), so there are now three independent treatments. Extensive discussions of examples and problems clarified the current state of the field and mapped out objectives for the next decade.

A Mini-Workshop on history entitled “Henri Poincaré and topology” was held during the same week. There was joint discussion of the early history of manifolds, and each group offered evening lectures on topics of interest to the other.

Several of the daytime history lectures also drew large numbers of homology manifold participants. The interaction between the two groups was very beneficial and should serve as a model for future such synergies.

We are grateful to the Oberwolfach Mathematics Institute for hosting the meeting, and to the participants, authors and the referees for their contributions.

F.Q., A.R.

August, 2005

Contents

Homological arc-homogeneous ENR's	
J. Bryant	1
Path concordances as detectors of codimension one manifold factors	
R. Daverman and D. Halverson	7
The Bryant-Ferry-Mio-Weinberger construction of generalized manifolds	
F. Hegenbarth and D. Repovs	16
The quadratic form E_8 and exotic homology manifolds	
W. Mio and A. Ranicki.....	33
Stability in controlled L-theory	
E. Pedersen and M. Yamasaki.....	67
Problems on homology manifolds	
F. Quinn	88
Controlled L -theory	
A. Ranicki and M. Yamasaki.....	106

Conference Photo



Banagl, Daverman, Dydak, Johnston, Mio, Halverson
Woolf, Quinn, Ranicki, Dranishnikov, Yamasaki
Edwards, Bryant, Pedersen (with Mickey), Lück, Repovs, Hegenbarth

Homologically arc-homogeneous ENR's

J. L. BRYANT

Abstract

In this note we prove that an arc-homogeneous ENR is a homology manifold.

AMS Classification 57P05 ; 55T05

Keywords homology manifolds, homogeneity

1 Introduction

The so-called Modified Bing-Borsuk Conjecture, which grew out of a question in [1], asserts that a homogeneous euclidean neighborhood retract is a homology manifold. At this miniworkshop on exotic homology manifolds, Frank Quinn asked whether a space that satisfies a similar property, which he calls *homological arc-homogeneity*, is a homology manifold. The purpose of this note is to show that the answer to this question is yes.

2 Statement and Proof of the Main Result

Theorem 2.1 *Suppose X is an n -dimensional homologically arc-homogeneous ENR. Then X is a homology n -manifold.*

Definitions. A *homology n -manifold* is a space X having the property that for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A *euclidean neighborhood retract* (ENR) is a space homeomorphic to a closed subset of euclidean space that is a retract of some neighborhood of itself.

A space X is *homologically arc-homogeneous* provided that for every path $\alpha: [0, 1] \rightarrow X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0), 0)) \rightarrow H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of α . The *local homology sheaf in dimension k* , \mathcal{H}_k , on a space X is the sheaf with stalks $H_k(X, X - x)$, $x \in X$.

By Theorem 15.2 of Bredon [2], if an n -dimensional space X is cohomologically locally connected (over \mathbb{Z}), has finitely generated local homology groups, $H_k(X, X - x)$, for each k , and if each \mathcal{H}_k is locally constant, then X is a homology manifold. We shall show that an n -dimensional, homologically arc-connected ENR satisfies the hypotheses of Bredon's theorem.

Assume from now on that X represents an n -dimensional, homologically arc-homogeneous ENR. Unless otherwise specified, all homology groups are assumed to have integer coefficients. The following lemma is a straightforward application of the definition and the Meyer-Vietoris theorem.

Lemma 2.2 *Given a path $\alpha: [0, 1] \rightarrow X$ and $t \in [0, 1]$, the inclusion induced map*

$$H_*(X \times t, X \times t - (\alpha(t), t)) \rightarrow H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism.

Given points $x, y \in X$, an arc $\alpha: I \rightarrow X$ from x to y , and an integer $k \geq 0$, let $\alpha_*: H_k(X, X - x) \rightarrow H_k(X, X - y)$ be defined by the composition

$$H_k(X, X - x) \xrightarrow{\times 0} H_*(X \times I, X \times I - \Gamma(\alpha)) \xleftarrow{\times 1} H_k(X, X - y).$$

Clearly $(\alpha^{-1})_* = \alpha_*^{-1}$ and $(\alpha\beta)_* = \beta_*\alpha_*$, whenever $\alpha\beta$ is defined.

Lemma 2.3 *Given $x \in X$ and $\eta \in H_k(X, X - x)$, there is a neighborhood U of x in X such that if α and β are paths in U from x to y , then $\alpha_*(\eta) = \beta_*(\eta) \in H_k(X, X - y)$.*

Proof We will prove the equivalent statement: for each $x \in X$ and $\eta \in H_k(X, X - x)$, there is a neighborhood U of x such that if α is a loop in U based at x , then $\alpha_*(\eta) = \eta$.

Suppose $x \in X$ and $\eta \in H_k(X, X - x)$. Since $H_k(X, X - x)$ is the direct limit of the groups $H_k(X, X - W)$, where W ranges over the (open) neighborhoods

of x in X , there is a neighborhood U of x and an $\eta_U \in H_k(X, X - U)$ that goes to η under the inclusion $H_k(X, X - U) \rightarrow H_k(X, X - x)$.

Suppose α is a loop in U based at x . Let $\eta_\alpha \in H_k(X \times I, X \times I - \Gamma(\alpha))$ correspond to η under the isomorphism $H_k(X, X - x) \xrightarrow{\times 0} H_k(X \times I, X \times I - \Gamma(\alpha))$ guaranteed by homological arc-homogeneity.

Let $\eta_{U \times I} = \eta_U \times 0 \in H_k(X \times I, X \times I - U \times I)$. Then the image of $\eta_{U \times I}$ in $H_k(X \times I, X \times I - \Gamma(\alpha))$ is η_α , as can be seen by chasing the following diagram around the lower square.

$$\begin{array}{ccc}
 H_k(X, X - U) & \longrightarrow & H_k(X, X - x) \\
 \cong \downarrow \times 1 & & \cong \downarrow \times 1 \\
 H_k(X \times I, X \times I - U \times I) & \longrightarrow & H_k(X \times I, X \times I - \Gamma(\alpha)) \\
 \cong \uparrow \times 0 & & \cong \uparrow \times 0 \\
 H_k(X, X - U) & \longrightarrow & H_k(X, X - x)
 \end{array}$$

But from the upper square we see that η_α must also come from η after including into $X \times 1$. That is, $\alpha_*(\eta) = \eta$. \square

Corollary Suppose the neighborhood U above is path connected and F is the cyclic subgroup of $H_k(X, X - U)$ generated by η_U . Then, for every $y \in U$, the inclusion $H_k(X, X - U) \rightarrow H_k(X, X - y)$ takes F one-to-one onto the subgroup F_y generated by $\alpha_*(\eta)$, where α is any path in U from x to y .

Lemma 2.4 Suppose $x, y \in X$ and α and β are path-homotopic paths in X from x to y . Then $\alpha_* = \beta_*: H_k(X, X - x) \rightarrow H_k(X, X - y)$.

Proof By a standard compactness argument it suffices to show that, for a given path α from x to y and element $\eta \in H_k(X, X - x)$, there is an $\epsilon > 0$ such that $\alpha_*(\eta) = \beta_*(\eta)$ for any path β from x to y ϵ -homotopic (rel $\{x, y\}$) to α .

Given a path α from x to y , $\eta \in H_k(X, X - x)$, and $t \in I$, let U_t be a path-connected neighborhood of $\alpha(t)$ associated with $(\alpha_t)_*(\eta) \in H_k(X, X - \alpha(t))$ given by Lemma 2.3, where α_t is the path $\alpha|_{[0, t]}$. There is a subdivision $\{0 = t_0 < t_1 < \dots < t_m = 1\}$ of I such that for each $i = 1, \dots, m$, $\alpha([t_{i-1}, t_i]) \subseteq U_i$ where $U_i = U_t$ for some t . There is an $\epsilon > 0$ so that if $H: I \times I \rightarrow X$ is an ϵ -path-homotopy from α to a path β , then $H([t_{i-1}, t_i] \times I) \subseteq U_i$.

For each $i = 1, \dots, m$, let $\alpha_i = \alpha|_{[t_{i-1}, t_i]}$ and $\beta_i = \beta|_{[t_{i-1}, t_i]}$, and for $i = 0, \dots, m$, let $\gamma_i = H|_{t_i \times I}$ and $\eta_i = (\alpha_{t_i})_*(\eta)$. By Corollary 2

$$(\alpha_i)_*(\eta_{i-1}) = (\gamma_{i-1}\beta_i\gamma_i^{-1})_*(\eta_{i-1}) = \eta_i$$

(where $\eta_0 = \eta$). Since γ_0 and γ_n are the constant paths, it follows easily that

$$\alpha_*(\eta) = (\alpha_n)_* \cdots (\alpha_1)_*(\eta) = (\beta_n)_* \cdots (\beta_1)_*(\eta) = \beta_*(\eta).$$

□

Proof of Theorem 2.1 As indicated at the beginning of this note, we need only show that the hypotheses of Theorem 15.2 of [2] are satisfied.

Since X is an ENR, X is locally contractible, hence, cohomologically locally connected over \mathbb{Z} .

Given $x \in X$, let W be a path-connected neighborhood of x such that W is contractible in X . If α and β are two paths in W from x to a point $y \in W$, then α and β are path-homotopic in X . Hence, by Lemma 2.4, $\alpha_*: H_k(X, X-x) \rightarrow H_k(X, X-y)$ is a well-defined isomorphism that is independent of α for every $k \geq 0$. Hence, $\mathcal{H}_k|_W$ is the constant sheaf, and so \mathcal{H}_k is locally constant.

Finally, we need to show that the local homology groups of X are finitely generated. This can be seen by working with a mapping cylinder neighborhood of X . Assume X is nicely embedded in \mathbb{R}^{n+m} , for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi: \partial N \rightarrow X$, with mapping cylinder projection $\pi: N \rightarrow X$ [3]. Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\dot{A} = \phi^{-1}(A)$.

Lemma 2.5 *If A is a closed subset of X , then $H_k(X, X-A) \cong \check{H}_c^{n+m-k}(A^*, \dot{A})$.*

Proof Suppose A is closed in X . Since $\pi: N \rightarrow X$ is a proper homotopy equivalence,

$$H_k(X, X-A) \cong H_k(N, N-A^*).$$

Since ∂N is collared in N ,

$$H_k(N, N-A^*) \cong H_k(\text{int}N, \text{int}N-A^*),$$

and by Alexander duality,

$$\begin{aligned} H_k(\text{int}N, \text{int}N-A^*) &\cong \check{H}_c^{n+m-k}(A^*-\dot{A}) \\ &\cong \check{H}_c^{n+m-k}(A^*, \dot{A}) \end{aligned}$$

(since \dot{A} is also collared in A^*). □

Since X is n -dimensional, we get the following

Corollary If A is a closed subset of X , then $\check{H}_c^q(A^*, \dot{A}) = 0$, if $q < m$ or $q > n + m$.

Thus, the local homology sheaf \mathcal{H}_k of X is isomorphic to the Leray sheaf \mathcal{H}^{n+m-k} of the map $\pi: N \rightarrow X$ whose stalks are $\check{H}^{n+m-k}(x^*, \dot{x})$. For each $k \geq 0$, this sheaf is also locally constant, so there is a path-connected neighborhood U of x such that $\mathcal{H}^q|_U$ is constant for all $q \geq 0$. Given such a U , there is a path-connected neighborhood V of x lying in U such that the inclusion of V into U is null-homotopic. Thus, for any coefficient group G , the inclusion $H_c^p(U, G) \rightarrow H_c^p(V, G)$ is zero if $p \neq 0$ and is an isomorphism for $p = 0$.

The Leray spectral sequences of $\pi|_{\pi^{-1}(U)}$ and $\pi|_{\pi^{-1}(V)}$ have E_2 terms

$$E_2^{p,q}(U) \cong H_c^p(U; \mathcal{H}^q), \quad E_2^{p,q}(V) \cong H_c^p(V; \mathcal{H}^q)$$

and converge to

$$E_\infty^{p,q}(U) \cong H_c^{p+q}(U^*, \dot{U}; \mathbb{Z}), \quad E_\infty^{p,q}(V) \cong H_c^{p+q}(V^*, \dot{V}; \mathbb{Z}),$$

respectively (Theorem 6.1 of [2]). Since the sheaf \mathcal{H}^q is constant on U and V , $H_c^p(U; \mathcal{H}^q)$ and $H_c^p(V; \mathcal{H}^q)$ represent ordinary cohomology groups with coefficients in $G_q \cong \check{H}^q(x^*, \dot{x})$.

By naturality, we have the commutative diagram

$$\begin{array}{ccc} E_2^{0,q}(U) & \longrightarrow & E_2^{2,q-1}(U) \\ \downarrow \cong & & \downarrow 0 \\ E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \end{array}$$

which implies that the differential $d_2: E_2^{0,q}(V) \rightarrow E_2^{2,q-1}(V)$ is the zero map. Hence,

$$E_3^{0,q}(V) = \ker(E_2^{0,q}(V) \rightarrow E_2^{2,q-1}(V)) / \text{im}(E_2^{-2,q+1}(V) \rightarrow E_2^{0,q}(V)) = E_2^{0,q}(V),$$

and, similarly, $E_3^{0,q}(V) = E_4^{0,q}(V) = \dots = E_\infty^{0,q}(V)$. Thus,

$$E_\infty^{0,q}(V) \cong H_c^q(V^*, \dot{V}; \mathbb{Z}) \cong E_2^{0,q}(V) \cong H_c^0(V; \mathcal{H}^q) \cong H_c^0(V; G_q) \cong G_q.$$

Applying the same argument to the inclusion $(x^*, \dot{x}) \subseteq (V^*, \dot{V})$ yields the commutative diagram

$$\begin{array}{ccc} E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \\ \downarrow \cong & & \downarrow 0 \\ E_2^{0,q}(x) & \longrightarrow & E_2^{2,q-1}(x) \end{array}$$

which, in turn, gives

$$\begin{array}{ccc} G_q \cong H_c^0(V; G_q) & \xrightarrow{\cong} & H_c^q(V^*, \dot{V}; \mathbb{Z}) \\ \cong \downarrow & & \downarrow \\ G_q \cong H^0(x; G_q) & \xrightarrow{\cong} & H^q(x^*, \dot{x}; \mathbb{Z}) \cong G_q \end{array}$$

from which it follows that the inclusion $H^q(V^*, \dot{V}; \mathbb{Z}) \rightarrow H^q(x^*, \dot{x}; \mathbb{Z}) \cong G_q$ is an isomorphism. Since (x^*, \dot{x}) is a compact pair in the manifold pair (V^*, \dot{V}) , it has a compact manifold pair neighborhood $(W, \partial W)$. Since the inclusion $H^q(V^*, \dot{V}) \rightarrow \check{H}^q(x^*, \dot{x})$ factors through $H^q(W, \partial W)$, its image is finitely generated for each q . Hence, $H_k(X, X - x) \cong \check{H}^{n+m-k}(x^*, \dot{x})$ is finitely generated for each k .

□

The following theorem, which may be of independent interest, emerges from the proof of Theorem 2.1.

Theorem 2.6 *Suppose X is an n -dimensional ENR whose local homology sheaf \mathcal{H}_k is locally constant for each $k \geq 0$. Then X is a homology n -manifold.*

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3005 Brandemere Drive
Tallahassee, FL 32312

Email: bryant@math.fsu.edu

Path concordances as detectors of codimension one manifold factors

ROBERT J. DAVERMAN

DENISE HALVERSON

Abstract We present a new property, the Disjoint Path Concordances Property, of an ENR homology manifold X which precisely characterizes when $X \times \mathbb{R}$ has the Disjoint Disks Property. As a consequence, $X \times \mathbb{R}$ is a manifold if and only if X is resolvable and it possesses this Disjoint Path Concordances Property.

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Keywords Disjoint Disks Property; Disjoint Homotopies Property; Concordance; Manifold factor; Homology manifold; Resolvable; Disjoint Arcs Property

1 Introduction

Back in the 1950s R. H. Bing showed that his nonmanifold "dogbone space" [1] is a Cartesian factor of Euclidean 4-space [2]. Since then topologists have sought to understand which spaces are factors of manifolds. It is now known that the manifold factors coincide with those ENR homology manifolds which admit a cell-like resolution by a manifold; equivalently, they are the ENR homology manifolds of trivial Quinn index [10] [13]. In particular, if X has trivial Quinn index then $X \times \mathbb{R}^k$ is a manifold for $k \geq 2$. Whether $X \times \mathbb{R}$ itself is necessarily a manifold stands as a fundamental unsettled question.

Several properties of a manifold factor X of dimension n assure that its product with \mathbb{R} is a manifold. Among them are: (1) the singular (or nonmanifold) subset $S(X)$ of X – namely, the complement of the maximal n -manifold contained in X – has dimension at most $n-2$, $n \geq 4$ [4, Theorem 10.1]; (2) there exists a cell-like map $f : M \rightarrow X$ defined on an n -manifold such that $\dim\{x \in X \mid f^{-1}(x) \neq \text{point}\} \leq n-3$ [7, Theorem 3.3]; (3) there exists a topologically embedded $(n-1)$ -complex or ENR homology $(n-1)$ -manifold K with $S(X) \subset K \subset X$ [8, Corollaries 26.12A-12B]; (4) X arises from a nested defining sequence, as

defined by Cannon and Daverman [5] [8, Chapter 34], for the decomposition into point inverses induced by a cell-like map $f : M \rightarrow X$, and (5) X has the Disjoint Arc-Disk Property of [8, p. 193]. Condition (5) is implied by either (1) or (2) but not by (3) or (4).

More pertinent to issues addressed in this manuscript, Halverson [11, Theorem 3.4] proved that if an ENR homology n -manifold X , $n \geq 4$, has a certain Disjoint Homotopies Property, defined in the next section and abbreviated as DHP, then $X \times \mathbb{R}$ has the more familiar Disjoint Disks Property, henceforth abbreviated as DDP. Because the DDP characterizes resolvable ENR homology manifolds of dimension $n \geq 5$ as manifolds [10] [8, Theorem 24.3], it follows that $X \times \mathbb{R}$ is a genuine manifold if X is resolvable and has this DHP. Still unknown are both whether all such ENR homology manifolds have DHP and whether X having said DHP is a necessary condition for $X \times \mathbb{R}$ to be a manifold.

Since an ENR homology n -manifold X , $n \geq 4$, has DHP if it satisfies any of the properties mentioned in the second paragraph except (3), the sort of homology manifolds that might fail to have it are the ghastly examples of [9]. Halverson [12] has constructed some related ghastly examples that do have DHP.

In hopes of better understanding the special codimension one manifold factors, this paper builds on Halverson's earlier work to present a necessary and sufficient condition, the Disjoint Path Concordances Property defined at the outset of Section 2, for $X \times \mathbb{R}$ to be a manifold (again provided $\dim X \geq 4$).

2 Preliminaries

Throughout what follows both D and I stand for the unit interval, $[0,1]$. A metric space X is said to have the *Disjoint Homotopies Property* if any pair of path homotopies $f_i : D \times I \rightarrow X$ ($i = 1, 2$) can be approximated, arbitrarily closely, by homotopies $g_i : D \times I \rightarrow X$ such that

$$g_1(D \times t) \cap g_2(D \times t) = \emptyset, \text{ for all } t \in I.$$

Definition A *path concordance* in a space X is a map $F : D \times I \rightarrow X \times I$ such that $F(D \times e) \subset X \times e$, $e \in \{0, 1\}$.

Let $proj_X : X \times I \rightarrow X$ denote projection.

Definition A metric space (X, ρ) satisfies the *Disjoint Path Concordances Property (DCP)* if, for any two path homotopies $f_i : D \times I \rightarrow X$ ($i = 1, 2$) and any $\epsilon > 0$, there exist path concordances $F_i : D \times I \rightarrow X \times I$ such that

$$F_1(D \times I) \cap F_2(D \times I) = \emptyset$$

and $\rho(f_i, \text{proj}_X F_i) < \epsilon$.

A *homology n -manifold* X is a locally compact metric space such that $H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{\text{origin}\}; \mathbb{Z})$ for all $x \in X$. An *ENR homology n -manifold* X is an homology n -manifold which is homeomorphic to a retract of an open subset of some Euclidean space (ENR is the abbreviation for "Euclidean neighborhood retract"); equivalently, X is a homology n -manifold which is both finite dimensional and locally contractible.

A homology n -manifold X is *resolvable* provided there exists a surjective, cell-like mapping $f : M \rightarrow X$ defined on an n -manifold M . Quinn [13] has shown that (connected) ENR homology n -manifolds X , $n \geq 4$, are resolvable if and only if a certain index $i(X) \in 1 + 8\mathbb{Z}$ equals 1.

A metric space X has the *Disjoint Arcs Property (DAP)* if every pair of maps $f_i : D \rightarrow X$ ($i = 1, 2$) can be approximated, arbitrarily closely, by maps $g_i : D \rightarrow X$ for which $g_1(D) \cap g_2(D) = \emptyset$. All ENR homology n -manifolds, $n \geq 3$, have the DAP.

The following Homotopy Extension Theorem is fairly standard. We state it here because it will be applied several times in our arguments.

Theorem 2.1 (Controlled Homotopy Extension Theorem(CHET)) *Suppose X is a metric ANR, C is a compact subset of X , $j : C \rightarrow X$ is the inclusion map, and $\epsilon > 0$. Then there exists $\delta > 0$ such that for each map $f : Y \rightarrow C$ defined on a normal space Y , each closed subset Z of Y , and each map $g_Z : Z \rightarrow X$ which is δ -close to $jf|_Z$, g_Z extends to a map $g : Y \rightarrow X$ which is ϵ -homotopic to jf . In particular, for any open set U for which $Z \subset U \subset Y$, there is a homotopy $H : Y \times I \rightarrow X$ such that*

- (1) $H_0 = jf$ and $H_1 = g$
- (2) $g|_Z = g_Z$
- (3) $H_t|_{Y-U} = jf|_{Y-U}$ for all $t \in I$
- (4) $\text{diam}(H(y \times I)) < \epsilon$ for all $y \in Y$.

3 Main Results

In this section we will demonstrate that DCP characterizes codimension one manifold factors among ENR homology n -manifolds, $n \geq 4$, of trivial Quinn

index. Essentially the DCP condition requires that any pair of level preserving path homotopies into $X \times I$ can be “approximated” by disjoint path concordances, where the approximation is measured in the X factor. The following crucial proposition demonstrates that the DCP condition implies that any pair of level preserving path homotopies can be approximated, as measured in $X \times I$, by disjoint path concordances.

Proposition 3.1 *Suppose that (X, ρ) is a metric ANR with DAP. Then X has DCP if and only if given any pair of level preserving maps $f_i : D \times I \rightarrow X \times I$ ($i = 1, 2$) and $\epsilon > 0$ there are maps $g_i : D \times I \rightarrow X \times I$ such that*

- (1) f_i and g_i are ϵ -close in $X \times I$,
- (2) $g_i|_{D \times \partial I}$ is level preserving, and
- (3) $g_1(D \times I) \cap g_2(D \times I) = \emptyset$.

Moreover, if $f_1(D \times \partial I) \cap f_2(D \times \partial I) = \emptyset$, then we also may require that $g_i|_{D \times \partial I} = f_i|_{D \times \partial I}$.

Proof Assume X has DCP. Use $\tilde{\rho}$ to denote the obvious sum metric on $X \times I$. Let $f_i : D \times I \rightarrow X \times I$ ($i = 1, 2$) be level preserving maps and let $\epsilon > 0$. Choose $0 = t_0 < t_1 < \dots < t_m = 1$ such that $t_k - t_{k-1} < \epsilon/2$ for $k = 1 \dots m$. Define $J_k = [t_k, t_{k-1}]$ and $f_i[k] = f_i|_{D \times J_k}$. Since X has DAP then applying Theorem 2.1 (CHET) (near $\text{proj}_X(f_1(D \times I) \cup f_2(D \times I))$) we may assume that $f_1(D \times t_k) \cap f_2(D \times t_k) = \emptyset$. Choose $\eta > 0$ so that $\tilde{\rho}(f_1(D \times t_k), f_2(D \times t_k)) > \eta$ for $k = 1, \dots, n$. Let $\delta > 0$ satisfy CHET for $X \times I$, a small compact neighborhood C of $f_1(D \times I) \cup f_2(D \times I)$ and $\min\{\eta/2, \epsilon/2\}$, ($i = 1, 2$). Then by DCP, applied to the intervals J_k , there are maps $g_i[k] : D \times J_k \rightarrow X \times J_k$ satisfying

- (1) $g_i[k](D \times J_k) \subset C$,
- (2) $\rho(\text{proj}_X f_i[k], \text{proj}_X g_i[k]) < \delta$,
- (3) $g_i[k]|_{D \times \partial J_k}$ is level preserving, and
- (4) $g_1[k](D \times J_k) \cap g_2[k](D \times J_k) = \emptyset$.

By CHET and the choice of δ there are maps $G_i[k] : D \times J_k \rightarrow X \times J_k$ satisfying

- (1) $\rho(\text{proj}_X f_i[k], \text{proj}_X G_i[k]) < \epsilon/2$,
- (2) $G_i[k]|_{D \times \partial J_k} = f_i[k]|_{D \times \partial J_k}$, and
- (3) $G_1[k](D \times J_k) \cap G_2[k](D \times J_k) = \emptyset$.

Set $G_i = \bigcup_k G_i[k]$. Confirming that G_1 and G_2 are the desired maps is straightforward.

The reverse direction is trivial. It merely requires treating any pair of path homotopies $D \times I \rightarrow X$ as level preserving maps $D \times I \rightarrow X \times I$. \square

Definition A metric space (X, ρ) satisfies the *Disjoint 1-Complex Concordances Property (DCP*)* if, for any two homotopies $f_i : K_i \times I \rightarrow X$ ($i = 1, 2$), where K_i is a finite 1-complex, and any $\epsilon > 0$ there exist concordances $F_i : K_i \times I \rightarrow X \times I$ such that

$$F_1(K_1 \times I) \cap F_2(K_2 \times I) = \emptyset,$$

$$F_i(K_i \times e) \subset X \times e \text{ for } e \in \partial I, \text{ and } \rho(f_i, \text{proj}_X F_i) < \epsilon.$$

Proposition 3.2 Suppose X is a locally compact, metrizable ANR with DAP. Then X has DCP if and only if X has DCP*.

Proof This argument is similar to the one showing the equivalence of the DDP with approximability of maps defined on finite 2-complexes by embeddings [8, Theorem 24.1], and also to another one showing the equivalence of DHP (for paths) and a Disjoint Homotopies Property for 1-complexes [11, Theorem 2.9]. We supply the short proof for completeness.

To show the forward direction, endow X with a complete metric ρ . Let K_i ($i = 1, 2$) be a finite 1-simplicial complex, and define

$$\mathcal{H} = \{(f_1, f_2) \in \text{Map}(K_1 \times I, X \times I) \times \text{Map}(K_2 \times I, X \times I) \mid f_i|_{D \times \partial I} \text{ is level preserving}\}$$

with the uniform metric. Note that \mathcal{H} is a complete metric space and, therefore, a Baire space. For $\sigma_i \in K_i$, let

$$\mathcal{O}(\sigma_1, \sigma_2) = \{(f_1, f_2) \in \mathcal{H} \mid f_1(\sigma_1 \times I) \cap f_2(\sigma_2 \times I) = \emptyset\}.$$

Clearly $\mathcal{O}(\sigma_1, \sigma_2)$ is open in \mathcal{H} . To see that $\mathcal{O}(\sigma_1, \sigma_2)$ is dense in \mathcal{H} , let $\epsilon > 0$. Choose $\delta > 0$ to satisfy CHET for $X \times I$, a small compact neighborhood there of $\bigcup_i f_i(\sigma_i \times I)$ and ϵ . Then choose $\eta > 0$ to satisfy CHET for $X, \text{proj}_X(\bigcup_i f_i(\sigma_i \times I))$ and δ . By Proposition 3.1 there are η -approximations g_i to $f_i|_{\sigma_i \times I}$ that are disjoint path concordances. First apply CHET to extend g_i over $(\sigma_i \times I) \cup (K_i \times \partial I)$ so that the new g_i is level preserving on $K_i \times \partial I$ and δ -close to

$f_i|_{(\sigma_i \times I) \cup (K_i \times \partial I)}$. Then apply CHET again to extend g_i over $K_i \times I$ so that g_i is now ϵ -close to f_i . Thus, $\mathcal{O}(\sigma_1, \sigma_2)$ is dense in \mathcal{H} . Since \mathcal{H} is a Baire space,

$$\mathcal{O} = \bigcap_{(\sigma_1, \sigma_2) \in K_1 \times K_2} \mathcal{O}(\sigma_1, \sigma_2)$$

is dense in \mathcal{H} . Note that if $(f_1, f_2) \in \mathcal{O}$ then $f_1(K_1 \times I) \cap f_2(K_2 \times I) = \emptyset$. Hence, X has DCP*.

The other direction is trivial. \square

Definition A topography on $D \times I$, Υ , consists of the following elements:

- (1) A set, $\{J_1, \dots, J_m\}$, of consecutive intervals in \mathbb{R} such that $J_j = [t_{j-1}, t_j]$ where $t_0 < t_1 < \dots < t_m$.
- (2) A finite set of complexes $\{L_0, \dots, L_m\}$ embedded in $D \times I$ of dimension at most one. These complexes are called the *transition levels*.
- (3) A finite set of 1-complexes $\{K_1, \dots, K_m\}$. These complexes are called the *level factors*.
- (4) A set of maps, $\{\phi_j : K_j \times J_j \rightarrow D \times I\}_{j=1, \dots, m}$, which satisfy the following conditions:
 - (a) $L_0 = \phi_1(K_1 \times \{t_0\})$
 $L_m = \phi_m(K_m \times \{t_m\})$
 $L_j = \phi_j(K_j \times \{t_j\}) \cup \phi_{j+1}(K_{j+1} \times \{t_j\})$ for $j = 1, \dots, m-1$
 - (b) $\phi_j|_{K_j \times \text{int } J_j}$ is an embedding for each $j = 1, \dots, m$
 - (c) $\bigcup_{j=1}^m \text{im } (\phi_j) = D \times I$

It is shown in [11] that a p.l. general position approximation of the projection of a map $f : D \times I \rightarrow X \times \mathbb{R}$ to the \mathbb{R} factor induces a topographical structure on the domain $D \times I$ of f .

Definition A map $f : D \times I \rightarrow X \times \mathbb{R}$ is a *topographical map* if there is a topography, Υ , on $D \times I$ which is level preserving in the sense that for each map $\phi_j : K_j \times J_j \rightarrow D \times I$ of the topography, $f \circ \phi_j(K_j \times t) \subset X \times t$ for all $t \in J_j$.

Note that a topographical structure on $D \times I$ is in no way related to the product structure of $D \times I$. The proof of the following lemma is provided in [11, Theorem 3.3]:

Lemma 3.3 Every map $f : D \times I \rightarrow X \times \mathbb{R}$ can be approximated by a topographical map $g : D \times I \rightarrow X \times \mathbb{R}$.

Theorem 3.4 (Disjoint Concordances Theorem) Suppose X is a locally compact, metric ANR with DAP. Then X has DCP if and only if $X \times \mathbb{R}$ has DDP.

Proof (\Leftarrow) Given two path homotopies $f_i : D \times I \rightarrow X$ ($i = 1, 2$), treat them as level preserving maps $f_i : D \times I \rightarrow X \times I$. Applying DAP, and using CHET as before, we may assume without loss of generality that $f_1(D \times \partial I) \cap f_2(D \times \partial I) = \emptyset$. Since $X \times (0, 1)$ has DDP, f_i can be approximated, fixing the actions on $D \times \{0, 1\}$, by ϵ -close maps g_i ($i = 1, 2$) for which the images of g_1, g_2 are disjoint. The DCP follows.

(\Rightarrow) Given maps $f_i : I^2 = D \times I \rightarrow X \times \mathbb{R}$ for $i = 1, 2$, by Lemma 3.3 we may assume that f_i is a topographical map with topography $\Upsilon[i]$. An object O in the definition of $\Upsilon[i]$ will be denoted as $O[i]$. Note that we also may assume the following:

- (1) The set of intervals $\{J_j[i]\}$ are the same for $i = 1, 2$. This follows from subdividing the intervals appropriately as outlined in [11, Theorem 3.4].
- (2) $f_1(\bigcup L_j[1]) \cap f_2(\bigcup L_j[2]) = \emptyset$. This follows from DAP and CHET.

By DCP* there are maps $\psi_j[i] : K_j[i] \times J_j$ approximating $\phi_j f_i$ such that $\text{im} \psi_j[1] \cap \text{im} \psi_j[2] = \emptyset$. Applying Theorem 2.1, we may assume that $\psi_j[i]|_{K_j[i] \times \partial J_j} = \phi_j f_i|_{K_j[i] \times \partial J_j}$. (Recall that $\phi_j f_i(K_j[i] \times \partial J_j) \subset (L_j \cup L_{j-1})$.) Set $g_i = \bigcup_j \psi_j[i]$.

Then g_1 and g_2 are the desired disjoint approximations of f_1 and f_2 . \square

Corollary 3.5 An ENR homology n -manifold $X, n \geq 4$, has DCP if and only if $X \times \mathbb{R}$ has DDP.

When $n=3$ Corollary 3.5 is formally but vacuously true: since no ENR homology 4-manifold has DDP, no such homology 3-manifold can have DCP.

Corollary 3.6 Let X be an ENR homology n -manifold, $n \geq 4$. Then $X \times \mathbb{R}$ is a manifold if and only if X has trivial Quinn index and satisfies DCP.

Theorem 3.4 demonstrates the equivalence in $X \times \mathbb{R}$ of DCP and DDP. By the standard combination of work Edwards [10] and of Quinn [13], $X \times \mathbb{R}$ is a manifold if and only if it has both this latter property and trivial Quinn index. Furthermore, by [13], $i(X \times \mathbb{R}) = i(X)$, without regard to any of these general position properties.

4 Questions

1. Is every finite-dimensional Busemann space (see [14]) X necessarily a manifold? Actually, there are two unsettled questions here: is $i(X)$, the Quinn index, trivial? When $\dim X > 4$, must X have DDP? It may be of interest to add that X is known to be a manifold if $\dim X \leq 4$ [14].

The same pair of concerns crops up in the following setting.

2. Suppose for any two points p, q of the compact, finite-dimensional metric space X , there is a homeomorphism from the suspension of a space Y onto X that carries the suspension points onto p, q . Is X a manifold?

3. Given an ENR homology n -manifold $X, n \geq 4$, can maps $f, g : I^2 \rightarrow X$ be approximated by $F, G : I^2 \rightarrow X$ for which there exists a 0-dimensional F_σ set $T \subset I^2$ such that

$$F(I^2 - T) \cap G(I^2 - T) = \emptyset?$$

If so, $X \times \mathbb{R}$ will satisfy DDP. Recall that $X \times \mathbb{R}$ satisfies the DDP when $S(X)$, the singular set of X , is at most $(n-2)$ -dimensional. A key to the argument is that then one can obtain $F, G : I^2 \rightarrow X$ and a compact, 0-dimensional T with $F(I^2 - T) \cap G(I^2 - T) = \emptyset$. Similar reasoning applies with F_σ -subsets T in place of compact subsets. It may be worth noting that, because X does satisfy the DAP, one can easily obtain (maps F, G and) such a T which is a 0-dimensional, G_δ -subset of I^2 , but the argument requires a 0-dimensional F_σ -subset, which is more "meager".

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Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA

Department of Mathematics Brigham Young University, Provo, UT 84602, USA

Email: `daverman@math.utk.edu` `deniseh@math.byu.edu`

The Bryant–Ferry–Mio–Weinberger construction of generalized manifolds

FRIEDRICH HEGENBARTH AND DUŠAN REPOVŠ

Abstract

Following Bryant, Ferry, Mio and Weinberger we construct generalized manifolds as limits of controlled sequences $\{X_i \xrightarrow{p_i} X_{i-1} \mid i = 1, 2, \dots\}$ of controlled Poincaré spaces. The basic ingredient is the ε - δ -surgery sequence recently proved by Pedersen, Quinn and Ranicki. Since one has to apply it not only in cases when the target is a manifold, but a controlled Poincaré complex, we explain this issue very roughly (Theorem 3.5). Specifically, it is applied in the inductive step to construct the desired controlled homotopy equivalence $p_{i+1} : X_{i+1} \rightarrow X_i$. Theorem 3.5 requires a sufficiently controlled Poincaré structure on X_i (over X_{i-1}). Our construction shows that this can be achieved. In fact, the Poincaré structure of X_i depends upon a homotopy equivalence used to glue two manifold pieces together (the rest is surgery theory leaving unaltered the Poincaré structure). It follows from the ε - δ -surgery sequence (more precisely from the Wall realization part) that this homotopy equivalence is sufficiently well controlled. In §4 we give additional explanation why the limit space of the X_i 's has no resolution.

AMS Classification Primary 57PXX; Secondary 55RXX

Keywords Generalized manifold, ANR, Poincaré duality, ε - δ -surgery, controlled, Poincaré complex, Quinn index, cell-like resolution

1 Preliminaries

A *generalized n -dimensional manifold* X is characterized by the following two properties:

- (i) X is a Euclidean neighborhood retract (ENR); and
- (ii) X has the local homology (with integer coefficients) of the Euclidean n -space \mathbb{R}^n , i.e.

$$H_*(X, X \setminus \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

Since we deal here with locally compact separable metric spaces of finite (covering) dimension, ENR's are the same as ANR's.

Generalized manifolds are Poincaré spaces, in particular they have the Spivak normal fibrations ν_X . The total space of ν_X is the boundary of a regular neighborhood $N(X) \subset \mathbb{R}^L$ of an embedding $X \subset \mathbb{R}^L$, for some large L . One can assume that $N(X)$ is a mapping cylinder neighborhood (see [5, Corollary 11.2]).

The global Poincaré duality of Poincaré spaces does not imply the local homology condition (ii) above. The local homology condition can be understood as the **controlled** global Poincaré duality (see [9, p. 270], and [1, Proposition 4.5]). More precisely, one has the following:

Theorem 1.1 *Let X be a compact ANR Poincaré duality space of finite (covering) dimension. Then X is a generalized manifold if and only if for every $\delta > 0$, X is a δ -Poincaré space (over X).*

The definition of the δ -Poincaré property is given below. The following basic fact about homology manifolds was proved by Ferry and Pedersen (see [4, Theorem 16.6]):

Theorem 1.2 *Let X be an ANR homology manifold. Then ν_X has a canonical TOP reduction.*

This statement is equivalent to existence of degree one normal maps $f : M^n \rightarrow X$, where M^n is a (closed) topological n -manifold, hence the structure set $\mathcal{S}^{TOP}(X)$ can be identified with $[X, G/TOP]$.

Let us denote the 4-periodic simply connected surgery spectrum by \mathbb{L} and let $\widehat{\mathbb{L}}$ be the connected covering of \mathbb{L} . There is a (canonical) map of spectra $\widehat{\mathbb{L}} \rightarrow \mathbb{L}$ given by the action of $\widehat{\mathbb{L}}$ on \mathbb{L} . Note that $\widehat{\mathbb{L}}_0$ is G/TOP .

If M^n is a topological manifold there exists a fundamental class $[M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet)$, where \mathbb{L}^\bullet is the symmetric surgery spectrum (see [11, Chapters 13 and 16]).

Theorem 1.3 *If M^n is a closed oriented topological n -manifold, then the cap product with $[M]_{\mathbb{L}}$ defines a Poincaré duality of \mathbb{L} -(co-)homology*

$$H^p(M; \mathbb{L}) \xrightarrow{\cong} H_{n-p}(M; \mathbb{L})$$

and $\widehat{\mathbb{L}}$ (co-)homology

$$H^p(M; \widehat{\mathbb{L}}) \xrightarrow{\cong} H_{n-p}(M; \widehat{\mathbb{L}}).$$

Since $H^0(M; \mathbb{L}) = [M, \mathbb{Z} \times G/TOP]$ and $H^0(M; \widehat{\mathbb{L}}) = [M, G/TOP]$, one has $H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$ and the map $\widehat{\mathbb{L}} \rightarrow \mathbb{L}$ has the property that the image of $H_n(M; \widehat{\mathbb{L}}) \rightarrow H_n(M; \mathbb{L}) = \mathbb{Z} \times H_n(M; \widehat{\mathbb{L}})$ is $\{1\} \times H_n(M; \widehat{\mathbb{L}})$ (see [11, Appendix C]). Moreover, the action of $H^0(M; \widehat{\mathbb{L}})$ on $H^0(M; \mathbb{L}) = \mathbb{Z} \times H^0(M; \widehat{\mathbb{L}})$, induced by the action of $\widehat{\mathbb{L}}$ on \mathbb{L} , preserves the \mathbb{Z} -sectors.

If X is a generalized n -manifold we get similar results by using the fundamental class $f_*([M]_{\mathbb{L}}) = [X]_{\mathbb{L}} \in H_n(X; \mathbb{L}^\bullet)$, where $f : M \rightarrow X$ is the canonical degree one normal map. So the composition map

$$\Theta : [X, G/TOP] \rightarrow H_n(X; \widehat{\mathbb{L}}) \rightarrow H_n(X; \mathbb{L}) = \mathbb{Z} \times H_n(X; \widehat{\mathbb{L}})$$

has the property that $\text{Im } \Theta$ belongs to a single \mathbb{Z} -sector, denoted by $I(X) \in \mathbb{Z}$.

The following is the fundamental result of Quinn on resolutions of generalized manifolds (see [10]):

Theorem 1.4 *Let X be a generalized n -manifold, $n \geq 5$. Then X has a resolution if and only if $I(X) = 1$.*

Remark The integer $I(X)$ is called the *Quinn index* of the generalized manifold X . Since the action of $\widehat{\mathbb{L}}$ on \mathbb{L} preserves the \mathbb{Z} -sectors, arbitrary degree one normal maps $g : N \rightarrow X$ can be used to calculate $I(X)$. Alternatively, we can define $I(X)$ using the fibration $\widehat{\mathbb{L}} \rightarrow \mathbb{L} \rightarrow \mathbb{K}(\mathbb{Z}, 0)$, where $\mathbb{K}(\mathbb{Z}, \cdot)$ is the Eilenberg–MacLane spectrum, and define $I(X)$ as the image of (see [11, Chapter 25]):

$$\{f : M \rightarrow X\} \in H_n(X; \mathbb{L}) \rightarrow H_n(X; \mathbb{K}(\mathbb{Z}, 0)) = H_n(X; \mathbb{Z}) = \mathbb{Z}.$$

We assume that X is oriented. Therefore $I(X)$ is also defined for Poincaré complexes, as long as we have a degree one normal map $f : M \rightarrow X$, determining an element in $H_n(X; \mathbb{L})$. In this case $I(X)$ is not a local index. In fact, for generalized manifolds one has local \mathbb{L} -Poincaré duality using locally finite chains, hence we can define $I(\mathcal{U})$ for any open set $\mathcal{U} \subset X$. It is also easy to see that $I(\mathcal{U}) = I(X)$. On the algebraic side $I(X)$ is an invariant of the controlled Poincaré duality type (see [11, p. 283]).

2 Constructing generalized manifolds from controlled sequences of Poincaré complexes

Beginning with a closed topological n -manifold M^n , $n \geq 5$, and $\sigma \in H_n(M; \mathbb{L})$, we shall construct a sequence of closed Poincaré duality spaces X_0, X_1, X_2, \dots and maps $p_i : X_i \rightarrow X_{i-1}$, $p_0 : X_0 \rightarrow M$.

We assume that M is a PL manifold, or that M has a cell structure. The X_i 's are built by gluing manifolds along boundaries with homotopy equivalences, and by doing some surgeries outside the singular sets. Hence all X_i 's have cell decompositions.

We can assume that the X_i 's lie in a (large enough) Euclidean space \mathbb{R}^L which induces the metric on X_i . So the cell chain complex $C_\#(X_i)$ can be considered as a geometric chain complex over X_{i-1} with respect to $p_i : X_i \rightarrow X_{i-1}$, i.e. the distance between two cells of X_i over X_{i-1} is the distance between the images of the centers of these two cells in X_{i-1} . Let us denote the distance function by d .

We now list five properties of the sequence $\{(X_i, p_i)\}_i$, including some definitions and comments. For each $i \geq 0$ we choose positive real numbers ξ_i and η_i .

- (i) $p_i : X_i \rightarrow X_{i-1}$ and $p_0 : X_0 \rightarrow M$ are \mathcal{UV}^1 -maps. This means that for every $\varepsilon > 0$ and for all diagrams

$$\begin{array}{ccc} K_0 & \xrightarrow{\alpha_0} & X_i \\ \cap & \nearrow \bar{\alpha} & \downarrow p_i \\ K & \xrightarrow{\alpha} & X_{i-1} \end{array}$$

K a 2-complex, $K_0 \subset K$ a subcomplex and maps α_0, α , there is a map $\bar{\alpha}$ such that $\alpha|_{K_0} = \alpha_0$ and $d(p_i \circ \bar{\alpha}, \alpha) < \varepsilon$. (This is also called $\mathcal{UV}^1(\varepsilon)$ property.)

- (ii) X_i is an η_i -Poincaré complex over X_{i-1} , i.e.
- (a) all cells of X_{i-1} have diameter $< \eta_i$ over X_{i-1} ; and
 - (b) there is an n -cycle $c \in C_n(X_i)$ inducing an η_i -chain equivalence $\cap c : C^\#(X_i) \rightarrow C_{n-\#}(X_i)$.

Equivalently, the diagonal $\Delta_\#(c) = \sum c' \otimes c'' \in C_\#(X) \otimes C_\#(X)$ has the property that $d(c', c'') < \eta_i$ for all tensor products appearing in $\Delta_\#(c)$.

- (iii) $p_i : X_i \rightarrow X_{i-1}$ is an ξ_i -homotopy equivalence over X_{i-2} , $i \geq 2$. In other words, there exist an inverse $p'_i : X_{i-1} \rightarrow X_i$ and homotopies $h_i : p'_i \circ p_i \simeq \text{Id}_{X_i}$, $h'_i : p_i \circ p'_i \simeq \text{Id}_{X_{i-1}}$ such that the tracks $\{(p_{i-1} \circ p_i \circ h_i)(x, t) \mid t \in [0, 1]\}$ and $\{(p_{i-1} \circ h'_i)(x', t) \mid t \in [0, 1]\}$ have diameter $< \xi_i$, for each $x \in X_i$ (resp. $x' \in X_{i-1}$). Note that p_0 need not be a homotopy equivalence.

- (iv) There is a regular neighborhood $W_0 \subset \mathbb{R}^L$ of X_0 such that $X_i \subset W_0$, $i = 0, 1, \dots$ and retractions $r_i : W_0 \rightarrow X_i$, satisfying $d(r_i, r_{i-1}) < \xi_i$ in \mathbb{R}^L .
- (v) There are **thin** regular neighborhoods $W_i \subset \mathbb{R}^L$, $\pi_i : W_i \rightarrow X_i$, with $W_i \subset \overset{\circ}{W}_{i-1}$ such that $W_{i-1} \setminus \overset{\circ}{W}_i$ is an ξ_i -thin h -cobordism with respect to $r_i : W_0 \rightarrow X_i$.

Let $W = W_{i-1} \setminus \overset{\circ}{W}_i$. Then there exist deformation retractions $r_t^0 : W \rightarrow \partial_0 W$ and $r_t^1 : W \rightarrow \partial_1 W$ with tracks of size $< \xi_i$ over X_{i-1} , i.e. the diameters of $\{(r_i \circ r_t^0)(w) \mid t \in [0, 1]\}$ and $\{(r_i \circ r_t^1)(w) \mid t \in [0, 1]\}$ are smaller than ξ_i . Moreover, we can choose η_i and ξ_i such that:

- (a) $\sum \eta_i < \infty$; and
- (b) $W_{i-1} \setminus \overset{\circ}{W}_i$ has a δ_i -product structure with $\sum \delta_i < \infty$, i.e. there is a homeomorphism $W = W_{i-1} \setminus \overset{\circ}{W}_i \xleftarrow{H} \partial_0 W \times I$ satisfying $\text{diam}\{(r_i \circ H)(w, t) \mid t \in I\} < \delta_i$, for every $w \in \partial_0 W$.

The property (v)(b) above follows from the **thin h -cobordism** theorem (see [8]). One can assume that $\sum \xi_i < \infty$. Let $X = \bigcap_i W_i$. We are going to show that X is a generalized manifold:

- (1) The map $r = \varinjlim r_i : W_0 \rightarrow X$ is well-defined and is a retraction, hence X is an ANR.
- (2) To show that X is a generalized manifold we shall apply the next two theorems. They also imply Theorem 1.1 above. The first one is due to Daverman and Husch [2], but it is already indicated in [8] (see the remark after Theorem 3.3.2):

Theorem 2.1 *Suppose that M^n is a closed topological n -manifold, B is an ANR, and $p : M \rightarrow B$ is proper and onto. Then B is a generalized manifold, provided that p is an approximate fibration.*

Approximate fibrations are characterized by the property that for every $\varepsilon > 0$ and every diagram

$$\begin{array}{ccc} K \times \{0\} & \xrightarrow{H_0} & M \\ \cap & \nearrow H & \downarrow p \\ K \times I & \xrightarrow{h} & B \end{array}$$

where K is a polyhedron, there exists a lifting H of h such that $d(p \circ H, h) < \varepsilon$. Here d is a metric on B . In other words, $p : M \rightarrow B$ has the ε -homotopy lifting property for all $\varepsilon > 0$.

We apply Theorem 2.1 to the map $\rho : \partial W_0 \rightarrow X$ defined as follows: Let $\rho : W_0 \rightarrow X$ be the map which associates to $w \in W_0$ the endpoint $\rho(x) \in X$ following the tracks defined by the **thin** product structures of the h -cobordism when decomposing

$$W_0 = (W_0 \setminus \overset{\circ}{W}_1) \cup (W_1 \setminus \overset{\circ}{W}_2) \cup \dots$$

The restriction to ∂W_0 will also be denoted by ρ . By (v)(b) above the map ρ is well-defined and continuous. We will show that it is an ε -approximate fibration for all $\varepsilon > 0$.

The map $\rho : W_0 \rightarrow X$ is the limit of maps $\rho_i : W_0 \rightarrow X_i$, where ρ_i is the composition given by the tracks $(W_0 \setminus \overset{\circ}{W}_1) \cup (W_1 \setminus \overset{\circ}{W}_2) \cup \dots \cup (W_{i-1} \setminus \overset{\circ}{W}_i)$ followed by $\pi_i : W_i \rightarrow X_i$. The second theorem is Proposition 4.5 of [1]:

Theorem 2.2 *Given n and B , there exist $\varepsilon_0 > 0$ and $T > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following holds: If $X \xrightarrow{p} B$ is an ε -Poincaré complex with respect to the UV^1 -map p and $W \subset \mathbb{R}^L$ is a regular neighborhood of $X \subset \mathbb{R}^L$, i.e. $\pi : W \rightarrow X$ is a neighborhood retraction, then $\pi|_{\partial W} : \partial W \rightarrow X$ has the $T\varepsilon$ -lifting property, provided that the codimension of X in \mathbb{R}^L is ≥ 3 .*

This is applied as follows: Let $B \subset \mathbb{R}^L$ be a (small) regular neighborhood of $X \subset \mathbb{R}^L$. Hence $X_k \subset W_k \subset B$ for sufficiently large k . It follows by property (ii) that X_i is an η_i -Poincaré complex over $X_i \xrightarrow{p_i} X_{i-1} \subset B$, hence (for i sufficiently large) we get the following:

Corollary 2.3 $\rho_i : \partial W_0 \rightarrow X_i$ is a $T\eta_i$ -approximate fibration over B .

Proof By the theorem above, $\pi_i : \partial W_i \rightarrow X_i$ is a $T\eta_i$ -approximate fibration over B , hence so is $\rho_i : \partial W_0 \cong \partial W_i \rightarrow X_i$. \square

It follows by construction that $\varprojlim_{p_i} X_i = X \subset B$, hence we have in the limit an approximate fibration $\rho : \partial W_0 \rightarrow X$ over $\text{Id} : X \rightarrow X$, i.e. X is a generalized manifold. We will show in §4 that $I(X)$ is determined by the \mathbb{Z} -sector of $\sigma \in H_n(M; \mathbb{L})$.

3 Construction of the sequence of controlled Poincaré complexes

Before we begin with the construction we need more fundamental results about controlled surgery and approximations.

3.1 $\varepsilon - \delta$ surgery theory

We recall the main theorem of [6]. Let B be a finite-dimensional compact ANR, and N^n a compact n -manifold (possibly with nonempty boundary ∂N), $n \geq 4$. Then there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exist $\delta > 0$ with the following property:

If $p : N \rightarrow B$ is a $\mathcal{UV}^1(\delta)$ map, then there exists a controlled exact surgery sequence:

3.1

$$H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon, \delta}(N, p) \rightarrow [N, \partial N; G/TOP, *] \xrightarrow{\Theta} H_n(B; \mathbb{L}).$$

The controlled structure set $\mathcal{S}_{\varepsilon, \delta}(N, p)$ is defined as follows: Elements of $\mathcal{S}_{\varepsilon, \delta}(N, p)$ are (equivalence) classes of (M, g) , where M is an n -manifold, $g : M \rightarrow N$ is a δ -homotopy equivalence over B and $g|_{\partial M} : \partial M \rightarrow \partial N$ is a homeomorphism. The pair (M, g) is related to (M', g') if there is a homeomorphism $h : M \rightarrow M'$, such that the diagram

$$\begin{array}{ccc} \partial M & \xrightarrow{h} & \partial M' \\ g \searrow & & \swarrow g' \\ & \partial N & \end{array}$$

commutes, and $g' \circ h$ is ε -homotopic to g over B . Since ε is fixed, this relation is not transitive. It is part of the assertion that it is actually an equivalence relation. Then $\mathcal{S}_{\varepsilon, \delta}(N, p)$ is the set of equivalence classes of pairs (M, g) .

As in the classical surgery theory, the map

3.2

$$H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon, \delta}(N, p)$$

is the controlled realization of surgery obstructions, and

3.3

$$\mathcal{S}_{\varepsilon,\delta}(N,p) \rightarrow [N, \partial N; G/TOP, *] \xrightarrow{\Theta} H_n(B; \mathbb{L})$$

is the actual (controlled) surgery part. The following discussion will show that (3.3) also holds for controlled Poincaré spaces (see Theorem 3.5 below). Moreover, δ is also of (arbitrary) small size, provided that such is also ε .

To see this we will go through some of the main points of the proof of Theorem 1 from [6]. For $\eta, \eta' > 0$ we denote by $L_n(B, \mathbb{Z}, \eta, \eta')$ the set of highly η -connected n -dimensional quadratic Poincaré complexes modulo highly η' -connected algebraic cobordisms. Then there is a well-defined obstruction map

$$\Theta_\eta : [N, G/TOP] \rightarrow L_n(B, \mathbb{Z}, \eta, \eta')$$

(for simplicity we shall assume that $\partial N = \emptyset$). If $(f, b) : M^n \rightarrow N^n$ is a degree one normal map one can do controlled surgery to obtain a highly η -connected normal map $(f', b') : M'^n \rightarrow N^n$ over B . If N^n is a manifold this can be done for every $\eta > 0$. If N^n is a Poincaré complex, it has to be η -controlled over B . By Theorem 1.1 above this holds in particular for generalized manifolds.

Given $\eta > 0$ there is an $\eta' > 0$ such that: if (f', b') and (f'', b'') are normally bordant highly η -connected degree one normal maps, there is then a highly η' -connected normal bordism between them. (Again this is true if N is an η -Poincaré complex over B .) This defines Θ_η .

To eventually complete surgeries in the middle dimension we assume that the map $p : N \rightarrow B$ is \mathcal{UV}^1 . Then one has the following (see [6, p.3, line 28f1]): Given $\delta > 0$ there exists $\eta > 0$ such that if $\Theta_\eta([f', b']) = 0$, then (f', b') is normally cobordant to a δ -homotopy equivalence. Moreover, if (f'', b'') and (f', b') are highly η -connected degree one normal maps being normally cobordant, then there is a highly connected η' -bordism between them (i.e. for given η there is such an η'). Then controlled surgery produces a controlled h -cobordism which gives an ε -homotopy by the thin h -cobordism theorem. This defines an element of $\mathcal{S}_{\varepsilon,\delta}(N,p)$, and shows the semi-exactness of the sequence

3.4

$$\mathcal{S}_{\varepsilon,\delta}(N,p) \rightarrow [N, G/TOP] \xrightarrow{\Theta_\eta} L_n(B, \mathbb{Z}, \eta, \eta'),$$

i.e. that $\mathcal{S}_{\varepsilon,\delta}(N,p)$ maps onto the kernel of Θ_η . We note that semi-exactness also holds for η -controlled Poincaré complexes over B .

One cannot expect the sequence (3.4) to be exact, i.e. that the composition map is zero, since passing from topology to algebra one loses control. As it

was noted in [6, p. 3], ε and δ are determined by the controlled Hurewicz and Whitehead theorems. Exactness of (3.4) will follow by the Squeezing Lemma (Lemma 4 of [7]).

The proof of (3.3) will be completed by showing that the assembly map

$$A : H_n(B; \mathbb{L}) \rightarrow L_n(B, \mathbb{Z}, \eta, \eta')$$

is bijective for sufficiently small η . This follows by splitting the controlled quadratic Poincaré complexes (i.e. the elements of $L_n(B', \mathbb{Z}, \eta, \eta')$) into small pieces over small simplices of B (we assume for simplicity that B is triangulated). If δ is given, and if we want a splitting where each piece is δ -controlled, we must start the subdivision with a sufficiently small η -controlled quadratic Poincaré complex (see Remark below). This can be done by Lemma 6 of [6] (see also [13, Lemma 2.5]). Since $A \circ \Theta = \Theta_\eta$, we get (3.3) from (3.4). The stability constant ε_0 is determined by the largest η for which A is bijective.

Remark Yamasaki has estimated the size of η in the Splitting Lemma. If one performs a splitting so that the two summands are δ -controlled, then one needs an η -controlled algebraic quadratic Poincaré complex with η of size $\delta/(an^k+b)$, where a, b, k depend on X (k is conjectured to be 1), and n is the length of the complex. Of course, squeezing also follows from the bijectivity of A for small η , but Lemma 3 of [7] is somehow a clean statement to apply (see Theorem 3.5 below). We also note that the bijectivity of A is of course, independent of whether N is a manifold or a Poincaré complex.

Theorem 3.5 *Suppose that $N \xrightarrow{p} B$ is a UV^1 map. Let $\delta > 0$ be given (sufficiently small, i.e. $\delta < \delta_0$ for some δ_0). Then there is $\eta > 0$ (small with respect to δ), such that if N is an η -Poincaré complex over B , and $(f, b) : M \rightarrow N$ is a degree one normal map, then $\Theta(f, b) = 0 \in H_n(B; \mathbb{L})$ if (and only if) (f, b) is normally bordant to a δ -equivalence.*

The **only if** part is more delicate and it follows by Lemma 3 of [7]: So let $f : M^n \rightarrow N^n$ be a δ -equivalence which defines a quadratic η_1 -Poincaré complex C in $L_n(B, \mathbb{Z}, \eta_1, \eta'_1)$ which is η_1 -cobordant to zero via $[N, G/TOP] \rightarrow L_n(B, \mathbb{Z}, \eta_1, \eta'_1)$.

Then C is $\kappa\eta_1$ -cobordant to an arbitrary small quadratic Poincaré complex (i.e. to a quadratic η -complex) which is $\kappa\eta'_1$ -cobordant to zero, with η_1 sufficiently small (i.e. η sufficiently small). In this case we can also assume that A is bijective. This proves the **only if** part.

Theorem 3.5 can also be stated as follows:

Theorem 3.5’ *Let N be a sufficiently fine η –Poincaré complex over a \mathcal{UV}^1 –map $p : N \rightarrow B$. Then there exist $\varepsilon > 0$ and $\delta > 0$, both sufficiently small, such that the sequence*

$$\mathcal{S}_{\varepsilon, \delta}(N, p) \rightarrow [N, G/TOP] \rightarrow H_n(B; \mathbb{L})$$

is exact. In particular, it holds for generalized manifolds.

3.2 \mathcal{UV}^1 approximation

Here we recall Proposition 4.3 and Theorem 4.4. of [1]:

Theorem 3.6 *Suppose that $f : (M^n, \partial M) \rightarrow B$ is a continuous map from a compact n –manifold with boundary such that the homotopy fiber of f is simply connected. If $n \geq 5$ then f is homotopic to a \mathcal{UV}^1 –map. In case that $f|_{\partial M}$ is already \mathcal{UV}^1 , the homotopy is relative ∂M .*

We state the second theorem in the form which we will need. We take it from Theorem 10.1 of [3]:

Theorem 3.7 (Ferry) *Let $p : N^n \rightarrow B$ be a map from a compact n –manifold into a polyhedron, $n \geq 5$.*

- (i) *Given $\varepsilon > 0$, there is a $\delta > 0$, such that if p is a $\mathcal{UV}^1(\delta)$ –map then p is ε –homotopic to a \mathcal{UV}^1 –map.*
- (ii) *Suppose that $p : N \rightarrow B$ is a \mathcal{UV}^1 map. Then for each $\varepsilon > 0$ there is a $\delta > 0$ (depending on p and ε) such that if $f : M \rightarrow N$ is a δ –1–connected map (over B) from a compact manifold M of $\dim M \geq 5$, then f is ε –close over B to a \mathcal{UV}^1 –map $g : M \rightarrow N$.*

3.3 Controlled gluing

The following is Proposition 4.6 of [1]:

Theorem 3.8 *Let $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ be (orientable) manifolds and $p_i : M_i \rightarrow B$ \mathcal{UV}^1 –maps. Then there exist $\varepsilon_0 > 0$ and $T > 0$ such that for ε with $0 < \varepsilon \leq \varepsilon_0$ and $h : \partial M_1 \rightarrow \partial M_2$ an (orientation preserving) ε –equivalence, $M_1 \cup_h M_2$ is a $T\varepsilon$ –Poincaré complex over B .*

3.4 Approximation of retractions

The following is Proposition 4.10 of [1]:

Theorem 3.9 *Let X and Y be finite polyhedra. Suppose that V is a regular neighborhood of X with $\dim V \geq 2 \dim Y + 1$ and $r : V \rightarrow X$ is the retraction. If $f : Y \rightarrow X$ is an ε -equivalence with respect to $p : X \rightarrow B$, then there exists an embedding $i : Y \rightarrow V$ and a retraction $s : V \rightarrow i(Y)$ with $d(p \circ r, p \circ s) < 2\varepsilon$.*

We now begin with the construction. Let M^n be a closed oriented (topological) manifold of dimension $n \geq 6$. Let $\sigma \in H_n(M; \mathbb{L})$ be fixed. Moreover, we assume that M is equipped with a simplicial structure. Then let $M = B \cup_D C$ be such that B is a regular neighborhood of the 2-skeleton, $D = \partial B$ is its boundary and C is the closure of the complement of B . So $D = \partial C = B \cap C$ is of dimension ≥ 5 .

By Part II above we can replace $(B, D) \subset M$ and $(C, D) \subset M$, by \mathcal{UV}^1 -maps $j : (B, D) \rightarrow M$ and $j : (C, D) \rightarrow M$, and we can realize σ according to $H_n(M; \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon, \delta}(D, j)$ by a degree one normal map $F_\sigma : V \rightarrow D \times I$ with $\partial_0 V = D$, $\partial_1 V = D'$, $F_\sigma|_{\partial_0 V} = \text{Id}$ and $f_\sigma = F_\sigma|_{\partial_1 V} : D' \rightarrow D$ a δ -equivalence over M .

We then define $X_0 = B \cup_{f_\sigma} -V \cup_{\text{Id}} C$, where $-V$ is the cobordism V turned upside down. We use the map $-F_\sigma \cup \text{Id} : -V \cup_{\text{Id}} C \rightarrow D \times I \cup C \cong C$ to extend j to a map $p_0 : X_0 \rightarrow M$.

The Wall realization $V \rightarrow D \times I$ is such that V is a cobordism built from D by adding high-dimensional handles (similarly beginning with D'). Therefore p_0 is a \mathcal{UV}^1 map: If (K, L) is a simplicial pair with K a 2-complex, and if there is given a diagram

$$\begin{array}{ccc} L & \xrightarrow{\alpha_0} & X_0 \\ \downarrow & & \downarrow p_0 \\ K & \xrightarrow{\alpha} & M \end{array}$$

then we first move (by an arbitrary small approximation) α and α_0 into B by general position arguments. Then one uses the \mathcal{UV}^1 -property of $j : B \rightarrow M$. By 3.8, X_0 is a $T\delta$ -Poincaré complex over M . Note that we can choose δ as

small as we want, hence we get an η_0 –Poincaré complex for a prescribed η_0 . This completes the first step.

To continue we define a manifold M_0^n and a degree one normal map $g_0 : M_0^n \rightarrow X_0$ as follows:

$$M_0 = B \underset{\text{Id}}{\cup} V \underset{\text{Id}}{\cup} -V \underset{\text{Id}}{\cup} C \rightarrow B \underset{\text{Id}}{\cup} D \times I \underset{f_\sigma}{\cup} -V \underset{\text{Id}}{\cup} C \cong X_0$$

using $F_\sigma \cup \text{Id} : V \underset{\text{Id}}{\cup} -V \rightarrow D \times I \underset{f_\sigma}{\cup} -V$. By construction it has a controlled surgery obstruction $\sigma \in H_n(M; \mathbb{L})$.

Moreover, there is $\bar{\sigma} \in H_n(X_0; \mathbb{L})$ with $p_{0*}(\bar{\sigma}) = \sigma$. This can be seen from the diagram:

$$\begin{array}{ccccc} H_n(M_0; \mathbb{L}) & \xrightarrow{g_{0*}} & H_n(X_0; \mathbb{L}) & \xrightarrow{p_{0*}} & H_n(M; \mathbb{L}) \\ \cong \uparrow & & & & \uparrow \cong \\ H^0(M_0; \mathbb{L}) & \xleftarrow{g_0^*} & H^0(X_0; \mathbb{L}) & \xleftarrow{p_0^*} & H^0(M; \mathbb{L}) \end{array}$$

The vertical isomorphisms are Poincaré dualities. Since p_0 is a \mathcal{UV}^1 map, $\bar{\sigma}$ belongs to the same \mathbb{Z} –sector as σ . We will again denote $\bar{\sigma}$ by σ .

We construct $p_1 : X_1 \rightarrow X_0$ as above: Let $M_0 = B_1 \underset{D_1}{\cup} C_1$, B_1 a regular neighborhood of the 2–skeleton (as fine as we want), C_1 the closure of the complement and $D_1 = C_1 \cap B_1 = \partial C_1 = \partial B_1$, $g_0 : D_1 \rightarrow X_0$ a \mathcal{UV}^1 map. Then we realize $\sigma \in H_n(X_0; \mathbb{L}) \rightarrow \mathcal{S}_{\varepsilon_1, \delta_1}(D_1, g_0)$ by $F_{1, \sigma} : V_1 \rightarrow D_1 \times I$ with $\partial_0 V_1 = D_1$, $\partial_1 V_1 = D'_1$, $F_{1, \sigma}|_{\partial_0 V_1} = \text{Id}$ and $f_{1, \sigma} = F_{1, \sigma}|_{\partial_1 V_1} : D'_1 \rightarrow D_1$ a δ_1 –equivalence over X_0 .

We define $p'_1 : X'_1 \rightarrow X_0$ as follows:

$$X'_1 = B_1 \underset{f_{1, \sigma}}{\cup} -V_1 \underset{\text{Id}}{\cup} C_1 \xrightarrow{f'_1} M_0 \cong B_1 \underset{\text{Id}}{\cup} D_1 \times I \underset{\text{Id}}{\cup} C_1,$$

using $-F_{1, \sigma} : -V_1 \rightarrow D_1 \times I$, and then $p'_1 = g_0 \circ f'_1 : X'_1 \rightarrow M_0 \rightarrow X_0$.

We now observe that:

- (i) by 3.8 X'_1 is a $T_1 \delta_1$ –Poincaré complex over X_0 ; and
- (ii) p'_1 is a degree one normal map with controlled surgery obstruction

$$-p_{0*}(\bar{\sigma}) + \sigma = 0 \in H_n(M; \mathbb{L}).$$

Let $\xi_1 > 0$ be given. We now apply Theorem 3.5 to produce a ξ_1 -homotopy equivalence by surgeries outside the singular set (note that the surgeries which have to be done are in the manifold part of X'_1). For this we need a sufficiently small η_0 -Poincaré structure on X_0 . However, this can be achieved as noted above. This finishes the second step.

We now proceed by induction: What we need for the third step in order to produce $p_2 : X_2 \rightarrow X_1$ is the following:

- (i) a degree one normal map $g_1 : M_1 \rightarrow X_1$ with controlled surgery obstruction $\sigma \in H_n(X_0; \mathbb{L})$; and
- (ii) $\bar{\sigma} \in H_n(X_1; \mathbb{L})$ with $p_{1*}(\bar{\sigma}) = \sigma$, being in the same \mathbb{Z} -sector as $\sigma \in H_n(X_0; \mathbb{L})$.

One can get $g_1 : M_1 \rightarrow X_1$ as follows: Consider $g'_1 : M'_1 \rightarrow X'_1$, where

$$M'_1 = B_1 \underset{\text{Id}}{\cup} V_1 \underset{\text{Id}}{\cup} -V_1 \underset{\text{Id}}{\cup} C_1 \rightarrow B_1 \underset{\text{Id}}{\cup} D_1 \times I \underset{f_{1,\sigma}}{\cup} -V_1 \underset{\text{Id}}{\cup} C_1 \cong X'_1$$

is induced by $F_{1,\sigma} : V_1 \rightarrow D_1 \times I$ and the identity. The map g'_1 is a degree one normal map. Then one performs the same surgeries on g'_1 as one has performed on $p'_1 : X'_1 \rightarrow X_0$ to obtain X_1 . This produces the desired g_1 . For (ii) we note that p_{1*} is a bijective map preserving the \mathbb{Z} -sectors (since p_1 is \mathcal{UV}^1).

So we have obtained the sequence of controlled Poincaré spaces $p_i : X_i \rightarrow X_{i-1}$ and $p_0 : X_0 \rightarrow M$ with degree one normal maps $g_i : M_i \rightarrow X_i$ and controlled surgery obstructions $\sigma \in H_n(X_{i-1}; \mathbb{L})$. The properties (iv) and (v) of §2 now follow by the thin h -cobordism theorem and approximation of retraction.

4 Nonresolvability, the DDP property and existence of generalized manifolds

4.1 Nonresolvability

At the beginning of the construction we have $\sigma \in H_n(M; \mathbb{L})$, where M is a closed (oriented) n -manifold with $n \geq 6$. For each m we constructed degree one normal maps $g_m : M_m \rightarrow X_m$ over $p_m : X_m \rightarrow X_{m-1}$, with controlled surgery obstructions $\sigma_m \in H_n(X_{m-1}; \mathbb{L})$, $p_{0*}(\sigma_1) = \sigma$, $p_{m*}(\sigma_{m+1}) = \sigma_m$, and all σ_m belong to the same \mathbb{Z} -sector as σ . So we will call all of them σ .

We consider the normal map $g_m : M_m \rightarrow X_m$ as a controlled normal map over $\text{Id} : X_m \rightarrow X_m$, and over $q_m : X_m \subset W_m \xrightarrow{\rho} X$ (see §2). Since $\rho|_{\partial W_m}$ is an

approximate fibration and $d(r_i, r_{i-1}) < \xi_i$, $\sum_{i=m+1}^{\infty} \xi_i < \varepsilon$, for large m , we can assume that q_m is $\mathcal{UV}^1(\delta)$ for large m , so $(q_m)_* : H_n(X_m; \mathbb{L}) \rightarrow H_n(X; \mathbb{L})$ maps σ to $(q_m)_*(\sigma) = \sigma'$, being in the same \mathbb{Z} -sector as σ . The map $(q_m)_*$ is a bijective, and we denote σ' by σ . In other words, we have a surgery problem over X :

$$\begin{array}{ccc} M_m & \xrightarrow{g_m} & X_m \\ & & \downarrow q_m \\ & & X \end{array}$$

with controlled surgery obstruction $\sigma \in H_n(X; \mathbb{L})$. Our goal is to consider the surgery problem:

$$\begin{array}{ccc} M_m & \xrightarrow{q_m \circ g_m} & X_m \\ & & \downarrow \text{Id} \\ & & X \end{array}$$

over $\text{Id} : X \rightarrow X$, and prove that $\sigma \in H_n(X; \mathbb{L})$ is its controlled surgery obstruction.

Observe that q_m is a δ -homotopy equivalence over $\text{Id} : X \rightarrow X$ if m is sufficiently large (for a given δ).

Let $\mathcal{N}(X) \cong [X, G/\text{TOP}]$ be the normal cobordism classes of degree one normal maps of X , and let $HE_\delta(X)$ be the set of δ -homotopy equivalences of X over $\text{Id} : X \rightarrow X$. Our claim will follow from the following:

Lemma 4.1 *Let $HE_{\delta'}(X) \times \mathcal{N}(X) \xrightarrow{\mu} \mathcal{N}(X)$ be the action map, i.e. $\mu(h, f) = h \circ f$. Then for sufficiently small $\delta' > 0$, the diagram*

$$\begin{array}{ccc} HE_{\delta'} \times \mathcal{N}(X) & \xrightarrow{\mu} & \mathcal{N}(X) \\ \downarrow pr & & \downarrow \Theta \\ \mathcal{N}(X) & \xrightarrow{\Theta} & H_n(X; \mathbb{L}) \end{array}$$

commutes.

Proof This follows from (3.5') since $HE_{\delta'}(X) \times \mathcal{S}_{\varepsilon'', \delta''}(X, \text{Id}) \rightarrow \mathcal{S}_{\varepsilon, \delta}(X, \text{Id})$ for sufficiently small δ' and δ'' . \square

We apply this lemma to the map $HE_{\delta}(X_m, X) \times \mathcal{N}(X_m) \rightarrow \mathcal{N}(X)$, which sends (h, g) to $h \circ g$, where $HE_{\delta}(X_m, X)$ are the δ -homotopy equivalences $X_m \rightarrow X$ over Id_X :

Let $\psi_m : X \rightarrow X_m$ be a controlled inverse of q_m . Then ψ_m induces $\psi_{m*} : HE_{\varepsilon}(X_m, X) \rightarrow HE_{\delta}(X)$, where δ is some multiple of ε .

One can then write the following commutative diagram (for sufficiently small δ).

$$\begin{array}{ccccc}
 HE_{\varepsilon}(X_m, X) \times H_n(X_m; \mathbb{L}) & & & & \\
 \uparrow \text{Id} \times \Theta & \searrow & & & \\
 HE_{\varepsilon}(X_m, X) \times \mathcal{N}(X_m) & \longrightarrow & \mathcal{N}(X) & \xrightarrow{\Theta} & H_n(X; \mathbb{L}) \\
 \downarrow (\psi_m)_* \times (q_m)_* & \nearrow \mu & & \nearrow \Theta & \\
 HE_{\delta}(X) \times \mathcal{N}(X) & \xrightarrow{\text{pr}} & \mathcal{N}(X) & &
 \end{array}$$

with $HE_{\varepsilon}(X) \times H_n(X; \mathbb{L}) \rightarrow H_n(X; \mathbb{L})$ given by $(h, \tau) \rightarrow h_*(\tau)$.

It follows from this that for large enough m , $q_m \circ g_m : M_m \rightarrow X$ has controlled surgery obstruction $\sigma \in H_n(X; \mathbb{L})$. Hence we get non-resolvable generalized manifolds if the \mathbb{Z} -sector of σ is $\neq 1$.

4.2 The DDP Property

The construction allows one to get the DDP property for X (see [1, section 8]). Roughly speaking, this can be seen as follows: The first step in the construction is to glue a highly connected cobordism V into a manifold M of dimension $n \geq 6$, in between the regular neighborhood of the 2-skeleton.

The result is a space which has the DDP. The other constructions are surgery on middle-dimensional spheres, which also preserves the DDP. But since we have to take the limit of the X_m 's, one must do it more carefully (see Definition 8.1 in [1]):

Definition 4.2 Given $\varepsilon > 0$ and $\delta > 0$, we say that a space Y has the (ε, δ) -DDP if for each pair of maps $f, g : D^2 \rightarrow Y$ there exist maps $\bar{f}, \bar{g} : D^2 \rightarrow Y$ such that $d(\bar{f}(D^2), \bar{g}(D^2)) > \delta$, $d(f, \bar{f}) < \varepsilon$ and $d(g, \bar{g}) < \varepsilon$.

Lemma 4.3 $\{X_m\}$ have the (ε, δ) –DDP for some $\varepsilon > \delta > 0$.

Proof The manifolds M_m^n , $n \geq 6$, have the (ε, δ) –DDP for all ε and δ . In fact, one can choose a sufficiently fine triangulation, such that any $f : D^2 \rightarrow M$ can be placed by arbitrary small moves into the 2–skeleton or into the dual $(n - 3)$ –skeleton. Then δ is the distance between these skeleta. The remarks above show that the X_m ’s have the (ε, δ) –DDP for some ε and δ . \square

It can then be shown that $X = \varprojlim X_i$ has the $(2\varepsilon, \delta/2)$ –DDP (see Proposition 8.4 in [1]).

4.3 Special cases

- (i) Let M^n and $\sigma \in H_n(M; \mathbb{L})$ be given as above. The first case which can occur is that σ goes to zero under the assembly map $A : H_n(M; \mathbb{L}) \rightarrow L_n(\pi_1 M)$. Then we can do surgery on the normal maps $F_\sigma : V \rightarrow D \times I$, $F_{1,\sigma} : V_1 \rightarrow D_1 \times I$ and so on, to replace them by products. In this case the generalized manifold X is homotopy equivalent to M .
- (ii) Suppose that A is injective (or is an isomorphism). Then X cannot be homotopy equivalent to any manifold, if the \mathbb{Z} –sector of σ is $\neq 1$. Suppose that $N^n \rightarrow X$ were a homotopy equivalence. It determines an element in $[X, G/TOP]$ which must map to $(1, 0) \in H_n(X; \mathbb{L})$, because its surgery obstruction in $L_n(\pi_1 X)$ is zero and A is injective. This contradicts our assumption that the index of $X \neq 1$. Examples of this type are given by the n –torus $M^n = T^n$.

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Department of Mathematics, University of Milano

Via C. Saldini 50, Milano, Italy 02130

and

Institute for Mathematics, Physics and Mechanics, University of Ljubljana

Jadranska 19, Ljubljana, Slovenia 1001

Email: friedrich.hegenbarth@mat.unimi.it and

dusan.repovs@fmf.uni-lj.si

URL: <http://pef.pef.uni-lj.si/~dusanr/index.htm>

Geometry & Topology Monographs

Volume X: Volume name goes here

Pages 33–66

The quadratic form E_8 and exotic homology manifolds

WASHINGTON MIO AND ANDREW RANICKI

Abstract An explicit $(-1)^n$ -quadratic form over $\mathbb{Z}[\mathbb{Z}^{2n}]$ representing the surgery problem $E_8 \times T^{2n}$ is obtained, for use in the Bryant-Ferry-Mio-Weinberger construction of $2n$ -dimensional exotic homology manifolds.

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Dedicated to John Bryant on his 60th birthday

1 Introduction

Exotic ENR homology n -manifolds, $n \geq 6$, were discovered in the early 1990s by Bryant, Ferry, Mio and Weinberger [2, 3]. In the 1970s, the existence of such spaces had become a widely debated problem among geometric topologists in connection with the works of Cannon and Edwards on the characterization of topological manifolds [5, 10, 9]. The *Resolution Conjecture*, formulated by Cannon in [4], implied the non-existence of exotic homology manifolds – compelling evidence supporting the conjecture was offered by the solution of the *Double Suspension Problem*. Quinn introduced methods of controlled K -theory and controlled surgery into the area. He associated with an ENR homology n -manifold X , $n \geq 5$, a local index $\iota(X) \in 8\mathbb{Z} + 1$ with the property that $\iota(X) = 1$ if and only if X is resolvable. A *resolution* of X is a proper surjection $f: M \rightarrow X$ from a topological manifold M such that, for each $x \in X$, $f^{-1}(x)$ is contractible in any of its neighborhoods in M . This led to the celebrated Edwards-Quinn characterization of topological n -manifolds, $n \geq 5$, as index-1 ENR homology manifolds satisfying the disjoint disks property (DDP) [19, 20, 9]. More details and historical remarks on these developments can be found in the survey articles [4, 10, 28, 15] and in [9].

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In [2, 3], ENR homology manifolds with non-trivial local indexes are constructed as inverse limits of ever finer Poincaré duality spaces, which are obtained from topological manifolds using controlled cut-paste constructions. In the simply-connected case, for example, topological manifolds are cut along the boundaries of regular neighborhoods of very fine 2-skeleta and pasted back together using ϵ -homotopy equivalences that “carry non-trivial local indexes” in the form of obstructions to deform them to homeomorphisms in a controlled manner. The construction of these ϵ -equivalences requires controlled surgery theory, the calculation of controlled surgery groups with trivial local fundamental group, and “Wall realization” of controlled surgery obstructions. The stability of controlled surgery groups is a key fact, whose proof was completed more recently by Pedersen, Quinn and Ranicki [16]; an elegant proof along similar lines was given by Pedersen and Yamasaki [17] at the 2003 Workshop on Exotic Homology Manifolds in Oberwolfach, employing methods of [29]. An alternative proof based on the α -Approximation Theorem is due to Ferry [11].

The construction of exotic homology manifolds presented in [3] is somewhat indirect. Along the years, many colleagues (notably Bob Edwards) voiced the desire to see – at least in one specific example – an explicit realization of the controlled quadratic form employed in the Wall realization of the local index. This became even clearer at the workshop in Oberwolfach. A detailed inspection of the construction of [3] reveals that it suffices to give this explicit description at the first (controlled) stage of the construction of the inverse limit, since fairly general arguments show that subsequent stages can be designed to inherit the local index. The main goal of this paper is to provide explicit realizations of controlled quadratic forms that lead to the construction of compact exotic homology manifolds with fundamental group \mathbb{Z}^{2n} , $n \geq 3$, which are not homotopy equivalent to any closed topological manifold. This construction was suggested in Section 7 of [3], but details were not provided. Starting with the rank 8 quadratic form E_8 of signature 8, which generates the Wall group $L_0(\mathbb{Z}) \cong \mathbb{Z}$, we explicitly realize its image in $L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ under the canonical embedding $L_0(\mathbb{Z}) \rightarrow L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$.

Let

$$\psi_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be the 8×8 matrix over \mathbb{Z} with symmetrization the unimodular 8×8 matrix of the E_8 -form

$$\psi_0 + \psi_0^* = E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Write

$$\begin{aligned} \mathbb{Z}[\mathbb{Z}^{2n}] &= \mathbb{Z}[z_1, z_1^{-1}, \dots, z_{2n}, z_{2n}^{-1}] \\ &= \mathbb{Z}[z_1, z_1^{-1}] \otimes \mathbb{Z}[z_2, z_2^{-1}] \otimes \cdots \otimes \mathbb{Z}[z_{2n}, z_{2n}^{-1}]. \end{aligned}$$

For $i = 1, 2, \dots, n$ define the 2×2 matrix over $\mathbb{Z}[z_{2i-1}, z_{2i-1}^{-1}, z_{2i}, z_{2i}^{-1}]$

$$\alpha_i = \begin{pmatrix} 1 - z_{2i-1} & z_{2i-1}z_{2i} - z_{2i-1} - z_{2i} \\ 1 & 1 - z_{2i} \end{pmatrix},$$

so that $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$ is a $2^n \times 2^n$ matrix over $\mathbb{Z}[\mathbb{Z}^{2n}]$. (See §6 for the geometric provenance of the matrices α_i).

Theorem 8.1 *The surgery obstruction $E_8 \times T^{2n} \in L_{2n}(\Lambda)$ ($\Lambda = \mathbb{Z}[\mathbb{Z}^{2n}]$) is represented by the nonsingular $(-1)^n$ -quadratic form (K, λ, μ) over Λ , with*

$$K = \mathbb{Z}^8 \otimes \Lambda^{2^n} = \Lambda^{2^{n+3}}$$

the f.g. free Λ -module of rank $8 \cdot 2^n = 2^{n+3}$ and

$$\begin{aligned} \lambda &= \psi + (-1)^n \psi^* : K \rightarrow K^* = \text{Hom}_\Lambda(K, \Lambda), \\ \mu(x) &= \psi(x)(x) \in Q_{(-1)^n}(\Lambda) = \Lambda / \{g + (-1)^{n+1} g^{-1} \mid g \in \mathbb{Z}^{2n}\} \quad (x \in K) \end{aligned}$$

with

$$\psi = \psi_0 \otimes \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n : K \rightarrow K^* .$$

□

Sections 2–8 contain background material on surgery theory and the arguments that lead to a proof of Theorem 8.1. Invariance of $E_8 \times T^{2n}$ under transfers to finite covers is proven in §9. In §10, using a large finite cover $T^{2n} \rightarrow T^{2n}$, we describe how to pass from the non-simply-connected surgery obstruction $E_8 \times T^{2n}$ to a controlled quadratic \mathbb{Z} -form over T^{2n} . Finally, in §11 we explain how the controlled version of $E_8 \times T^{2n}$ is used in the construction of exotic homology $2n$ -manifolds X with Quinn index $\iota(X) = 9$.

2 The Wall groups

We begin with some recollections of surgery obstruction theory – we only need the details in the even-dimensional oriented case.

Let Λ be a ring with an involution, that is a function

$$- : \Lambda \rightarrow \Lambda ; a \mapsto \bar{a}$$

satisfying

$$\overline{a+b} = \bar{a} + \bar{b} , \overline{ab} = \bar{b}.\bar{a} , \bar{\bar{a}} = a , \bar{1} = 1 \in \Lambda .$$

Example In the applications to topology $\Lambda = \mathbb{Z}[\pi]$ is a group ring, with the involution

$$- : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi] ; \sum_{g \in \pi} a_g g \mapsto \sum_{g \in \pi} a_g g^{-1} \quad (a_g \in \mathbb{Z}) .$$

□

The involution is used to define a left Λ -module structure on the dual of a left Λ -module K

$$K^* := \text{Hom}_{\Lambda}(K, \Lambda) ,$$

with

$$\Lambda \times K^* \rightarrow K^* ; (a, f) \mapsto (x \mapsto f(x).\bar{a}) .$$

The $2n$ -dimensional surgery obstruction group $L_{2n}(\Lambda)$ is defined by Wall [27, §5] to be the Witt group of *nonsingular* $(-1)^n$ -quadratic forms (K, λ, μ) over Λ , with K a finitely generated free (left) Λ -module together with

(i) a pairing

$$\lambda: K \times K \rightarrow \Lambda$$

such that

$$\lambda(x, ay) = a\lambda(x, y), \lambda(x, y+z) = \lambda(x, y) + \lambda(x, z), \lambda(y, x) = (-1)^n \overline{\lambda(x, y)}$$

and the adjoint Λ -module morphism

$$\lambda: K \rightarrow K^*; x \mapsto (y \mapsto \lambda(x, y))$$

is an isomorphism,

(ii) a $(-1)^n$ -quadratic function

$$\mu: K \rightarrow Q_{(-1)^n}(\Lambda) = \Lambda / \{a + (-1)^{n+1}\bar{a} \mid a \in \Lambda\}$$

with

$$\lambda(x, x) = \mu(x) + (-1)^n \overline{\mu(x)}, \mu(x+y) = \mu(x) + \mu(y) + \lambda(x, y), \mu(ax) = a\mu(x)\bar{a}.$$

For a f.g. free Λ -module $K = \Lambda^r$ with basis $\{e_1, e_2, \dots, e_r\}$ the pair (λ, μ) can be regarded as an equivalence class of $r \times r$ matrices over Λ

$$\psi = (\psi_{ij})_{1 \leq i, j \leq r} \quad (\psi_{ij} \in \Lambda)$$

such that $\psi + (-1)^n \psi^*$ is invertible, with $\psi^* = (\bar{\psi}_{ji})$, and

$$\psi \sim \psi' \text{ if } \psi' - \psi = \chi + (-1)^{n+1} \chi^* \text{ for some } r \times r \text{ matrix } \chi = (\chi_{ij}).$$

The relationship between (λ, μ) and ψ is given by

$$\begin{aligned} \lambda(e_i, e_j) &= \psi_{ij} + (-1)^n \bar{\psi}_{ji} \in \Lambda, \\ \mu(e_i) &= \psi_{ii} \in Q_{(-1)^n}(\Lambda), \end{aligned}$$

and we shall write

$$(\lambda, \mu) = (\psi + (-1)^n \psi^*, \psi).$$

The detailed definitions of the odd-dimensional L -groups $L_{2n+1}(\Lambda)$ are rather more complicated, and are not required here. The quadratic L -groups are 4-periodic

$$L_m(\Lambda) = L_{m+4}(\Lambda).$$

The simply-connected quadratic L -groups are given by

$$L_m(\mathbb{Z}) = P_m = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \pmod{4} \\ 0 & \text{if } m \equiv 1 \pmod{4} \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \pmod{4} \\ 0 & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

(Kervaire-Milnor). In particular, for $m \equiv 0 \pmod{4}$ there is defined an isomorphism

$$L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z} ; (K, \lambda, \mu) \mapsto \frac{1}{8} \text{signature}(K, \lambda) .$$

The *kernel form* of an n -connected normal map $(f, b) : M^{2n} \rightarrow X$ from a $2n$ -dimensional manifold M to an oriented $2n$ -dimensional geometric Poincaré complex X is the nonsingular $(-1)^n$ -quadratic form over $\mathbb{Z}[\pi_1(X)]$ defined in [27, §5]

$$(K_n(M), \lambda, \mu)$$

with

$$K_n(M) = \ker(\tilde{f}_* : H_n(\tilde{M}) \rightarrow H_n(\tilde{X}))$$

the kernel (stably) f.g. free $\mathbb{Z}[\pi_1(X)]$ -module, \tilde{X} the universal cover of X , $\tilde{M} = f^* \tilde{X}$ the pullback cover of M and (λ, μ) given by geometric (intersection, self-intersection) numbers. The *surgery obstruction* of [27]

$$\sigma_*(f, b) = (K_n(M), \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

is such that $\sigma_*(f, b) = 0$ if (and for $n \geq 3$ only if) (f, b) is bordant to a homotopy equivalence.

The Realization Theorem of [27, §5] states that for a finitely presented group π and $n \geq 3$ every nonsingular $(-1)^n$ -quadratic form (K, λ, μ) over $\mathbb{Z}[\pi]$ is the kernel form of an n -connected $2n$ -dimensional normal map $f : M \rightarrow X$ with $\pi_1(X) = \pi$.

3 The instant surgery obstruction

Let $(f, b) : M^m \rightarrow X$ be an m -dimensional normal map with $f_* : \pi_1(M) \rightarrow \pi_1(X)$ an isomorphism, and let $\tilde{f} : \tilde{M} \rightarrow \tilde{X}$ be a $\pi_1(X)$ -equivariant lift of f to the universal covers of M, X . The $\mathbb{Z}[\pi_1(X)]$ -module morphisms $\tilde{f}_* : H_r(\tilde{M}) \rightarrow H_r(\tilde{X})$ are split surjections, with the Umkehr maps

$$f^! : H_r(\tilde{X}) \cong H^{m-r}(\tilde{X}) \xrightarrow{\tilde{f}^*} H^{m-r}(\tilde{M}) \cong H_r(\tilde{M}) ,$$

such that

$$\tilde{f}_* f^! = 1 : H_r(\tilde{X}) \rightarrow H_r(\tilde{X}) .$$

The kernel $\mathbb{Z}[\pi_1(X)]$ -modules

$$K_r(M) = \ker(\tilde{f}_* : H_r(\tilde{M}) \rightarrow H_r(\tilde{X}))$$

are such that

$$H_r(\widetilde{M}) = K_r(M) \oplus H_r(\widetilde{X}) , \quad K_r(M) = H_{r+1}(\widetilde{f}) .$$

By the Hurewicz theorem, (f, b) is k -connected if and only if

$$K_r(M) = 0 \text{ for } r < k ,$$

in which case $K_k(M) = \pi_{k+1}(f)$. If $m = 2n$ or $2n+1$ then by Poincaré duality (f, b) is $(n+1)$ -connected if and only if it is a homotopy equivalence. In the even-dimensional case $m = 2n$ the surgery obstruction of (f, b) is defined to be

$$\sigma_*(f, b) = \sigma_*(f', b') = (K_n(M'), \lambda', \mu') \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

with $(f', b') : M' \rightarrow X$ any bordant n -connected normal map obtained from (f, b) by surgery below the middle dimension. The instant surgery obstruction of Ranicki [21] is an expression for such a form $(K_n(M'), \lambda', \mu')$ in terms of the kernel $2n$ -dimensional quadratic Poincaré complex (C, ψ) such that $H_*(C) = K_*(M)$. In §8 we below we shall use a variant of the instant surgery obstruction to obtain an explicit $(-1)^n$ -quadratic form over $\mathbb{Z}[\mathbb{Z}^{2n}]$ representing $E_8 \times T^{2n} \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$.

Given a ring with involution Λ and an m -dimensional f.g. free Λ -module chain complex

$$C : C_m \xrightarrow{d} C_{m-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d} C_0$$

let C^{m-*} be the dual m -dimensional f.g. free Λ -module chain complex, with

$$\begin{aligned} d_{C^{m-*}} &= (-1)^r d^* : \\ (C^{m-*})_r &= C^{m-r} = (C_{m-r})^* = \text{Hom}_\Lambda(C_{m-r}, \Lambda) \rightarrow C^{m-r+1} . \end{aligned}$$

Define a duality involution on $\text{Hom}_\Lambda(C^{m-*}, C)$ by

$$T : \text{Hom}_\Lambda(C^p, C_q) \rightarrow \text{Hom}_\Lambda(C^q, C_p) ; \quad \phi \mapsto (-1)^{pq} \phi^* .$$

An m -dimensional quadratic Poincaré complex (C, ψ) over Λ is an m -dimensional f.g. free Λ -module chain complex C together with Λ -module morphisms

$$\psi_s : C^r \rightarrow C_{m-r-s} \quad (s \geq 0)$$

such that

$$d\psi_s + (-1)^r \psi_s d^* + (-1)^{m-s-1} (\psi_{s+1} + (-1)^{s+1} T\psi_{s+1}) = 0 : C^{m-r-s-1} \rightarrow C_r \quad (s \geq 0)$$

and such that $(1 + T)\psi_0 : C^{m-*} \rightarrow C$ is a chain equivalence. The cobordism group of m -dimensional quadratic Poincaré complexes over Λ was identified

in Ranicki [21] with the Wall surgery obstruction $L_m(\Lambda)$, and the surgery obstruction of an m -dimensional normal map $(f, b) : M \rightarrow X$ was identified with the cobordism class

$$\sigma_*(f, b) = (\mathcal{C}(f^!), \psi_b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

of the kernel quadratic Poincaré complex $(\mathcal{C}(f^!), \psi_b)$, with $\mathcal{C}(f^!)$ the algebraic mapping cone of the Umkehr $\mathbb{Z}[\pi_1(X)]$ -module chain map

$$f^! : C(\tilde{X}) \simeq C(\tilde{X})^{m-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{m-*} \simeq C(\tilde{M}) .$$

The homology $\mathbb{Z}[\pi_1(X)]$ -modules of $\mathcal{C}(f^!)$ are the kernel $\mathbb{Z}[\pi_1(X)]$ -modules of f

$$H_*(\mathcal{C}(f^!)) = K_*(M) = \ker(\tilde{f}_* : H_*(\tilde{M}) \rightarrow H_*(\tilde{X})) .$$

Definition 3.1 The *instant form* of a $2n$ -dimensional quadratic Poincaré complex (C, ψ) over Λ is the nonsingular $(-1)^n$ -quadratic form over Λ

$$(K, \lambda, \mu) = \left(\operatorname{coker} \left(\begin{pmatrix} d^* & 0 \\ (-1)^{n+1}(1+T)\psi_0 & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \rightarrow C^n \oplus C_{n+1} \right), \begin{bmatrix} \psi_0 + (-1)^n \psi_0^* & d \\ (-1)^n d^* & 0 \end{bmatrix}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix} \right) .$$

□

If C_r is f.g. free with $\operatorname{rank}_\Lambda C_r = c_r$ then K is (stably) f.g. free with

$$\operatorname{rank}_\Lambda K = \sum_{r=0}^n (-1)^r (c_{n-r} + c_{n+r+1}) \in \mathbb{Z} .$$

If $(1+T)\psi_0 : C^{2n-*} \rightarrow C$ is an isomorphism then

$$c_{n+r+1} = c_{n-r-1} , \operatorname{rank}_\Lambda K = c_n ,$$

with

$$(K, \lambda, \mu) = (C^n, \psi_0 + (-1)^n \psi_0^*, \psi_0) .$$

Proposition 3.2 (Instant surgery obstruction [21, Proposition I.4.3])

(i) The cobordism class of a $2n$ -dimensional quadratic Poincaré complex (C, ψ) over Λ is the Witt class

$$(C, \psi) = (K, \lambda, \mu) \in L_{2n}(\Lambda)$$

of the instant nonsingular $(-1)^n$ -quadratic form (K, λ, μ) over Λ .

(ii) The surgery obstruction of a $2n$ -dimensional normal map $(f, b) : M \rightarrow X$ is represented by the instant form (K, λ, μ) of any quadratic Poincaré complex (C, ψ) chain equivalent to the kernel $2n$ -dimensional quadratic Poincaré complex $(C(f^\dagger), \psi_b)$

$$\sigma_*(f, b) = (K, \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)]) .$$

□

Remark (i) If (f, b) is n -connected then C is chain equivalent to the chain complex concentrated in dimension n

$$C : 0 \rightarrow \cdots \rightarrow 0 \rightarrow K_n(M) \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

and the instant form is just the kernel form $(K_n(M), \lambda, \mu)$ of Wall [27].

(ii) More generally, if (f, b) is k -connected for some $k \leq n$ then C is chain equivalent to a chain complex concentrated in dimensions $k, k+1, \dots, 2n-k$

$$C : 0 \rightarrow \cdots \rightarrow 0 \rightarrow C_{2n-k} \rightarrow \cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_k \rightarrow 0 \rightarrow \cdots \rightarrow 0 .$$

For $n \geq 3$ the effect of surgeries killing the c_{2n-k} generators of $H^{2n-k}(C) = K_k(M)$ represented by a basis of C^{2n-k} is a bordant $(k+1)$ -connected normal map $(f', b') : M' \rightarrow X$ with $\mathcal{C}(f' : C(\tilde{X}) \rightarrow C(\tilde{M}'))$ chain equivalent to a chain complex of the type

$$C' : 0 \rightarrow \cdots \rightarrow 0 \rightarrow C'_{2n-k-1} \rightarrow \cdots \rightarrow C'_n \rightarrow \cdots \rightarrow C'_{k+1} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

with

$$C'_r = \begin{cases} \ker((d(1+T)\psi_0) : C_{k+1} \oplus C^{2n-k} \rightarrow C_k) & \text{if } r = k+1 \\ C_r & \text{if } k+2 \leq r \leq 2n-k-1 . \end{cases}$$

Proceeding in this way, there is obtained a sequence of bordant j -connected normal maps

$$(f_j, b_j) : M_j \rightarrow X \quad (j = k, k+1, \dots, n)$$

with

$$(f_k, b_k) = (f, b) , (f_{j+1}, b_{j+1}) = (f_j, b_j)' .$$

The instant form of (C, ψ) is precisely the kernel $(-1)^n$ -quadratic form $(K_n(M_n), \lambda_n, \mu_n)$ of the n -connected normal map $(f_n, b_n) : M_n \rightarrow X$, so that the surgery obstruction of (f, b) is given by

$$\begin{aligned} \sigma_*(f, b) &= \sigma_*(f_k, b_k) = \dots = \sigma_*(f_n, b_n) \\ &= (K_n(M_n), \lambda_n, \mu_n) \in L_{2n}(\mathbb{Z}[\pi_1(X)]) . \end{aligned}$$

□

4 The quadratic form E_8

For $m \geq 2$ let M_0^{4m} be the $(2m-1)$ -connected $4m$ -dimensional PL manifold obtained from the Milnor E_8 -plumbing of 8 copies of $\tau_{S^{2m}}$ by coning off the (exotic) $(4m-1)$ -sphere boundary, with intersection form E_8 of signature 8. (For $m=1$ we can take M_0 to be the simply-connected 4-dimensional Freedman topological manifold with intersection form E_8). The surgery obstruction of the corresponding $2m$ -connected normal map $(f_0, b_0) : M_0^{4m} \rightarrow S^{4m}$ represents the generator

$$\sigma_*(f_0, b_0) = (K_{2m}(M_0), \lambda, \mu) = (\mathbb{Z}^8, E_8) = 1 \in L_{4m}(\mathbb{Z}) = L_0(\mathbb{Z}) = \mathbb{Z}$$

with

$$K_{2m}(M_0) = H_{2m}(M_0) = \mathbb{Z}^8 ,$$

$$\lambda = E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} ,$$

$$\mu(0, \dots, 1, \dots, 0) = 1 .$$

5 The surgery product formula

Surgery product formulae were originally obtained in the simply-connected case, notably by Sullivan. We now recall the non-simply-connected surgery product formula of Ranicki [21] involving the Mishchenko symmetric L -groups. In §6 we shall recall the variant of the surgery product formula involving almost symmetric L -groups of Clauwens, which will be used in Theorem 8.1 below to write down an explicit nonsingular $(-1)^n$ -quadratic form over $\mathbb{Z}[\mathbb{Z}^{2n}]$ ($n \geq 1$) representing the image of the generator

$$1 = E_8 \in L_{4m}(\mathbb{Z}) \cong \mathbb{Z} \ (m \geq 0)$$

under the canonical embedding

$$\begin{aligned} & - \times T^{2n} : L_{4m}(\mathbb{Z}) \rightarrow L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) ; \\ & \sigma_*((f_0, b_0) : M_0 \rightarrow S^{4m}) = E_8 \mapsto \sigma_*((f_0, b_0) \times 1 : M_0 \times T^{2n} \rightarrow S^{4m} \times T^{2n}) . \end{aligned}$$

An n -dimensional symmetric Poincaré complex (C, ϕ) over a ring with involution Λ is an n -dimensional f.g. free Λ -module chain complex

$$C : C_n \xrightarrow{d} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d} C_0$$

together with Λ -module morphisms

$$\phi_s : C^r = \text{Hom}_\Lambda(C_r, \Lambda) \rightarrow C_{n-r+s} \quad (s \geq 0)$$

such that

$$\begin{aligned} d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) &= 0 : \\ C^{n-r+s-1} &\rightarrow C_r \quad (s \geq 0, \phi_{-1} = 0) \end{aligned}$$

and $\phi_0 : C^{n-*} \rightarrow C$ is a chain equivalence. The cobordism group of n -dimensional symmetric Poincaré complexes over Λ is denoted by $L^n(\Lambda)$ – see [21] for a detailed exposition of symmetric L -theory. Note that the symmetric L -groups $L^*(\Lambda)$ are not 4-periodic in general

$$L^n(\Lambda) \neq L^{n+4}(\Lambda) .$$

The symmetric L -groups of \mathbb{Z} are given by

$$L^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{4} \\ \mathbb{Z}_2 & \text{if } n \equiv 1 \pmod{4} \\ 0 & \text{if } n \equiv 2 \pmod{4} \\ 0 & \text{if } n \equiv 3 \pmod{4} . \end{cases}$$

For $m \equiv 0 \pmod{4}$ there is defined an isomorphism

$$L^{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z} ; (C, \phi) \mapsto \text{signature}(H^{2k}(C), \phi_0) .$$

A CW structure on an oriented n -dimensional manifold with $\pi_1(N) = \rho$ and universal cover \tilde{N} and the Alexander-Whitney-Steenrod diagonal construction on the cellular complex $C(\tilde{N})$ determine an n -dimensional symmetric Poincaré complex $(C(\tilde{N}), \phi)$ over $\mathbb{Z}[\rho]$ with

$$\phi_0 = [N] \cap - : C(\tilde{N})^{n-*} \rightarrow C(\tilde{N}) .$$

The Mishchenko *symmetric signature* of N is the cobordism class

$$\sigma^*(N) = (C, \phi) \in L^n(\mathbb{Z}[\rho]) .$$

For $n = 4k$ the image of $\sigma^*(N)$ in $L^{4k}(\mathbb{Z}) = \mathbb{Z}$ is just the usual signature of N .

For any rings with involution Λ, Λ' there are defined products

$$L^n(\Lambda) \otimes L^{n'}(\Lambda') \rightarrow L^{n+n'}(\Lambda \otimes \Lambda') ; (C, \phi) \otimes (C', \phi') \mapsto (C \otimes C', \phi \otimes \phi') ,$$

$$L_n(\Lambda) \otimes L^{n'}(\Lambda') \rightarrow L_{n+n'}(\Lambda \otimes \Lambda') ; (C, \psi) \otimes (C', \phi') \mapsto (C \otimes C', \psi \otimes \phi')$$

as in [23]. The tensor product of group rings is given by

$$\mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi'] = \mathbb{Z}[\pi \times \pi'] .$$

Theorem 5.1 (Symmetric L -theory surgery product formula [21])

(i) *The symmetric signature of a product $N \times N'$ of an n -dimensional manifold N and an n' -dimensional manifold N' is the product of the symmetric signatures*

$$\sigma^*(N \times N') = \sigma^*(N) \otimes \sigma^*(N') \in L^{n+n'}(\mathbb{Z}[\pi_1(N) \times \pi_1(N')]) .$$

(ii) *The product of an m -dimensional normal map $(f, b) : M \rightarrow X$ and an n -dimensional manifold N is an $(m+n)$ -dimensional normal map*

$$(g, c) = (f, b) \times 1 : M \times N \rightarrow X \times N$$

with surgery obstruction

$$\sigma_*(g, c) = \sigma_*(f, b) \otimes \sigma^*(N) \in L_{m+n}(\mathbb{Z}[\pi_1(X) \times \pi_1(N)]) .$$

Proof These formulae already hold on the chain homotopy level, and chain equivalent symmetric/quadratic Poincaré complexes are cobordant. In somewhat greater detail:

(i) The symmetric Poincaré complex of a product $N'' = N \times N'$ is the product of the symmetric Poincaré complexes of N and N'

$$(C(\tilde{N}''), \phi'') = (C(\tilde{N}) \otimes C(\tilde{N}'), \phi \otimes \phi') .$$

(ii) The kernel quadratic Poincaré complex $(\mathcal{C}(g^!), \psi_c)$ of the product normal map $(g, c) = (f, b) \times 1 : M \times N \rightarrow X \times N$ is the product of the kernel quadratic Poincaré complex $(\mathcal{C}(f^!), \psi_b)$ of (f, b) and the symmetric Poincaré complex $(C(\tilde{N}), \phi)$ of N

$$(\mathcal{C}(g^!), \psi_c) = (\mathcal{C}(f^!) \otimes C(\tilde{N}), \psi_b \otimes \phi) .$$

□

Theorem 5.2 (i) (Shaneson [26], Wall [27]) *The quadratic L -groups of $\mathbb{Z}[\mathbb{Z}^n]$ are given by*

$$L_m(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{m-r}(\mathbb{Z}) \quad (m \geq 0) ,$$

interpreting $L_{m-r}(\mathbb{Z})$ for $m - r < 0$ as $L_{m-r+4*}(\mathbb{Z})$.

(ii) (Milgram and Ranicki [14], Ranicki [22, §19]) The symmetric L -groups of $\mathbb{Z}[\mathbb{Z}^n]$ are given by

$$L^m(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L^{m-r}(\mathbb{Z}) \quad (m \geq 0)$$

interpreting $L^{m-r}(\mathbb{Z})$ for $m < r$ as

$$L^{m-r}(\mathbb{Z}) = \begin{cases} 0 & \text{if } m = r - 1, r - 2 \\ L_{m-r}(\mathbb{Z}) & \text{if } m < r - 2. \end{cases}$$

□

The symmetric signature of T^n

$$\sigma^*(T^n) = (C(\tilde{T}^n), \phi) = (0, \dots, 0, 1) \in L^n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L^{n-r}(\mathbb{Z}),$$

is the cobordism class of the n -dimensional symmetric Poincaré complex $(C(\tilde{T}^n), \phi)$ over $\mathbb{Z}[\mathbb{Z}^n]$ with

$$C(\tilde{T}^n) = \bigotimes_n C(\tilde{S}^1), \quad \text{rank}_{\mathbb{Z}[\mathbb{Z}^n]} C_r(\tilde{T}^n) = \binom{n}{r}.$$

The surgery obstruction

$$E_8 \times T^n = (0, \dots, 0, 1) \in L_n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{n-r}(\mathbb{Z})$$

is the cobordism class of the n -dimensional quadratic Poincaré complex over $\mathbb{Z}[\mathbb{Z}^n]$

$$(C, \psi) = (\mathbb{Z}^8, E_8) \otimes (C(\tilde{T}^n), \phi)$$

with

$$\text{rank}_{\mathbb{Z}[\mathbb{Z}^n]} C_r = 8 \binom{n}{r}.$$

6 Almost $(-1)^n$ -symmetric forms

The surgery obstruction of the $(4m + 2n)$ -dimensional normal map

$$(f, b) = (f_0, b_0) \times 1 : M_0^{4m} \times T^{2n} \rightarrow S^{4m} \times T^{2n}$$

is given by the instant surgery obstruction of §3 and the surgery product formula of §5 to be the Witt class

$$\sigma_*(f, b) = (K, \lambda, \mu) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

of the instant form (K, λ, μ) of the $2n$ -dimensional quadratic Poincaré complex

$$(C, \psi) = (\mathbb{Z}^8, E_8) \otimes (C(\tilde{T}^{2n}), \phi) ,$$

with

$$\text{rank}_{\mathbb{Z}[\mathbb{Z}^{2n}]} K = 8 \text{rank}_{\mathbb{Z}[\mathbb{Z}^{2n}]} C_n(\tilde{T}^{2n}) = 8 \binom{2n}{n} .$$

In principle, it is possible to compute (λ, μ) directly from the $(4m + 2n)$ -dimensional symmetric Poincaré complex $E_8 \otimes (C(\tilde{T}^n), \phi)$. In practice, we shall use the almost symmetric form surgery product formula of Clauwens [6],[7],[8], which is the analogue for symmetric Poincaré complexes of the instant surgery obstruction of §3. We establish a product formula for almost symmetric forms which will be used in §7 to obtain an almost $(-1)^n$ -symmetric form for T^{2n} of rank $2^n \leq \binom{2n}{n}$, and hence a representative $(-1)^n$ -quadratic form for $\sigma_*(f, b) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$ of rank $2^{n+3} \leq 8 \binom{2n}{n}$.

Definition 6.1 Let R be a ring with involution.

(i) An *almost $(-1)^n$ -symmetric form* (A, α) over R is a f.g. free R -module A together with a nonsingular pairing $\alpha : A \rightarrow A^*$ such that the endomorphism

$$\beta = 1 + (-1)^{n+1} \alpha^{-1} \alpha^* : A \rightarrow A$$

is nilpotent, i.e. $\beta^N = 0$ for some $N \geq 1$.

(ii) A *sublagrangian* of an almost $(-1)^n$ -symmetric form (A, α) is a direct summand $L \subset A$ such that $L \subseteq L^\perp$, where

$$L^\perp := \{b \in A \mid \alpha(b)(A) = \alpha(A)(b) = \{0\}\} .$$

A *lagrangian* is a sublagrangian L such that

$$L = L^\perp .$$

(iii) The *almost $(-1)^n$ -symmetric Witt group* $AL^{2n}(R)$ is the abelian group of isomorphism classes of almost $(-1)^n$ -symmetric forms (A, α) over R with relations

$$(A, \alpha) = 0 \text{ if } (A, \alpha) \text{ admits a lagrangian}$$

and addition by

$$(A, \alpha) + (A', \alpha') = (A \oplus A', \alpha \oplus \alpha') .$$

□

Example A nonsingular $(-1)^n$ -symmetric form (A, α) is an almost $(-1)^n$ -symmetric form such that

$$\alpha = (-1)^n \alpha^* : A \rightarrow A^*$$

so that $1 + (-1)^{n+1} \alpha^{-1} \alpha^* = 0 : A \rightarrow A$. \square

An almost $(-1)^n$ -symmetric form (R^q, α) on a f.g. free R -module of rank q is represented by an invertible $q \times q$ matrix $\alpha = (\alpha_{rs})$ such that the $q \times q$ matrix $1 + (-1)^{n+1} \alpha^{-1} \alpha^*$ is nilpotent.

Definition 6.2 The *instant form* of a $2n$ -dimensional symmetric Poincaré complex (C, ϕ) over R is the almost $(-1)^n$ -symmetric form over R

$$(A, \alpha) = \left(\text{coker} \left(\begin{pmatrix} d^* & 0 \\ -\phi_0^* & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \rightarrow C^n \oplus C_{n+1} \right), \begin{bmatrix} \phi_0 + d\phi_1 & d \\ (-1)^n d^* & 0 \end{bmatrix} \right). \quad \square$$

Example If $\phi_0 : C^{2n-*} \rightarrow C$ is an isomorphism the instant almost $(-1)^n$ -symmetric form is

$$(A, \alpha) = (C^n, \phi_0 + d\phi_1). \quad \square$$

Every $2n$ -dimensional symmetric Poincaré complex (C, ϕ) over a ring with involution R is chain equivalent to a complex (C', ϕ') such that $\phi'_0 : C'^{2n-*} \rightarrow C'$ is an isomorphism, with

$$C' : C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^{n-1} \rightarrow A^* \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0$$

and

$$\phi'_0 + d'\phi'_1 = \alpha : C'^n = A \rightarrow C'_n = A^*.$$

(We shall not actually need this chain equivalence, since $\phi_0 : C^{2n-*} \rightarrow C$ is an isomorphism for $C = C(\tilde{T}^{2n})$, so Example 6 will apply). The instant form defines a forgetful map

$$L^{2n}(R) \rightarrow AL^{2n}(R) ; (C, \phi) \mapsto (A, \alpha).$$

Proposition 6.3 (Ranicki [24, 36.3]) The almost $(-1)^n$ -symmetric Witt group of \mathbb{Z} is given by

$$AL^{2n}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod{2} \\ 0 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

with $L^{4k}(\mathbb{Z}) \rightarrow AL^{4k}(\mathbb{Z})$ an isomorphism. The Witt class of an almost symmetric form (A, α) over \mathbb{Z} is

$$(A, \alpha) = \text{signature}(\mathbb{Q} \otimes A, (\alpha + \alpha^*)/2) \in AL^{4k}(\mathbb{Z}) = L^{4k}(\mathbb{Z}) = \mathbb{Z}.$$

□

The almost $(-1)^n$ -symmetric L -group $AL^{2n}(R)$ was denoted $L\text{Asy}_{h, S_{(-1)^n}}^0(R)$ in [24].

Definition 6.4 The *almost symmetric signature* of a $2n$ -dimensional manifold N^{2n} with $\pi_1(N) = \rho$ is the Witt class

$$\sigma^*(N) = (A, \alpha) \in AL^{2n}(\mathbb{Z}[\rho])$$

of the instant almost $(-1)^n$ -symmetric form (A, α) over $\mathbb{Z}[\rho]$ of the $2n$ -dimensional symmetric Poincaré complex $(C(\tilde{N}), \phi)$ over $\mathbb{Z}[\rho]$. □

The forgetful map $L^{2n}(\mathbb{Z}[\rho]) \rightarrow AL^{2n}(\mathbb{Z}[\rho])$ sends the symmetric signature $\sigma^*(N) \in L^{2n}(\mathbb{Z}[\rho])$ to the almost symmetric signature $\sigma^*(N) \in AL^{2n}(\mathbb{Z}[\rho])$.

For any rings with involution R_1, R_2 there is defined a product

$$\begin{aligned} AL^{2n_1}(R_1) \otimes AL^{2n_2}(R_2) &\rightarrow AL^{2n_1+2n_2}(R_1 \otimes R_2); \\ (A_1, \alpha_1) \otimes (A_2, \alpha_2) &\mapsto (A_1 \otimes A_2, \alpha_1 \otimes \alpha_2). \end{aligned}$$

Proposition 6.5 The *almost symmetric signature* of a product $N = N_1 \times N_2$ of $2n_i$ -dimensional manifolds N_i with $\pi_1(N_i) = \rho_i$ and almost $(-1)^{n_i}$ -symmetric forms $(\mathbb{Z}[\rho_i]^{q_i}, \alpha_i)$ ($i = 1, 2$) is the product

$$\begin{aligned} \sigma^*(N_1 \times N_2) &= \sigma^*(N_1) \otimes \sigma^*(N_2) \\ &\in \text{im}(AL^{2n_1}(\mathbb{Z}[\rho_1]) \otimes AL^{2n_2}(\mathbb{Z}[\rho_2]) \rightarrow AL^{2n_1+2n_2}(\mathbb{Z}[\rho_1 \times \rho_2])). \end{aligned}$$

Proof The almost $(-1)^{n_1+n_2}$ -symmetric form (A, α) of $N_1 \times N_2$ is defined on

$$A = C^{n_1+n_2}(\tilde{N}_1 \times \tilde{N}_2) = \bigoplus_{(p_1, p_2) \in S} C^{p_1}(\tilde{N}_1) \otimes C^{p_2}(\tilde{N}_2)$$

with

$$S = \{(p_1, p_2) \mid p_1 + p_2 = n_1 + n_2\}.$$

Define an involution

$$T : S \rightarrow S; (p_1, p_2) \mapsto (2n_1 - p_1, 2n_2 - p_2),$$

and let $U \subset S \setminus \{(n_1, n_2)\}$ be any subset such that S decomposes as a disjoint union

$$S = \{(n_1, n_2)\} \cup U \cup T(U) .$$

The submodule

$$L = \bigoplus_{(p_1, p_2) \in U} C^{p_1}(\tilde{N}_1) \otimes C^{p_2}(\tilde{N}_2) \subseteq A$$

is a sublagrangian of (A, α) such that

$$(L^\perp/L, [\alpha]) = (C^{m_1}(\tilde{N}_1), \alpha_1) \otimes (C^{m_2}(\tilde{N}_2), \alpha_2) .$$

The submodule

$$\Delta_{L^\perp} = \{(b, [b]) \mid b \in L^\perp\} \subset A \oplus (L^\perp/L)$$

is a lagrangian of $(A, \alpha) \oplus (L^\perp/L, -[\alpha])$, and

$$\begin{aligned} (A, \alpha) &= (L^\perp/L, [\alpha]) = (C^{n_1}(\tilde{N}_1), \alpha_1) \otimes (C^{n_2}(\tilde{N}_2), \alpha_2) \\ &\in \text{im}(AL^{2n_1}(\mathbb{Z}[\rho_1]) \otimes AL^{2n_2}(\mathbb{Z}[\rho_2]) \rightarrow AL^{2(n_1+n_2)}(\mathbb{Z}[\rho_1 \times \rho_2])) . \end{aligned}$$

□

The product of a nonsingular $(-1)^m$ -quadratic form (K, λ, μ) over Λ and a $2n$ -dimensional symmetric Poincaré complex (C, ϕ) over R is a $2(m+n)$ -dimensional quadratic Poincaré complex $(K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$ over $\Lambda' = \Lambda \otimes R$, as in [21], with K_{*-m} the $2m$ -dimensional f.g. free Λ -module chain complex concentrated in degree m

$$K_{*-m} : 0 \rightarrow \cdots \rightarrow 0 \rightarrow K \rightarrow 0 \rightarrow \cdots \rightarrow 0 .$$

The pairing

$$L_{2m}(\Lambda) \otimes L^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R) ; (K, \lambda, \mu) \otimes (C, \phi) \mapsto (K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$$

has the following generalization.

Definition 6.6 The *product* of a nonsingular $(-1)^m$ -quadratic form (K, λ, μ) over Λ and an almost $(-1)^n$ -symmetric form (A, α) over R is the nonsingular $(-1)^{m+n}$ -quadratic form over $\Lambda' = \Lambda \otimes R$

$$(K', \lambda', \mu') = (K, \lambda, \mu) \otimes (A, \alpha)$$

with

$$K' = K \otimes A , (\lambda', \mu') = (\lambda, \mu) \otimes \alpha = (\psi' + (-1)^{m+n} \psi'^*, \psi')$$

determined by the Λ' -module morphism

$$\psi' = \psi \otimes \alpha : K' = K \otimes A \rightarrow K'^* = K^* \otimes A^*$$

with $\psi : K \rightarrow K^*$ a Λ -module morphism such that

$$(\lambda, \mu) = (\psi + (-1)^m \psi^*, \psi) .$$

□

In particular, if $K = \Lambda^p$ then ψ is given by a $p \times p$ matrix $\psi = \{\psi_{ij}\}$ over Λ , and if $A = R^q$ then $\alpha = \{\alpha_{rs}\}$ is given by a $q \times q$ matrix over R , so that

$$\psi' = \psi \otimes \alpha$$

is the $pq \times pq$ matrix over Λ' with

$$\psi'_{tu} = \psi_{ij} \otimes \alpha_{rs} \text{ if } t = (i-1)p + r, \ u = (j-1)p + s .$$

If (A, α) is an almost $(-1)^n$ -symmetric form over R with a sublagrangian $L \subset A$ the induced almost $(-1)^n$ -symmetric form $(L^\perp/L, [\alpha])$ over R is such that

$$\Delta_{L^\perp} = \{(b, [b]) \mid b \in L^\perp\} \subset A \oplus (L^\perp/L)$$

is a lagrangian of $(A, \alpha) \oplus (L^\perp/L, -[\alpha])$, and

$$(K, \lambda, \mu) \otimes (A, \alpha) = (K, \lambda, \mu) \otimes (L^\perp/L, [\alpha]) \in L_{2m+2n}(\Lambda') .$$

In particular, if L is a lagrangian of (A, α) then

$$(K, \lambda, \mu) \otimes (A, \alpha) = 0 \in L_{2m+2n}(\Lambda') ,$$

so that the product

$$L_{2m}(\Lambda) \otimes AL^{2n}(R) \rightarrow L_{2m+2n}(\Lambda') ; (K, \lambda, \mu) \otimes (A, \alpha) \mapsto (K \otimes A, (\lambda, \mu) \otimes \alpha)$$

is well-defined.

Theorem 6.7 (Almost symmetric L -theory surgery product formula, Clauwens [6])

(i) *The product*

$$L_{2m}(\Lambda) \otimes L^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R) ; (K, \lambda, \mu) \otimes (C, \phi) \mapsto (K_{*-m} \otimes C, (\lambda, \mu) \otimes \phi)$$

factors through the product

$$L_{2m}(\Lambda) \otimes AL^{2n}(R) \rightarrow L_{2m+2n}(\Lambda \otimes R) ; (K, \lambda, \mu) \otimes (A, \alpha) \mapsto (K \otimes A, (\lambda, \mu) \otimes \alpha) .$$

(ii) Let $(f, b) : M \rightarrow X$ be a $2m$ -dimensional normal map with surgery obstruction

$$\sigma_*(f, b) = (\mathbb{Z}[\pi]^p, \lambda, \mu) \in L_{2m}(\mathbb{Z}[\pi]) \quad (\pi = \pi_1(X)) ,$$

and let N be a $2n$ -dimensional manifold with almost $(-1)^n$ -symmetric signature

$$\sigma^*(N) = (\mathbb{Z}[\rho]^q, \alpha) \in AL^{2n}(\mathbb{Z}[\rho]) \quad (\rho = \pi_1(N)) .$$

The surgery obstruction of the $(2m + 2n)$ -dimensional normal map

$$(g, c) = (f, b) \times 1 : M \times N \rightarrow X \times N$$

is given by

$$\begin{aligned} \sigma_*(g, c) &= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha) \\ &\in \text{im}(L_{2m}(\mathbb{Z}[\pi]) \otimes AL^{2n}(\mathbb{Z}[\rho]) \rightarrow L_{2m+2n}(\mathbb{Z}[\pi \times \rho])) . \end{aligned}$$

(iii) The surgery obstruction of the product $2(m + n_1 + n_2)$ -dimensional normal map

$$(g, c) = (f, b) \times 1 : M \times N_1 \times N_2 \rightarrow X \times N_1 \times N_2$$

is given by

$$\sigma_*(g, c) = (\mathbb{Z}[\pi \times \rho_1 \times \rho_2]^{pq_1q_2}, (\lambda, \mu) \otimes \alpha_1 \otimes \alpha_2) \in L_{2(m+n_1+n_2)}(\mathbb{Z}[\pi \times \rho_1 \times \rho_2]) .$$

Proof (i) By construction.

(ii) It may be assumed that $(f, b) : M \rightarrow X$ is an m -connected $2m$ -dimensional normal map, with kernel $(-1)^m$ -quadratic form over $\mathbb{Z}[\pi]$

$$(K_m(M), \lambda, \mu) = (\mathbb{Z}[\pi]^p, \lambda, \mu) .$$

The product $(g, c) = (f, b) \times 1 : M \times N \rightarrow X \times N$ is m -connected, with quadratic Poincaré complex

$$(C, \psi) = (K_m(M), \lambda, \mu) \otimes (C(\tilde{N}), \phi)$$

and kernel $\mathbb{Z}[\pi \times \rho]$ -modules

$$K_*(M \times N) = K_m(M) \otimes H_{*-m}(\tilde{N}) .$$

Let $(f', b') : M' \rightarrow X \times N$ be the bordant $(m + n)$ -connected normal map obtained from (g, c) by surgery below the middle dimension, using (C, ψ) as in Remark 3 (ii). The kernel $(-1)^{m+n}$ -quadratic form over $\mathbb{Z}[\pi \times \rho]$ of (f', b') is

the instant form of (C, ψ) , which is just the product of $(K_m(M), \lambda, \mu)$ and the almost $(-1)^n$ -symmetric form $(\mathbb{Z}[\rho]^q, \alpha)$

$$\begin{aligned} (K_{m+n}(M'), \lambda', \mu') &= \\ \left(\text{coker} \left(\begin{pmatrix} d^* & 0 \\ (-1)^{m+n+1}(1+T)\psi_0 & d \end{pmatrix} : C^{m+n-1} \oplus C_{m+n+2} \rightarrow C^{m+n} \oplus C_{m+n+1} \right), \right. \\ &\quad \left. \begin{bmatrix} \psi_0 + (-1)^{m+n}\psi_0^* & d \\ (-1)^{m+n}d^* & 0 \end{bmatrix}, \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix} \right) \\ &= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha) . \end{aligned}$$

The surgery obstruction of (g, c) is thus given by

$$\begin{aligned} \sigma_*(g, c) &= \sigma_*(f', b') = (K_{m+n}(M'), \lambda', \mu') \\ &= (\mathbb{Z}[\pi \times \rho]^{pq}, (\lambda, \mu) \otimes \alpha) \in L_{2m+2n}(\mathbb{Z}[\pi \times \rho]) . \end{aligned}$$

(iii) Combine (i) and (ii) with Proposition 6.5. □

7 The almost $(-1)^n$ -symmetric form of T^{2n}

Geometrically, $-\times T^{2n}$ sends the surgery obstruction $\sigma_*(f_0, b_0) = E_8 \in L_{4m}(\mathbb{Z})$ to the surgery obstruction

$$E_8 \times T^{2n} = \sigma_*(f_n, b_n) \in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

of the $(4m + 2n)$ -dimensional normal map

$$(f_n, b_n) = (f_0, b_0) \times 1 : M_0^{4m} \times T^{2n} \rightarrow S^{4m} \times T^{2n}$$

given by product with the almost symmetric signature of

$$\begin{aligned} T^{2n} &= S^1 \times S^1 \times \cdots \times S^1 \quad (2n \text{ factors}) \\ &= T^2 \times T^2 \times \cdots \times T^2 \quad (n \text{ factors}) . \end{aligned}$$

In order to apply the almost symmetric surgery product formula 6.7 for $N^{2n} = T^{2n}$ it therefore suffices to work out the almost (-1) -symmetric form $(C^1(\tilde{T}^2), \alpha)$ of T^2 .

The symmetric Poincaré structure $\phi = \{\phi_s | s \geq 0\}$ of the universal cover $\tilde{S}^1 = \mathbb{R}$ of S^1 is given by

$$\begin{aligned} d &= 1 - z : C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] , \\ \phi_0 &= \begin{cases} 1 : C^0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \\ z : C^1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_0(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] , \end{cases} \\ \phi_1 &= -1 : C^1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] \rightarrow C_1(\mathbb{R}) = \mathbb{Z}[z, z^{-1}] . \end{aligned}$$

Write

$$\Lambda = \mathbb{Z}[\pi_1(T^2)] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}] .$$

The Poincaré duality of $\tilde{T}^2 = \mathbb{R}^2$ is the Λ -module chain isomorphism given by the chain-level Künneth formula to be

$$\begin{array}{ccccccc} C(\tilde{T}^2)^{2-*} : C^0 = \Lambda & \xrightarrow{d^* = \begin{pmatrix} z_2 - 1 \\ 1 - z_1^{-1} \end{pmatrix}} & C^1 = \Lambda \oplus \Lambda & \xrightarrow{d^* = \begin{pmatrix} 1 - z_1^{-1} & 1 - z_2 \end{pmatrix}} & C^2 = \Lambda & & \\ \downarrow \phi_0 & & \downarrow \begin{pmatrix} 0 & -z_1 \\ z_2^{-1} & 0 \end{pmatrix} & & \downarrow -z_1 z_2^{-1} & & \\ C(\tilde{T}^2) : C_2 = \Lambda & \xrightarrow{d = \begin{pmatrix} 1 - z_1 \\ 1 - z_2^{-1} \end{pmatrix}} & C_1 = \Lambda \oplus \Lambda & \xrightarrow{d = \begin{pmatrix} z_2^{-1} - 1 & 1 - z_1 \end{pmatrix}} & C_0 = \Lambda & & \end{array}$$

The chain homotopy

$$\phi_1 : \phi_0 \simeq T\phi_0 : C(\tilde{T}^2)^{2-*} \rightarrow C(\tilde{T}^2)$$

is given by

$$\phi_1 = \begin{cases} \begin{pmatrix} 1 & -z_2 \end{pmatrix} : C^1 = \Lambda \oplus \Lambda \rightarrow C_2 = \Lambda \\ \begin{pmatrix} -z_1 \\ 1 \end{pmatrix} : C^2 = \Lambda \rightarrow C_1 = \Lambda \oplus \Lambda . \end{cases}$$

Proposition 7.1 *The almost (-1) -symmetric form of T^2 is given by (C^1, α) with*

$$\alpha = \phi_0 - \phi_1 d^* = \begin{pmatrix} 1 - z_1 & z_1 z_2 - z_1 - z_2 \\ 1 & 1 - z_2 \end{pmatrix} : C^1 = \Lambda \oplus \Lambda \rightarrow C_1 = \Lambda \oplus \Lambda .$$

Proof By construction, noting that

$$1 + \alpha^{-1} \alpha^* = \begin{pmatrix} -(1 - z_1)(1 - z_2^{-1}) & z_1(1 - z_2)(1 - z_2^{-1}) \\ -z_2^{-1}(1 - z_1)(1 - z_1^{-1}) & (1 - z_1)(1 - z_2^{-1}) \end{pmatrix} : \\ C^1 = \Lambda \oplus \Lambda \rightarrow C^1 = \Lambda \oplus \Lambda$$

is nilpotent, with

$$(1 + \alpha^{-1} \alpha^*)^2 = 0 : C^1 = \Lambda \oplus \Lambda \rightarrow C^1 = \Lambda \oplus \Lambda .$$

□

Remark An almost $(-1)^n$ -symmetric form (R^q, α) over R determines a nonsingular $(-1)^n$ -quadratic form $(R[1/2]^q, \lambda, \mu)$ over $R[1/2]$, with

$$\lambda(x, y) = (\alpha(x, y) + (-1)^n \overline{\alpha(y, x)})/2, \quad \mu(x) = \alpha(x)(x)/2.$$

In particular, the almost (-1) -symmetric form $(\Lambda \oplus \Lambda, \alpha)$ of T^2 determines the nonsingular (-1) -quadratic form $(\Lambda[1/2] \oplus \Lambda[1/2], \lambda, \mu)$ over $\Lambda[1/2] = \mathbb{Z}[\mathbb{Z}^2][1/2]$, with

$$\begin{aligned} \lambda &= (\alpha - \alpha^*)/2 \\ &= \begin{pmatrix} ((z_1)^{-1} - z_1)/2 & (1 - z_1 z_2 - z_1 - z_2)/2 \\ (-1 + (z_1)^{-1}(z_2)^{-1} + (z_1)^{-1} + (z_2)^{-1})/2 & ((z_2)^{-1} - z_2)/2 \end{pmatrix} \end{aligned}$$

the invertible skew-symmetric 2×2 matrix exhibited in [12][Example, p.120].

□

8 An explicit form representing $E_8 \times T^{2n} \in L_{4*+2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$

Write the generators of the free abelian group $\pi_1(T^{2n}) = \mathbb{Z}^{2n}$ as $z_1, z_2, \dots, z_{2n-1}, z_{2n}$, so that

$$\mathbb{Z}[\mathbb{Z}^{2n}] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_{2n}, z_{2n}^{-1}].$$

The expression of T^{2n} as an n -fold cartesian product of T^2 's

$$T^{2n} = T^2 \times T^2 \times \dots \times T^2$$

gives

$$\mathbb{Z}[\mathbb{Z}^{2n}] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}] \otimes \mathbb{Z}[z_3, z_3^{-1}, z_4, z_4^{-1}] \otimes \dots \otimes \mathbb{Z}[z_{2n-1}, z_{2n-1}^{-1}, z_{2n}, z_{2n}^{-1}].$$

For $i = 1, 2, \dots, n$ define the invertible 2×2 matrix over $\mathbb{Z}[z_{2i-1}, z_{2i-1}^{-1}, z_{2i}, z_{2i}^{-1}]$

$$\alpha_i = \begin{pmatrix} 1 - z_{2i-1} & z_{2i-1} z_{2i} - z_{2i-1} - z_{2i} \\ 1 & 1 - z_{2i} \end{pmatrix}.$$

The generator $1 = E_8 \in L_0(\mathbb{Z}) = \mathbb{Z}$ is represented by the nonsingular quadratic form (\mathbb{Z}^8, ψ_0) over \mathbb{Z} with

$$\psi_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 8.1 The $2^{n+3} \times 2^{n+3}$ matrix over $\mathbb{Z}[\mathbb{Z}^{2n}]$

$$\psi_n = \psi_0 \otimes \alpha_1 \otimes \alpha_2 \cdots \otimes \alpha_n$$

is such that

$$E_8 \times T^{2n} = (\mathbb{Z}[\mathbb{Z}^{2n}]^{2^{n+3}}, \psi_n) \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) .$$

Proof A direct application of the almost symmetric surgery product formula 6.7, noting that $\alpha_1, \alpha_2, \dots, \alpha_n$ are copies of the almost (-1) -symmetric form of T^2 obtained in 7.1. \square

9 Transfer invariance

A covering map $p : T^n \rightarrow T^n$ induces an injection of the fundamental group in itself

$$p_* : \pi_1(T^n) = \mathbb{Z}^n \rightarrow \pi_1(T^n) = \mathbb{Z}^n$$

as a subgroup of finite index, say $q = [\mathbb{Z}^n : p_*(\mathbb{Z}^n)]$. Given a $\mathbb{Z}[\mathbb{Z}^n]$ -module K let $p^!K$ be the $\mathbb{Z}[\mathbb{Z}^n]$ -module defined by the additive group of K with

$$\mathbb{Z}[\mathbb{Z}^n] \times p^!K \rightarrow p^!K ; (a, b) \mapsto p_*(a)b .$$

In particular

$$p^!\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[\mathbb{Z}^n]^q .$$

The restriction functor

$$p^! : \{\mathbb{Z}[\mathbb{Z}^n]\text{-modules}\} \rightarrow \{\mathbb{Z}[\mathbb{Z}^n]\text{-modules}\} ; K \mapsto p^!K$$

induces transfer maps in the quadratic L -groups

$$p^! : L_m(\mathbb{Z}[\mathbb{Z}^n]) \rightarrow L_m(\mathbb{Z}[\mathbb{Z}^n]) ; (C, \psi) \mapsto p^!(C, \psi) .$$

Proposition 9.1 The image of the (split) injection

$$L_0(\mathbb{Z}) \rightarrow L_n(\mathbb{Z}[\mathbb{Z}^n]) = \sum_{r=0}^n \binom{n}{r} L_{n-r}(\mathbb{Z}) ; E_8 \mapsto E_8 \times T^n$$

is the subgroup of the transfer-invariant elements

$$L_n(\mathbb{Z}[\mathbb{Z}^n])^{INV} = \{x \in L_n(\mathbb{Z}[\mathbb{Z}^n]) \mid p^!x = x \text{ for all } p : T^n \rightarrow T^n\} .$$

Proof See Chapter 18 of Ranicki [22]. \square

Example (i) Write

$$\Lambda = \mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}] .$$

Here is an explicit verification that

$$p^!(E_8 \times T^2) = E_8 \times T^2 \in L_2(\Lambda)$$

for the double cover

$$p : T^2 = S^1 \times S^1 \rightarrow T^2 ; (w_1, w_2) \mapsto ((w_1)^2, w_2)$$

with

$$p_* : \pi_1(T^2) = \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 ; z_1 \mapsto (z_1)^2, z_2 \mapsto z_2$$

the inclusion of a subgroup of index 2. For any $j_1, j_2 \in \mathbb{Z}$ the transfer of the Λ -module morphism $z_1^{j_1} z_2^{j_2} : \Lambda \rightarrow \Lambda$ is given by the Λ -module morphism

$$p^!(z_1^{j_1} z_2^{j_2}) = \begin{cases} \begin{pmatrix} (z_1)^{j_1/2} z_2^{j_2} & 0 \\ 0 & (z_1)^{j_1/2} z_2^{j_2} \end{pmatrix} : \\ p^! \Lambda = \Lambda \oplus \Lambda \rightarrow p^! \Lambda = \Lambda \oplus \Lambda \quad \text{if } j_1 \text{ is even} \\ \\ \begin{pmatrix} 0 & (z_1)^{(j_1+1)/2} z_2^{j_2} \\ (z_1)^{(j_1-1)/2} z_2^{j_2} & 0 \end{pmatrix} : \\ p^! \Lambda = \Lambda \oplus \Lambda \rightarrow p^! \Lambda = \Lambda \oplus \Lambda \quad \text{if } j_1 \text{ is odd} . \end{cases}$$

The transfer of the almost (-1) -symmetric form of T^2 over Λ

$$(C^1(\tilde{T}^2), \alpha) = (\Lambda \oplus \Lambda, \begin{pmatrix} 1 - z_1 & z_1 z_2 - z_1 - z_2 \\ 1 & 1 - z_2 \end{pmatrix})$$

is the almost (-1) -symmetric form over Λ

$$p^!(C^1(\tilde{T}^2), \alpha) = (\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda, \begin{pmatrix} 1 & -z_1 & -z_2 & z_1 z_2 - z_1 \\ -1 & 1 & z_2 - 1 & -z_2 \\ 1 & 0 & 1 - z_2 & 0 \\ 0 & 1 & 0 & 1 - z_2 \end{pmatrix})$$

The Λ -module morphisms

$$i = \begin{pmatrix} z_1 - z_1 z_2 \\ 0 \\ -z_1 \\ 1 \end{pmatrix} : \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda ,$$

$$j = \begin{pmatrix} 1 & 0 & z_1 - z_1 z_2 \\ z_1^{-1} & 0 & 0 \\ 0 & 1 & -z_1 \\ 0 & 0 & 1 \end{pmatrix} : \Lambda \oplus \Lambda \oplus \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$$

are such that $i = j|_{0 \oplus 0 \oplus \Lambda}$ and there is defined a (split) exact sequence

$$0 \longrightarrow \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{j} \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{i^* p^! \alpha} \Lambda \longrightarrow 0$$

with

$$j^*(p^! \alpha)j = \begin{pmatrix} 1 - z_1 & z_1 z_2 - z_1 - z_2 & 0 \\ 1 & 1 - z_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \Lambda \oplus \Lambda \oplus \Lambda \rightarrow \Lambda \oplus \Lambda \oplus \Lambda .$$

The submodule

$$L = i(\Lambda) \subset p^!(\Lambda \oplus \Lambda) = \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$$

is thus a sublagrangian of the almost (-1) -symmetric form $p^!(C^1(\tilde{T}^2), \alpha)$ over $\mathbb{Z}[\mathbb{Z}^2]$ such that

$$(L^\perp/L, [p^! \alpha]) = (C^1(\tilde{T}^2), \alpha)$$

and

$$\begin{aligned} p^!(E_8 \times T^2) &= E_8 \otimes p^!(C^1(\tilde{T}^2), \alpha) \\ &= E_8 \otimes (L^\perp/L, [p^! \alpha]) \\ &= E_8 \otimes (C^1(\tilde{T}^2), \alpha) = E_8 \times T^2 \in L_2(\Lambda) . \end{aligned}$$

(ii) For any $n \geq 1$ replace p by

$$p_n = p \times 1 : T^{2n} = T^2 \times T^{2n-2} \rightarrow T^{2n} = T^2 \times T^{2n-2}$$

to likewise obtain an explicit verification that

$$p_n^!(E_8 \times T^{2n}) = E_8 \times T^{2n} \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) .$$

□

10 Controlled surgery groups

A *geometric $\mathbb{Z}[\pi]$ -module* over a metric space B is a pair (K, φ) , where $K = \mathbb{Z}[\pi]^r$ is a free $\mathbb{Z}[\pi]$ -module with basis $S = \{e_1, \dots, e_r\}$ and $\varphi: S \rightarrow B$ is a map. The (ϵ, δ) -*controlled surgery group* $L_n(B; \mathbb{Z}, \epsilon, \delta)$ (with trivial local fundamental group) is defined as the group of n -dimensional quadratic \mathbb{Z} -Poincaré complexes (see [21]) over B of radius $< \delta$, modulo $(n+1)$ -dimensional quadratic \mathbb{Z} -Poincaré bordisms of radius $< \epsilon$. Elements of $L_{2n}(B; \mathbb{Z}, \epsilon, \delta)$ are represented by non-singular $(-1)^n$ -quadratic forms (K, λ, μ) , where $K = \mathbb{Z}^r$ is a geometric \mathbb{Z} -module over B , and λ has radius $< \delta$, i.e., $\lambda(e_i, e_j) = 0$

if $d(\varphi(e_i), \varphi(e_j)) \geq \delta$. In matrix representation (K, ψ) , this is equivalent to $\psi_{ij} = 0$ if $d(\varphi(e_i), \varphi(e_j)) \geq \delta$. The radius of a bordism is defined similarly.

In effect, Yamasaki [29] defined an assembly map $H_n(B; \mathbb{L}) \rightarrow L_n(B; \mathbb{Z}, \epsilon, \delta)$, where $H_*(B; \mathbb{L})$ denotes homology with coefficients in the 4-periodic simply-connected surgery spectrum \mathbb{L} of Chapter 25 of Ranicki [23].

The following Stability Theorem is a key ingredient in the construction of exotic ENR homology manifolds.

Theorem 10.1 (Stability) (Pedersen, Quinn and Ranicki [16], Ferry [11], Pedersen and Yamasaki [17])

Let $n \geq 0$ and suppose B is a compact metric ENR. Then there exist constants $\epsilon_0 > 0$ and $\kappa > 1$, which depend on n and B , such that the assembly map $H_n(B; \mathbb{L}) \rightarrow L_n(B; \mathbb{Z}, \epsilon, \delta)$ is an isomorphism if $\epsilon_0 \geq \epsilon \geq \kappa\delta$, so that

$$\lim_{\epsilon \rightarrow \infty} \lim_{\delta \rightarrow 0} L_n(B; \mathbb{Z}, \epsilon, \delta) = H_n(B; \mathbb{L}) .$$

□

We are interested in controlled surgery over the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ equipped with the usual geodesic metric. Let (K, ψ) represent an element of $L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$, where $K = \mathbb{Z}[\mathbb{Z}^{2n}]^r$. Our next goal is to show that passing to a sufficiently large covering space $p: T^{2n} \rightarrow T^{2n}$, (K, ψ) defines an element of $L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta)$. For simplicity, we assume that

$$p_*: \pi_1(T^{2n}) \cong \mathbb{Z}^{2n} \rightarrow \pi_1(T^{2n}) \cong \mathbb{Z}^{2n}$$

is given by multiplication by $k > 0$, so that p is a k^{2n} -sheeted covering space.

Let $(\bar{K}, \bar{\psi}) = \mathbb{Z}[\mathbb{Z}_k^{2n}] \otimes_{\mathbb{Z}[\mathbb{Z}^{2n}]} (K, \psi)$, where the (right) $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module structure on $\mathbb{Z}[\mathbb{Z}_k^{2n}]$ is induced by reduction modulo k . The \mathbb{Z} -module \tilde{K} underlying \bar{K} has basis $\mathbb{Z}_k^{2n} \times S$; if $g \in \mathbb{Z}_k^{2n}$ and $e_i \in S$, we write $(g, e_i) = g e_i$. Pick a point x_0 in the covering torus T^{2n} viewed as a \mathbb{Z}_k^{2n} -space under the action of the group of deck transformations. Let $\varphi(e_i) = x_0$, for every $e_i \in S$, and extend it \mathbb{Z}_k^{2n} -equivariantly to obtain $\varphi: \mathbb{Z}_k^{2n} \times S \rightarrow T^{2n}$. Then, the pair (\tilde{K}, φ) is a geometric \mathbb{Z} -module over T^{2n} of dimension rk^{2n} .

We now describe the quadratic \mathbb{Z} -module $(\tilde{K}, \tilde{\psi})$ induced by (K, ψ) and the covering p . Write

$$\tilde{\psi} = \sum_{g \in \mathbb{Z}_k^{2n}} g \bar{\psi}_g ,$$

where each $\bar{\psi}_g$ is a matrix with integer entries. For basis elements $ge_i, fe_j \in \mathbb{Z}_k^{2n} \times S$, let $\tilde{\psi}(ge_i, fe_j) = \bar{\psi}_{g^{-1}f}(e_i, e_j)$; this defines a bilinear \mathbb{Z} -form on the

geometric \mathbb{Z} -module \tilde{K} . For a given quadratic $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module (K, ψ) , we show that $(\tilde{K}, \tilde{\psi})$ has diameter $< \delta$ over the (covering) torus T^{2n} , if k is sufficiently large.

Elements of \mathbb{Z}^{2n} can be expressed uniquely as monomials

$$z^i = z_1^{i_1} \dots z_{2n}^{i_{2n}}$$

where $i = (i_1, \dots, i_{2n}) \in \mathbb{Z}^{2n}$ is a multi-index. We use the notation

$$|i| = \max \{|i_1|, \dots, |i_{2n}|\} .$$

Any $z \in \mathbb{Z}[\mathbb{Z}^{2n}]$ can be expressed uniquely as

$$z = \sum_{i \in \mathbb{Z}^{2n}} \alpha_i z^i ,$$

where $\alpha_i \in \mathbb{Z}$ is zero for all but finitely many values of i . We define the *order* of z to be

$$o(z) = \max \{|i| : \alpha_i \neq 0\}$$

and let

$$|\psi| = \max \{o(\psi_{ij}), 1 \leq i, j \leq r\} .$$

Then, $(\tilde{K}, \tilde{\psi})$ is a quadratic \mathbb{Z} -module over T^{2n} of radius $< \delta$, provided that $k > 2|\psi|/\delta$. Similarly, quadratic $\mathbb{Z}[\mathbb{Z}^{2n}]$ -Poincaré bordisms induce quadratic \mathbb{Z} -Poincaré ϵ -bordisms for k large.

10.1 The forgetful map

We give an algebraic description of the forget-control map

$$\mathcal{F}: L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta) \rightarrow L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]),$$

for ϵ and δ small. Let $\sigma \in L(T^{2n}; \mathbb{Z}, \epsilon, \delta)$ be represented by the $(-1)^n$ -quadratic \mathbb{Z} -module (K, ψ) over T^{2n} of radius $< \delta$, where K has basis $S = \{e_1, \dots, e_r\}$ and projection $\varphi: S \rightarrow T^{2n}$. Consider the free $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module \tilde{K} of rank r generated by $\tilde{S} = \{\tilde{e}_1, \dots, \tilde{e}_r\}$ and let $\tilde{\varphi}: \tilde{S} \rightarrow \mathbb{R}^{2n}$ be a map satisfying $q \circ \tilde{\varphi}(\tilde{e}_i) = \varphi(e_i)$, $1 \leq i \leq r$, where $q: \mathbb{R}^{2n} \rightarrow T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is the universal cover. If $\psi_{ij} \neq 0$ and δ is small, there is a unique element g_{ij} of \mathbb{Z}^{2n} such that $d(\tilde{\varphi}(\tilde{e}_j) + g_{ij}, \tilde{\varphi}(\tilde{e}_i)) < \delta$, where d denotes Euclidean distance. Let $\tilde{\psi} = (\tilde{\psi}_{ij})$, $1 \leq i, j \leq r$ be the matrix whose entries in $\mathbb{Z}[\mathbb{Z}^{2n}]$ are

$$\tilde{\psi}_{ij} = \begin{cases} 0, & \text{if } \psi_{ij} = 0; \\ \psi_{ij} g_{ij}, & \text{if } \psi_{ij} \neq 0. \end{cases} \quad (1)$$

The quadratic $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module $(\tilde{K}, \tilde{\psi})$ represents $\mathcal{F}(\sigma) \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$. Likewise, quadratic \mathbb{Z} -Poincaré ϵ -bordisms over T^{2n} induce quadratic $\mathbb{Z}[\mathbb{Z}^{2n}]$ -Poincaré bordisms.

10.2 Controlled E_8 over T^{2n}

Starting with the $(-1)^n$ -quadratic $\mathbb{Z}[\mathbb{Z}^{2n}]$ -module $E_8 \times T^{2n}$, pass to a large covering space $p: T^{2n} \rightarrow T^{2n}$ to obtain a δ -controlled quadratic \mathbb{Z} -module \tilde{E}_8 over T^{2n} representing an element of $L_{2n}(T^{2n}; \mathbb{Z}, \epsilon, \delta)$. It is simple to verify that $\mathcal{F}(\tilde{E}_8) = p^!(E_8 \times T^{2n})$, where $p^!$ is the L -theory transfer. The transfer invariance results discussed in Section 9 imply that $\mathcal{F}(\tilde{E}_8) = E_8 \times T^{2n}$. Thus, \tilde{E}_8 gives a δ -controlled realization of the form E_8 over T^{2n} .

10.3 Controlled surgery obstructions

Definition 10.2 Let $p: X \rightarrow B$ be a map to a metric space B and $\epsilon > 0$. A map $f: Y \rightarrow X$ is an ϵ -homotopy equivalence over B , if there exist a map $g: X \rightarrow Y$ and homotopies H_t from $g \circ f$ to 1_Y and K_t from $f \circ g$ to 1_X such that $\text{diam}(p \circ f \circ H_t(y)) < \epsilon$ for every $y \in Y$, and $\text{diam}(p \circ K_t(x)) < \epsilon$, for every $x \in X$. This means that the tracks of H and K are ϵ -small as viewed from B . \square

Controlled surgery theory addresses the question of the existence and uniqueness of controlled manifold structures on a space. Polyhedra homotopy equivalent to compact topological manifolds satisfy the Poincaré duality isomorphism. Likewise, there is a notion of ϵ -Poincaré duality satisfied by polyhedra finely equivalent to a manifold. Poincaré duality can be estimated by the diameter of cap product with a fundamental class as a chain homotopy equivalence.

Definition 10.3 Let $p: X \rightarrow B$ be a map, where X is a polyhedron and B is a metric space. X is an ϵ -Poincaré complex of formal dimension n over B if there exist a subdivision of X such that simplices have diameter $\ll \epsilon$ in B and an n -cycle y in the simplicial chains of X so that $\cap y: C^\sharp(X) \rightarrow C_{n-\sharp}(X)$ is an ϵ -chain homotopy equivalence in the sense that $\cap y$ and the chain homotopies have the property that the image of each generator σ only involves generators whose images under p are within an ϵ -neighborhood of $p(\sigma)$ in B . \square

To formulate simply-connected controlled surgery problems, the notion of locally trivial fundamental group from the viewpoint of the control space is needed. This can be formalized using the notion of UV^1 maps as follows.

Definition 10.4 Given $\delta > 0$, a map $p: X \rightarrow B$ is called δ - UV^1 if for any polyhedral pair (P, Q) , with $\dim(P) \leq 2$, and maps $\alpha_0: Q \rightarrow X$ and $\beta: P \rightarrow B$ such that $p \circ \alpha_0 = \beta|_Q$,

$$\begin{array}{ccc} Q & \xrightarrow{\alpha_0} & X \\ \downarrow i & \nearrow \alpha & \downarrow p \\ P & \xrightarrow{\beta} & B \end{array}$$

there is a map $\alpha: P \rightarrow X$ extending α_0 so that $p \circ \alpha$ is δ -homotopic to β over B . The map p is UV^1 if it is δ - UV^1 , for every $\delta > 0$. \square

Let B be a compact metric ENR and $n \geq 5$. Given $\epsilon > 0$, there is a $\delta > 0$ such that if $p: X \rightarrow B$ is a δ -Poincaré duality space over B of formal dimension n , $(f, b): M^n \rightarrow X$ is a surgery problem, and p is δ - UV^1 . By the Stability Theorem 10.1 there is a well-defined surgery obstruction

$$\sigma_*(f, b) \in \varprojlim_{\epsilon} \varprojlim_{\delta} L_n(B; \mathbb{Z}, \epsilon, \delta) = H_n(B; \mathbb{L})$$

such that (f, b) is normally cobordant to an ϵ -homotopy equivalence for any $\epsilon > 0$ if and only if $\sigma_*(f, b) = 0$. See Ranicki and Yamasaki [25] for an exposition of controlled L -theory.

The main theorem of [16] is the following controlled surgery exact sequence (see also [11], [25]).

Theorem 10.5 Suppose B is a compact metric ENR and $n \geq 4$. There is a stability threshold $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there is $\delta > 0$ with the property that if $p: N \rightarrow B$ is a δ - UV^1 map, with N is a compact n -manifold, there is an exact sequence

$$H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\epsilon, \delta}(N, f) \rightarrow [N, \partial N; G/TOP, *] \rightarrow H_n(B; \mathbb{L}).$$

\square

Here, $\mathcal{S}_{\epsilon, \delta}$ is the controlled structure set defined as the set of equivalence classes of pairs (M, g) , where M is a topological manifold and $g: (M, \partial M) \rightarrow (N, \partial N)$ restricts to a homeomorphism on ∂N and is a δ -homotopy equivalence relative to the boundary. The pairs (M_1, g_1) and (M_2, g_2) are equivalent if there is a homeomorphism $h: M_1 \rightarrow M_2$ such that g_1 and $h \circ g_2$ are ϵ -homotopic rel boundary. As in classical surgery, the map $H_{n+1}(B; \mathbb{L}) \rightarrow \mathcal{S}_{\epsilon, \delta}(N, f)$ is defined using controlled Wall realization.

11 Exotic homology manifolds

In [2], exotic ENR homology manifolds of dimensions greater than 5 are constructed as limits of sequences of controlled Poincaré complexes $\{X_i, i \geq 0\}$. These complexes are related by maps $p_i: X_{i+1} \rightarrow X_i$ such that X_{i+1} is ϵ_{i+1} -Poincaré over X_i , $i \geq 0$, and p_i is an ϵ_i -homotopy equivalence over X_{i-1} , $i \geq 1$, where $\sum \epsilon_i < \infty$. Beginning, say, with a closed manifold X_0 , the sequence $\{X_i\}$ is constructed iteratively using cut-paste constructions on closed manifolds. The gluing maps are obtained using the Wall realization of controlled surgery obstructions, which emerge as a non-trivial local index in the limiting ENR homology manifold. As pointed out in the Introduction, our main goal is to give an explicit construction of the first controlled stage X_1 of this construction using the quadratic form E_8 , beginning with the $2n$ -dimensional torus $X_0 = T^{2n}$, $n \geq 3$. The construction of subsequent stages follows from fairly general arguments presented in [2] and leads to an index-9 ENR homology manifold not homotopy equivalent to any closed topological manifold. Since an explicit algebraic description of the controlled quadratic module \tilde{E}_8 over T^{2n} has already been given in Section 10.2, we conclude the paper with a review of how this quadratic module can be used to construct X_1 .

Let P be the 2-skeleton of a fine triangulation of T^{2n} , and C a regular neighborhood of P in T^{2n} . The closure of the complement of C in T^{2n} will be denoted D , and the common boundary $N = \partial C = \partial D$ (see Figure 1). Given $\delta > 0$, we may assume that the inclusions of C, D and N into T^{2n} are all δ -UV¹ by taking a fine enough triangulation.

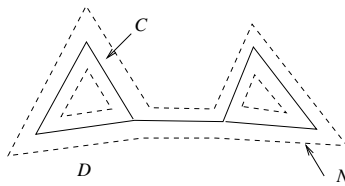


Figure 1:

Let (K, φ) be a geometric \mathbb{Z} -module over T^{2n} representing the controlled quadratic form \tilde{E}_8 , where $K \cong \mathbb{Z}^r$ is a free \mathbb{Z} -module with basis $S = \{e_1, \dots, e_r\}$ and $\varphi: S \rightarrow T^{2n}$ is a map. If $Q \subset T^{2n}$ is the dual complex of P , after a small perturbation, we can assume that $\varphi(S) \cap (P \cup Q) = \emptyset$. Composing this deformation with a retraction $T^{2n} \setminus (P \cup Q) \rightarrow N$, we can assume that φ factors through N , that is, the geometric module is actually realized over N .

Using a controlled analogue of the Wall Realization Theorem (Theorem 5.8

of [27]) applied to the identity map of N , realize this quadratic module over $N \subset T^{2n}$ to obtain a degree-one normal map $F: (V, N, N') \rightarrow (N \times I, N \times \{0\}, N \times \{1\})$ satisfying:

- (a) $F|_N = 1_N$.
- (b) $f = F|_{N'}: N' \rightarrow N$ is a fine homotopy equivalence over T^{2n} .
- (c) The controlled surgery obstruction of F rel ∂ over T^{2n} is $\tilde{E}_8 \in H_{2n}(T^{2n}; \mathbb{L})$.

The map F can be assumed to be δ - UV^1 using controlled analogues of UV^1 deformation results of Bestvina and Walsh [13].

Let C_f be the mapping cylinder of f . Form a Poincaré complex X_1 by pasting $C_f \cup_{N'} (-V)$ into T^{2n} along N , that is,

$$X_1 = C \cup_N C_f \cup_{N'} (-V) \cup_N D,$$

as shown in Figure 2. Our next goal is to define the map $p_1: X_1 \rightarrow X_0 = T^{2n}$.

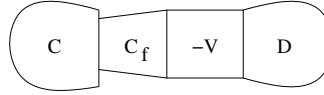


Figure 2: The Poincaré complex X_1 .

Let $g: N \rightarrow N'$ be a controlled homotopy inverse of f . Composing f and g , and using an estimated version of the Homotopy Extension Theorem (see e.g. [2]) and the controlled Bestvina-Walsh Theorem, one can modify F to a δ - UV^1 map $G: V \rightarrow C_g$, so that $G|_{N'} = 1_{N'}$ and $G|_N = 1_N$.

Let $X'_1 = C \cup_N C_f \cup_{N'} C_g \cup_N D$ and $p_1^*: X_1 \rightarrow X'_1$ be as indicated in Figure 3. Crushing $C_f \cup_{N'} C_g$ to $N = \partial C$, we obtain the desired map $p_1: X_1 \rightarrow T^{2n} =$

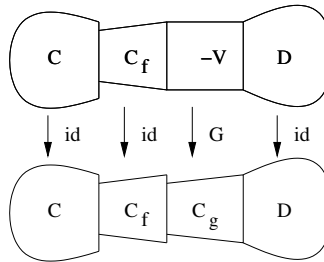
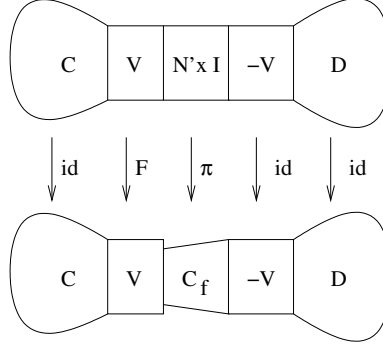


Figure 3: The map $p_1^*: X_1 \rightarrow X'_1$.

$C \cup_N D$.

Figure 4: The map $\phi: M \rightarrow X_1$.

To conclude, as in [3], we argue that X_1 is not homotopy equivalent to any closed topological manifold. To see this, consider the closed manifold

$$M = C \cup_N V \cup_{N'} N' \times I \cup_{N'} (-V) \cup_N D$$

and the degree-one normal map $\phi: M \rightarrow X_1$ depicted in Figure 4, where $\pi: N' \times I \rightarrow C_f$ is induced by $f: N' \rightarrow N$. The controlled surgery obstruction of ϕ over T^{2n} is the generator

$$\begin{aligned} \sigma_*(\phi) &= E_8 \times T^{2n} = (0, \dots, 0, 1) \\ &\in L_0(\mathbb{Z}) = L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])^{INV} \subset H_{2n}(T^{2n}; \mathbb{L}) = L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = \sum_{r=0}^{2n} \binom{2n}{r} L_{2n-r}(\mathbb{Z}) \end{aligned}$$

of the subgroup of the transfer invariant elements (9.1). Let $\mathbb{L}\langle 1 \rangle$ be the 1-connective cover of \mathbb{L} , the simply-connected surgery spectrum with 0th space (homotopy equivalent to) G/TOP . Now

$$L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = H_{2n}(T^{2n}; \mathbb{L}) = H_{2n}(T^{2n}; \mathbb{L}\langle 1 \rangle) \oplus L_0(\mathbb{Z})$$

with

$$H_{2n}(T^{2n}; \mathbb{L}\langle 1 \rangle) = [T^{2n}, G/TOP] = \sum_{r=1}^{2n} \binom{2n}{r} L_{2n-r}(\mathbb{Z}) \subset L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

the subgroup of the surgery obstructions of normal maps $M_1 \rightarrow T^{2n}$. The surgery obstruction of any normal map $\phi_1: M_1 \rightarrow X_1$ is of the type

$$\sigma_*(\phi_1) = (\tau, 1) \neq 0 \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = [T^{2n}, G/TOP] \oplus L_0(\mathbb{Z})$$

for some $\tau \in [T^{2n}, G/TOP]$, since the variation of normal invariant only changes the component of the surgery obstruction in $[T^{2n}, G/TOP] \subset L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$.

Thus, X_1 is not homotopy equivalent to any topological manifold. In the terminology of Chapter 17 of [23] the total surgery obstruction $s(X_1) \in \mathcal{S}_{2n}(X_1)$ has image

$$(p_1)_*s(X_1) = 1 \in \mathcal{S}_{2n}(T^{2n}) = L_0(\mathbb{Z}) .$$

The Bryant-Ferry-Mio-Weinberger procedure for constructing an ENR homology manifold starting with $p_1: X_1 \rightarrow T^{2n}$ leads to a homology manifold homotopy equivalent to X_1 . Thus, from the quadratic form E_8 , we obtained a compact index-9 ENR homology $2n$ -manifold \mathfrak{X}_8 which is not homotopy equivalent to any closed topological manifold.

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Department of Mathematics, Florida State University
Tallahassee, FL 32306-4510 USA

and

School of Mathematics, University of Edinburgh
Edinburgh EH9 3JZ, Scotland, UK

Email: mio@math.fsu.edu a.ranicki@ed.ac.uk

Stability in controlled L -Theory

ERIK KJÆR PEDERSEN

MASAYUKI YAMASAKI

Abstract We prove a squeezing/stability theorem for delta-epsilon controlled L -groups when the control map is a polyhedral stratified system of fibrations on a finite polyhedron. A relation with boundedly-controlled L -groups is also discussed.

AMS Classification 18F25; 57R67

Keywords Controlled L -groups.

1 Introduction

Let us fix an integer $n \geq 0$, a continuous map $p_X: M \rightarrow X$ to a metric space X , and a ring R with involution. For each pair of positive numbers $\epsilon \leq \delta$, the delta-epsilon controlled L -group $L_n^{\delta, \epsilon}(X; p_X, R)$ is defined to be the set of equivalence classes of n -dimensional quadratic Poincaré R -module complexes on p_X of radius ϵ ($= n$ -dimensional ϵ Poincaré ϵ quadratic R -module complexes on p_X), where the equivalence relation is generated by Poincaré cobordisms of radius δ ($= \delta$ Poincaré δ cobordisms) [9] [10] [12]. If $\delta \leq \delta'$ and $\epsilon \leq \epsilon'$, there is a natural homomorphism

$$L_n^{\delta, \epsilon}(X; p_X, R) \rightarrow L_n^{\delta', \epsilon'}(X; p_X, R)$$

defined by relaxation of control. In general, this map is neither surjective nor injective. None the less, if X is a finite polyhedron and p_X is a polyhedral stratified system of fibrations in the sense of [5], the map above turns out to be an isomorphism for certain values of δ , δ' , ϵ , ϵ' :

Theorem 1 (Stability in Controlled L -groups) *For each integer $n \geq 0$ and a finite polyhedron X , there exist constants $\delta_0 > 0$ and $\kappa > 1$ such that the following hold: If*

- (1) $\kappa\epsilon \leq \delta \leq \delta_0$, $\kappa\epsilon' \leq \delta' \leq \delta'_0$, $\delta \leq \delta'$, $\epsilon \leq \epsilon'$,
- (2) $p_X: M \rightarrow X$ is a polyhedral stratified system of fibrations, and

(3) R is a ring with involution,

then the relax-control map $L_n^{\delta, \epsilon}(X; p_X, R) \rightarrow L_n^{\delta', \epsilon'}(X; p_X, R)$ is an isomorphism.

It follows that all the groups $L_n^{\delta, \epsilon}(X; p_X, R)$ with $\kappa\epsilon \leq \delta \leq \delta_0$ are isomorphic and are equal to the controlled L -group $L_n^c(X; p_X, R)$ of p_X with coefficient ring R .

Stability is a consequence of squeezing; squeezing/stability for controlled K_0 and K_1 -groups were known [3]. ‘Splitting’ was the key idea there. In section 2, we discuss splitting in the controlled L -theory. An element of a controlled L -group is represented by a quadratic Poincaré complex on a space. If it splits into small pieces lying over cone-shaped sets (*e.g.* simplices), then we can shrink all the pieces at the same time to obtain a squeezed complex. But splitting in L -theory requires a change of K -theoretic decoration; if you split a free quadratic Poincaré complex, then you get a projective one in the middle. Since the controlled reduced projective class group is known to vanish when the coefficient ring is \mathbb{Z} and the control map is UV^1 , we do not need to worry about the controlled K -theory and squeezing holds in this case [4].

Several years ago the first named author proposed an approach to squeezing/stability in controlled L -groups imitating the method of [3]. The idea was to use projective complexes to split and to eventually eliminate the projective pieces using the Eilenberg swindle :

$$\begin{aligned} [P] &= [P] + (-[P] + [P]) + (-[P] + [P]) + (-[P] + [P]) + \cdots \\ &= ([P] - [P]) + ([P] - [P]) + ([P] - [P]) + ([P] - [P]) + \cdots = 0 \quad . \end{aligned}$$

This approach works for any R if X is a circle; we will briefly discuss the proof in section 3.

The method used in section 3 does not generalize to higher dimensions, because it requires repeated application of splitting but that is not easy to do with projective complexes. This means that we should not try to shrink the complex globally, but should try to shrink a small part of the complex lying over a cone neighborhood of some point at a time. Such a local shrinking construction is possible when the control map is a polyhedral stratified system of fibrations, and is called an Alexander trick. We study its effect in section 4, and use it repeatedly to prove Theorem 1 in section 5. Note that we do one splitting of the whole complex for each application of an Alexander trick; we are not splitting the split pieces.

In section 6, we discuss several variations of Theorem 1.

Finally, in section 7, we relate the delta-epsilon controlled L -groups to the bounded L -groups in a special case.

The authors would like to thank Frank Connolly, Jim Davis, Frank Quinn and Andrew Ranicki for invaluable suggestions.

2 Glueing and Splitting

In this section we review techniques called glueing and splitting. If $p_X: M \rightarrow X$ is a control map and Y is a subset of X , then we denote the restriction $p_X|_Y$ of p_X by p_Y . A closed ϵ neighborhood of Y in X is denoted by Y^ϵ . We refer the reader to [9] [10] for terms and notations in controlled L -theory.

We first discuss the glueing operation; it is to take the union of two objects with common pieces of boundary. Suppose there are consecutive Poincaré cobordisms of radius δ , one from (C, ψ) to (C', ψ') and the other from (C', ψ') to (C'', ψ'') . Then their union is a Poincaré cobordism of radius 100δ from (C, ψ) to (C'', ψ'') (Proposition 2.8 of [10]). We will encounter this factor “100” many times in this article, and will denote it by μ at several places of section 5. For example, we will need the following, which is a special case of Proposition 3.7 of [10].

Proposition 2 *If $[C, \psi] = 0$ in $L_n^{\delta, \epsilon}(X; p_X, R)$, then there is a Poincaré cobordism of radius 100δ from (C, ψ) to 0.*

Proof By definition, there is a sequence of consecutive Poincaré cobordisms starting from (C, ψ) and ending at 0. Their union can be regarded as the union of the even-numbered ones and the odd-numbered ones, so it is 100δ Poincaré. \square

Next we discuss splitting. Before stating the splitting lemma, let us recall a minor technicality from §6 of [8]: Suppose X is the union of two closed subsets A and B with intersection $Y = A \cap B$. If a path $\gamma: [0, s] \rightarrow M$ with $p_X \gamma(0) \in A$ is contained in $p_X^{-1}(\{\gamma(0)\}^\epsilon)$, then it lies in $p_X^{-1}(A \cup Y^{2\epsilon})$. Of course it is contained also in $p_X^{-1}(A^\epsilon)$, but this is slightly less useful.

Lemma 3 (Splitting Lemma) *For any integer $n \geq 2$, there exists a positive number $\lambda \geq 1$ such that the following holds: If $p_X: M \rightarrow X$ is a map to a metric space X , X is the union of two closed subsets A and B with intersection Y ,*

and R is a ring with involution, then for any n -dimensional quadratic Poincaré R -module complex $c = (C, \psi)$ on p_X of radius ϵ , there exist a Poincaré cobordism of radius $\lambda\epsilon$ from c to the union $c' \cup c''$ of an n -dimensional quadratic Poincaré pair $c' = (f': P \rightarrow C', (\delta\bar{\psi}', -\bar{\psi}))$ on $p_{A \cup Y^{\lambda\epsilon}}$ of radius $\lambda\epsilon$ and an n -dimensional quadratic Poincaré pair $c'' = (f'': P \rightarrow C'', (\delta\bar{\psi}'', \bar{\psi}))$ on $p_{B \cup Y^{\lambda\epsilon}}$ of radius $\lambda\epsilon$, where $(P, \bar{\psi})$ is an $(n-1)$ -dimensional quadratic Poincaré projective R -module complex on $p_{Y^{\lambda\epsilon}}$ and P is $\lambda\epsilon$ chain equivalent to an $(n-1)$ -dimensional free chain complex on $p_{A \cup Y^{\lambda\epsilon}}$ and also to an $(n-1)$ -dimensional free chain complex on $p_{B \cup Y^{\lambda\epsilon}}$.

Proof This is an epsilon-control version of Ranicki's argument for the bounded control case [7]. For a given (C, ψ) of radius ϵ , pick up a subcomplex $C' \subset C$ such that C' is identical with C over A and C' lies over some neighborhood of A . Let $p: C \rightarrow C/C'$ be the quotient map and define C'' by the n -dual $(C/C')^{n-*}$. Define a complex E by the desuspension $\Omega\mathcal{C}(p\mathcal{D}_\psi p^*)$ of the algebraic mapping cone of the following map:

$$C'' = (C/C')^{n-*} \xrightarrow{p^*} C^{n-*} \xrightarrow{\mathcal{D}_\psi} C \xrightarrow{p} C/C' ,$$

where \mathcal{D}_ψ is the duality map $(1+T)\psi_0$ for ψ . There are natural maps $g': E \rightarrow C'$, $g'': E \rightarrow C''$ and adjoining quadratic Poincaré structures on them such that the union along the common boundary is homotopy equivalent to the original complex c . We should note that E is non-trivial in degrees -1 and n and that it lies over B .

Since \mathcal{D}_ψ is a small chain equivalence, its mapping cone is contractible. Therefore, E is contractible away from the union of A and a small neighborhood of Y , and it is chain equivalent to a projective chain complex P lying over a small neighborhood of Y by 5.1 and 5.2 of [8]. Note that \mathbb{Z} is used as the coefficient ring in [8], but the same argument works when the coefficient ring is replaced by R . Since $n \geq 2$, we can assume that P is strictly $(n-1)$ -dimensional (*i.e.* $C_i = 0$ for $i < 0$ and $i > n-1$) by the standard folding argument, and the chain equivalence induces a desired cobordism.

There is a quadratic Poincaré structure on a chain map $f': P \rightarrow C'$; therefore, the duality map gives a chain equivalence $C'^{n-*} \rightarrow \mathcal{C}(f')$, where $\mathcal{C}(f')$ denotes the algebraic mapping cone of $f': P \rightarrow C'$. Therefore

$$[P] = -([C'] - [P]) = -[\mathcal{C}(f')] = -[C'^{n-*}] = 0$$

in the epsilon controlled reduced projective class group of the union of A and a small neighborhood of Y with coefficient in R , and hence P is chain equivalent to a free $(n-1)$ -dimensional complex F' lying over the union of A and a small neighborhood of Y . \square

Remarks. (1) Suppose that X is a finite polyhedron or a finite cell complex in the sense of [11] more generally. Then there exist positive numbers $\epsilon_X > 0$, $\mu_X \geq 1$ and a homotopy $\{f_t\}: X \rightarrow X$ such that

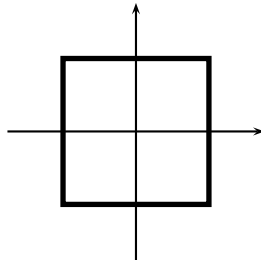
- $f_0 = 1_X$,
- $f_t(\Delta) \subset \Delta$ for each cell Δ and for each $t \in [0, 1]$,
- f_t is Lipschitz with Lipschitz constant μ_X for each $t \in [0, 1]$, and
- $f_1((X^{(i)})^{\epsilon_X}) \subset X^{(i)}$ for every i , where $X^{(i)}$ is the i -skeleton of X .

Suppose $\{f_t\}$ is covered by a homotopy $\{F_t: M \rightarrow M\}$ and set $\delta_X = \epsilon_X \lambda$. If $\epsilon \leq \delta_X$ and A and B are subcomplexes of X , then by applying F_1 to the splitting given in the above lemma, we may assume that the pieces lie over A , B and $A \cap B$ respectively, instead of their neighborhoods, since the homotopy gives small isomorphisms between the corresponding pieces. But λ is now replaced by $\mu_X \lambda$ and it depends not only on n but also on X . We call such a deformation $\{f_t\}$ a *rectification* for X .

(2) The splitting formula for pairs given in [12] can be combined with 5.1 and 5.2 of [8] to prove a similar splitting lemma for pairs of dimension ≥ 3 : a sufficiently small Poincaré pair splits into two adjoining quadratic Poincaré triads whose common boundary piece is possibly a projective pair.

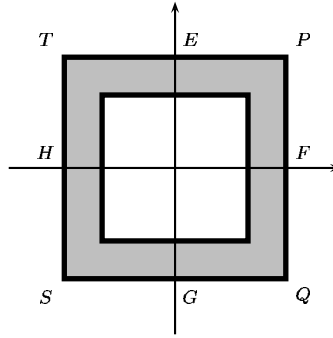
3 Squeezing over a Circle

We discuss squeezing over the unit circle. We use the maximum metric of \mathbb{R}^2 , so the unit circle looks like a square:



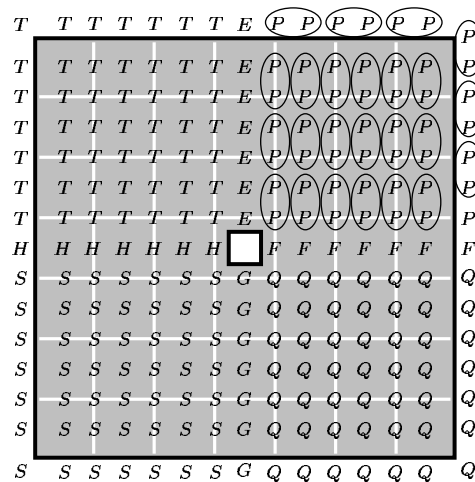
Consider a quadratic Poincaré R -module complex on the unit circle. We assume that its radius is sufficiently small so that it splits into four free pieces E , F , G , H with projective boundary pieces P , Q , S , T as shown in the picture below. The shadowed region is a cobordism between the original complex and the union of E , F , G , H . Although we actually measure the radius using the

radial projection to the unit circle (i.e. the square), we pretend that complexes and cobordisms are over the plane.



We extend this cobordism in the following way. On the right vertical edge, we have a quadratic pair $P \oplus Q \rightarrow F$. (We are omitting the quadratic structure from notation.) Take the tensor product of this with the symmetric complex of the unit interval $[0, 1]$. Make many copies of such a product and consecutively glue them one after the other to the cobordism. Do the same thing with the other three edges. Then fill in the four quadrants by copies of P , Q , S , T multiplied by the symmetric complex of $[0, 1]^2$ so that the whole picture looks like a huge square with a square hole at the center.

Although this cobordism is made up of free complexes and projective complexes, the projective complexes sitting on the white edges are shifted up 1 dimension, and the projective complexes sitting at the lattice points are shifted up 2 dimension in the union.



We can make pairs of these (as shown in the picture above for P 's) so that each pair contributes the trivial element in the controlled reduced projective class group. Replace each pair by a free complex.

Unlike the real Eilenberg swindle, there are four projective complexes left which do not make pairs. We may assume that they are the boundary pieces of F and H on the outer end. Since the two pairs $P \oplus Q \rightarrow F$, $S \oplus T \rightarrow H$ are Poincaré, the unions $P \oplus Q$ and $S \oplus T$ are locally chain equivalent to free complexes. Thus we can replace them by free complexes, and now everything is free.

Now recall that we actually measure things by a radial projection to the square. Thus we have a cobordism from the original complex to another complex of very small radius. If we increase the number of layers in the construction, the radius of the outer end becomes arbitrarily small. This is the squeezing in the case of S^1 .

4 Alexander Trick

The method in the previous section does not work for higher dimensional complexes, because we cannot inductively split the projective pieces. But the proof suggests an alternative way toward squeezing/stability. This is the topic of this section. Although we used a radial projection to measure the size in the previous section, we draw pictures of things in their real sizes in this section.

Let us fix an integer $n \geq 2$ and a finite polyhedron X . All the complexes below are R -module complexes, where R is a ring with involution. We assume that the control map $p_X: M \rightarrow X$ is a polyhedral stratified system of fibrations [5]; p_X is fiber homotopy equivalent to a map $q_X: N \rightarrow X$ which has an iterated mapping cylinder decomposition in the sense of Hatcher [2]: there is a partial order on the set of the vertices of X such that, for each simplex Δ of X ,

- (1) the partial order restricts to a total order of the vertices of Δ

$$v_0 < v_1 < \cdots < v_k ,$$

- (2) $q_X^{-1}(\Delta)$ is the iterated mapping cylinder of a sequence of maps

$$F_{v_0} \longrightarrow F_{v_1} \longrightarrow \cdots \longrightarrow F_{v_k} ,$$

- (3) the restriction $q_X|_{q_X^{-1}(\Delta)}$ is the natural map induced from the iterated mapping cylinder structure of $q_X^{-1}(\Delta)$ above and the iterated mapping cylinder structure of Δ coming from the sequence

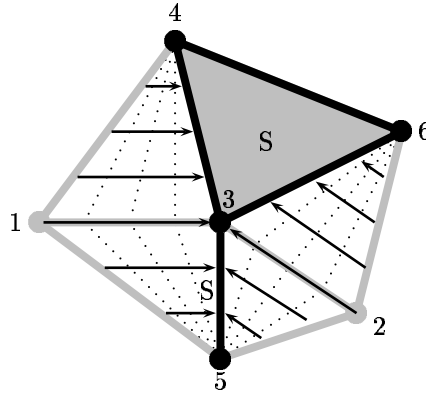
$$\{v_0\} \longrightarrow \{v_1\} \longrightarrow \cdots \longrightarrow \{v_k\} .$$

An order on the set of the vertices of X is said to be **compatible with p_X** if it is compatible with this partial order. Let us fix an order compatible with p_X .

Pick a vertex v of X , and let A be the star neighborhood of v , B be the closure of the complement of A in X , and S be the union of the simplices in A whose vertices are all $\geq v$ with respect to the chosen order. This will be called the *stable set* at v . Let $s: A \rightarrow S$ be the simplicial retraction defined by

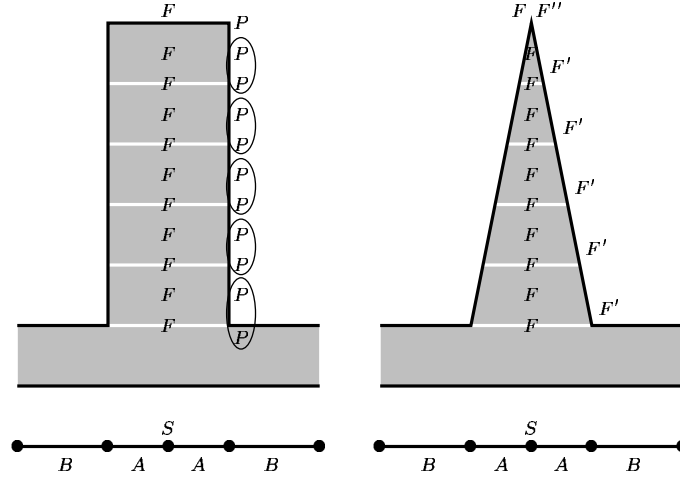
$$s(v') = \begin{cases} v & \text{if } v' < v, \\ v' & \text{if } v' \geq v, \end{cases}$$

for vertices v' of A . A strong deformation retraction $s_t: A \rightarrow A$ is defined by $s_t(a) = (1-t)a + ts(a)$ for $a \in A$ and $t \in [0, 1]$. Note that this strong deformation retraction s_t is covered by a deformation \tilde{s}_t on M , since p_X is a polyhedral stratified system of fibrations.



Given a sufficiently small n -dimensional quadratic Poincaré complex $c = (C, \psi)$ on p_X , one can split it according to the splitting of X into A and B : c is cobordant (actually homotopy equivalent) to the union c' of a projective quadratic Poincaré pair $a = (f: P \rightarrow F, (\delta\psi', \psi'))$ on p_A and a projective quadratic Poincaré pair $b = (g: P \rightarrow G, (\delta\psi'', -\psi'))$ on p_B , where F is an n -dimensional chain complex on p_A , G is an n -dimensional chain complex on p_B , and P is an $(n-1)$ -dimensional projective chain complex on $p_{A \cap B}$. Here we again used the assumption on p_X . See the remark after the splitting lemma.

Make many copies of the product cobordism from the pair a to itself, and successively glue them to the cobordism between c and c' . This gives us a cobordism from c to a (possibly) projective complex as in the left picture below.



We will remedy the situation by replacing the projective end by a free complex as follows. The copies of P connecting the layers are actually shifted up 1 dimension in the union, so the marked pairs of P 's contribute the trivial element of the controlled \tilde{K}_0 group of $A \cap B$, and we can replace each pair with a free module by adding chain complexes of the form

$$0 \longrightarrow Q_i \xrightarrow{1} Q_i \longrightarrow 0$$

lying over $A \cap B$, where Q_i is a projective module such that $P_i \oplus Q_i$ is free. Therefore, these pairs are all chain equivalent to some free chain complex F' . The last P remaining at the top of the picture can be replaced by some free complex F'' lying over A as stated in the splitting lemma.

We deform the tower, which is now free, toward S using \tilde{s}_t as in the picture above so that the top of the tower is completely deformed to S .

Summary *There exist constants $\delta > 0$ and $\lambda \geq 1$ which depend on n and X such that any n -dimensional quadratic Poincaré complex of radius $\epsilon \leq \delta$ on p_X is $\lambda\epsilon$ Poincaré cobordant to another complex which is small in the track direction of s_t . The more layers we use, the smaller the result becomes in the track direction.*

Remarks. (1) We cannot take $\lambda = 1$ in general, since the radius of the complexes gets bigger during the splitting/glueing processes.

(2) This construction will be referred to as the *Alexander trick* at v .

(3) There is also an *Alexander trick for pairs*. If we use the splitting lemma for pairs, then instead of a pair we get a Poincaré triad

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & \searrow & \downarrow \\ E & \longrightarrow & F \end{array}$$

over A , where P, Q are projective and E, F are free. Since both P and Q are free over A , we can carry out the construction exactly in the same manner as above. The effect on the boundary is exactly the same as the absolute Alexander trick.

(4) Take a simplex Δ of X with ordered vertices $v_0 < v_1 < \cdots < v_n$. Let $(\lambda_0, \dots, \lambda_n)$ be the barycentric coordinates of a point $x \in \Delta$, i.e. $x = \sum \lambda_i v_i$. Then we define the *pseudo-coordinates* (x_1, \dots, x_n) of x by $x_i = \lambda_i / (\lambda_0 + \cdots + \lambda_i)$. Actually x_i is indeterminate if $\lambda_0 = \cdots = \lambda_i$. Let $s_{i,t}: \Delta \rightarrow \Delta$ be the restriction to Δ of the deformation retraction used for an Alexander trick at v_i ; then $s_{0,t} = 1_\Delta$ for every $t \in [0, 1]$, and $s_{i,t}$ preserves the pseudo-coordinate x_j for j not equal to i . This means that, roughly speaking, an Alexander trick at v_i improves the radius control in the x_i direction and changes the radius control in the x_j direction ($j \neq i$) only up to multiplications by the constant λ given in the Splitting Lemma and by the Lipschitz constant of $s_{i,t}$ which is uniform with respect to t . Thus, if we can perform appropriate Alexander tricks at all the vertices of Δ , then we can obtain an arbitrarily fine control over Δ . A more detailed discussion will be given in the next section.

Let us state a lemma on Lipschitz properties related to the homotopy s_t above, for future use.

Lemma 4 *Let X be a subset of \mathbb{R}^N with diameter d and $s: X \rightarrow X$ be a Lipschitz map with Lipschitz constant $K \geq 1$. Suppose X contains the line segment $xs(x)$ for every $x \in X$ and let $s_t(x) = ts(x) + (1-t)x$ for $t \in [0, a]$. Then $s_t: X \rightarrow X$ has Lipschitz constant K , and the map*

$$H: X \times [0, a] \rightarrow X \times [0, a]; \quad H(x, t) = (s_{t/a}(x), t)$$

has Lipschitz constant $\max\{d/a, 1\} + K$ with respect to the maximum metric on $X \times [0, a]$.

Proof Let x, y be points in X . Then

$$\begin{aligned} d(s_t(x), s_t(y)) &= \|t(s(x) - s(y)) + (1 - t)(x - y)\| \\ &\leq td(s(x), s(y)) + (1 - t)d(x, y) \\ &\leq tKd(x, y) + (1 - t)Kd(x, y) = Kd(x, y) . \end{aligned}$$

Next, take two points $p = (x, t)$, $q = (y, u)$ of $X \times [0, a]$, and let $p' = (x, u)$. Then we have

$$\begin{aligned} d(H(p), H(q)) &\leq d(H(p), H(p')) + d(H(p'), H(q)) \\ &= \max\{d(s_{t/a}(x), s_{u/a}(x)), |t - u|\} + d(s_{u/a}(x), s_{u/a}(y)) \\ &\leq \max\{|t - u|d(s(x), x)/a, |t - u|\} + Kd(x, y) \\ &\leq |t - u| \max\{d/a, 1\} + Kd(x, y) \\ &\leq (\max\{d/a, 1\} + K) \max\{d(x, y), |t - u|\} \\ &= (\max\{d/a, 1\} + K)d(p, q) . \end{aligned}$$

□

5 Proof of Theorem 1

The algebraic theory of surgery on quadratic Poincaré complexes in an additive category [6] carries over nicely to the controlled setting, and can be used to prove a stable periodicity of the controlled L -groups. Therefore, we give a proof of the stability in the case $n \geq 2$. The stability for $n = 0, 1$ follows from the stability for $n = 4, 5$.

We first state the squeezing lemma for quadratic Poincaré complexes:

Lemma 5 (Squeezing of Quadratic Poincaré Complexes) *Let $n \geq 2$ be an integer and X be a finite polyhedron. There exist constants $\delta_0 > 0$ and $\kappa > 1$ such that the following hold: If $\epsilon < \epsilon' \leq \delta_0$, then any n -dimensional quadratic Poincaré R -module complex of radius ϵ' on a polyhedral stratified system of fibrations over X is $\kappa\epsilon'$ Poincaré cobordant to a quadratic Poincaré complex of radius ϵ .*

Proof Let X be a polyhedron in \mathbb{R}^N , and $p_X: M \rightarrow X$ be a polyhedral stratified system of fibrations. Order the vertices of X compatibly with p_X :

$$v_0 < v_1 < \cdots < v_m .$$

The basic idea is to apply the Alexander trick at each v_i . This should make the complex arbitrarily small in X as noted in the previous section. The problem is that an Alexander trick is made up of two steps: the first step is to make a tower using splitting, and the second step is to squeeze the tower, and estimating the effect of the splitting used in the first step is very difficult especially near the vertex when the object is getting smaller in a non-uniform way. To avoid this difficulty, we blow up the metric around each vertex so that the ordinary control on the new metric space insures us that the result has a desired small control measured on the original metric space X . Note that we are implicitly using this approach in the circle case.

Let us start from a complex c of radius $\epsilon' > 0$ on X . Since X is a finite polyhedron, there exist $\delta > 0$ and $\lambda \geq 1$ such that if $\epsilon' \leq \delta$ then c is $\lambda\epsilon'$ cobordant to the union of two pieces according to the splitting of X into two subpolyhedra as in the remark after Lemma 3. Recall that δ and λ depends on X . Set $\mu = 100$, and set $\delta_0 = \delta/(\mu\lambda^2)^{m-1}$. The factor 100 comes from 2.8 of [10] as was mentioned in §2. We claim that if $\epsilon' \leq \delta_0$, then a successive application of Alexander tricks produces a cobordism from c to a complex of radius ϵ .

Let us fix some more notation. V_1, \dots, V_m are the star neighborhoods of v_1, \dots, v_m , and L_1, \dots, L_m are the links of v_1, \dots, v_m ; V_i is the cone over L_i with vertex v_i for each i . S_1, \dots, S_m are the stable sets at v_1, \dots, v_m . $K \geq 1$ is the Lipschitz constant which works for every retraction $s_i: V_i \rightarrow S_i$ used for the Alexander trick at v_i . Let d denote the diameter of X , and let $\sharp(X)$ denote the number of simplices of X . Now fix a number $H \geq 1$ such that

$$H > d \quad \text{and} \quad 4\mu\sharp(X)(K+1)^m(\mu\lambda^2)^m\epsilon' \cdot \frac{d}{H} < \epsilon.$$

We inductively define metric spaces and subsets

$$X_*^{i,j} \supset X^{i,j} \supset V_k^{i,j} \supset L_k^{i,j} \quad (1 \leq i \leq j < k \leq m)$$

together with control maps $p_*^{i,j}: M_*^{i,j} \rightarrow X_*^{i,j}$ as follows.

Identify \mathbb{R}^N with the subset $\mathbb{R}^N \times \{0\}$ of $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ with the maximum product metric. For each $i = 1, \dots, m$, define $X_*^{i,i}$ and its subsets $X^{i,i}$, $V_k^{i,i}$,

$L_k^{i,i}$ ($k = i + 1, \dots, m$) by

$$\begin{aligned} X_*^{i,i} &= X \cup (V_i \times [0, H]) , \\ X^{i,i} &= (X - V_i) \cup (L_i \times [0, H]) \cup (V_i \times \{H\}) , \\ V_k^{i,i} &= (V_k - V_i) \cup (V_k \cap L_i \times [0, H]) \cup (V_k \cap V_i \times \{H\}) , \\ L_k^{i,i} &= (L_k - V_i) \cup (L_k \cap L_i \times [0, H]) \cup (L_k \cap V_i \times \{H\}) . \end{aligned}$$

The projection $\mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ restricts to a retraction $r_{i,i}: X_*^{i,i} \rightarrow X$. We define the control map $p_*^{i,i}: M_*^{i,i} \rightarrow X_*^{i,i}$ to be the pullback of $p_X: M \rightarrow X$ via $r_{i,i}$, and define the control map $p^{i,i}: M^{i,i} \rightarrow X^{i,i}$ to be the restriction of $p_*^{i,i}$ to $M^{i,i}$. Note that the stereographic projection from $(v_i, -H) \in \mathbb{R}^N \times \mathbb{R}$ defines a homeomorphism $X \rightarrow X^{i,i}$ sending V_k and L_k to $V_k^{i,i}$ and $L_k^{i,i}$ respectively, since V_i is the cone on L_i with center v_i .

Next, for each $i = 1, \dots, m-1$, define $X_*^{i,i+1} \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ and its subsets $X^{i,i+1}$, $V_k^{i,i+1}$, $L_k^{i,i+1}$ ($k = i+2, \dots, m$) by

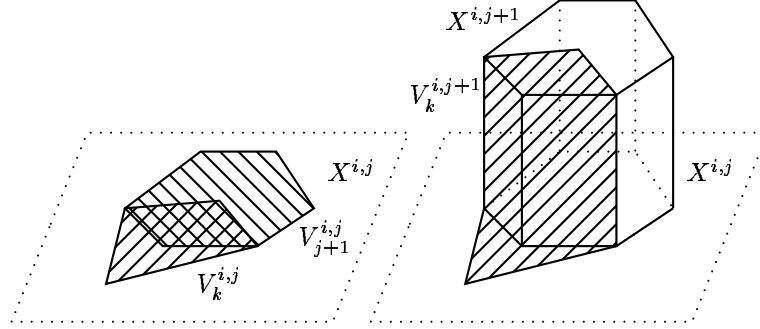
$$\begin{aligned} X_*^{i,i+1} &= X^{i,i} \cup (V_{i+1}^{i,i} \times [0, H]) , \\ X^{i,i+1} &= (X^{i,i} - V_{i+1}^{i,i}) \cup (L_{i+1}^{i,i} \times [0, H]) \cup (V_{i+1}^{i,i} \times \{H\}) \subset X^{i,i} \times \mathbb{R} , \\ V_k^{i,i+1} &= (V_k^{i,i} - V_{i+1}^{i,i}) \cup (V_k^{i,i} \cap L_{i+1}^{i,i} \times [0, H]) \cup (V_k^{i,i} \cap V_{i+1}^{i,i} \times \{H\}) , \\ L_k^{i,i+1} &= (L_k^{i,i} - V_{i+1}^{i,i}) \cup (L_k^{i,i} \cap L_{i+1}^{i,i} \times [0, H]) \cup (L_k^{i,i} \cap V_{i+1}^{i,i} \times \{H\}) . \end{aligned}$$

Again we use the product metric of $\mathbb{R}^N \times \mathbb{R}$ and \mathbb{R} . The projection $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^N \times \mathbb{R}$ restricts to a retraction $r_{i,i+1}: X_*^{i,i+1} \rightarrow X^{i,i}$. The control maps $p_*^{i,i+1}: M_*^{i,i+1} \rightarrow X_*^{i,i+1}$ and $p^{i,i+1}: M^{i,i+1} \rightarrow X^{i,i+1}$ are defined to be the pullbacks of $p_*^{i,i}$ via $r_{i,i+1}$ and $r_{i,i+1}|_{X^{i,i+1}}$, respectively. Although $V_{i+1}^{i,i}$ is not a cone, it is homeomorphic to V_{i+1} and has a topological cone structure. So one can construct a homeomorphism from $X^{i,i+1}$ to $X^{i,i}$ sending $V_k^{i,i+1}$ and $L_k^{i,i+1}$ to $V_k^{i,i}$ and $L_k^{i,i}$ respectively, and hence a homeomorphism to X .

We can continue this to inductively obtain the metric space

$$X_*^{i,j} = X^{i,j-1} \cup (V_j^{i,j-1} \times [0, H])$$

as a subset of $\mathbb{R}^N \times \mathbb{R}^{j-i+1}$, and its subsets $X^{i,j} \supset V_k^{i,j} \supset L_k^{i,j}$ ($k = j+1, \dots, m$), together with control maps $p_*^{i,j}: M_*^{i,j} \rightarrow X_*^{i,j}$, and $p^{i,j}: M^{i,j} \rightarrow X^{i,j}$. Topologically all the spaces $X^{i,j}$'s are equal to X , and all the sets $V_k^{i,j}$'s are equal to V_k . We are only changing the metric, the cell structure of X , and the control map.



Our next task is to do Alexander tricks at v_1, \dots, v_m on these spaces instead of X . Since $\epsilon' \leq \delta_0 \leq \delta$, we can split the original complex c into two pieces on V_1 and the closure of its complements by a $\lambda\epsilon'$ cobordism. Now we construct a tower: we make copies of the trivial cobordism from the pair on (V_1, L_1) to itself and successively attach them to the cobordism along $V_1 \times [0, H]$. This is actually done on $M_*^{1,1}$.

We use enough layers so that the result is a projective cobordism of radius $\mu\lambda\epsilon'$ measured on $X_*^{1,1}$ from $c = \bar{c}_0$ to a complex c'_1 on $p^{1,1}$. Recall $\mu = 100$ and it comes from taking a union of Poincaré cobordisms. As described in previous sections, we can replace this by a free cobordism of radius $\mu\lambda^2\epsilon'$ from c to a free complex \bar{c}_1 on $p^{1,1}$.

We postpone the squeezing to a later stage and go ahead to perform Alexander trick over $V_2^{1,1} \subset X^{1,1}$ on \bar{c}_1 . Although $X^{1,1}$ has a different metric from X , the difference lies along the cylinder $L_1 \times [0, H]$. If H is sufficiently large, then a rectification for $X^{1,1}$ can be easily constructed from those for X and $[0, H]$, and the δ and λ for X works also for $X^{1,1}$. Since $\mu\lambda^2\epsilon' \leq \delta$, we can do splitting and cut out the portion on $V_2^{1,1}$ by a $\mu\lambda^3\epsilon'$ cobordism. Again use enough copies of this to get a $\mu^2\lambda^3\epsilon'$ cobordism on $p_*^{1,2}$ to a complex \bar{c}'_2 on $p^{1,2}$ and then replace this by free $\mu^2\lambda^4\epsilon'$ cobordism to a free complex \bar{c}_2 on $p^{1,2}$. Since $\epsilon' \leq \delta_0$, we can continue this process to obtain a consecutive sequence of free cobordisms:

$$c = \bar{c}_0 \xrightarrow[X_*^{1,1}]{\mu\lambda^2\epsilon'} \bar{c}_1 \xrightarrow[X_*^{1,2}]{(\mu\lambda^2)^2\epsilon'} \bar{c}_2 \cdots \bar{c}_{m-2} \xrightarrow[X_*^{1,m-1}]{(\mu\lambda^2)^{m-1}\epsilon'} \bar{c}_{m-1} \xrightarrow[X_*^{1,m}]{(\mu\lambda^2)^m\epsilon'} \bar{c}_m$$

Now we construct a map $S^{1,m}: X^{1,m} \rightarrow X$ and a map $\tilde{S}^{1,m}: M^{1,m} \rightarrow M$ which covers $S^{1,m}$ so that the functorial image of \bar{c}_m has the desired property. This is done by inductively constructing maps $S_*^{i,j}: X_*^{i,j} \rightarrow X$ and its restriction

$S^{i,j}: X^{i,j} \rightarrow X$ covered by maps $\widetilde{S}_*^{i,j}: M_*^{i,j} \rightarrow M$ and $\widetilde{S}^{i,j}: M^{i,j} \rightarrow M$, respectively, for certain pairs $j \geq i$.

First we define $S_*^{i,i}: X_*^{i,i} \rightarrow X$. Let us recall that $S_i \subset V_i$ denotes the stable set at v_i . Using the strong deformation retraction $s_{i,t}: V_i \rightarrow V_i$, define a map $S'_i: X_*^{i,i} \rightarrow X_*^{i,i}$ by:

$$(x, h) \mapsto \begin{cases} (x, 0) & \text{if } x \in X \text{ and } h = 0, \\ (s_{i,h/H}(x), h) & \text{if } x \in V_i \text{ and } h > 0. \end{cases}$$

This map is covered by a map $\widetilde{S}'_i: M_*^{i,i} \rightarrow M_*^{i,i}$.

Lemma 6 S'_i has Lipschitz constant $K + 1$.

Proof This is obtained by applying Lemma 4 to the sets of the form $\{x\} \cup V_i$ for $x \in X - V_i$, extending the map s_i on x by $s_i(x) = x$. \square

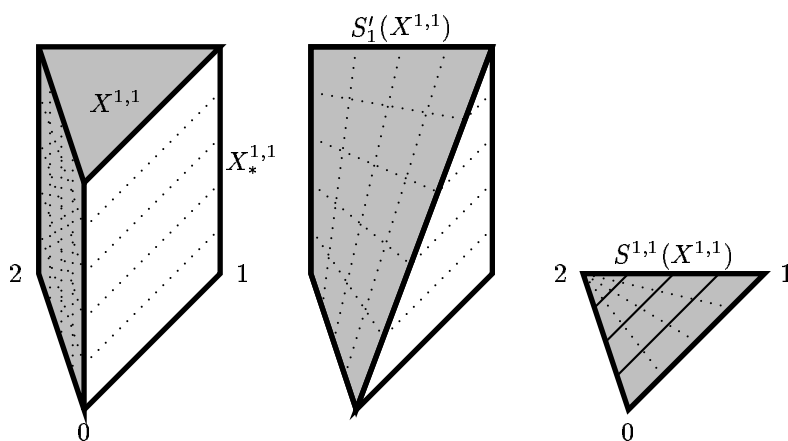
$S_*^{i,i}: X_*^{i,i} \rightarrow X$ is defined by composing S'_i with the projection $r_{i,i}: X_*^{i,i} \rightarrow X$. It has Lipschitz constant $K + 1$. Since $r_{i,i}$ is obviously covered by a map $M_*^{i,i} \rightarrow M$, the map $S_*^{i,i}$ is covered by a map $\widetilde{S}_*^{i,i}: M_*^{i,i} \rightarrow M$. Define $S^{i,i}: X^{i,i} \rightarrow X$ to be the restriction of $S_*^{i,i}$.

Now recall that $X_*^{1,2}$ and $X^{2,2}$ are obtained by attaching $V_2^{1,1} \times [0, H]$ and $V_2 \times [0, H]$ to $X^{1,1}$ and X , respectively. Since $S^{1,1}: X^{1,1} \rightarrow X$ maps $V_2^{1,1}$ to V_2 , the product map $S^{1,1} \times 1|_{[0,H]}: X^{1,1} \times [0, H] \rightarrow X \times [0, H]$ restricts to a map $S^{1,1} \times 1|: X_*^{1,2} \rightarrow X_*^{2,2}$. Compose this with $S_*^{2,2}: X_*^{2,2} \rightarrow X$ to define $S_*^{1,2}: X^{1,2} \rightarrow X$ which is covered by a map $\widetilde{S}_*^{1,2}: M_*^{1,2} \rightarrow M$. Continue this process as in the following diagram to eventually get the desired map $S^{1,m}: X^{1,m} \rightarrow X$.

$$\begin{array}{ccccc} X^{1,j-1} & \xrightarrow{S^{1,j-1}} & X & & \\ \downarrow & & \downarrow & & \\ X_*^{1,j} & \xrightarrow{S^{1,j-1} \times 1|} & X_*^{j,j} & \xrightarrow{S_*^{j,j}} & X \\ & \searrow S_*^{1,j} & & & \parallel \\ X^{1,j} & \xrightarrow{S^{1,j}} & X & & \end{array}$$

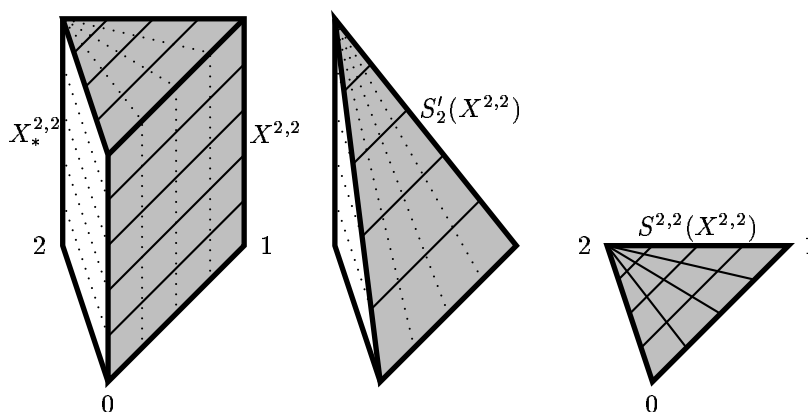
Recall that there is a topological identification of $X^{1,m}$ with X . So we can think of $S^{1,m}$ to be a map from X to X equipped with different metrics. Although

it is not a homeomorphism, it preserves all the simplices, *i.e.* $S^{1,m}(\Delta) = \Delta$ for every simplex Δ of X . When restricted to a simplex, $S^{1,m}$ has Lipschitz constant $(K+1)^m d/H$.



The three pictures above illustrate the application of $S^{1,1}$ to $X^{1,1}$. The thin solid lines in the rightmost picture indicate the direction in which controls are obtained.

The three pictures below illustrate the application of $S^{1,2}$ to $X^{1,2}$. The leftmost picture shows the image $(S^{1,1} \times 1)(X^{1,2}) = X^{2,2}$. Again the thin solid lines on the faces indicate the directions in which controls are obtained.



Let us consider the functorial image c_m of \bar{c}_m by the map $\tilde{S}^{1,m}: M^{1,m} \rightarrow M$. Recall that \bar{c}_m has radius $\epsilon'' = (\mu\lambda^2)^m \epsilon'$. Take a ball B of radius ϵ'' with in $X^{1,m}$. B is the union of subsets $B \cap \Delta$ each having diameter $2\epsilon''$, where Δ are the simplices of $X^{1,m}$. The images of $B \cap \Delta$ in X by $S^{1,m}$ all have diameter $2(K+1)^m \epsilon'' d/H$ and their union $S^{1,m}(B)$ is connected. Therefore $S^{1,m}(B)$ has diameter $2\sharp(X)(K+1)^m \epsilon'' d/H$. Thus c_m has radius

$$4\sharp(X)(K+1)^m (\mu\lambda^2)^m \epsilon' d/H ,$$

and this is smaller than ϵ by the choice of H .

It remains to find a constant κ such that c and c_m are $\kappa\epsilon'$ cobordant. Define a complex c_i on p_X to be the functorial image of \bar{c}_i by the map $\tilde{S}^{1,i}: M^{1,i} \rightarrow M$. The functorial image of the $(\mu\lambda^2)^i \epsilon'$ cobordism between \bar{c}_{i-1} and \bar{c}_i by the map $\tilde{S}_*^{1,i}$ gives a $4\sharp(X)(K+1)^i (\mu\lambda^2)^i \epsilon'$ cobordism between c_{i-1} and c_i . Composing these we get a $4\mu\sharp(X)(K+1)^m (\mu\lambda^2)^m \epsilon'$ cobordism between c and c_m . Thus $\kappa = 4\mu\sharp(X)(K+1)^m (\mu\lambda^2)^m$ works. This completes the proof. \square

Note that Lemma 5 implies that the relax-control map in Theorem 1 is surjective: Take an element $[c'] \in L_n^{\delta', \epsilon'}(X; p_X, R)$ with $\delta' \leq \delta_0$. Then the inequality $\epsilon' \leq \delta_0$ holds and therefore there is a Poincaré cobordism of radius $\kappa\epsilon'$ ($\leq \delta_0$) from c' to a quadratic Poincaré complex c of radius ϵ , determining an element $[c] \in L_n^{\delta, \epsilon}(X; p_X, R)$ whose image under the relax-control map is $[c']$.

Squeezing for complexes can be generalized to squeezing for pairs.

Lemma 7 (Squeezing of Quadratic Poincaré Pairs) *Let $n \geq 2$ be an integer and X be a finite polyhedron. There exist constants $\delta_0 > 0$ and $\kappa > 1$ such that the following hold: If $\delta < \epsilon' \leq \delta' \leq \delta_0$, then any $(n+1)$ -dimensional quadratic Poincaré R -module pair of radius δ' on a polyhedral stratified system of fibrations over X with ϵ' Poincaré boundary is $\kappa\delta'$ Poincaré cobordant to a quadratic Poincaré pair of radius ϵ . The cobordism between the boundary is $\kappa\epsilon'$ Poincaré.*

Proof Same as the proof of Lemma 5. Use the Alexander trick for pairs. \square

Corollary 8 (Relative Squeezing of Quadratic Poincaré Pairs) *Let $n \geq 2$ be an integer and X be a finite polyhedron. There exist constants $\delta_0 > 0$ and $\kappa > 1$ such that the following hold: If $\kappa\epsilon < \delta' \leq \delta_0$, then any $(n+1)$ -dimensional quadratic Poincaré R -module pair of radius δ' on a polyhedral stratified system of fibrations over X with an ϵ Poincaré boundary is $\kappa\delta'$ Poincaré cobordant fixing the boundary to a quadratic Poincaré pair of radius $\kappa\epsilon$.*

Proof Temporarily choose δ_0 and κ as in Lemma 7. Suppose $\kappa\epsilon < \delta' \leq \delta_0$, and let $d = (f: C \rightarrow D, (\delta\psi, \psi))$ be an $(n+1)$ -dimensional quadratic Poincaré pair of radius δ' , and assume that (C, ψ) is ϵ Poincaré. Choose a positive number $\epsilon' < \epsilon$. By Lemma 7, d is $\kappa\delta'$ cobordant to a quadratic Poincaré pair $d' = (f': C' \rightarrow D', (\delta\psi', \psi'))$ of radius ϵ . Glue d' to the $\kappa\epsilon$ Poincaré cobordism between (C, ψ) and (C', ψ') to get a quadratic Poincaré pair $d'' = (f'': C \rightarrow D'', (\delta\psi'', \psi))$ of radius $100\kappa\epsilon$. By construction, $d \cup -d''$ is $100\kappa\delta'$ Poincaré null-cobordant. Thus 100κ works as the κ in the statement of the lemma. \square

The injectivity of the relax-control map follows from this: Temporarily let δ_0 and κ be as in Corollary 8, and suppose $\delta \leq \delta'$, $\epsilon \leq \epsilon'$, $\kappa\epsilon \leq \delta$. Take an element $[c]$ in the kernel of the relax control map

$$L_n^{\delta, \epsilon}(X; p_X, R) \rightarrow L_n^{\delta', \epsilon'}(X; p_X, R) .$$

By Proposition 2, the quadratic complex $c = (C, \psi)$ of radius ϵ' is the boundary of an $(n+1)$ -dimensional quadratic Poincaré pair $(f: C \rightarrow D, (\delta\psi, \psi))$ of radius $100\delta'$. If $\delta' \leq \delta_0/100$, then $\kappa\epsilon \leq 100\delta' \leq \delta_0$, and by Corollary 8 the element $[c]$ is 0 in $L_n^{\kappa\epsilon, \epsilon}(X; p_X, R)$, and hence also in $L_n^{\delta, \epsilon}(X; p_X, R)$. So, by replacing δ_0 with $\delta_0/100$, we established the desired injectivity. This finishes the proof of Theorem 1.

6 Variations

6.1 Projective L -groups

There is a controlled analogue of projective L^p -groups. $L_n^{p, \delta, \epsilon}(X; p_X, R)$ is defined using ϵ Poincaré ϵ quadratic projective R -module complexes on p_X and δ Poincaré δ projective cobordisms. Similar stability results hold for these.

To get a squeezing result in the L^p -group case, we first take the tensor product of the given projective quadratic Poincaré complex c with the symmetric complex $\sigma(S^1)$ of the circle S^1 . By replacing it with a finite cover if necessary, we may assume that the radius of $\sigma(S^1)$ is sufficiently small. If the radius of c is also sufficiently small, we can construct a cobordism to a squeezed complex. Split the cobordism along $X \times \{\text{two points}\} \subset X \times S^1$ to get a projective cobordism from the original complex to a squeezed projective complex.

6.2 UV^1 control maps

When the control map is UV^1 , there is no need to use paths to define morphisms between geometric modules [4]. This simplifies the situation quite a lot, and we have:

Proposition 9 *Let $p_X: M \rightarrow X$ be a UV^1 map to a finite polyhedron. Then for any pair of positive numbers $\delta \geq \epsilon$, there is an isomorphism*

$$L_n^{\delta, \epsilon}(X; p_X, R) \cong L_n^{\delta, \epsilon}(X; 1_X, R)$$

for any ring with involution R and any integer $n \geq 0$.

By Theorem 1, the stability holds for $L_n^{\delta, \epsilon}(X; 1_X, R)$ and hence the stability holds also for $L_n^{\delta, \epsilon}(X; p_X, R)$.

6.3 Compact metric ANR's

Squeezing and stability also hold when X is a compact metric ANR, and the control map is a fibration. To see this, embed X in the Hilbert cube I^∞ . There is a closed neighborhood U of X of the form $P \times I^{\infty-N}$, where P is a polyhedron in I^N . Use the fact that the retraction from U to X is uniformly continuous to deduce the desired stability from the stability on P and U .

7 Relations to Bounded L -Theory

In this section we shall identify the controlled L -theory groups with a bounded L -theory group, at least in the case of constant coefficients. The main advantage to having a bounded controlled description, is that it facilitates computations.

Definition 10 Let X be a finite polyhedron and R a ring with involution. Let $p_X: X \times K \rightarrow X$ be a trivial fibration. We denote the common value of $L_n^{\delta, \epsilon}(X; p_X, R)$ for small values of δ and ϵ , which exists by Theorem 1, by $L_n^{h, c}(X; p_X, R)$. Here the h signifies that we have no simpleness condition and the c stands for controlled.

We may embed the finite polyhedron X in a large dimensional sphere S^n and consider the open cone $O(X) = \{t \cdot x \in R^{n+1} | t \in [0, \infty), x \in X\}$. We denote X with a disjoint basepoint added by X_+ .

Theorem 11 *Let $p_X: X \times K \rightarrow X$ be as above, $\pi = \pi_1(K)$, R a ring with involution. Then*

$$L_n^{c,h}(X; p_X, R) \cong L_{n+1}^s(\mathcal{C}_{O(X_+)}(R[\pi]))$$

where $\mathcal{C}_{O(X_+)}(R[\pi])$ denotes the category of free $R[\pi]$ modules parameterized by $O(X_+)$ and bounded morphisms.

Proof Given an element in $L_n^{c,h}(X; p_X, R)$, we can choose a stable (δ, ϵ) representative. Crossing with the symmetric chain complex of $(-\infty, 0]$ produces a bounded quadratic chain complex when we parameterize it by $O(+)$, which is obviously a half line, with $+$ being the extra basepoint. According to Theorem 1, we may produce a sequence of bordisms to increasingly smaller representatives of the given element in $L_n^{c,h}(X; p_X, R)$. These bordisms may be parameterized by $\{t \cdot x | x \in X, a_i < t < a_{i+1}\}$ where the sequence of a_i 's is chosen such that when these bordisms are glued together, we obtain a bounded quadratic complex parameterized by $O(X_+)$. We get an s -decoration because obviously we can split the bounded quadratic complex. The map in the opposite direction is given by a splitting obtained the same way as in Lemma 3. \square

One advantage of a categorical description is computational. We have as close an analogue to excision as is possible in the following: Let Y be a subcomplex of X , and S a ring with involution. We then get a sequence of categories

$$\mathcal{C}_{O(Y_+)}(S) \rightarrow \mathcal{C}_{O(X_+)}(S) \rightarrow \mathcal{C}_{O(X/Y)}(S)$$

which leads to a long exact sequence

$$\dots \rightarrow L_n^a(\mathcal{C}_{O(Y_+)}(S)) \rightarrow L_n^b(\mathcal{C}_{O(X_+)}(S)) \rightarrow L_n^c(\mathcal{C}_{O(X/Y)}(S)) \rightarrow \dots$$

where the rule to determine the decorations is that b can be chosen to be any involution preserving subgroup of $K_i(\mathcal{C}_{O(X_+)}(S))$, $i \leq 2$, but then c has to be the image in $K_i(\mathcal{C}_{O(X/Y)}(S))$, and a has to be the preimage in $K_i(\mathcal{C}_{O(Y_+)}(S))$. See [1] for a derivation of these exact sequences. This makes it possible to do extensive calculations with controlled L -groups.

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Department of Mathematical Sciences, SUNY at Binghamton, Binghamton, New York 13901, USA

Department of Applied Science, Okayama University of Science, Okayama, Okayama 700-0005, Japan

Email: erik@math.binghamton.edu masayuki@mdas.ous.ac.jp

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Problems on homology manifolds

FRANK QUINN

Abstract A compilation of questions for the proceedings of the Workshop on Exotic Homology Manifolds, Oberwolfach June 29 – July 5, 2003

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In these notes “homology manifold” means ENR (Euclidean neighborhood retract) Z -coefficient homology manifold, unless otherwise specified, and “exotic” means not a manifold factor (i.e. local or “Quinn” index $\neq 1$). We use the multiplicative version of the local index, taking values in $1 + 8Z$. In the last decade exotic homology manifolds have been shown to exist and quite a bit of structure theory has been developed. However they have not yet appeared in other areas of mathematics. The first groups of questions suggest ways this might happen. Later questions are more internal to the subject.

Sections 1, 2, and 3 concern possible “natural” appearances of homology manifolds: as aspherical geometric objects; as Gromov-Hausdorff limits; and as boundaries of compactifications. Section 4 discusses group actions, where the use of homology manifold fixed sets may give simpler classification results. Sections 5 and 6 consider possible generalizations to non-ANR and “approximate” homology manifolds. Section 7 concerns spaces with special metric structures. Section 8 describes still-open low dimensional cases of the current theory. The final section, 9, collects problems related to homeomorphisms and the “disjoint disk property” for exotic homology manifolds.

1 Aspherical homology manifolds

Geometric structures on aspherical spaces seem to be rigid. The “Borel conjecture” is that closed aspherical manifolds are determined up to homeomorphism by their fundamental groups, and this has been verified in many cases, see Farrell [23] for a survey. More generally it is expected that aspherical homology

manifolds should be determined up to s-cobordism by their fundamental groups, so in particular the fundamental group should determine the local index. So far, however, there are no exotic examples.

Problem 1.1 *Is there a closed aspherical homology manifold with local index $\neq 1$?*

If so then exotic homology manifolds would be required for a full analysis of the aspherical question. See §6 for a “approximate” version of the problem.

2 Gromov-Hausdorff Limits

Differential geometers have investigated limits of Riemannian manifolds with various curvature and other constraints. As constraints are weakened one sees:

- 1) first, smooth manifold limits diffeomorphic to nearby manifolds (Anderson-Cheeger [2], Petersen [42]);
- 2) next, topological manifold limits homeomorphic to nearby manifolds (Grove-Peterson-Wu [30]);
- 3) then topologically stratified limits (Perel'man [39], [40], Perel'man-Petrinin [41]); and finally
- 4) more-singular limits currently not good for much.

Limits in the homeomorphism case (2) were first only known to be homology manifolds, and nearby manifolds were analyzed using controlled topology. However Perel'man [39] later used the Alexandroff curvature structure to show the limits in (2) are in fact manifolds, and extended this to stratifications with manifold strata in some singular cases (3). This considerably simplified the analysis and removed the need for homology manifolds. An approach to singular cases using Ricci curvature is given by Cheeger-Colding [11], see also Zhu [57]. We might still hope for a role for the more sophisticated topology:

Problem 2.1 *Are there differential-geometric conditions or processes that give exotic homology manifold limits?*

Such conditions must involve something other than diameter, volume, and sectional curvature bounds. Exotic ENR homology manifolds cannot arise this way so the most interesting outcome would be to get infinite-dimensional limits. An analysis of manifolds near such limits has been announced by Dranishnikov and Ferry and apparently these can vary quite a lot, see also §5.

2.1 Stratified Gromov-Hausdorff limits

The most immediately promising problems about limits concern stratifications.

Problem 2.2 *Are there differential-geometric conditions on smooth stratified sets that imply Gromov-Hausdorff limits are homotopy stratified sets with homology-manifold strata? What is the structure of the nearby smooth stratified sets?*

There are two phenomena here: “collapse” in which new strata are generated, and convergence that is in some sense stratum-wise. The first case is typified by “volume collapse” of manifolds to stratified sets. The Cheeger-Fukaya-Gromov structure theory of collapsed manifolds suggests the nearby manifolds should be total spaces of stratified systems of fibrations with nilmanifold fibers. Note however this structure should only be topological: smooth structures on the limit and bundles are unlikely in general. In cases where curvature is bounded below Perel’man’s analysis of the Alexandroff structure of the limit gives a topologically stratified space, solving the first part of the problem.

In the second case (stratum-wise convergence) the given smooth stratifications need not converge, but some sort of “homotopy intrinsic” stratifications should converge. More detail and an elaborate proposal for the topological part of this question is given in Quinn [46]. In this case if strata in the limits are ENR homology manifolds then one expects nearby stratified sets to be stratified s-cobordant. Again cases where Perel’man’s Alexandroff-space results apply should be much more accessible.

The motivation for this question is to study compactifications of collections of algebraic varieties or of stratifications arising in singularity theory. Therefore to be useful the “differential-geometric” hypotheses should have reasonable interpretations in these contexts. Other possibilities are to relate this to limits of special processes, e.g. the Ricci flow (Glickenstein [29]) or special limits defined by logical constraints (van den Dries [54]).

3 Compactifications

Negatively curved spaces and groups (in the sense of Gromov) have compactifications with “boundaries” defined by equivalence classes of geodesics. “Hyperbolization” procedures that mass-produce examples are described by Davis-Januszkiewicz [18], Davis-Januszkiewicz-Weinberger [19] and Charney-Davis

[12]. “Visible boundaries” can be defined for nonpositively curved spaces using additional geometric information. Bestvina [4] has given axioms for compactifications and shown compactifications of Poincaré duality groups satisfying his axioms give homology manifolds.

In classical cases the space on which the group acts is homeomorphic to Euclidean space, the boundary is a sphere compactifying the space to a disk. Behavior of limits in the sphere are more interesting than the sphere itself. More interesting examples arise with “Davis” manifolds: contractible nonpositively curved manifolds not simply connected at infinity. Fischer [26] shows that a class of these have boundaries that are

- 1) finite dimensional cohomology manifolds with the (Čech) homology of a sphere;
- 2) not locally 1-connected (so not ENR);
- 3) homogeneous, and
- 4) the double of the compactification along the boundary is a genuine sphere.

This connects nicely with the non-ENR questions raised in §5. However it seems unlikely that interesting ENR examples will arise this way: boundaries can be exotic only if the input space is exotic, for example the universal cover of an exotic closed aspherical manifold as in §1, and this is probably not compatible with nonpositive curvature assumptions, see §7.

To get more exotic behavior probably will require going outside the nonpositive curvature realm:

Problem 3.1 *Find non-curvature constructions for limits at infinity of Poincaré duality groups, and find (or verify) criteria for these to be homology manifolds.*

See Davis [17] for a survey of Poincaré duality groups. This question may provide an approach to closed aspherical exotic homology manifolds: first construct the “sphere at infinity” of the universal cover, then somehow fill in.

A variation on this idea is suggested by a proof of cases of the Novikov conjecture by Farrell-Hsiang [24] and many others since. They use a compactification of the universal cover to construct a fiberwise compactification of the tangent bundle. This suggests directly constructing completions of a bundle rather than a single fiber. The bundle might include parameters, for instance to resolve ambiguities arising in constructing limits without negative curvature. The context for this is discussed in section 6.

Problem 3.2 *Construct “approximate” limits of duality groups, as “fibers” of the total space of an approximate fibration over a parameter space.*

4 Group actions and non- Z coefficients

This topic probably has the greatest potential for profound applications, but also has severe technical difficulty. Smith theory shows fixed sets of actions of p -groups on homology manifolds must be Z/pZ homology manifolds. In the PL case there is a remarkable near converse: Jones [34] shows PL Z/pZ homology submanifolds satisfying the Smith conditions are frequently fixed sets of a Z/pZ action. Better results are likely for topological actions.

Problem 4.1 *Extend the Jones analysis to determine when 1-LC embedded Z/pZ homology submanifolds are fixed sets of Z/pZ actions.*

If the submanifold is an ENR then there are tools available (e.g. mapping cylinder neighborhoods) that should bring this within reach. Unfortunately the non-ENR case is likely to be the one with powerful applications. As a test case we formulate a stable version of 4.1 in which some difficulties should be avoided:

Problem 4.2 *Suppose $X \subset R^n$ is an even-codimension properly embedded possibly non-ENR Z/pZ homology manifold and is Z/pZ acyclic. Is there a Z/pZ action on R^{n+2k} for some k with fixed set $X \times \{0\}$?*

Extending 4.1 to a systematic classification theory for topological group actions will require a good understanding of the corresponding homology manifolds:

Problem 4.3 *Are there “surgery theories” for Z/nZ and rational homology manifolds?*

Surgery for PL manifolds up to Z/pZ homology equivalence was developed in the 70s Quinn [47], Anderson [3], Taylor-Williams [52] and a speculative sketch for PL Z/pZ homology manifolds is given in Quinn [47]. There are two serious difficulties for a topological version. The first problem is that the local and “normal” structures do not decouple. The boundary of a regular neighborhood in Euclidean space is the appropriate model for the Spanier-Whitehead dual of a space. Z -Poincaré spaces are characterized by this neighborhood being equivalent to a spherical fibration over the space. When manipulating a Poincaré space within its homotopy type (e.g. while constructing homology manifolds) the bundle gives easy and controlled access to the Spanier-Whitehead dual. Z/pZ Poincaré spaces have regular neighborhoods that are Z/nZ spherical fibrations, but this only specifies the Z/pZ homotopy type. Local structure at

other primes can vary from place to place, and the normal structure must conform to this. Some additional structure is probably needed, but this is unclear.

The second problem is that constructions of Z/pZ homology manifolds are unlikely to give ENRs. In the Z case ENRs are obtained as limits of sequences of controlled homotopy equivalences. Homotopy equivalences are obtained because obstructions to constructing these can be identified with global data (essentially the topological structure on the normal bundle). In the Z/pZ case there will probably only be enough data to get controlled Z/pZ homology equivalences. It seems likely that n -dimensional homology manifolds can be arranged to have covering dimension n and have nice $\lfloor \frac{n-1}{2} \rfloor$ skeleta, but above the middle dimension infinitely generated homology prime to p is likely to be common.

Z/pZ homology manifolds might geometrically implement some of the remarkable but formal “ p -complete” manifold theory proposed in Sullivan [51].

Dranishnikov [21] gives constructions of rational homology 5-manifolds with large but still finite covering and cohomological dimension. If this complicates the development it may be appropriate to consider only homology manifolds with covering dimension equal to the duality dimension.

4.1 Circle actions

A group action is “semifree” if points are either fixed by the whole group or moved freely. In this case the fixed set is also the fixed set of the Z/pZ subgroups, all p , so it follows from Smith theory that it is a Z homology manifold. Problems 4.1 and 4.2 therefore have analogs for semifree S^1 actions and Z -coefficient homology manifolds:

Problem 4.4 *Determine when Z coefficient homology submanifolds satisfying Smith conditions are fixed sets of semifree S^1 actions.*

This setting has the advantages that there are fewer obstructions, and in the ENR case the analog of 4.3 is already available. Again the significance of non-ENR case depends on how many non-ENR homology manifolds there are (see §5). If they all occur as boundaries with ENR interior then it seems likely a general action will be concordant to one with ENR fixed set. At the other extreme if there are fixed sets with non-integer local index then a full treatment of group actions will probably need non-ENRs.

5 Non-ENR homology manifolds

It is a folk theorem that a homology manifold that is finite dimensional and locally 1-connected is an ENR. The proof goes as follows: duality shows homology manifolds are homologically locally n -connected, all n , and a local Whitehead theorem shows local 1-connected and homological local n -connected implies local n -connected in the usual homotopic sense. Finally finite-dimensional and locally n -connected for large n implies ENR. The point here is that the ENR condition can fail in two ways: failure of finite dimensionality or failure of local 1-connectedness. These are discussed separately.

5.1 Infinite dimensional homology manifolds

Here we consider locally compact metric spaces that are locally contractible (or at least locally 1-connected) and homology manifolds in the usual finite-dimensional sense, but with infinite covering dimension. This is not related to manifolds modeled on infinite dimensional spaces.

Problem 5.1 *Is there a “surgery theory” of infinite dimensional homology manifolds?*

These were shown to arise as cell-like images of manifolds by Dranishnikov [20], following a proposal of Edwards. Recently Dranishnikov and Ferry have announced that there are examples with arbitrarily close (in the Gromov-Hausdorff sense) topological manifolds with different homotopy types. This contrasts with the ENR case where sufficiently close manifolds are all homeomorphic, and suggests this is a way to loosen the strait-jacket constraints of homotopy type in standard surgery. In particular the “surgery theory” should *not* follow the usual pattern of fixing a homotopy type, and “structures” should include manifolds of different homotopy type. This might be done by following Dranishnikov-Ferry in assuming existence of metrics that are sufficiently Gromov-Hausdorff close. See 3.1.

The source dimension for infinite-dimensionality is not quite settled:

Problem 5.2 *Are there infinite-dimensional Z -homology 4-manifolds? Are there infinite-dimensional homology 4- or 5-manifolds with nearby manifolds of different homotopy type?*

Walsh [55] has shown homology manifolds of homological dimension ≤ 3 are finite dimensional. Dydak-Walsh [15] produced infinite-dimensional examples of homology 5-manifolds but these do not connect with the Dranishnikov-Ferry analysis. It may be that an interesting “surgery theory” does not start until dimension 6.

5.2 Non locally-1-connected homology manifolds

Now consider finite dimensional metric homology manifolds that may fail to be locally 1-connected. These arise as “spheres at infinity” for certain groups, see §3. The first question seeks to locate these spaces relative to ENR and “virtual” homology manifolds. This is important for applications to group actions, see 4.4.

Problem 5.3 *Extend the definition of local index to finite dimensional non-ENR homology manifolds*

If the extended definition still takes values in $1 + 8\mathbb{Z}$ then the next question (motivated by §3) would be:

Problem 5.4 *Does every finite dimensional homology manifold arise as the “weakly tame” boundary of one with ENR interior? Is the union of two such extensions along their boundary an ENR?*

Here “weakly tame” should be as close to “locally 1-connected complement” as possible. An affirmative answer to this question would suggest thinking of non-ENR homology manifolds as “puffed up” versions not much different from ENRs.

At another extreme “approximate” homology manifolds are defined in section 6 in terms of approximate fibrations with homology manifold base and total space. These behave as though they have “fibers” that are homology manifolds with local indices in $1 + 8\mathbb{Z}_{(2)}$. If an extension of the local index to non ENRs can take non-integer values then the ENR boundary question above is wrong and we should ask:

Problem 5.5 *Does every finite dimensional homology manifold occur as a fiber of an approximate fibration with ENR homology manifold base and total space? Conversely is any such approximate fibration concordant to one with such a fiber?*

An appropriate relative version of this would show approximate homology manifolds are equivalent to finite dimensional non-ENR homology manifolds.

6 Approximate homology manifolds

The intent is that approximate homology manifolds should be fibers of approximate fibrations with base and total space ENR homology manifolds. Actual point-inverses are not topologically well-defined and do not encode the interesting information, so we use a germ approach. For simplicity we restrict to the compact case (fibers of proper maps).

A *compact approximate homology manifold* is a pair $(f: E \rightarrow B, b)$ where f is a proper approximate fibration with homology manifold base and total space, and b is a point in B . “Concordance” is the equivalence relation generated by

- 1) changing the basepoint by an arc in the base;
- 2) restricting to a neighborhood of the basepoint; and
- 3) product with identity maps of homology manifolds.

6.1 Basic structure

Suppose F is a compact virtual homology manifold defined by an proper approximate fibration $f: E \rightarrow B$ with E, B connected homology manifolds, and $b \in B$.

- 1) F has a well-defined homotopy type (the homotopy fiber of the map) that is a Poincaré space (with universal coefficients);
- 2) this Poincaré space has a canonical topological reduction of the normal fibration, given by restriction of the difference of the canonical reductions of E and B ; and
- 3) there is a local index defined by $i(F) = i(E)/i(B)$.

If f is a locally trivial bundle then the fiber is a ENR homology manifold. Multiplicativity of the local index shows the formula in (3) does give the local index of the fiber in this case. In general the quotients in (3) lie in $1 + 8Z_{(2)}$, where $Z_{(2)}$ is the rationals with odd denominator.

6.2 Example

Suppose X is a homology manifold, and choose a 1-LC embedding in a manifold of dimension at least 5. This has a mapping cylinder neighborhood; let $f: \partial N \rightarrow X$ be the map. Duality shows this is an approximate fibration with

fiber the homotopy type of a sphere. As an approximate homology manifold the index is $1/i(X)$, which is not an integer unless $i(X) = 1$.

Products of these examples with genuine homology manifolds show that all elements of $1+8\mathbb{Z}_{(2)}$ are realized as indices of approximate homology manifolds.

6.3 Example

A tame end of a manifold has an “approximate collar” in the sense of a neighborhood of the end that approximately fibers over R . In the controlled case the local fundamental group is required to be stratified; see Hughes [32] for a special case and Quinn [48] in general.

There is a finiteness obstruction to finding a genuine collar. In some cases it follows that the approximate homology manifold appearing as the “fiber” of the approximate collar does not have the homotopy type of a finite complex.

Problem 6.1 *Show that the exotic behavior of the examples are the only differences: if an approximate homology manifold has integral local index and is homotopy equivalent to a finite complex then it is concordant to an ENR homology manifold.*

Problem 6.2 *Define “approximate transversality” of homology manifolds, and determine when a map from one homology manifold to another can be made approximate transverse to a submanifold.*

The examples give maps approximately transverse to a point, but for which more geometric forms of transversality are obstructed. Transversality theories restricted to situations where indices must be integers have been developed by Johnston [34], Johnston-Ranicki [35] and Bryant-Mio [9], and a finiteness-obstruction case has been investigated by Bryant-Kirby [8]. The hope is that a more complete approximate transversality theory is possible. There still will be restrictions however: a degree-1 map of homology manifolds of different index cannot be made geometrically transverse to a point in any useful sense.

Problem 6.3 *Develop a surgery theory for approximate homology manifolds.*

The obstructions should lie in the $L^{-\infty}$ groups introduced by Yamasaki [56].

7 Special metric spaces

Several special classes of metric spaces have been developed, particularly by the Russian school, as general settings for some of the results of differential geometry. It is natural to ask how these hypotheses relate to manifolds and homology manifolds, but for the question to have much real significance it is necessary to have sources of examples not a priori known to be manifolds. Gromov-Hausdorff convergence gives Alexandroff spaces, see §1.

A Busemann space is a metric space in which geodesic (locally length-minimizing) segments can be extended, and small metric balls are cones parameterized by geodesics starting at the center point. The standard question is:

Problem 7.1 *Must a Busemann space be a manifold?*

This is true in dimensions ≤ 4 ; the 4-dimensional case was done by Thurston [53] and is not elementary.

An Alexandroff space is a metric space in which geodesics and curvature constraints make sense, but with less structure than Busemann spaces. These need not be homology manifolds, so the appropriate question seems to be:

Problem 7.2 *Is an Alexandroff space that is a homology manifold in fact a manifold in the complement of a discrete set?*

The problematic discrete set should be detectable by local fundamental groups of complements, as with the cone on a non-simply-connected homology sphere. The answer is “yes” when there is a lower curvature bound, because the analysis in Perel’man [39], [40] and Perel’man-Petrinin [41] shows it is a topological stratified set and topological stratified sets have this property (Quinn [49]).

8 Low dimensions

1- and 2-dimensional ENR homology manifolds are manifolds. In dimensions ≥ 5 exotic homology manifolds of arbitrary local index exist, and there are “many” of them in the senses that

- there is a “full surgery theory” given by Bryant-Ferry-Mio-Weinberger [7] for dimensions ≥ 6 and announced by Ferry-Johnston for dimension 5; and

- in dimensions ≥ 6 Bryant-Ferry-Mio-Weinberger have announced a proof that an arbitrary homology manifold is the cell-like image of one with the DDP.

The 5-dimensional case of (2) is still open:

Problem 8.1 *Can 5-dimensional exotic homology manifolds be resolved by ones with the DDP?*

In dimension 4:

Problem 8.2 *Are there exotic 4-dimensional homology manifolds?*

The expected answer is “yes.” In a little more detail the possibilities seem to be:

- 1) exotic homology manifolds don’t exist; or
- 2) sporadic examples exist; or
- 3) there is a “full surgery theory”.

Even in higher dimensions there are currently no methods for getting isolated examples: to get anything one essentially has to go through the full surgery theory. More-direct examples in higher dimensions would be useful in approaching (2) as well as interesting in their own right. In (3) note there is currently a fundamental group restriction in the manifold case Freedman-Quinn [27], Freedman-Teichner [28], Krushkal-Quinn [37]. Homology-manifold surgery would imply manifold surgery so “full surgery theory” should be interpreted to mean “as full as the manifold case.”

Finally in dimension 3:

Problem 8.3 *Are there exotic 3-dimensional homology manifolds?*

The expected answer is “no”. See Repovš [50] for special conclusions in the resolvable case.

9 Homeomorphisms and the DDP

The basic question is: do the key homeomorphism theorems for manifolds extend to homology manifolds? The question should include a nondegeneracy

condition that gives manifolds in the index = 1 case. Here we use the DDP (Disjoint Disk Property): any two maps $i, j: D^2 \rightarrow X$ can be arbitrarily closely approximated by maps with disjoint images. However see 9.6.

There is a feeling that the first three problems should be roughly equivalent in the sense that one good idea could resolve them all.

Problem 9.1 *Is the “ α approximation theorem” of Chapman-Ferry [10] true for DDP homology manifolds?*

This expected answer is “yes”, and current techniques suggest a proof might break into two sub-problems:

- A compact metric homology manifold X has $\epsilon > 0$ so if X' is ϵ homotopy equivalent to X then there is a DDP Y with cell-like maps onto both X and X' ; and
- if X , X' , or both, have the DDP then the corresponding cell-like maps can be chosen to be homeomorphisms.

As a testbed for technique for the second part it would be valuable to have a surgery-type proof of Edwards’ theorem: a cell-like map from a (genuine) manifold to a homology manifold with DDP can be arbitrarily closely approximated by homeomorphisms.

Problem 9.2 *Is the h -cobordism theorem true for DDP homology manifolds?*

h -cobordisms appear in a natural way in the definition of “homology manifold structure sets”, among other places, and can be produced by surgery.

Problem 9.3 *Is a homology manifold with the DDP arc-homogeneous?*

“Arc-homogeneous” means if x, y are in the same component of M then there is a homeomorphism $M \times I \rightarrow M \times I$ that is the identity on one end and on the other takes x to y . “Isotopy-homogeneous” is the sharper version in which the homeomorphism is required to preserve the I coordinate, so gives an ambient isotopy taking x to y . The “arc” in the terminology refers to the track of the point under the homeomorphism or isotopy.

An affirmative answer to 9.3 would show DDP homology manifolds have coordinate charts homeomorphic to subsets of standard models in the same way manifolds have Euclidean charts. Note this is consistent with a number of different models in each index: only one model could occur in a *connected* homology manifold but different components might have different models.

The “Bing-Borsuk conjecture” is that a homogeneous ENR is a manifold, where “homogeneous” is used in the traditional sense that any two points have homeomorphic neighborhoods. A version more in line with current expectations, and avoiding low dimensional problems, is that the homogeneous ENRs of dimension at least 5 are exactly the DDP homology manifolds. For applications and philosophical reasons we prefer arc versions of homogeneity, and split the question into 9.3 and a homological question:

Problem 9.4 *Is a locally 1-connected homologically arc-homogeneous space a homology manifold (possibly infinite-dimensional)?*

A space is *homologically arc-homogeneous* if for any arc $f: I \rightarrow X$ the induced maps

$$H_i(X \times \{0\}, X \times \{0\} - (f(0), 0); Z) \rightarrow H_i(X \times I, X \times I - \text{graph}(f); Z)$$

is an isomorphism. This is clearly an analog of “arc-homogeneous” as defined above. A homology manifold satisfies this by Alexander duality. In fact it holds for $(I, 0)$ replaced by a n -disk and a point in the boundary.

It was shown by Bredon that homogeneous (in the traditional point sense) ENRs are homology manifolds *provided* the local homology groups are finitely generated, see Bryant [5], Dydak-Walsh [15]. The problem is to show the local homology groups form a locally constant sheaf. The arc version of homogeneity gives local isomorphisms so the problem becomes showing these are locally well-defined. This would follow immediately from a “homologically 2-disk-homogeneous” hypothesis, so is equivalent to this condition. The question is whether this follows from arc-homogeneity and local 1-connectedness. Bryant [6] has recently proved 9.4 under the assumption that the space is an ENR. Note that a finite dimensional locally 1-connected homology manifold is an ENR, so the question remaining is whether “ENR” can be shifted from hypothesis to conclusion.

In a somewhat different direction the following is still unknown even in the manifold case:

Problem 9.5 *Is the product of a homology manifold and R homogeneous?*

This can be disengaged from the homogeneity questions by asking “does $X \times R$ have DDP?”, but see the discussion of the DDP in 9.6. $X \times R^2$ does have DDP (Daverman [14] and there are quite a number of properties of X that imply $X \times R$ has DDP, see Halverson [32], Daverman-Halverson [16]. However there

are ghastly (in the technical sense) examples of homology manifolds that show none of these properties holds in general, see Daverman-Walsh [15], Halverson [32].

The final question is vague but potentially important:

Problem 9.6 *Is there a weaker condition than DDP that implies $\text{index} = 1$ homology manifolds are manifolds?*

If so then this condition should be substituted for the DDP in the other problems in this section. This could make some of them significantly easier, and may also help with understanding dimension 4. A good way to approach this would be to find a surgery-based proof of Edwards' approximation theorem (see 9.1), then inspect it closely to find the minimum needed to make it work. Edwards' proof (see Daverman [13]) uses unobstructed cases of engulfing and approximation theorems. Surgery by contrast proceeds by showing an obstruction vanishes. Potentially-obstructed proofs (when they work) are often more flexible and have led to sharper results.

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Mathematics, Virginia Tech, Blacksburg VA 24061-0123

Email: quinnmath.vt.edu

Controlled L -Theory

ANDREW RANICKI
MASAYUKI YAMASAKI

Abstract We develop an epsilon-controlled algebraic L -theory, extending our earlier work on epsilon-controlled algebraic K -theory. The controlled L -theory is very close to being a generalized homology theory; we study analogues of the homology exact sequence of a pair, excision properties, and the Mayer-Vietoris exact sequence. As an application we give a controlled L -theory proof of the classic theorem of Novikov on the topological invariance of the rational Pontrjagin classes.

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Keywords Controlled algebraic L -theory, homology theory.

1 Introduction.

The purpose of this article is to develop a controlled algebraic L -theory, of the type first proposed by Quinn [8] in connection with the resolution of homology manifolds by topological manifolds.

We define and study the epsilon-controlled L -groups $L_n^{\delta, \epsilon}(X; p_X, R)$, extending to L -theory the controlled K -theory of Ranicki and Yamasaki [14]. When the control map p_X is a fibration and X is a compact ANR, these groups are stable in the sense that $L_n^{\delta, \epsilon}(X; p_X, R)$ depends only on p_X and R and not on δ or ϵ as long as δ is sufficiently small and $\epsilon \ll \delta$ [5].

These are the candidates of the controlled surgery obstruction groups; in fact, such a controlled surgery theory has been established when the control map p_X is UV^1 ([4], [2]).

Although epsilon controlled L -groups do not produce a homology theory in general, they have the features of a generalized homology modulo controlled K -theory problems. In this article we study the controlled L -theory analogues of the homology exact sequence of a pair (5.2), excision properties (§6), and the Mayer-Vietoris sequence (7.3).

In certain cases when there are no controlled K -theoretic difficulties, we can actually show that controlled L -groups are generalized homology groups. This is discussed in §8.

In the last two sections, we study locally-finite analogues and as an application give a controlled L -theory proof of the classic theorem of Novikov [3] on the topological invariance of the rational Pontrjagin classes.

2 Epsilon-controlled quadratic structures.

In this section we study several operations concerning quadratic Poincaré complexes with geometric control. These will be used to define epsilon controlled L -groups in the next section.

In [14] we discussed various aspects of geometric modules and morphisms and geometric control on them, and studied K -theoretic properties of geometric (=free) and projective module chain complexes with geometric control. There we considered only \mathbb{Z} -coefficient geometric modules, but the material in §§1–7 remains valid if we use any ring R with unity as the coefficient. To incorporate the coefficient ring into the notation, the group $\tilde{K}_0(X, p_X, n, \epsilon)$ defined using the coefficient ring R will be denoted $\tilde{K}_0^{n, \epsilon}(X; p_X, R)$ in this article.

To deal with L -theory, we need to use duals. Fix the control map $p_X : M \rightarrow X$ from a space M to a metric space X and let R be a ring with involution [10]. The dual G^* of a geometric R -module G is G itself. Recall that a geometric morphism is a linear combination of paths in M with coefficient in R . The dual f^* of a geometric morphism $f = \sum_{\lambda} a_{\lambda} \rho_{\lambda}$ is defined by $f^* = \sum_{\lambda} \bar{a}_{\lambda} \bar{\rho}_{\lambda}$, where $\bar{a}_{\lambda} \in R$ is the image of a by the involution of R and $\bar{\rho}_{\lambda}$ is the path obtained from ρ_{λ} by reversing the orientation. Note that if f has radius ϵ then so does its dual f^* and that $f \sim_{\epsilon} g$ implies $f^* \sim_{\epsilon} g^*$, by our convention. For a geometric R -module chain complex C , its n -dual C^{n-*} is defined using the formula in [9].

For a subset S of a metric space X , S^{ϵ} will denote the closed ϵ neighborhood of S in X when $\epsilon \geq 0$. When $\epsilon < 0$, S^{ϵ} will denote the set $X - (X - S)^{-\epsilon}$.

Let C be a free R -module chain complex on $p_X : M \rightarrow X$. An n -dimensional ϵ quadratic structure ψ on C is a collection $\{\psi_s | s \geq 0\}$ of geometric morphisms

$$\psi_s : C^{n-r-s} = (C_{n-r-s})^* \rightarrow C_r \quad (r \in \mathbb{Z})$$

of radius ϵ such that

$$(*) \quad d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1} (\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \sim_{3\epsilon} 0 : C^{n-r-s-1} \rightarrow C_r,$$

for $s \geq 0$. An n -dimensional free ϵ chain complex C on p_X equipped with an n -dimensional ϵ quadratic structure is called an n -dimensional ϵ quadratic R -module complex on p_X .

Let $f : C \rightarrow D$ be a chain map between free chain complexes on p_X . An $(n+1)$ -dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on f is a collection $\{\delta\psi_s, \psi_s | s \geq 0\}$ of geometric morphisms $\delta\psi_s : D^{n+1-r-s} \rightarrow D_r$, $\psi_s : C^{n-r-s} \rightarrow C_r$ ($r \in \mathbb{Z}$) of radius ϵ such that the following holds in addition to $(*)$:

$$d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^* \sim_{3\epsilon} 0 \\ : D^{n-r-s} \rightarrow D_r,$$

for $s \geq 0$. An ϵ chain map $f : C \rightarrow D$ between an n -dimensional free ϵ chain complex C on p_X and an $(n+1)$ -dimensional free ϵ chain complex D on p_X equipped with an $(n+1)$ -dimensional ϵ quadratic structure is called an $(n+1)$ -dimensional ϵ quadratic R -module pair on p_X . Obviously its boundary (C, ψ) is an n -dimensional ϵ quadratic R -module complex on p_X . We will suppress references to the coefficient ring R unless we need to emphasize the coefficient ring.

An ϵ cobordism of n -dimensional ϵ quadratic structures ψ on C and ψ' on C' is an $(n+1)$ -dimensional ϵ quadratic structure $(\delta\psi, \psi \oplus -\psi')$ on some chain map $C \oplus C' \rightarrow D$. An ϵ cobordism of n -dimensional ϵ quadratic complexes (C, ψ) , (C', ψ') on p_X is an $(n+1)$ -dimensional ϵ quadratic pair on p_X

$$((f \quad f') : C \oplus C' \rightarrow D, \quad (\delta\psi, \psi \oplus -\psi'))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. The union of adjoining cobordisms is defined using the formula in [9]. The union of adjoining ϵ cobordisms is a 2ϵ cobordism.

ΣC and ΩC will denote the suspension and the desuspension of C respectively, and $\mathcal{C}(f)$ will denote the algebraic mapping cone of a chain map f .

Definition 2.1 Let W be a subset of X . An n -dimensional ϵ quadratic structure ψ on C is ϵ Poincaré (over W) if the algebraic mapping cone of the duality 3ϵ chain map

$$\mathcal{D}_\psi = (1 + T)\psi_0 : C^{n-*} \longrightarrow C$$

is 4ϵ contractible (over W). A quadratic complex (C, ψ) is ϵ Poincaré (over W) if ψ is ϵ Poincaré (over W). Similarly, an $(n+1)$ -dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : C \rightarrow D$ is ϵ Poincaré (over W) if the algebraic mapping cone of the duality 4ϵ chain map

$$\mathcal{D}_{(\delta\psi, \psi)} = \begin{pmatrix} (1 + T)\delta\psi_0 \\ (-)^{n+1-r}(1 + T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \rightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is 4ϵ contractible (over W) (or equivalently the algebraic mapping cone of the 4ϵ chain map

$$\bar{\mathcal{D}}_{(\delta\psi, \psi)} = ((1+T)\delta\psi_0 \quad f(1+T)\psi_0) : \mathcal{C}(f)^{n+1-*} \longrightarrow D$$

is 4ϵ contractible (over W)) and ψ is ϵ Poincaré (over W). A quadratic pair $(f, (\delta\psi, \psi))$ is ϵ Poincaré (over W) if $(\delta\psi, \psi)$ is ϵ Poincaré (over W). We will also use the notation $\mathcal{D}_{\delta\psi} = (1+T)\delta\psi_0$, although it does not define a chain map from D^{n+1-*} to D in general.

This definition is slightly different from the one given in [16] (especially when W is a proper subset of X). There a quadratic complex/pair was defined to be ϵ Poincaré over W if the duality map is an ϵ chain equivalence over W . If $\mathcal{C}(\mathcal{D}_{\psi})$ (resp. $\mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})$) is 4ϵ contractible (over W), then \mathcal{D}_{ψ} (resp. $\mathcal{D}_{(\delta\psi, \psi)}$) is only a “weak” 8ϵ chain equivalence over W .

Definition 2.2 A chain map $f : C \rightarrow D$ is a *weak ϵ chain equivalence* over W if

- (1) f is an ϵ chain map,
- (2) there exists a family $g = \{g_r : D_r \rightarrow C_r\}$ of geometric morphisms of radius ϵ such that the following holds for all r :
 - dg_r and g_rd have radius ϵ , and
 - $dg_r \sim_{\epsilon} g_{r-1}d$ over W
- (3) there exist two families $h = \{h_r : C_r \rightarrow C_{r+1}\}$ and $k = \{k_r : D_r \rightarrow D_{r+1}\}$ of ϵ morphisms such that the following holds for all r :
 - $dh_r + h_{r-1}d \sim_{2\epsilon} 1 - g_rf_r$ over W , and
 - $dk_r + k_{r-1}d \sim_{2\epsilon} 1 - f_rg_r$ over W .

In other words a weak chain equivalence satisfies all the properties of a chain equivalence except that its inverse may not be a chain map outside of W .

Weak chain equivalences behave quite similarly to chain equivalences. For example, 2.3(3) and 2.4 of [14] can be easily generalized as follows:

Proposition 2.3 *If $f : C \rightarrow D$ is a weak δ chain equivalence over V and $f' : D \rightarrow E$ is a weak ϵ chain equivalence over W , then $f'f$ is a weak $\delta + \epsilon$ chain equivalence over $V^{-\delta-\epsilon} \cap W^{-\delta}$. If we further assume that f is a δ chain equivalence, then $f'f$ is a weak $\delta + \epsilon$ chain equivalence over $V^{-\epsilon} \cap W^{-\delta}$.*

Proposition 2.4 *Let $f : C \rightarrow D$ be an ϵ chain map. If the algebraic mapping cone $\mathcal{C}(f)$ is ϵ contractible over W , then f is a weak 2ϵ chain equivalence over W . If f is a weak ϵ chain equivalence over W , then $\mathcal{C}(f)$ is 3ϵ contractible over $W^{-2\epsilon}$.*

We have employed the definition of Poincaré complexes/pairs using local contractibility of the algebraic mapping cone of the duality map, because algebraic mapping cones are easier to handle than chain equivalences. For example, consider a triad Γ :

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ g \downarrow & \rightsquigarrow h & \downarrow g' \\ C' & \xrightarrow{f'} & D' \end{array} \quad dh + hd \sim f'g - g'f$$

and assume

- (1) f (resp. f') is a δ (resp. δ') chain map,
- (2) g (resp. g') is an ϵ (resp. ϵ') chain map,
- (3) $h : g'f \simeq f'g$ is a γ chain homotopy.

Then there are induced a $\max\{\delta, \delta', 2\gamma\}$ chain map

$$F = \begin{pmatrix} f' & (-)^r h \\ 0 & -f \end{pmatrix} : \mathcal{C}(-g)_r = C'_r \oplus C_{r-1} \rightarrow \mathcal{C}(g')_r = D'_r \oplus D_{r-1}$$

and a $\max\{\epsilon, \epsilon', 2\gamma\}$ chain map

$$G = \begin{pmatrix} g' & (-)^r h \\ 0 & g \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \rightarrow \mathcal{C}(f')_r = D'_r \oplus C'_{r-1}.$$

It is easily seen that $\mathcal{C}(F) = \mathcal{C}(G)$.

Proposition 2.5 *If $\mathcal{C}(g : C \rightarrow C')$ is ϵ contractible over W , then $\mathcal{C}(-g)$ is ϵ contractible over W .*

Proof Suppose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathcal{C}(g)_r = C_r \oplus C'_{r-1} \longrightarrow \mathcal{C}(g)_{r+1} = C_{r+1} \oplus C'_r$$

is an ϵ chain contraction over W of $\mathcal{C}(g)$, then

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

is an ϵ chain contraction over W of $\mathcal{C}(-g)$. □

Proposition 2.6 *Let Γ be as above, and further assume that $\mathcal{C}(g)$ is ϵ contractible over W and $\mathcal{C}(g')$ is ϵ' contractible over W , then $\mathcal{C}(G)$ is $3 \max\{\epsilon, \epsilon', \delta, \delta', 2\gamma\}$ contractible over $W^{-2 \max\{\epsilon, \epsilon', \delta, \delta', 2\gamma\}}$.*

Proof By 2.5, $\mathcal{C}(-g)$ is ϵ contractible over W . Therefore $F : \mathcal{C}(-g) \rightarrow \mathcal{C}(g')$ is a $\max\{\epsilon, \epsilon', \delta, \delta', 2\gamma\}$ chain equivalence over W , and the proposition is proved by applying 2.4 to F . \square

Corollary 2.7 *Let C and D be free ϵ chain complexes, and let $(\delta\psi, \psi)$ be an ϵ quadratic structure on an ϵ chain map $f : C \rightarrow D$. If $\mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})$ is 4ϵ contractible over W , then $\mathcal{C}(\mathcal{D}_\psi)$ is 100ϵ contractible over $W^{-100\epsilon}$.*

Proof Consider the triad Γ :

$$\begin{array}{ccc} \Omega\mathcal{C}(f)^{n+1-*} & \xrightarrow{\alpha=(1 \ 0)} & \Omega\mathcal{D}^{(n+1-*)} \\ \mathcal{D}_{(\delta\psi, \psi)} \downarrow & \searrow h & \downarrow \bar{\mathcal{D}}_{(\delta\psi, \psi)} \\ \Omega\mathcal{D} & \xrightarrow{\beta=\iota(1 \ 0)} & \Omega\mathcal{C}(f) \end{array} \quad h = \begin{pmatrix} 0 & 0 \\ 0 & (-)^{r+1}(1+T)\psi_0 \end{pmatrix}$$

and consider the chain map $G : \mathcal{C}(\alpha) \rightarrow \mathcal{C}(\beta)$ induced from Γ as above. Then $\mathcal{C}(G)$ is 12ϵ contractible over $W^{-8\epsilon}$ by the previous proposition. Therefore G is a weak 24ϵ chain equivalence over $W^{-8\epsilon}$. $(1+T)\psi_0$ is equal to the following composite of G with two ϵ chain equivalences:

$$C^{n-*} \xrightarrow[\simeq_\epsilon]{\iota(0 \ 0 \ 1)} \mathcal{C}(\alpha) \xrightarrow{G} \mathcal{C}(\beta) \xrightarrow[\simeq_\epsilon]{(0 \ 1 \ 0)} C,$$

and the claim follows from 2.3. \square

Next we describe various constructions on quadratic complexes with some size estimates. Firstly a direct calculation shows the following. (See the non-controlled case [9].)

Proposition 2.8 *If adjoining ϵ cobordisms c and c' are ϵ Poincaré over W , then $c \cup c'$ is 100ϵ Poincaré over $W^{-100\epsilon}$.*

The following proposition gives us a method to construct quadratic structures and cobordisms.

Proposition 2.9 Suppose $g : C \rightarrow C'$ is a δ chain map of δ chain complexes and ψ is an n -dimensional ϵ quadratic structure on C .

(1) $g\% \psi = \{(g\% \psi)_s = g\psi_s g^*\}$ is a $2\delta + \epsilon$ quadratic structure on C' , and $(0, \psi \oplus -g\% \psi)$ is a $2\delta + \epsilon$ quadratic structure on the δ chain map $(g \ 1) : C \oplus C' \rightarrow C'$.

(2) If ψ is ϵ Poincaré over W and g is a weak δ chain equivalence over W , then $g\% \psi$ and $(0, \psi \oplus -g\% \psi)$ are $2\delta + 6\epsilon$ Poincaré over $W^{-(6\delta+24\epsilon)}$.

(3) If ψ is ϵ Poincaré, g is a δ chain equivalence, $\Psi' = (\delta\psi, \psi' = g\% \psi)$ is an ϵ' Poincaré δ' quadratic structure on a δ' chain map $f' : C' \rightarrow D$, and D is a γ chain complex, then $\Psi = (\delta\psi, \psi)$ is an $\epsilon' + 3 \max\{9\delta, 6\delta', 4\epsilon, 3\gamma\}$ Poincaré $\max\{\delta, \epsilon\}$ quadratic structure on the $\delta' + \epsilon$ chain map $f = f' \circ g : C \rightarrow D$.

Proof (1) This can be checked easily.

(2) This holds because the duality maps for $(C', g\% \psi)$ and c split as follows:

$$\begin{aligned} (C')^{n-*} &\xrightarrow{g^*} C^{n-*} \xrightarrow{(1+T)\psi_0} C \xrightarrow{g} C', \\ \mathcal{C}((g \ 1))^{n+1-*} &\xrightarrow[\simeq_\delta]{(0 \ 1 - g^*)} C^{n-*} \xrightarrow{(1+T)\psi_0} C \xrightarrow{g} C'. \end{aligned}$$

(3) We study the duality map for Ψ' . Since g is a δ chain equivalence and $\mathcal{C}(1 : D \rightarrow D)$ is γ contractible, the algebraic mapping cone of the $\max\{\gamma, \delta\}$ chain map

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow \mathcal{C}(f') = D_r \oplus C'_{r-1}$$

is $3 \max\{3\delta, \delta + \delta', \gamma\}$ contractible, and so is $\mathcal{C}(\tilde{g}^* : \mathcal{C}(f')^{n+1-*} \rightarrow \mathcal{C}(f)^{n+1-*})$. Therefore, the chain map $\mathcal{C}(\mathcal{D}_{\Psi'}) \rightarrow \mathcal{C}(\mathcal{D}_{\Psi})$ defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{g}^* \end{pmatrix} : \mathcal{C}(\mathcal{D}_{\Psi'}) = D_r \oplus \mathcal{C}(f')^{n+2-r} \longrightarrow \mathcal{C}(\mathcal{D}_{\Psi}) = D_r \oplus \mathcal{C}(f)^{n+2-r}$$

is a $6 \max\{9\delta, 6\delta', 4\epsilon, 3\gamma\}$ chain equivalence. The claim now follows from the next lemma. \square

Lemma 2.10 If a chain complex A is ϵ chain equivalent to a chain complex B which is δ contractible over $X - Y$, then A is $(2\epsilon + \delta)$ contractible over $X - Y^\epsilon$.

Proof Let $f : A \rightarrow B$ be an ϵ chain equivalence, g an ϵ chain homotopy inverse, $h : gf \simeq_\epsilon 1$ an ϵ chain homotopy, and Γ a δ chain contraction of B over $X - Y$. Then $g\Gamma f + h$ gives a $(2\epsilon + \delta)$ chain contraction of A over $X - Y$. \square

Remarks (1) An ϵ chain equivalence $f : C \rightarrow C'$ such that $f\% \psi = \psi'$ will be called an ϵ *homotopy equivalence* from (C, ψ) to (C', ψ') . By 2.9, a homotopy equivalence between quadratic Poincaré complexes induces a Poincaré cobordism between them.

(2) The estimates given in 2.9 and 2.10 are, of course, not acute in general. For example, consider an ϵ quadratic complex (C, ψ) which is ϵ Poincaré over W . Then a direct calculation shows that the cobordism between (C, ψ) and itself induced by the identity map of C is an ϵ quadratic pair and is ϵ Poincaré over W . This cobordism will be called the *trivial cobordism* from (C, ψ) to itself.

3 Epsilon-controlled L -groups.

In this section we review the boundary construction of the first-named author and then introduce epsilon-controlled L -groups

$$L_n^{\delta, \epsilon}(X; p_X, R) \quad \text{and} \quad L_n^{\delta, \epsilon}(X, Y; p_X, R)$$

for $p_X : M \rightarrow X$, $Y \subset X$, $n \geq 0$, $\delta \geq \epsilon \geq 0$, and a ring R with involution. These are defined using geometric R -module chain complexes with quadratic Poincaré structures discussed in the previous section.

Let (C, ψ) be an n -dimensional ϵ quadratic R -module complex on p_X , where $n \geq 1$. Define a (possibly non-positive) 2ϵ chain complex ∂C by $\Omega\mathcal{C}(\mathcal{D}_\psi)$. Then an $(n-1)$ -dimensional 2ϵ Poincaré 2ϵ quadratic structure $\partial\psi$ on ∂C is defined by:

$$\begin{aligned} \partial\psi_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \partial C^{n-r-1} = C^{n-r} \oplus C_{r+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{m-r} \\ \partial\psi_s &= \begin{pmatrix} (-)^{n-r-s-1} T\psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} : \\ &\quad \partial C^{n-r-s-1} = C^{m-r-s} \oplus C_{r+s+1} \longrightarrow \partial C_r = C_{r+1} \oplus C^{m-r} \quad (s \geq 1). \end{aligned}$$

This is the *boundary construction* of Ranicki [9]. The structure $\Psi_1 = (0, \partial\psi)$ is an n -dimensional 2ϵ Poincaré 2ϵ quadratic structure on the ϵ chain map

$$i_C = \text{projection} : \partial C \longrightarrow C^{n-*}$$

of 2ϵ chain complexes. This is called the *algebraic Poincaré thickening* [9].

Example 3.1 Consider an n -dimensional ϵ chain complex F , and give ΣF the trivial $(n+1)$ -dimensional quadratic structure $\theta_s = 0$ ($s \geq 0$). Its algebraic Poincaré thickening

$$(i_{\Sigma F} : \partial \Sigma F = F \oplus F^{n-*} \longrightarrow (\Sigma F)^{n+1-*} = F^{n-*}, \quad (0, \partial\theta))$$

is an $(n+1)$ -dimensional ϵ Poincaré ϵ null-cobordism of $(\partial \Sigma F, \partial\theta)$.

There is an inverse operation up to homotopy equivalence. Given an n -dimensional ϵ Poincaré ϵ quadratic pair $c = (f : C \rightarrow D, (\delta\psi, \psi))$, take the union $(\tilde{C}, \tilde{\psi})$ of c with the ϵ quadratic pair $(C \rightarrow 0, (0, -\psi))$. \tilde{C} is equal to $\mathcal{C}(f)$. $(\tilde{C}, \tilde{\psi})$ is an n -dimensional 2ϵ quadratic complex and is called the *algebraic Thom complex* of c . The algebraic Poincaré thickening of $(\tilde{C}, \tilde{\psi})$ is “homotopy equivalent” to the original pair c (as pairs). Since we will not use this full statement, we do not define homotopy equivalences of pairs here and only mention that the chain map

$$g = \begin{pmatrix} 0 & 1 & 0 & -\psi_0 \end{pmatrix} : \partial \tilde{C}_r = D_{r+1} \oplus C_r \oplus D^{n-r} \oplus C^{n-r-1} \longrightarrow C_r$$

gives an 1ϵ chain equivalence such that $g_{\%}(\partial\tilde{\psi}) = \psi$. If we start with an n -dimensional ϵ quadratic complex (C, ψ) on p_X , then the algebraic Thom complex of the algebraic Poincaré thickening $(i_C : \partial C \longrightarrow C^{n-*}, (0, \partial\psi))$ of (C, ψ) is 3ϵ homotopy equivalent to (C, ψ) ; 3ϵ homotopy equivalences are given by

$$\begin{aligned} f &= \begin{pmatrix} -\mathcal{D}_\psi & 1 & 0 \end{pmatrix} : \mathcal{C}(i_C)_r = C^{n-r} \oplus C_r \oplus C^{n-r+1} \longrightarrow C_r, \\ f_{\%}(\Psi_1 \cup_{\partial\psi} -\Psi_2) &= \psi, \\ f' &= {}^t \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} : C_r \longrightarrow \mathcal{C}(i_C)_r = C^{n-r} \oplus C_r \oplus C^{n-r+1}, \\ f'_{\%}\psi &= \Psi_1 \cup_{\partial\psi} -\Psi_2, \end{aligned}$$

where $\Psi_2 = (0, \partial\psi)$ is the n -dimensional ϵ quadratic structure on the trivial chain map $0 : \partial C \longrightarrow 0$.

The boundary construction described above generalizes to quadratic pairs. For an $(n+1)$ -dimensional ϵ quadratic pair $(f : C \rightarrow D, (\delta\psi, \psi))$ on p_X , define a (possibly non-positive) 2ϵ chain complex ∂D by $\Omega\mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})$ and define an

n -dimensional 3ϵ Poincaré 2ϵ quadratic structure $\Psi_3 = (\partial\delta\psi, \partial\psi)$ on the 2ϵ chain map of 2ϵ chain complexes

$$\partial f = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} : \partial C_r = C_{r+1} \oplus C^{n-r} \longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r}$$

by

$$\begin{aligned} \partial\delta\psi_0 &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial D^{n-r} = D^{n-r+1} \oplus D_{r+1} \oplus C_r \\ &\longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r} \\ \partial\delta\psi_s &= \begin{pmatrix} (-)^{n-r-s-1} T\delta\psi_{s-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \partial D^{n-r-s} = D^{n-r-s+1} \oplus D_{r+s+1} \oplus C_{r+s} \\ &\longrightarrow \partial D_r = D_{r+1} \oplus D^{n-r+1} \oplus C^{n-r} \quad (s \geq 1). \end{aligned}$$

$\partial\psi$ is the same as above. Then $(0, \Psi_1 \cup_{\partial\psi} (-\Psi_3))$ gives an $(n+1)$ -dimensional 300ϵ Poincaré 4ϵ quadratic structure on the ϵ chain map

$$\begin{aligned} (i \quad 0 \quad i_D) : (C^{n-*} \cup_{\partial C} \partial D)_r &= C^{n-r} \oplus \partial C_{r-1} \oplus \partial D_r \\ &\longrightarrow (\mathcal{C}(f)^{n+1-*})_r = \mathcal{C}(f)^{n+1-r} \end{aligned}$$

of 2ϵ chain complexes, where $i : C^{n-*} \rightarrow \mathcal{C}(f)^{n+1-*}$ is the inclusion map and $i_D : \partial D \rightarrow \mathcal{C}(f)^{n+1-*}$ is the projection map.

If (C, ψ) (resp. $(f : C \rightarrow D, (\delta\psi, \psi))$) is ϵ Poincaré, then ∂C is (resp. ∂C and ∂D are) 4ϵ contractible, and hence chain homotopic to a positive chain complex (resp. positive chain complexes). But in general ∂C (and ∂D) may not be chain homotopic to a positive chain complex. This leads us to the following definition. The non-controlled version is described in [9].

Definition 3.2 (1) A positive geometric chain complex C ($C_i = 0$ for $i < 0$) is ϵ connected if there exists a 4ϵ morphism $h : C_0 \rightarrow C_1$ such that $dh \sim_{8\epsilon} 1_{C_0}$.

(2) A chain map $f : C \rightarrow D$ of positive chain complexes is ϵ connected if $\mathcal{C}(f)$ is ϵ connected.

(3) A quadratic complex (C, ψ) is ϵ connected if \mathcal{D}_ψ is ϵ connected.

(4) A quadratic pair $(f : C \rightarrow D, (\delta\psi, \psi))$ is ϵ connected if \mathcal{D}_ψ and $\mathcal{D}_{(\delta\psi, \psi)}$ are ϵ connected.

Lemma 3.3 (1) The composition of a δ connected chain map and an ϵ connected chain map is $\delta + \epsilon$ connected.

(2) Quadratic complexes and pairs that are ϵ Poincaré are ϵ connected.

(3) If ψ is an ϵ connected quadratic structure on C and $g : C \rightarrow C'$ is a δ connected chain map, then $\mathcal{D}_{(0, \psi \oplus -g\% \psi)}$ is $\epsilon + 2\delta$ connected.

Proof (1) is similar to 2.3. (2) is immediate from definition. (3) is similar to 2.9 (2). \square

Remark In general the ϵ connectivity of g does not imply the ϵ connectivity of g^* (or δ connectivity for any δ). Therefore we do not have any estimate on the connectivity of $g\% \psi$ in (3) above. It should be checked by an *ad hoc* method in each case. For example, see section 6.

If the desuspension ΩC of a positive complex C on p_X is ϵ chain equivalent to a positive complex, then C is $\epsilon/4$ connected. On the other hand, we have:

Proposition 3.4 Let $n \geq 1$.

(1) Suppose an n -dimensional ϵ quadratic complex (C, ψ) on p_X is ϵ connected. Then ∂C is 12ϵ chain equivalent to an $(n-1)$ -dimensional (resp. a 1-dimensional) 4ϵ chain complex $\hat{\partial}C$ if $n > 1$ (resp. if $n = 1$).

(2) Suppose an $(n+1)$ -dimensional ϵ quadratic pair $(f : C \rightarrow D, (\delta\psi, \psi))$ is ϵ connected. Then ∂D is 24ϵ chain equivalent to an n -dimensional 5ϵ chain complex $\hat{\partial}D$.

(3) When $n = 1$, the free 1-dimensional chain complex $(\hat{\partial}C, 1)$ given in (1) and (2), viewed as a projective chain complex, is 32ϵ chain equivalent to a 0-dimensional 32ϵ projective chain complex $(\tilde{\partial}C, p)$ and there is a 32ϵ isomorphism

$$(\hat{\partial}C_1, 1) \oplus (\tilde{\partial}C_0, p) \longrightarrow (\hat{\partial}C_0, 1),$$

and hence the controlled reduced projective class $[\tilde{\partial}C, p]$ vanishes in $\tilde{K}_0^{0, 32\epsilon}(X; p_X, R)$.

Proof (1) There exists a 4ϵ morphism $h : \partial C_{-1} \rightarrow \partial C_0$ such that $dh \sim_{8\epsilon} 1$. Define a 4ϵ morphism $h' : \partial C_{n-1} \rightarrow \partial C_n$ by the composite:

$$h' : \partial C_{n-1} = C_n \oplus C^1 \xrightarrow{\begin{smallmatrix} 0 & (-)^n \\ (-)^n & 0 \end{smallmatrix}} C^1 \oplus C_n \xrightarrow{h^*} C^0 = \partial C_n,$$

then $h'd \sim_{8\epsilon} 1$. Now one can use the folding argument from the bottom [16] using h and, if $n > 1$, from the top [14] using h' to construct a desired chain equivalence.

(2) There exists a 4ϵ morphism $h : \partial D_{-1} \rightarrow \partial D_0$ such that $dh \sim_{8\epsilon} 1$. Define a 5ϵ morphism $h' : \partial D_n \rightarrow \partial D_{n+1}$ by the composite of

$$\begin{pmatrix} 0 & (-)^{n+1} & 0 \\ (-)^{n+1} & 0 & 0 \\ 0 & 0 & (1+T)\psi_0 \end{pmatrix} : \partial D_n = D_{n+1} \oplus D^1 \oplus C^0 \rightarrow \partial D^0 = D^1 \oplus D_{n+1} \oplus C_n$$

and $h^* : \partial D^0 \rightarrow \partial D^{-1} = \partial D_{n+1}$, then $h'd \sim_{8\epsilon} 1$. Use the folding argument again.

(3) The boundary map $\hat{\partial}C_1 = \partial C_1 \oplus \partial C_{-1} \rightarrow \hat{\partial}C_0 = \partial C_0$ is given by the matrix $\begin{pmatrix} d_{\partial C} & h \end{pmatrix}$. Therefore

$$s = \begin{pmatrix} h' - h'hd_{\partial C} \\ d_{\partial C} \end{pmatrix} : \partial C_0 \longrightarrow \partial C_1 \oplus \partial C_{-1}$$

defines a 12ϵ morphism $s : \hat{\partial}C_0 \rightarrow \hat{\partial}C_1$ such that $sd_{\hat{\partial}C} \sim_{16\epsilon} 1$. Define $\tilde{\partial}C_0$ to be $\hat{\partial}C_0$ and define a 16ϵ morphism $p : \tilde{\partial}C_0 \rightarrow \tilde{\partial}C_0$ by $1 - d_{\hat{\partial}C}$, then $p^2 \sim_{32\epsilon} p$ and $p : (\tilde{\partial}C_0, 1) \rightarrow (\tilde{\partial}C_0, p)$ defines the desired 32ϵ chain equivalence. The isomorphism can be obtained by combining the following isomorphisms:

$$\begin{aligned} (\hat{\partial}C_1, 1) &\xrightarrow{d} (\hat{\partial}C_0, 1 - p) \\ &\xleftarrow{s} \\ (\hat{\partial}C_0, 1 - p) \oplus (\hat{\partial}C_0, p) &\xrightarrow[\begin{smallmatrix} (q & p) \\ \leftarrow & \end{smallmatrix}]{\begin{smallmatrix} (q & p) \\ \rightarrow & \end{smallmatrix}} (\hat{\partial}C_0, 1) \end{aligned}$$

□

Controlled connectivity is preserved under union operation in the following manner.

Proposition 3.5 *If adjoining ϵ cobordisms c and c' are ϵ connected, then $c \cup c'$ is 100ϵ connected.*

Proof Similar to 2.8. □

Now we define the epsilon-controlled L -groups. Let Y be a subset of X .

Definition 3.6 For an integer $n \geq 0$, pair of non-negative numbers $\delta \geq \epsilon \geq 0$, and a ring with involution R , $L_n^{\delta, \epsilon}(X, Y; p_X, R)$ is defined to be the equivalence classes of finitely generated n -dimensional ϵ connected ϵ quadratic complexes on p_X that are ϵ Poincaré over $X - Y$. The equivalence relation is generated by finitely generated δ connected δ cobordisms that are δ Poincaré over $X - Y$.

Remark We use the following abbreviations for simplicity:

- $L_n^{\delta, \epsilon}(X; p_X, R) = L_n^{\delta, \epsilon}(X, \emptyset; p_X, R)$
- $L_n^{\epsilon}(X, Y; p_X, R) = L_n^{\epsilon, \epsilon}(X, Y; p_X, R)$
- $L_n^{\epsilon}(X; p_X, R) = L_n^{\epsilon, \epsilon}(X; p_X, R)$

Proposition 3.7 Direct sum $(C, \psi) \oplus (C', \psi') = (C \oplus C', \psi \oplus \psi')$ induces an abelian group structure on $L_n^{\delta, \epsilon}(X, Y; p_X, R)$. Furthermore if $[C, \psi] = [C', \psi'] \in L_n^{\delta, \epsilon}(X, Y; p_X, R)$, then there is a finitely generated 100δ connected 2δ cobordism between (C, ψ) and (C', ψ') that is 100δ Poincaré over $X - Y^{100\delta}$.

Proof The inverse of an element $[C, \psi]$ is given by $[C, -\psi]$. In fact, as in 2.9 and 3.3 (with $g = 1$),

$$((1 \ 1) : C \oplus C \longrightarrow C, (0, \psi \oplus -\psi))$$

gives an ϵ connected ϵ null-cobordism of $(C, \psi) \oplus (C, -\psi)$ that is ϵ Poincaré over $X - Y$. The second claim follows from 2.8 and 3.5, because we can glue a sequence of cobordisms at once. \square

If $\delta' \geq \delta$ and $\epsilon' \geq \epsilon$, then there is a homomorphism

$$L_n^{\delta, \epsilon}(X, Y; p_X, R) \longrightarrow L_n^{\delta', \epsilon'}(X, Y; p_X, R)$$

which sends $[C, \psi]$ to $[C, \psi]$. This is called the *relax-control map*.

In the study of controlled L -groups, we need an analogue of 2.9 for pairs:

Proposition 3.8 Suppose there is a triad of ϵ chain complexes on p_X

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ g \downarrow & \text{\scriptsize wavy } g & \downarrow h \\ C' & \xrightarrow{f'} & D' \end{array} \quad dk + kd \sim f'g - hf$$

where f, f', g, h are ϵ chain maps and k is an ϵ chain homotopy, and suppose $(\delta\psi, \psi)$ is an $(n+1)$ -dimensional ϵ quadratic structure on f .

(1) There is induced a 4ϵ quadratic structure on f' :

$$(g, h; k)_{\%}(\delta\psi, \psi) = (h\delta\psi_s h^* + (-)^{n+1} k\psi_s f^* h^* + (-)^{n-r+1} f' g\psi_s k^* \\ + (-)^{r+1} kT\psi_{s+1} k^* : (D'^{n+1-r-s}, q'^*) \rightarrow (D'_r, q'_r), g\psi_s g^*)_{s \geq 0}.$$

(2) Suppose g and h are ϵ chain equivalences.

- (a) If $(\delta\psi, \psi)$ is ϵ Poincaré over $X - Y$, then $(g, h; k)_{\%}(\delta\psi, \psi)$ is 30ϵ Poincaré over $X - Y^{81\epsilon}$.
- (b) If $(f, (\delta\psi, \psi))$ is ϵ connected, then $(f', (g, h; k)_{\%}(\delta\psi, \psi))$ is 30ϵ connected.

Proof (1) is easy to check. (2) can be checked by showing that

$$\begin{pmatrix} (-)^{n+1-r} k\psi_0 k^* & k(1+T)\psi_0 g^* \end{pmatrix} : \mathcal{C}(f')^{n+1-r} = D'^{n+1-r} \oplus C'^{n-r} \\ \longrightarrow D'_{r+1}$$

is a 3ϵ chain homotopy between the duality map for $(g, h; k)_{\%}(\delta\psi, \psi)$ and the following chain map:

$$h((1+T)\delta\psi_0 \quad f(1+T)\psi_0) \begin{pmatrix} h^* & 0 \\ (-)^{n+1-r} k^* & g^* \end{pmatrix} : \mathcal{C}(f')^{n+1-r} \longrightarrow (D'_r, q'_r),$$

which is a weak 27ϵ chain equivalence over $X - Y^{18\epsilon}$ in case (2a), and 16ϵ connected in case (2b). \square

Corollary 3.9 Suppose $f : C \rightarrow D$ is an ϵ chain map, $(\delta\psi, \psi)$ is an $(n+1)$ -dimensional ϵ quadratic structure on f , $g : C \rightarrow C'$ is a γ chain equivalence, and $h : D \rightarrow D'$ is a δ chain equivalence. Let $\epsilon' = \gamma + \delta + \epsilon$ and g^{-1} be a γ chain homotopy inverse of g .

- (1) There is an $(n+1)$ -dimensional $4\epsilon'$ quadratic structure $(\delta\psi', \psi' = g_{\%}\psi)$ on the ϵ' chain map $f' = hf g^{-1} : (C', p') \rightarrow (D', q')$.
- (2) If $(\delta\psi, \psi)$ is ϵ Poincaré over $X - Y$, then $(\delta\psi', \psi')$ is $30\epsilon'$ Poincaré over $X - Y^{81\epsilon'}$.
- (3) If $(\delta\psi, \psi)$ is ϵ connected, then $(\delta\psi', \psi')$ is $30\epsilon'$ connected.

Proof Let $\Gamma : g^{-1}g \simeq 1$ be a γ chain homotopy. Define an ϵ' chain homotopy $k : hf \simeq f'g$ by $k = -hf\Gamma$, and apply 3.8 \square

The last topic of this section is the functoriality. A *map between control maps* $p_X : M \rightarrow X$ and $p_Y : N \rightarrow Y$ means a pair of continuous maps $(f : M \rightarrow N, \bar{f} : X \rightarrow Y)$ which makes the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{\bar{f}} & Y. \end{array}$$

For example, given a control map $p_Y : N \rightarrow Y$ and a subset $X \subset Y$, let us denote the control map $p_Y|_{p_Y^{-1}(X)} : p_Y^{-1}(X) \rightarrow X$ by $p_X : M \rightarrow X$. Then the inclusion maps $j : M \rightarrow N$, $\bar{j} : X \rightarrow Y$ form a map from p_X to p_Y .

Epsilon controlled L -groups are functorial with respect to maps and relaxation of control in the following sense.

Proposition 3.10 *Let $F = (f, \bar{f})$ be a map from $p_X : M \rightarrow X$ to $p_Y : N \rightarrow Y$, and suppose that \bar{f} is Lipschitz continuous with Lipschitz constant λ , i.e., there exists a constant $\lambda > 0$ such that*

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \quad (x_1, x_2 \in X).$$

Then F induces a homomorphism

$$F_* : L_n^{\delta, \epsilon}(X, X'; p_X, R) \longrightarrow L_n^{\delta', \epsilon'}(Y, Y'; p_Y, R)$$

if $\delta' \geq \lambda\delta$, $\epsilon' \geq \lambda\epsilon$ and $\bar{f}(X') \subset Y'$. If two maps $F = (f, \bar{f})$ and $G = (g, \bar{g})$ are homotopic through maps $H_t = (h_t, \bar{h}_t)$ such that each \bar{h}_t is Lipschitz continuous with Lipschitz constant λ , $\delta' > \lambda\delta$, $\epsilon' \geq \lambda\epsilon$, and $\bar{h}_t(X') \subset Y'$, then F and G induce the same homomorphism :

$$F_* = G_* : L_n^{\delta, \epsilon}(X, X'; p_X, R) \longrightarrow L_n^{\delta', \epsilon'}(Y, Y'; p_Y, R) .$$

Proof The direct image construction for geometric modules and morphisms [14] (p.7) can be used to define the direct images $f_{\#}(C, \psi)$ of quadratic complexes and the direct images of cobordisms. And this induces the desired F_* .

For the second part, split the homotopy into thin layers to construct small cobordisms. The size of the cobordism may be slightly bigger than the size of the object itself. \square

Remark The above is stated for Lipschitz continuous maps to simplify the statement. For specific $\delta \geq \epsilon$ and $\delta' \geq \epsilon'$, the following condition, instead of

the Lipschitz condition above, is sufficient for the existence of F_* :

$$\begin{aligned} d(\bar{f}(x_1), \bar{f}(x_2)) &\leq k\epsilon' && \text{whenever } d(x_1, x_2) \leq k\epsilon, \text{ and} \\ d(\bar{f}(x_1), \bar{f}(x_2)) &\leq k\delta' && \text{whenever } d(x_1, x_2) \leq k\delta, \end{aligned}$$

for $k = 1, 3, 4, 8$. The second part of the proposition also holds under this condition. When X is compact and $\delta' \geq \epsilon'$ are given, the uniform continuity of \bar{f} implies that this condition is satisfied for sufficiently small pairs $\delta \geq \epsilon$.

4 Epsilon-controlled projective L -groups.

Fix a subset Y of X , and let \mathcal{F} be a family of subsets of X such that $Z \supset Y$ for each $Z \in \mathcal{F}$. In this section we introduce intermediate epsilon-controlled L -groups $L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$, which will appear in the stable-exact sequence of a pair (§5) and also in the Mayer-Vietoris sequence (§7). Roughly speaking, these are defined using “controlled projective quadratic chain complexes” $((C, p), \psi)$ with vanishing ϵ -controlled reduced projective class $[C, p] = 0 \in \tilde{K}_0^{n, \epsilon}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$.

$\tilde{K}_0^{n, \epsilon}(Z; p_Z, R)$ is an abelian group defined as the set of equivalence classes $[C, p]$ of finitely generated ϵ projective chain complexes on p_Z . See [14] for the details. The following is known (3.1 and 3.5 of [14]) :

Proposition 4.1 *If $[C, p] = 0 \in \tilde{K}_0^{n, \epsilon}(Z; p_Z, R)$, then there is an n -dimensional free ϵ chain complex $(E, 1)$ such that $(C, p) \oplus (E, 1)$ is 3ϵ chain equivalent to an n -dimensional free ϵ chain complex on p_Z . If we further assume that $n \geq 1$, then (C, p) itself is 60ϵ chain equivalent to an n -dimensional free 30ϵ chain complex on p_Z .*

All the materials in the previous two sections (except for 3.4(3)) have obvious analogues in the category of projective chain complexes with the identity maps in the formulae replaced by appropriate projections. So we shall only describe the basic definitions and omit stating the obvious analogues of 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 3.3, 3.5, 3.8, 3.9, and we refer them by 2.3', 2.4', An analogue of 3.7 will be explicitly stated in 4.4 below.

For a projective module (A, p) on p_X , its dual $(A, p)^*$ is the projective module (A^*, p^*) on p_X . If $f : (A, p) \rightarrow (B, q)$ is an ϵ morphism [14], then $f^* : (B, q)^* \rightarrow (A, p)^*$ is also an ϵ morphism. For an ϵ projective chain complex on p_X

$$(C, p) : \dots \longrightarrow (C_r, p_r) \xrightarrow{d_r} (C_{r-1}, p_{r-1}) \xrightarrow{d_{r-1}} \dots$$

in the sense of [14], $(C, p)^{n-*}$ will denote the ϵ projective chain complex on p_X defined by:

$$\dots \longrightarrow (C^{n-r}, p_{n-r}^*) \xrightarrow{(-)^r d_r^*} (C^{n-r+1}, p_{n-r+1}^*) \longrightarrow \dots$$

Before we go on to define ϵ projective quadratic complexes, we need to define basic notions for projective chain complexes. For $Y = X$ or for free chain complexes, these are already defined in [14].

Suppose $f : (A, p) \rightarrow (B, q)$ is a morphism between projective modules on p_X , and let Y be a subset of X . The restriction $f|Y$ of f to Y will mean the restriction of f in the sense of [14](p.21) with f viewed as a geometric morphism from A to B ; i.e., $f|Y$ is the sum of the paths (with coefficients) that start from points in $p_X^{-1}(Y)$. $f|Y$ can be viewed as a geometric morphism from A to B and also as a geometric morphism from $A(Y)$ to $B(Y)$, where $A(Y)$ denotes the restriction of A to Y in the sense of [14], i.e. the geometric submodule of A generated by the basis elements of A that are in $p_X^{-1}(Y)$. But, in general, it does not give a morphism from (A, p) to (B, q) . Also note that there is no obvious way to “restrict” a projection $p : A \rightarrow A$ to a projection on $A(Y)$.

The following four paragraphs are almost verbatim copies of the definitions for free chain complexes [14](p.22).

Let $f, g : (A, p) \rightarrow (B, q)$ be morphisms; f is said to be *equal to g over Y* ($f = g$ over Y) if $f|Y = g|Y$, and f is said to be ϵ *homotopic to g over Y* ($f \sim_\epsilon g$ over Y) if $f|Y \sim_\epsilon g|Y$.

Let $f, g : (C, p) \rightarrow (D, q)$ be chain maps between projective chain complexes on p_X . A collection $\{h_r : (C_r, p_r) \rightarrow (D_{r+1}, q_{r+1})\}$ of ϵ morphisms is said to be an ϵ *chain homotopy over Y between f and g* if $dh + hd \sim_{2\epsilon} g - f$ over Y .

An ϵ chain map $f : (C, p) \rightarrow (D, q)$ is said to be an ϵ *chain equivalence over Y* if there exist an ϵ chain map $g : (D, q) \rightarrow (C, p)$ and ϵ chain homotopies over Y between gf and p and between fg and q .

A chain complex (C, p) is said to be ϵ *contractible over Y* if there is an ϵ chain homotopy over Y between $0 : (C, p) \rightarrow (C, p)$ and $p : (C, p) \rightarrow (C, p)$; such a chain homotopy over Y is called an ϵ *chain contraction of (C, p) over Y* .

The definition 2.2 of *weak ϵ chain equivalences over Y* (for chain maps between free chain complexes) can be rewritten for maps between projective chain complexes in the obvious manner.

The following is the most important technical proposition in the theory of controlled projective chain complexes.

Proposition 4.2 (5.1 and 5.2 of [14]) *If an n -dimensional free ϵ chain complex C on p_X is ϵ contractible over $X - Y$, then $(C, 1)$ is $(6n + 15)\epsilon$ chain equivalent to an n -dimensional $(3n + 12)\epsilon$ projective chain complex on $p_{Y(4n+14)\epsilon}$. Conversely, if an n -dimensional free chain complex $(C, 1)$ on p_X is ϵ chain equivalent to a projective chain complex (D, r) on p_Y , then C is ϵ contractible over $X - Y^\epsilon$.*

Now we introduce quadratic structures on projective chain complexes and pairs. An n -dimensional ϵ quadratic structure on a projective chain complex (C, p) on p_X is an n -dimensional ϵ quadratic structure ψ on C (in the sense of §2) such that $\psi_s : (C^{n-r-s}, p^*) \rightarrow (C_r, p)$ is an ϵ morphism for every $s \geq 0$ and $r \in \mathbb{Z}$. Similarly, an $(n + 1)$ -dimensional ϵ quadratic structure on a chain map $f : (C, p) \rightarrow (D, q)$ is an $(n + 1)$ -dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : C \rightarrow D$ such that $\delta\psi_s : (D^{n+1-r-s}, q^*) \rightarrow (D_r, q)$ and $\psi_s : (C^{n-r-s}, p^*) \rightarrow (C_r, p)$ are ϵ morphisms for every $s \geq 0$ and $r \in \mathbb{Z}$. An n -dimensional ϵ projective chain complex (C, p) on p_X equipped with an n -dimensional ϵ quadratic structure is called an n -dimensional ϵ projective quadratic complex on p_X , and an ϵ chain map $f : (C, p) \rightarrow (D, q)$ between an n -dimensional ϵ projective chain complex (C, p) on p_X and an $(n + 1)$ -dimensional ϵ projective chain complex (D, q) on p_X equipped with an $(n + 1)$ -dimensional ϵ quadratic structure is called an $(n + 1)$ -dimensional ϵ projective quadratic pair on p_X .

An ϵ cobordism of n -dimensional ϵ projective quadratic complexes $((C, p), \psi)$, $((C', p'), \psi')$ on p_X is an $(n + 1)$ -dimensional ϵ projective quadratic pair on p_X

$$((f \ f') : (C, p) \oplus (C', p') \longrightarrow (D, q), (\delta\psi, \psi \oplus -\psi'))$$

with boundary $((C, p) \oplus (C', p'), \psi \oplus -\psi')$.

Boundary constructions, algebraic Poincaré thickenings, algebraic Thom complexes, ϵ connectedness are defined as in the previous section.

An n -dimensional ϵ quadratic structure ψ on (C, p) is ϵ Poincaré (over Y) if

$$\partial(C, p) = \Omega\mathcal{C}((1 + T)\psi_0 : (C^{n-*}, p^*) \longrightarrow (C, p))$$

is 4ϵ contractible (over Y). $((C, p), \psi)$ is ϵ Poincaré (over Y) if ψ is ϵ Poincaré (over Y). Similarly, an $(n + 1)$ -dimensional ϵ quadratic structure $(\delta\psi, \psi)$ on $f : (C, p) \rightarrow (D, q)$ is ϵ Poincaré (over Y) if $\partial(C, p)$ and

$$\partial(D, q) = \Omega\mathcal{C}(((1 + T)\delta\psi_0 \quad f(1 + T)\psi_0) : \mathcal{C}(f)^{n+1-*} \longrightarrow (D, q))$$

are both 4ϵ contractible (over Y). A pair $(f, (\delta\psi, \psi))$ is ϵ Poincaré (over Y) if $(\delta\psi, \psi)$ is ϵ Poincaré (over Y).

Let Y and be a subset of X and \mathcal{F} be a family of subsets of X such that $Z \supset Y$ for every $Z \in \mathcal{F}$.

Definition 4.3 Let $n \geq 0$ and $\delta \geq \epsilon \geq 0$. $L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$ is the equivalence classes of finitely generated n -dimensional ϵ Poincaré ϵ projective quadratic complexes $((C, p), \psi)$ on p_Y such that $[C, p] = 0$ in $\tilde{K}_0^{n, \epsilon}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$. The equivalence relation is generated by finitely generated δ Poincaré δ cobordisms $((f \ f') : (C, p) \oplus (C', p') \rightarrow (D, q), (\delta\psi, \psi \oplus -\psi'))$ on p_Y such that $[D, q] = 0$ in $\tilde{K}_0^{n+1, \delta}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$.

Remark We use the following abbreviation: $L_n^{\mathcal{F}, \epsilon}(Y; p_X, R) = L_n^{\mathcal{F}, \epsilon, \epsilon}(Y; p_X, R)$.

Proposition 4.4 Direct sum induces an abelian group structure on $L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$. Furthermore if $[(C, p), \psi] = [(C', p'), \psi'] \in L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$, then there is a finitely generated 100δ Poincaré 2δ cobordism on p_Y

$$((f \ f') : (C, p) \oplus (C', p') \rightarrow (D, q), (\delta\psi, \psi \oplus -\psi'))$$

such that $[D, q] = 0$ in $\tilde{K}_0^{n+1, 9\delta}(Z; p_Z, R)$ for each $Z \in \mathcal{F}$.

Proof The first part is similar to the proof of 3.7. $[D, q] = 0$ in $\tilde{K}_0^{n+1, 9\delta}(Z; p_Z, R)$, because $[\mathcal{C}(g : (E, r) \rightarrow (F, s))] = [F, s] - [E, r] \in \tilde{K}_0^{n+1, 9\delta}(Z; p_Z, R)$ for any δ chain map g between δ projective chain complexes (E, r) (of dimension n) and (F, s) (of dimension $n+1$) on p_Z . See p.18 of [14]. \square

A functoriality with respect to maps and relaxation of control similar to 3.10 holds for epsilon-controlled projective L -groups.

Proposition 4.5 Let $F = (f, \bar{f})$ be a map from $p_X : M \rightarrow X$ to $p_Y : N \rightarrow Y$, and suppose that \bar{f} is Lipschitz continuous with Lipschitz constant λ , i.e., there exists a constant $\lambda > 0$ such that

$$d(\bar{f}(x_1), \bar{f}(x_2)) \leq \lambda d(x_1, x_2) \quad (x_1, x_2 \in X).$$

If $\delta' \geq \lambda\delta$, $\epsilon' \geq \lambda\epsilon$, $\bar{f}(A) \subset B$, and there exists a $Z \in \mathcal{F}$ satisfying $\bar{f}(Z) \subset Z'$ for each $Z' \in \mathcal{F}'$, then F induces a homomorphism

$$F_* : L_n^{\mathcal{F}, \delta, \epsilon}(A; p_X, R) \longrightarrow L_n^{\mathcal{F}', \delta', \epsilon'}(B; p_Y, R).$$

It is λ -Lipschitz-homotopy invariant if $\delta' > \lambda\delta$ in addition.

Remark As in the remark to 3.10, for a specific δ and ϵ , we do not need the full Lipschitz condition to guarantee the existence of F_* .

There is an obvious homomorphism from free L -groups to projective L -groups:

$$\iota : L_n^{\delta, \epsilon}(Y; p_Y, R) \longrightarrow L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R); \quad [C, \psi] \mapsto [(C, 1), \psi] .$$

On the other hand, the controlled K -theoretic condition posed in the definition can be used to construct homomorphisms from projective L -groups to free L -groups :

Proposition 4.6 *There exist a constant $\alpha > 1$ such that the following holds true: for any control map $p_X : M \rightarrow X$, any subset $Y \subset X$, any family of subsets \mathcal{F} of X containing Y , any element Z of \mathcal{F} , any number $n \geq 0$, and any pair of positive numbers $\delta \geq \epsilon$ and $\delta' \geq \epsilon$ with $\delta' \geq \alpha\delta$, $\epsilon' \geq \alpha\epsilon$, there is a well-defined homomorphism functorial with respect to relaxation of control:*

$$\iota_Z : L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R) \longrightarrow L_n^{\delta', \epsilon'}(Z; p_Z, R)$$

such that the following composite maps are equal to the ones induced from inclusion maps:

$$\begin{aligned} L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R) &\xrightarrow{\iota_Z} L_n^{\delta', \epsilon'}(Z; p_Z, R) \xrightarrow{\iota} L_n^{p, \delta', \epsilon'}(Z; p_Z, R) , \\ L_n^{\delta, \epsilon}(Y; p_Y, R) &\xrightarrow{\iota} L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R) \xrightarrow{\iota_Z} L_n^{\delta', \epsilon'}(Z; p_Z, R) . \end{aligned}$$

Remark Actually $\alpha = 20000$ works. In the rest of the paper, we always assume that $\alpha = 20000$.

Proof Let $[(C, p), \psi]$ be an element of $L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$, and fix $Z \in \mathcal{F}$. Recall that $[C, p] = 0 \in \tilde{K}_0^{n, \epsilon}(Z; p_Z, R)$. By 4.1, there exists an n -dimensional free ϵ chain complex $(E, 1)$ on p_Z such that $(C, p) \oplus (E, 1)$ is 3ϵ chain equivalent to some n -dimensional free ϵ chain complex $(\bar{F}, 1)$ on p_Z . Add $1 : (E^{n-*}, 1) \rightarrow (E^{n-*}, 1)$ to this chain equivalence to get a 3ϵ chain equivalence

$$g : (C, p) \oplus (\partial \Sigma E, 1) \longrightarrow (\bar{F}, 1) \oplus (E^{n-*}, 1) = (F, 1)$$

of projective chain complexes on p_Z , where ΣE is defined using the trivial $(n+1)$ -dimensional quadratic structure $\theta = 0$ on ΣE . See 3.1. We set

$$\iota_Z[(C, p), \psi] = [F, g_*(\psi \oplus \partial \theta)] .$$

Let us show that this defines a well-defined map. Suppose $[(C, p), \psi] = [(C', p'), \psi']$ in $L_n^{\mathcal{F}, \delta, \epsilon}(Y; p_X, R)$, and let E and E' be n -dimensional free ϵ chain complexes on p_Z together with 3ϵ chain equivalences

$$\begin{aligned} g : (\bar{C}, \bar{p}) &= (C, p) \oplus (\partial \Sigma E, 1) \rightarrow (F, 1) \\ g' : (\bar{C}', \bar{p}') &= (C', p') \oplus (\partial \Sigma E', 1) \rightarrow (F', 1) \end{aligned}$$

to free ϵ chain complexes F and F' on p_Z . By 4.4 above and 4.1, there is a 100δ Poincaré 2δ null-cobordism

$$(f : (C, p) \oplus (C', p') \longrightarrow (D, q), \quad (\delta\psi, \psi \oplus -\psi'))$$

such that (D, q) is 540δ chain equivalent to an $(n+1)$ -dimensional free 270δ chain complex $(G, 1)$ (as a projective chain complex on p_Z). Take the direct sum with the null-cobordisms

$$\begin{aligned} (i_{\Sigma E} : (\partial\Sigma E, 1) &\longrightarrow (E^{n-*}, 1), \quad (0, \partial\theta)) , \\ (i_{\Sigma E'} : (\partial\Sigma E', 1) &\longrightarrow (E'^{n-*}, 1), \quad (0, -\partial\theta')) . \end{aligned}$$

Now the claim follows from 3.9'.

$$\begin{array}{ccc} (\bar{C}, \bar{p}) \oplus (\bar{C}', \bar{p}') & \xrightarrow{100\delta \text{ Poincaré}} & (D, q) \oplus (E^{n-*}, 1) \oplus (E'^{n-*}, 1) \\ g \oplus g' \downarrow \simeq_{3\epsilon} & & \downarrow \simeq_{540\delta} \\ (F, 1) \oplus (F', 1) & \xrightarrow{20000\delta \text{ Poincaré}} & (G, 1) \oplus (E^{n-*}, 1) \oplus (E'^{n-*}, 1) \end{array}$$

□

5 Stably-exact sequence of a pair.

Let Y be a subset of X . We discuss relations between the various controlled L -groups of X , Y , and (X, Y) by fitting them into a stably-exact sequence. Two of the three kinds of maps forming the sequence have already appeared. The first is the map

$$i_* = \iota_X : L_n^{\{X\}, \delta, \epsilon}(Y; p_X, R) \longrightarrow L_n^{\delta', \epsilon'}(X; p_X, R)$$

defined when $\delta' \geq \alpha\delta$ and $\epsilon' \geq \alpha\epsilon$. The second is the homomorphism induced by the inclusion map $j : (X, \emptyset) \rightarrow (X, Y)$:

$$j_* : L_n^{\delta, \epsilon}(X; p_X, R) \rightarrow L_n^{\delta', \epsilon'}(X, Y; p_X, R).$$

defined for positive numbers $\delta' \geq \delta$ and $\epsilon' \geq \epsilon$. The third map ∂ is described in the next proposition.

Proposition 5.1 *For $n \geq 1$, there exists a constant $\kappa_n > 1$ such that the following holds true: If $Y' \supset Y^{\kappa_n\delta}$, $\delta' \geq \kappa_n\delta$, and $\epsilon' \geq \kappa_n\epsilon$, $\partial([C, \psi]) = [(E, q), \beta_{\%}\partial\psi]$ defines a well-defined homomorphism :*

$$\partial : L_n^{\delta, \epsilon}(X, Y; p_X, R) \rightarrow L_{n-1}^{\{X\}, \delta', \epsilon'}(Y'; p_X, R) ,$$

where

$$\beta : (\partial C, 1) \longrightarrow (E, q)$$

is any $(200n+300)\epsilon$ chain equivalence from $(\partial C, 1)$ to some $(n-1)$ -dimensional $(100n+300)\epsilon$ projective chain complex on $p_{Y'}$.

Remark Actually $\kappa_n = 150000(n+2)$ works. In the rest of the paper, we always assume that $\kappa_n \geq 150000(n+2)$.

Proof We first show the existence of such β . Take $[C, \psi] \in L_n^{\delta, \epsilon}(X, Y; p_X, R)$. Suppose $n > 1$. By 3.4(1), there is a 12ϵ chain equivalence between ∂C and an $(n-1)$ -dimensional 4ϵ chain complex $\hat{\partial}C$ on p_X . Since ∂C is 4ϵ contractible over $X-Y$, $\hat{\partial}C$ is 28ϵ contractible over $X-Y^{12\epsilon}$ by 2.10. Now by 4.2, $(\hat{\partial}C, 1)$ is $(168n+252)\epsilon$ ($= (6(n-1)+15) \times 28\epsilon$) chain equivalent to an $(n-1)$ -dimensional $(84n+252)\epsilon$ projective chain complex on $p_{Y^{(112n+292)\epsilon}}$.

Next suppose $n = 1$. By 3.4(1) and (3), there is a 44ϵ chain equivalence between $(\partial C, 1)$ and a 0-dimensional 32ϵ chain complex $(\tilde{\partial}C, p)$. Since ∂C is 4ϵ contractible over $X-Y$, $(\tilde{\partial}C, p)$ is 92ϵ contractible over $X-Y^{44\epsilon}$, i.e., $p \sim_{184\epsilon} 0$ over $X-Y^{44\epsilon}$. Let $E = \tilde{\partial}C|Y^{76\epsilon}$ and $q = p|Y^{44\epsilon}$, then $p - q = p|(X-Y^{44\epsilon}) \sim_{184\epsilon} 0$. Therefore

$$q \sim_{184\epsilon} p \sim_{32\epsilon} p^2 \sim_{216\epsilon} pq \sim_{216\epsilon} q^2,$$

and (E, q) is a 0-dimensional 216ϵ projective chain complex on $p_{Y^{292\epsilon}}$. The 32ϵ morphism q defines a 216ϵ isomorphism between $(\tilde{\partial}C, p)$ and (E, q) in each direction. Therefore $(\partial C, 1)$ is 260ϵ chain equivalent to (E, q) . This completes the proof of the existence of β .

Suppose $[C, \psi] = [C', \psi'] \in L_n^{\delta, \epsilon}(X, Y; p_X, R)$ and suppose $\beta : (\partial C, 1) \rightarrow (E, q)$ and $\beta' : (\partial C', 1) \rightarrow (E', q')$ are chain equivalences satisfying the condition, and suppose Y' , δ' , and ϵ' satisfy the hypothesis. We show that $((E, q), \beta_{\%} \partial \psi)$ and $((E', q'), \beta'_{\%} \partial \psi')$ represent the same element in $L_{n-1}^{\{X\}, \delta', \epsilon'}(Y'; p_X, R)$. Without loss of generality, we may assume that there is an ϵ connected ϵ cobordism

$$((f \quad f') : C \oplus C' \longrightarrow D, (\partial \psi, \psi \oplus -\psi'))$$

which is ϵ Poincaré over $X-Y$. Apply the boundary construction (§3) to this pair to get a 3ϵ Poincaré 2ϵ quadratic structure $(\partial \delta \psi, \partial \psi \oplus -\partial \psi')$ on the 2ϵ chain map $(\partial C, 1) \oplus (\partial C', 1) \longrightarrow (\partial D, 1)$ of 2ϵ chain complexes. By 2.10', 3.4 and 4.2, $(\partial D, 1)$ is $(312n+904)\epsilon$ chain equivalent to an n -dimensional $(156n+624)\epsilon$ projective chain complex (F, r) on $p_{Y^{(208n+752)\epsilon}}$. Now, by 3.9', we can obtain a $(15360n+36210)\epsilon$ Poincaré cobordism

$$(E, q) \oplus (E', q') \longrightarrow (F, r), \quad (\chi, \beta_{\%} \partial \psi \oplus (-\beta'_{\%} \partial \psi')) .$$

Since such a structure involves $8(15360n + 36210)\epsilon$ homotopies, this cobordism can be viewed to be on $p_{Y(123088n+290432)\epsilon}$. Also $[F, r] = [\hat{\partial}D, 1] = 0$ in $\tilde{K}_0^{n, \epsilon'}(X; p_X, R)$, and similarly $[E, q] = [E', q'] = 0$ in $\tilde{K}_0^{n-1, \epsilon'}(X; p_X, R)$. Therefore $[(E, q), \beta_{\%}\partial\psi] = [(E', q'), \beta'_{\%}\partial\psi']$ in $L_{n-1}^{\{X\}, \delta', \epsilon'}(Y'; p_X, R)$.

□

Theorem 5.2 *For any integer $n \geq 0$, there exists a constant $\lambda_n > 1$ which depends only on n such that the following holds true for any control map p_X and a subset Y of X :*

(1) Suppose $\delta' \geq \alpha\delta$, $\epsilon' \geq \alpha\epsilon$, $\delta'' \geq \delta'$, and $\epsilon'' \geq \epsilon'$ so that the following two maps are defined:

$$L_n^{\{X\}, \delta, \epsilon}(Y; p_X, R) \xrightarrow{i_*} L_n^{\delta', \epsilon'}(X; p_X, R) \xrightarrow{j_*} L_n^{\delta'', \epsilon''}(X, W; p_X, R).$$

If $W \supset Y^{\alpha\epsilon}$, then j_*i_* is zero.

(2) Suppose $\delta'' \geq \delta'$, $\epsilon'' \geq \epsilon'$ so that $j_* : L_n^{\delta', \epsilon'}(X; p_X, R) \rightarrow L_n^{\delta'', \epsilon''}(X, W; p_X, R)$ is defined. If $\delta \geq \lambda_n\delta''$ and $Y \supset W^{\lambda_n\delta''}$, then the relax-control image of the kernel of j_* in $L_n^{\alpha\delta}(X; p_X, R)$ is contained in the image of i_* below:

$$\begin{array}{ccc} L_n^{\delta', \epsilon'}(X; p_X, R) & \xrightarrow{j_*} & L_n^{\delta'', \epsilon''}(X, W; p_X, R) \\ \downarrow & & \\ L_n^{\{X\}, \delta}(Y; p_X, R) & \xrightarrow{i_*} & L_n^{\alpha\delta}(X; p_X, R) \end{array}$$

(3) Suppose $n \geq 1$, $\delta' \geq \delta$, $\epsilon' \geq \epsilon$, $W \supset Y^{\kappa_n\delta'}$, $\delta'' \geq \kappa_n\delta'$, and $\epsilon'' \geq \kappa_n\epsilon'$ so that the following two maps are defined :

$$L_n^{\delta, \epsilon}(X; p_X, R) \xrightarrow{j_*} L_n^{\delta', \epsilon'}(X, Y; p_X, R) \xrightarrow{\partial} L_{n-1}^{\{X\}, \delta'', \epsilon''}(W; p_X, R).$$

Then ∂j_* is zero.

(4) Suppose $n \geq 1$, $W \supset Y^{\kappa_n\delta'}$, $\delta'' \geq \kappa_n\delta'$, and $\epsilon'' \geq \kappa_n\epsilon'$ so that the map $\partial : L_n^{\delta', \epsilon'}(X, Y; p_X, R) \rightarrow L_{n-1}^{\{X\}, \delta'', \epsilon''}(W; p_X, R)$ is defined. If $\delta \geq \lambda_n\delta''$ and $Y' \supset W^{\lambda_n\delta''}$, then the relax-control image of the kernel of ∂ in $L_n^{\delta}(X, Y'; p_X, R)$ is contained in the image of j_* below :

$$\begin{array}{ccc} L_n^{\delta', \epsilon'}(X, Y; p_X, R) & \xrightarrow{\partial} & L_{n-1}^{\{X\}, \delta'', \epsilon''}(W; p_X, R) \\ \downarrow & & \\ L_n^{\delta}(X; p_X, R) & \xrightarrow{j_*} & L_n^{\delta}(X, Y'; p_X, R) \end{array}$$

(5) Suppose $n \geq 1$, $Y' \supset Y^{\kappa_n \delta}$, $\delta' \geq \kappa_n \delta$, $\epsilon' \geq \kappa_n \epsilon$, $\delta'' \geq \alpha \delta'$, and $\epsilon'' \geq \alpha \epsilon'$ so that the following two maps are defined :

$$L_n^{\delta, \epsilon}(X, Y; p_X, R) \xrightarrow{\partial} L_{n-1}^{\{X\}, \delta', \epsilon'}(Y'; p_X, R) \xrightarrow{i_*} L_{n-1}^{\delta'', \epsilon''}(X; p_X, R).$$

Then $i_* \partial$ is zero.

(6) Suppose $n \geq 1$, $\delta'' \geq \alpha \delta'$, and $\epsilon'' \geq \alpha \epsilon'$ so that $i_* : L_{n-1}^{\{X\}, \delta', \epsilon'}(Y; p_X, R) \rightarrow L_{n-1}^{\delta'', \epsilon''}(X; p_X, R)$ is defined. If $\delta \geq \lambda_n \delta''$ and $W \supset Y^{\lambda_n \delta''}$, then the relax-control image of the kernel of i_* in $L_{n-1}^{\{X\}, \kappa_n \delta}(W^{\kappa_n \delta}; p_X, R)$ is contained in the image of ∂ below :

$$\begin{array}{ccc} L_{n-1}^{\{X\}, \delta', \epsilon'}(Y; p_X, R) & \xrightarrow{i_*} & L_{n-1}^{\delta'', \epsilon''}(X; p_X, R) \\ \downarrow & & \\ L_n^{\delta}(X, W; p_X, R) & \xrightarrow{\partial} & L_{n-1}^{\{X\}, \kappa_n \delta}(W^{\kappa_n \delta}; p_X, R) \end{array}$$

Proof (1) Let $[(C, p), \psi] \in L_n^{\{X\}, \delta, \epsilon}(Y; p_X, R)$. There is a 3ϵ chain equivalence $g : (C, p) \oplus (\partial \Sigma E) \rightarrow (F, 1)$ for some n -dimensional free ϵ chain complexes E and F on p_X , and $j_* i_* [(C, p), \psi] \in L_n^{\delta'', \epsilon''}(X, W; p_X, R)$ is represented by $(F, g_{\%}(\psi \oplus \partial \theta))$, where θ is the trivial quadratic structure on ΣE . Take the sum of

$$(0 : (C, p) \rightarrow 0, (0, \psi)), \quad \text{and} \quad (i_{\Sigma E} : (\partial \Sigma E, 1) \rightarrow (E^{n-*}, 1), (0, \partial \theta)).$$

$(0, \psi \oplus \partial \theta)$ is a 2ϵ connected 2ϵ quadratic structure, and it is 2ϵ Poincaré over $X - Y$. Use the chain equivalence g and 3.9' to get a 180ϵ connected 24ϵ null-cobordism

$$(F \longrightarrow E^{n-*}, (\chi, g_{\%}(\psi \oplus \partial \theta)))$$

that is 180ϵ Poincaré over $X - Y^{486\epsilon}$.

(2) Let $[C, \psi] \in L_n^{\delta', \epsilon'}(X; p_X, R)$ and assume $j_* [C, \psi] = 0 \in L_n^{\delta'', \epsilon''}(X, W; p_X, R)$. By 3.7, there is a $100\delta''$ connected $2\delta''$ null-cobordism

$$(f : C \rightarrow D, (\delta \psi, \psi))$$

that is $100\delta''$ Poincaré over $X - W^{100\delta''}$. Apply the boundary construction to this null-cobordism to get a $4\delta''$ chain map ∂f of $4\delta''$ chain complexes and an n -dimensional $6\delta''$ Poincaré $6\delta''$ quadratic structure on it:

$$\partial f : \partial C \rightarrow \partial D, \quad \Psi_3 = (\partial \delta \psi, \partial \psi).$$

$(\partial C, \partial \psi)$ also appears as the boundaries of

- an n -dimensional $2\epsilon'$ Poincaré $2\epsilon'$ quadratic structure $\Psi_1 = (0, \partial\psi)$ on the ϵ' chain map $i_C : \partial C \rightarrow C^{n-*}$, and
- an n -dimensional ϵ' quadratic structure $\Psi_2 = (0, \partial\psi)$ on the 0 chain map $0 : \partial C \rightarrow 0$, which is ϵ' Poincaré because ∂C is $4\epsilon'$ contractible.

The union $\Psi_2 \cup_{\partial C} \Psi_3$ is a $600\delta''$ Poincaré $7\delta''$ quadratic structure on $0 \cup_{\partial C} \partial D = \mathcal{C}(\partial f)$. By 3.4(2), there is a $2400\delta''$ chain equivalence between ∂D and an n -dimensional $500\delta''$ chain complex $\hat{\partial}D$. This chain equivalence, together with the $4\epsilon'$ chain contraction of ∂C , induces a $43200\delta''$ chain equivalence $g : 0 \cup_{\partial C} \partial D \rightarrow \hat{\partial}D$. Define a $43200\delta''$ Poincaré $3 \cdot 43200\delta''$ quadratic structure $\hat{\psi}$ on $\hat{\partial}D$ by $g_*(\Psi_2 \cup_{\partial C} \Psi_3)$. By 2.9, there is a $43200\delta''$ Poincaré $3 \cdot 43200\delta''$ quadratic structure on a $43200\delta''$ chain map

$$(0 \cup_{\partial C} \partial D) \oplus \hat{\partial}D \longrightarrow \hat{\partial}D,$$

and, therefore, the right square in the picture below is filled with a cobordism.

$$\begin{array}{ccccc}
 & \Psi_1 & \partial\psi & \Psi_3 & \\
 & C^{n-*} & \partial C & \partial D & \\
 \vdots & & \bullet & & \vdots \\
 & C & 0 : \Psi_2 & \hat{\partial}D & \\
 \vdots & & & & \vdots \\
 & C, \psi & & \hat{\partial}D, \hat{\psi} &
 \end{array}$$

The left square can also be filled in with a cobordism. There is a $3\epsilon'$ homotopy equivalence:

$$(C^{n-*} \cup_{\partial C} 0 = \mathcal{C}(i_C), \Psi_1 \cup_{\partial C} \Psi_2) \longrightarrow (C, \psi),$$

and again by 2.9, this induces a $30\epsilon'$ Poincaré $9\epsilon'$ quadratic structure on a $3\epsilon'$ chain map

$$(C^{n-*} \cup_{\partial C} 0) \oplus C \longrightarrow C.$$

Glue these along the pair $(\partial C \rightarrow 0, \Psi_2)$, and we get a chain map

$$(C^{n-*} \cup_{\partial C} \partial D) \oplus C \oplus \hat{\partial}D \longrightarrow C \oplus \hat{\partial}D$$

and a $43200000\delta''$ Poincaré $6 \cdot 43200\delta''$ quadratic structure on it. Since ∂C is $4\epsilon'$ contractible and ∂D is $2400\delta''$ chain equivalent to $\hat{\partial}D$, there is a $43200\delta''$ chain equivalence

$$G : C^{n-*} \cup_{\partial C} \partial D \longrightarrow E = C^{n-*} \oplus \hat{\partial}D,$$

and hence, by 3.9, there is a $30 \cdot 43300000\delta''$ Poincaré $4 \cdot 43300000\delta''$ null-cobordism of $(E, G_{\%}(\Psi_1 \cup_{\partial\psi} - \Psi_3)) \oplus (C, -\psi) \oplus (\hat{\partial}D, -\hat{\psi})$. Therefore

$$[C, \psi] + [\hat{\partial}D, \hat{\psi}] = [E, G_{\%}(\Psi_1 \cup_{\partial\psi} - \Psi_3)]$$

in $L_n^{13 \cdot 10^8 \delta''}(X; p_X, R)$.

On the other hand, there is a $600\delta''$ Poincaré null-cobordism of $\Psi_1 \cup_{\partial\psi} - \Psi_3$ on the chain map

$$C^{n-*} \cup_{\partial C} \partial D \longrightarrow \mathcal{C}(f)^{n+1-*}.$$

Using G and 3.9, we obtain a $30(600 + 43200 + 4)\delta''$ Poincaré null-cobordism

$$(E \rightarrow \mathcal{C}(f)^{n+1-*}, \quad (\chi, G_{\%}(\Psi_1 \cup_{\partial\psi} - \Psi_3))),$$

and this implies

$$[E, G_{\%}(\Psi_1 \cup_{\partial\psi} - \Psi_3)] = 0 \in L_n^{13 \cdot 10^8 \delta''}(X; p_X, R)$$

and hence

$$[C, \psi] = -[\hat{\partial}D, \hat{\psi}] \in L_n^{13 \cdot 10^8 \delta''}(X; p_X, R).$$

Since ∂D is $400\epsilon''$ contractible over $X - W^{100\epsilon''}$ and $\hat{\partial}D$ is $2400\delta''$ chain equivalent to ∂D , $\hat{\partial}D$ is $5200\delta''$ contractible over $X - W^{2500\delta}$, by 2.10. By 4.2, there is a $(6n + 15) \cdot 5200\delta''$ chain equivalence h from $(\hat{\partial}D, 1)$ to an n -dimensional $(3n + 12) \cdot 5200\delta''$ projective chain complex (F, p) on $p_{W^{(20800n+75300)\delta''}}$. Suppose $\lambda_n \geq 10^5(4n + 50)$. If $\delta \geq \lambda_n\delta''$ and $Y \supset W^{\lambda_n\delta''}$, then $((F, p), h_{\%}(\hat{\psi}))$ represents an element of $L_n^{\{X\}, \delta}(Y; p_Y, R)$ by 2.9, and its image

$$i_*([(F, p), h_{\%}(\hat{\psi})]) \in L_n^{\alpha\delta}(X; p_X, R)$$

is represented by $(\hat{\partial}D, (h^{-1})_{\%}(h_{\%}(\hat{\psi})) = (h^{-1}h)_{\%}(\hat{\psi}))$. Since $h^{-1}h$ is 2δ chain homotopic to the identity map,

$$[\hat{\partial}D, \hat{\psi}] = [\hat{\partial}D, (h^{-1}h)_{\%}(\hat{\psi})] \in L_n^{\alpha\delta}(X; p_X, R).$$

Since $\alpha\delta \geq 13 \cdot 10^8 \delta''$, we have

$$i_*(-[(F, p), g_{\%}(\hat{\psi})]) = [C, \psi] \in L_n^{\alpha\delta}(X; p_X, R).$$

(3) Let $[C, \psi] \in L_n^{\delta, \epsilon}(X; p_X, R)$, then ∂C is 4ϵ contractible. Thus $(\partial C, 1)$ is 4ϵ chain equivalent to $(E = 0, q = 0)$.

(4) Let $[C, \psi] \in L_n^{\delta', \epsilon'}(X, Y; p_X, R)$ and suppose $\partial[C, \psi] = 0$ in $L_{n-1}^{\{X\}, \delta'', \epsilon''}(W; p_X, R)$. Let $\beta : (\partial C, 1) \rightarrow (E, q)$ be a $(200n + 300)\epsilon'$ chain equivalence to an $(n - 1)$ -dimensional $(100n + 300)\epsilon'$ projective chain complex on p_W posited in the definition of ∂ . By assumption, $[(E, q), \beta_{\%} \partial \psi] = 0$ in $L_{n-1}^{\{X\}, \delta'', \epsilon''}(W; p_X, R)$. By 4.4, there is an $(n - 1)$ -dimensional $100\delta''$ Poincaré $2\delta''$ null-cobordism on p_W

$$(f' : (E, q) \longrightarrow (D, p), \quad (\delta\psi', \beta_{\%} \partial \psi))$$

such that $[D, p] = 0$ in $\tilde{K}_0^{n, 9\delta''}(X; p_X, R)$. By 2.10', $(\delta\psi', \partial\psi)$ is a $125\delta''$ Poincaré $2\delta''$ quadratic structure on the $3\delta''$ chain map

$$f = f' \circ \beta : (\partial C, 1) \longrightarrow (D, p).$$

On the other hand, $(0, \partial\psi)$ is a $2\epsilon'$ Poincaré $2\epsilon'$ quadratic structure on the ϵ' chain map

$$i_C : (\partial C, 1) \longrightarrow (C^{n-*}, 1).$$

Gluing these together, we obtain a $12500\delta''$ Poincaré $4\delta''$ quadratic structure

$$\psi' = (0, \partial\psi) \cup_{\partial\psi} -(\delta\psi', \partial\psi)$$

on $(C', p') = (C^{n-*}, 1) \cup_{(\partial C, 1)} (D, p)$. Since $n \geq 1$, (D, p) is $540\delta''$ chain equivalent to an n -dimensional free $270\delta''$ chain complex $(F, 1)$ on p_X by 4.1.

Assume $n \geq 2$. In this case ∂C is $12\epsilon'$ chain equivalent to an $(n - 1)$ -dimensional $4\epsilon'$ chain complex $\hat{\partial}C$, by 3.4. Using these chain equivalences and 2.6, we can construct a $6528\delta''$ chain equivalence

$$\gamma : (C', p') \longrightarrow (C'' = C^{n-*} \cup_{\hat{\partial}C} F, 1).$$

If $\delta \geq 9 \cdot 10^5 \delta''$, then $(C'', \psi'' = \gamma_{\%} \psi')$ determines an element of $L_n^{\delta}(X; p_X, R)$. Suppose $\delta \geq 4 \cdot 10^6 \delta''$ and $Y' \supset W^{12 \cdot 10^6 \delta''}$. We shall show that its image by j_* is equal to the relax-control image of $[C, \psi]$ in $L_n^{\delta}(X, Y'; p_X, R)$.

Since (D, p) lies over W , it is 0 contractible over $X - W$. Therefore, by 2.6, the chain map $G : (C', p') \rightarrow (C^{n-*} \cup_{\partial C} 0, 1)$ defined by

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : (C^{n-r}, 1) \oplus (\partial C_{r-1}, 1) \oplus (D_r, p) \longrightarrow (C^{n-r}, 1) \oplus (\partial C_{r-1}, 1)$$

is a $18\delta''$ chain equivalence over $X - W^{6\delta''}$. Furthermore one can easily check that G is 0 connected and that $G_{\%}(\psi') = (0, \partial\psi) \cup_{\partial\psi} - (0, \partial\psi)$. Compose G with a $3\epsilon'$ homotopy equivalence

$$((C^{n-*} \cup_{\partial C} 0, 1), (0, \partial\psi) \cup_{\partial\psi} - (0, \partial\psi)) \longrightarrow ((C, 1), \psi)$$

to get a $3\epsilon'$ connected $19\delta''$ chain equivalence over $X - W^{7\delta''}$:

$$H : (C', p') \longrightarrow (C, 1); \quad H_{\%}(\psi') = \psi.$$

By 2.9, there is a $3 \cdot 12500\delta''$ connected $3 \cdot 19\delta''$ quadratic structure $(0, \psi' \oplus -\psi)$ on a chain map $(C', p') \oplus (C, 1) \rightarrow (C, 1)$ that is $125000\delta''$ Poincaré over $X - W^{375007\delta''}$. Now use the $6528\delta''$ chain equivalence $\gamma : (C', p') \rightarrow (C'', 1)$ and 3.9' to this cobordism to obtain an $(n+1)$ -dimensional δ cobordism between (C'', ψ'') and (C, ψ) that is δ connected and δ Poincaré over $X - Y'$.

In the $n = 1$ case, use the non-positive chain complex obtained from ∂C by applying the folding argument from top instead of $\hat{\partial}C$. See the proof of 3.4(1).

(5) Let $[C, \psi] \in L_n^{\delta, \epsilon}(X, Y; p_X, R)$ and let $\beta : (\partial C, 1) \rightarrow (E, q)$ be as in the definition of ∂ ; $\partial[C, \psi]$ is given by $[(E, q), \beta_{\%}\partial\psi]$. There exist $(n-1)$ -dimensional free ϵ' chain complexes E', F on p_X and a $3\epsilon'$ chain equivalence

$$g : (E, q) \oplus (\partial\Sigma E', 1) \longrightarrow (F, 1)$$

with $\Sigma E'$ given the trivial quadratic structure θ , and $i_*[(E, q), \beta_{\%}\partial\psi]$ is represented by $(F, g_{\%}((\beta_{\%}\partial\psi) \oplus \partial\theta))$. We construct a δ'' Poincaré null-cobordism of this.

Take the direct sum of the algebraic Poincaré thickenings of (C, ψ) and $(\Sigma E', \theta)$ to get an ϵ' Poincaré pair

$$(\partial C \oplus \partial\Sigma E' \longrightarrow C^{n-*} \oplus E'^{n-1-*}, \quad (0 \oplus 0, \partial\psi \oplus \partial\theta)).$$

Apply the $4\epsilon'$ chain equivalence

$$\partial C \oplus \partial\Sigma E' = (\partial C, 1) \oplus (\partial\Sigma E', 1) \xrightarrow{\beta \oplus 1} (E, q) \oplus (\partial\Sigma E', 1) \xrightarrow{g} (F, 1) = F,$$

to this pair, to obtain an ϵ'' Poincaré null-cobordism of $(F, g_{\%}((\beta_{\%}\partial\psi) \oplus \partial\theta))$. (If $n \geq 2$, then we may assume $E' = 0$, and the proof can be much simplified.)

(6) Take an element $[(C, p), \psi] \in L_{n-1}^{\{X\}, \delta', \epsilon'}(Y; p_X, R)$ and assume $i_*[(C, p), \psi] = 0$ in $L_{n-1}^{\delta'', \epsilon''}(X; p_X, R)$. By definition of i_* , there exist $(n-1)$ -dimensional free ϵ' chain complexes E, F on p_X and a $3\epsilon'$ chain equivalence $g : (C, p) \oplus (\partial\Sigma E, 1) \rightarrow (F, 1)$ such that $i_*[(C, p), \psi] = [F, g_{\%}(\psi \oplus \partial\theta)]$. Here θ is the trivial quadratic structure on ΣE . By 3.7, there is an n -dimensional $100\delta''$ Poincaré $2\delta''$ null-cobordism on p_X of $(F, g_{\%}(\psi \oplus \partial\theta))$:

$$(f : F \longrightarrow D, \quad (\delta\psi, g_{\%}(\psi \oplus \partial\theta))).$$

By 2.10', we obtain a $127\delta''$ Poincaré $3\delta''$ null-cobordism:

$$(f \circ g : (C, p) \oplus (\partial\Sigma E, 1) \longrightarrow (D, 1), \quad (\delta\psi, \psi \oplus \partial\theta)).$$

Take the union of this with the 0 connected ϵ' projective quadratic pair

$$((C, p) \longrightarrow 0, \quad (0, -\psi)),$$

which is 0 Poincaré over $X - Y$, and the $3\epsilon'$ Poincaré $3\epsilon'$ quadratic pair

$$(i_{\Sigma E} : (\partial \Sigma E, 1) \longrightarrow (E^{n-*}, 1), \quad (0, -\partial \theta))$$

to get a $6\delta''$ projective quadratic complex $((\hat{C}, \hat{r}), \hat{\psi})$ which is $12700\delta''$ Poincaré over $X - Y^{12700\delta''}$ and is $12700\delta''$ connected.

The $3\epsilon'$ chain equivalence g induces a $48\delta''$ chain equivalence $\tilde{g} : (\hat{C}, \hat{r}) \rightarrow (\tilde{C}, 1)$ to an n -dimensional free chain complex $(\tilde{C}, 1) = (D, 1) \cup_{(F, 1)} (E^{n-*}, 1)$, and the $18\delta''$ quadratic structure $\tilde{\psi} = \tilde{g}_{\%}(\hat{\psi})$ is $2 \cdot 10^5\delta''$ Poincaré over $X - Y^{4 \cdot 10^5\delta''}$ and is $2 \cdot 10^5\delta''$ connected. Suppose $W \supset Y^{10^6\delta''}$ and $\delta \geq 10^6\delta''$. Then $(\tilde{C}, \tilde{\psi})$ defines an element in $L_n^\delta(X, W; p_X, R)$.

We shall show that $\partial[\tilde{C}, \tilde{\psi}] = [(C, p), \psi]$ in $L_{n-1}^{\{X\}, \kappa_n\delta}(W^{\kappa_n\delta}; p_X, R)$. By the definition of ∂ , there is a $(200n + 300)\delta$ chain equivalence $\beta : (\partial \tilde{C}, 1) \rightarrow (\tilde{E}, \tilde{q})$ to an $(n-1)$ -dimensional $(100n + 300)\delta$ projective chain complex on $p_{W^{(150n+300)\delta}}$, and $\partial[\tilde{C}, \tilde{\psi}]$ is represented by $((\tilde{E}, \tilde{q}), \beta_{\%}\partial\tilde{\psi})$. We construct a cobordism between $((C, p), \psi)$ and $((\tilde{E}, \tilde{q}), \beta_{\%}\partial\tilde{\psi})$.

By 2.9', \tilde{g} induces an $(n+1)$ -dimensional $3 \cdot 48\delta''$ cobordism:

$$((\tilde{g} \ 1) : (\hat{C}, \hat{r}) \oplus (\tilde{C}, 1) \longrightarrow (\tilde{C}, 1), \quad \tilde{\Psi} = (0, \hat{\psi} \oplus -\tilde{\psi})).$$

Let us apply the boundary construction to this to get a $6 \cdot 48\delta''$ chain map

$$\partial(\tilde{g} \ 1) : \partial(\hat{C}, \hat{r}) \oplus \partial(\tilde{C}, 1) \longrightarrow (G, q)$$

and a $9 \cdot 48\delta''$ Poincaré $6 \cdot 48\delta''$ quadratic structure $(\chi, \partial\hat{\psi} \oplus -\partial\tilde{\psi})$ on it. We modify this to get the desired cobordism.

Firstly, note that $((\hat{C}, \hat{r}), \hat{\psi})$ is the algebraic Thom complex of a $12700\delta''$ Poincaré $6\delta''$ quadratic pair with boundary equal to $((C, p), \psi)$. Therefore there is a $11 \cdot 12700\delta''$ chain equivalence $\gamma : \partial(\hat{C}, \hat{r}) \rightarrow (C, p)$ such that $\gamma_{\%}(\partial\hat{\psi}) = \psi$.

Secondly, there is a chain equivalence $\beta : (\partial \tilde{C}, 1) \rightarrow (\tilde{E}, \tilde{q})$ as noted above.

Thirdly, recall that (G, q) is equal $\Omega\mathcal{C}(\mathcal{D}_{\tilde{\Psi}})$ and $\partial(\tilde{C}, 1)$ is equal to $\Omega\mathcal{C}(\mathcal{D}_{\tilde{\psi}})$, and note that there is a $96\delta''$ chain equivalence

$$\mathcal{C}((\tilde{g} \ 1))^{n+1-*} \xrightarrow{(0 \ 1 \ -\tilde{g}^*)} (\hat{C}, \hat{r}) \xrightarrow{(\tilde{g}^{-1})^*} (\tilde{C}, 1)$$

and that it induces a 6δ chain equivalence from (G, q) to $\partial(\tilde{C}, 1)$. Compose this with β to get a $(200n + 306)\delta$ chain equivalence $\beta' : (G, q) \rightarrow (\tilde{E}, \tilde{q})$.

Now, by 3.9', one can conclude that the chain equivalences γ , β , β' induce an n -dimensional $\kappa_n \delta$ Poincaré cobordism on $p_{W^{\kappa_n \delta}}$:

$$((C, p) \oplus (\tilde{E}, \tilde{q}) \longrightarrow (\tilde{E}, \tilde{q}), \quad (\chi, \psi \oplus -\beta_{\%}(\partial \tilde{\psi}))).$$

Since $[C, p] = 0$ in $\tilde{K}_0^{n, \epsilon'}(X; p_X, R)$ and $[\tilde{E}, \tilde{q}] = [\partial \tilde{C}, 1] = 0$ in $\tilde{K}_0^{n, \kappa_n \delta}(X; p_X, R)$, this implies that $[(C, p), \psi] = \partial[\tilde{C}, \tilde{\psi}]$ in $L_{n-1}^{\{X\}, \kappa_n \delta}(W^{\kappa_n \delta}; p_X, R)$.

□

6 Excision.

In this section we study the excision property of epsilon-controlled L -theory. Suppose that X is the union of two closed subsets A and B with intersection $M = A \cap B$. There is an inclusion-induced homomorphism

$$i_* : L_n^{\delta, \epsilon}(A, M; p_A, R) \rightarrow L_n^{\delta, \epsilon}(X, B; p_X, R).$$

For $n \geq 1$, we construct its stable inverse

$$\text{exc} : L_n^{\delta, \epsilon}(X, B; p_X, R) \rightarrow L_n^{\delta, \epsilon}(A, A \cap M^{(n+5)4\delta}; p_A, R).$$

First we define geometric subcomplexes and quotient complexes of free chain complexes. Let C be a free chain complex on p_X . When each C_r is the direct sum $C_r = C'_r \oplus C''_r$ of two geometric submodules and d_C is of the form

$$\begin{pmatrix} d_{C'} & * \\ 0 & d_{C''} \end{pmatrix} : C'_r \oplus C''_r \longrightarrow C'_{r-1} \oplus C''_{r-1},$$

C' is said to be a *geometric subcomplex* of C , and C'' (together with $d_{C''}$) is said to be the *quotient* of C by C' and is denoted C/C' . If C is a free ϵ chain complex, then any geometric subcomplex C' and the quotient C/C' are both free ϵ chain complexes. The obvious projection map $p : C \rightarrow C/C'$ is 0 connected.

Next suppose we are given an n -dimensional ϵ quadratic complex (C, ψ) on p_X and C' is a geometric subcomplex of C . The projection $p : C \rightarrow C/C'$ induces an n -dimensional ϵ quadratic complex $(C/C', p_{\%}\psi)$ and there is an ϵ cobordism between (C, ψ) and $(C/C', p_{\%}\psi)$. For a morphism $g : G \rightarrow H$ between geometric modules and geometric submodules $G' \subset G$ and $H' \subset H$, we write $g(G') \subset H'$ when every path with non-zero coefficient in g starting from a generator of G' ends at a generator of H' .

Proposition 6.1 Let (C, ψ) , C' , and p be as above, and suppose (C, ψ) is ϵ connected. If $\mathcal{D}_\psi(C'^n) \subset C'_0$, then $(C/C', p\% \psi)$ and the cobordism between (C, ψ) and $(C/C', p\% \psi)$ induced by p are both ϵ connected.

Proof Let us write $C'' = C/C'$. By assumption, the morphism $d_{\mathcal{C}(\mathcal{D}_\psi)} : \mathcal{C}(\mathcal{D}_\psi)_1 \rightarrow \mathcal{C}(\mathcal{D}_\psi)_0$ can be expressed by a matrix of the form

$$\begin{pmatrix} d_{C'} & * & * & * \\ 0 & d_{C''} & 0 & \mathcal{D}_{p\% \psi} \end{pmatrix} : C'_1 \oplus C''_1 \oplus C'^n \oplus C''^n \rightarrow C'_0 \oplus C''_0.$$

Let $h : C_0 = \mathcal{C}(\mathcal{D}_\psi)_0 \rightarrow \mathcal{C}(\mathcal{D}_\psi)_1$ be a 4ϵ morphism such that $d_{\mathcal{C}(\mathcal{D}_\psi)} h \sim_{8\epsilon} 1_{C_0}$, and define 4ϵ morphisms $h_1 : C''_0 \rightarrow C''_1$ and $h_2 : C''_0 \rightarrow C''^n$ by

$$h = \begin{pmatrix} * & * \\ * & h_1 \\ * & * \\ * & h_2 \end{pmatrix} : C'_0 \oplus C''_0 \rightarrow C'_1 \oplus C''_1 \oplus C'^n \oplus C''^n.$$

Then we get a homotopy

$$d_{C''} h_1 + \mathcal{D}_{p\% \psi} h_2 \sim_{8\epsilon} 1_{C''_0}.$$

Therefore

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} : C''_0 \longrightarrow C''_1 \oplus C''^n$$

gives a desired splitting of the boundary morphism $\mathcal{C}(\mathcal{D}_{p\% \psi})_1 \rightarrow \mathcal{C}(\mathcal{D}_{p\% \psi})_0$. Therefore $(C'', p\% \psi)$ is ϵ connected. Now the ϵ connectivity of cobordism induced by p follows from 3.3. \square

Example 6.2 Let (C, ψ) be an n -dimensional ϵ quadratic complex on p_X and Y be a subset of X . Fix $\delta(> 0)$ and $l(\geq 0)$, and define a geometric submodule C'_r of C_r to be the restriction $C_r(Y^{(n+l-r)\delta})$ of C_r to $Y^{(n+l-r)\delta}$. If $\delta \geq \epsilon$, $\{C'_r\}$ is a geometric subcomplex of C , and we can form the quotient C/C' of C by C' and the natural projection $p : C \rightarrow C/C'$. $(C/C')_r$ is equal to $C_r(X - Y^{(n+l-r)\delta})$. Suppose further that (C, ψ) is ϵ connected, $\delta \geq 4\epsilon$, and $n \geq 1$; then $\mathcal{D}_\psi(C'^n) \subset C'_0$ holds, and $(C/C', p\% \psi)$ and the cobordism between (C, ψ) and $(C/C', p\% \psi)$ induced by p are both ϵ connected.

Next we consider pairs. Suppose $(f : C \rightarrow D, (\delta\psi, \psi))$ is an $(n+1)$ -dimensional ϵ quadratic pair on p_X and C', D' are geometric subcomplexes of C, D , respectively such that $f(C'_r) \subset (D'_r)$ for every r . Define an ϵ chain map $\bar{f} : C/C' \rightarrow D/D'$ by

$$f = \begin{pmatrix} * & * \\ 0 & \bar{f} \end{pmatrix} : C'_r \oplus (C/C')_r \longrightarrow D'_r \oplus (D/D')_r,$$

then the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ p \downarrow & & \downarrow q \\ C/C' & \xrightarrow{\bar{f}} & D/D' \end{array}$$

commutes strictly, where p and q are the natural projections, and

$$(\bar{f} : C/C' \rightarrow D/D', (q_{\%}\delta\psi, p_{\%}\psi))$$

is an $(n+1)$ -dimensional ϵ quadratic pair.

Proposition 6.3 *If $(f, (\delta\psi, \psi))$ is ϵ connected, $\mathcal{D}_{\psi}(C'^n) \subset C'_0$, and $\mathcal{D}_{\delta\psi}(D'^{n+1}) \subset D'_0$, then $(\bar{f}, (q_{\%}\delta\psi, p_{\%}\psi))$ is ϵ connected.*

Proof We check the ϵ connectivity of the duality map $\mathcal{D}_{(q_{\%}\delta\psi, p_{\%}\psi)}$. Let us use the notation $C'' = C/C'$ and $D'' = D/D'$. The boundary morphism $d_{\mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})} : \mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})_1 \rightarrow \mathcal{C}(\mathcal{D}_{(\delta\psi, \psi)})_0$ can be expressed by a matrix of the form

$$\begin{pmatrix} * & * & * & * & * & * \\ 0 & d_{D''} & 0 & \mathcal{D}_{q_{\%}\delta\psi} & 0 & \bar{f}\mathcal{D}_{p_{\%}\psi} \end{pmatrix} : D'_1 \oplus D''_1 \oplus D'^{n+1} \oplus D''^{n+1} \oplus C'^n \oplus C''^n \longrightarrow D'_0 \oplus D''_0.$$

The desired ϵ connectivity follows from this as in 6.1 □

Proposition 6.4 *Let Y be a subset of X , and let $[(C, d), \psi]$ and $[(\hat{C}, \hat{d}), \hat{\psi}]$ be elements of $L_n^{\delta, \epsilon}(X, Y; p_X, R)$ ($n \geq 1$). If*

- (1) $C_r(X - Y) = \hat{C}_r(X - Y)$,
- (2) $d_r|X - Y^{4\epsilon} = \hat{d}_r|X - Y^{4\epsilon}$, and
- (3) $\psi_s|X - Y^{4\epsilon} = \hat{\psi}_s|X - Y^{4\epsilon}$

for every r and s (≥ 0), then $[(C, d), \psi] = [(\hat{C}, \hat{d}), \hat{\psi}]$ in $L_n^{\delta, \epsilon}(X, Y^{(n+3)4\epsilon}; p_X, R)$.

Proof Define a geometric subcomplex C' of C by $C'_r = C_r(Y^{(n+1-r)4\epsilon})$, and let $p : C \rightarrow C/C'$ be the projection. Then $(C/C', p_{\%}\psi)$ is an ϵ connected ϵ quadratic complex by 6.1. The boundary maps for $\mathcal{C}(\mathcal{D}_{\psi})$ have radius 4ϵ and are of the form

$$\begin{pmatrix} d_{C'} & * & * & * \\ 0 & d_{C''} & * & (-)^{r-1}\mathcal{D}_{p_{\%}\psi} \\ 0 & 0 & * & * \\ 0 & 0 & * & (-)^{r-1}d_{C''}^* \end{pmatrix} : C'_r \oplus C''_r \oplus C'^{n+1-r} \oplus C''^{n+1-r} \longrightarrow C'_{r-1} \oplus C''_{r-1} \oplus C'^{n+2-r} \oplus C''^{n+2-r}.$$

Therefore $\mathcal{C}(\mathcal{D}_{p\% \psi})$ and $\mathcal{C}(\mathcal{D}_\psi)$ are exactly the same over $X - Y^{(n+2)4\epsilon}$, and $\mathcal{C}(\mathcal{D}_{p\% \psi})$ is 4ϵ contractible over $X - Y^{(n+3)4\epsilon}$, i.e., $p\% \psi$ is ϵ Poincaré over $X - Y^{(n+3)4\epsilon}$. In fact, if Γ is a 4ϵ chain contraction over $X - Y$ of $\mathcal{C}(\mathcal{D}_\psi)$, then $\Gamma|_{X - Y^{(n+2)4\epsilon}}$ gives a 4ϵ chain contraction over $X - Y^{(n+3)4\epsilon}$ of $\mathcal{C}(\mathcal{D}_{p\% \psi})$. Thus $(C/C', p\% \psi)$ determines an element of $L_n^{\delta, \epsilon}(X, Y^{(n+3)4\epsilon}; p_X, R)$.

By 3.3, the cobordism between (C, ψ) and $(C/C', p\% \psi)$ induced by p is an ϵ connected ϵ quadratic pair. Since this cobordism is exactly the same over $X - Y^{(n+2)4\epsilon}$ as the trivial cobordism from (C, ψ) to itself, it is ϵ Poincaré over $X - Y^{(n+3)4\epsilon}$. Therefore,

$$[C, \psi] = [C/C', p\% \psi] \in L_n^{\delta, \epsilon}(X, Y^{(n+3)4\epsilon}; p_X, R).$$

The same construction for $(\hat{C}, \hat{\psi})$ yields the same element as this, and we can conclude that

$$[C, \psi] = [\hat{C}, \hat{\psi}] \in L_n^{\delta, \epsilon}(X, Y^{(n+3)4\epsilon}; p_X, R).$$

□

Now suppose X is the union of two closed subsets A, B with intersection $N = A \cap B$.

Lemma 6.5 *Let G, H be geometric modules on p_X , and $f : G \rightarrow H$ be a morphism of radius δ . Then, for any $\gamma \geq 0$,*

$$f(G(B \cup N^\gamma)) \subset H(B \cup N^{\max\{\gamma+\delta, 2\delta\}}).$$

Proof This can be deduced from the following two claims:

- (1) $f(G(N^\gamma)) \subset H(N^{\gamma+\delta})$,
- (2) $f(G(B)) \subset H(B \cup N^{2\delta})$.

The first claim is obvious. To prove the second claim, take a generator of $G(B)$ and a path c starting from a with non-zero coefficient in f . By its continuity, the path $p_X \circ c$ in X either stays inside of B or passes through a point in N , and hence its image is contained in $B \cup N^{2\delta}$. This proves the second claim. □

Now let us define the excision map:

$$\text{exc} : L_n^{\delta, \epsilon}(X, B; p_X, R) \rightarrow L_n^{\delta, \epsilon}(A, A \cap N^{(n+5)4\delta}; p_A, R),$$

Take an element $[C, \psi] \in L_n^{\delta, \epsilon}(X, B; p_X, R)$. Define a geometric subcomplex C' of C by

$$C'_r = C_r(B \cup N^{(n+2-r)4\epsilon}),$$

and let $p : C \rightarrow C/C'$ denote the projection. Then, by 6.5 and 6.1, $(C/C', p_{\%}\psi)$ is an ϵ connected ϵ quadratic complex on p_A and is ϵ Poincaré over $A - N^{(n+4)4\epsilon}$. We define $\text{exc}([C, \psi])$ to be the element

$$[C/C', p_{\%}\psi] \in L_n^{\delta, \epsilon}(A, A \cap N^{(n+5)4\epsilon}; p_A, R).$$

The excision map is well-defined. Suppose

$$[C, \psi] = [\hat{C}, \hat{\psi}] \in L_n^{\delta, \epsilon}(X, B; p_X, R).$$

Without loss of generality we may assume that there is a δ connected δ cobordism

$$(f : C \oplus \hat{C} \rightarrow D, (\delta\psi, \psi \oplus -\hat{\psi}))$$

between (C, ψ) and $(\hat{C}, \hat{\psi})$ that is δ Poincaré over $X - B$. Let us construct $(C/C', p_{\%}\psi)$ and $(\hat{C}/\hat{C}', \hat{p}_{\%}\hat{\psi})$ as above, define a geometric subcomplex D' of D by

$$D'_r = D_r(B \cup N^{(n+3-r)4\delta}),$$

and let $q : D \rightarrow D/D'$ denote the projection. By 6.5 and 6.3, we obtain an δ connected δ cobordism

$$(\bar{f} : C/C' \oplus \hat{C}/\hat{C}' \rightarrow D/D', (q_{\%}\delta\psi, p_{\%}\psi \oplus -\hat{p}_{\%}\hat{\psi}))$$

which is δ Poincaré over $A - B \cup N^{(n+5)4\delta}$. Therefore exc is well-defined.

By using 6.4, we can check that the homomorphisms i_* and exc are stable inverses; *i.e.*, the following diagram commutes:

$$\begin{array}{ccc} L_n^{\delta, \epsilon}(A, N; p_A, R) & \xrightarrow{i_*} & L_n^{\delta, \epsilon}(X, B; p_X, R) \\ \downarrow & & \parallel \\ L_n^{\delta, \epsilon}(A, A \cap N^{(n+5)4\delta}; p_A, R) & \xleftarrow{\text{exc}} & L_n^{\delta, \epsilon}(X, B; p_X, R) \\ \parallel & & \downarrow \\ L_n^{\delta, \epsilon}(A, A \cap N^{(n+5)4\delta}; p_A, R) & \xrightarrow{i_*} & L_n^{\delta, \epsilon}(X, B \cup N^{(n+5)4\delta}; p_X, R) \end{array}$$

where the vertical maps are the homomorphisms induced by inclusion maps.

7 Mayer-Vietoris type sequence

We continue to assume that X is the union of two closed subsets A, B with intersection $N = A \cap B$, and will present a Mayer-Vietoris type stably exact sequence.

Replace κ_n by $\kappa_n + 4(n + 5)$, and suppose $\delta \geq \epsilon > 0$. Let W be a subset of X containing $N^{\kappa_n \delta}$ and assume $\delta' \geq \kappa_n \delta$, $\epsilon' \geq \kappa_n \epsilon$. Then a homomorphism

$$\bar{\partial} : L_n^{\delta, \epsilon}(X; p_X, R) \longrightarrow L_{n-1}^{\{A \cup W\}, \delta', \epsilon'}(W; p_{A \cup W}, R)$$

is obtained by composing the following maps:

$$\begin{aligned} L_n^{\delta, \epsilon}(X; p_X, R) &\longrightarrow L_n^{\delta, \epsilon}(X, B; p_X, R) \xrightarrow{\text{exc}} L_n^{\delta, \epsilon}(A, A \cap N^{(n+5)4\delta}; p_A, R) \\ &\xrightarrow{\partial} L_{n-1}^{\{A\}, \delta', \epsilon'}(A \cap W; p_A, R) \longrightarrow L_{n-1}^{\{A \cup W\}, \delta', \epsilon'}(W; p_{A \cup W}, R). \end{aligned}$$

If $[C, \psi] \in L_n^{\delta, \epsilon}(X; p_X, R)$, then its image $\bar{\partial}[C, \psi]$ is represented by $((E, q), \psi')$ which is homotopy equivalent to the boundary $\hat{\partial}(C/C', p_{\%}\psi)$, where $C' \subset C$ and $p : C \rightarrow C/C'$ are as in the definition of exc. This is exactly the projective quadratic Poincaré complex $(Q, \bar{\psi})$ which appears in the Splitting Lemma:

Lemma 7.1 ([5]) *For any integer $n \geq 2$, there exists a positive number $\kappa_n \geq 1$ such that the following holds: Suppose $p_X : M \rightarrow X$ is a map to a metric space X , X is the union of two closed subsets A and B with intersection $N = A \cap B$, and R is a ring with involution. Let ϵ be any positive number, and set $\epsilon' = \kappa_n \epsilon$, $N' = N^{\epsilon'}$, $A' = A \cup N'$, and $B' = B \cup N'$. Then for any n -dimensional quadratic Poincaré R -module complex $c = (C, \psi)$ on p_X of radius ϵ , there exist a Poincaré cobordism of radius ϵ' from c to the union $c' \cup c''$ of an n -dimensional quadratic Poincaré pair $c' = (f' : Q \rightarrow C', (\delta\bar{\psi}', -\bar{\psi}))$ on $p_{A'}$ of radius ϵ' and an n -dimensional quadratic Poincaré pair $c'' = (f'' : Q \rightarrow C'', (\delta\bar{\psi}'', \bar{\psi}))$ on $p_{B'}$ of radius ϵ' along an $(n-1)$ -dimensional quadratic Poincaré projective R -module complex $(Q, \bar{\psi})$ on $p_{N'}$, where Q is ϵ' chain equivalent to an $(n-1)$ -dimensional free chain complex on $p_{A'}$ and also to an $(n-1)$ -dimensional free chain complex on $p_{B'}$.*

From this and its relative version, we obtain the following:

Proposition 7.2 *If $n \geq 2$, the map $\bar{\partial}$ factors through a homomorphism*

$$\partial : L_n^{\delta, \epsilon}(X; p_X, R) \longrightarrow L_{n-1}^{\mathcal{F}, \delta', \epsilon'}(W; p_X, R),$$

where $\mathcal{F} = \{A \cup W, A \cup W\}$. Moreover the image $\partial[C, \psi]$ is given by $[Q, \bar{\psi}]$ which appears in any splitting (up to cobordism) of (C, ψ) according to the closed subsets A, B .

Now we present the Mayer-Vietoris type stably-exact sequence. It is made up of three kinds of maps. The first is the map

$$i_* : L_n^{\{A, B\}, \delta, \epsilon}(N; p_X, R) \rightarrow L_n^{\delta', \epsilon'}(A; p_A, R) \oplus L_n^{\delta', \epsilon'}(B; p_B, R)$$

defined by $i_*(x) = (\iota_A(x), -\iota_B(x))$ when $\delta' \geq \alpha\delta$ and $\epsilon' \geq \alpha\epsilon$. The second is the map

$$j_* : L_n^{\delta, \epsilon}(A; p_A, R) \oplus L_n^{\delta, \epsilon}(B; p_B, R) \rightarrow L_n^{\delta', \epsilon'}(X; p_X, R)$$

defined by $j_*(x, y) = j_{A*}(x) + j_{B*}(y)$ when $\delta' \geq \delta$ and $\epsilon' \geq \epsilon$. Here $j_A : A \rightarrow X$ and $j_B : B \rightarrow X$ are inclusion maps. The third is the map ∂ given in 7.2:

$$\partial : L_n^{\delta, \epsilon}(X; p_X, R) \longrightarrow L_{n-1}^{\{A \cup W, B \cup W\}, \delta', \epsilon'}(W; p_X, R),$$

where $W \supset N^{\kappa_n \delta}$, $\delta' \geq \kappa_n \delta$, and $\epsilon' \geq \kappa_n \epsilon$.

In the rest of this section, we omit the control map and the coefficient ring from notation.

Theorem 7.3 *For any integer $n \geq 2$, there exists a constant $\lambda_n > 1$ which depends only on n such that the following holds true for any control map p_X and two closed subsets A, B of X satisfying $X = A \cup B$:*

(1) Suppose $\delta' \geq \alpha\delta$, $\epsilon' \geq \alpha\epsilon$, $\delta'' \geq \delta'$, and $\epsilon'' \geq \epsilon'$ so that the following two maps are defined:

$$L_n^{\{A, B\}, \delta, \epsilon}(N) \xrightarrow{i_*} L_n^{\delta', \epsilon'}(A) \oplus L_n^{\delta', \epsilon'}(B) \xrightarrow{j_*} L_n^{\delta'', \epsilon''}(X).$$

Then $j_* i_*$ is zero.

(2) Suppose $\delta'' \geq \delta'$, $\epsilon'' \geq \epsilon'$ so that $j_* : L_n^{\delta', \epsilon'}(A) \oplus L_n^{\delta', \epsilon'}(B) \rightarrow L_n^{\delta'', \epsilon''}(X)$ is defined. If $\delta \geq \lambda_n \delta''$ and $W \supset N^{\lambda_n \delta''}$, then the relax-control image of the kernel of j_* in $L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W)$ is contained in the image of i_* below:

$$\begin{array}{ccc} L_n^{\delta', \epsilon'}(A) \oplus L_n^{\delta', \epsilon'}(B) & \xrightarrow{j_*} & L_n^{\delta'', \epsilon''}(X) \\ \downarrow & & \\ L_n^{\{A \cup W, B \cup W\}, \delta}(W) & \xrightarrow{i_*} & L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W) \end{array}$$

(3) Suppose $\delta' \geq \delta$, $\epsilon' \geq \epsilon$, $W \supset N^{\kappa_n \delta'}$, $\delta'' \geq \kappa_n \delta'$, and $\epsilon'' \geq \kappa_n \epsilon'$ so that the following two maps are defined :

$$L_n^{\delta, \epsilon}(A) \oplus L_n^{\delta, \epsilon}(B) \xrightarrow{j_*} L_n^{\delta', \epsilon'}(X) \xrightarrow{\partial} L_{n-1}^{\{A \cup W, B \cup W\}, \delta'', \epsilon''}(W).$$

Then ∂j_* is zero.

(4) Suppose $W \supset N^{\kappa_n \delta'}$, $\delta'' \geq \kappa_n \delta'$, and $\epsilon'' \geq \kappa_n \epsilon'$ so that the map $\partial : L_n^{\delta', \epsilon'}(X) \rightarrow L_{n-1}^{\{A \cup W, B \cup W\}, \delta'', \epsilon''}(W)$ is defined. If $\delta \geq \lambda_n \delta''$, then the relax-control image of the kernel of ∂ in $L_n^\delta(X)$ is contained in the image of j_* below :

$$\begin{array}{ccc} L_n^{\delta', \epsilon'}(X) & \xrightarrow{\partial} & L_{n-1}^{\{A \cup W, B \cup W\}, \delta'', \epsilon''}(W) \\ \downarrow & & \\ L_n^\delta(A \cup W) \oplus L_n^\delta(B \cup W) & \xrightarrow{j_*} & L_n^\delta(X) \end{array}$$

(5) Suppose $W \supset N^{\kappa_n \delta}$, $\delta' \geq \kappa_n \delta$, $\epsilon' \geq \kappa_n \epsilon$, $\delta'' \geq \alpha \delta'$, and $\epsilon'' \geq \alpha \epsilon'$ so that the following two maps are defined :

$$L_n^{\delta, \epsilon}(X) \xrightarrow{\partial} L_{n-1}^{\{A \cup W, B \cup W\}, \delta', \epsilon'}(W) \xrightarrow{i_*} L_{n-1}^{\delta'', \epsilon''}(A \cup W) \oplus L_{n-1}^{\delta'', \epsilon''}(B \cup W).$$

Then $i_* \partial$ is zero.

(6) Suppose $\delta'' \geq \alpha \delta'$, and $\epsilon'' \geq \alpha \epsilon'$ so that $i_* : L_{n-1}^{\{A, B\}, \delta', \epsilon'}(N) \rightarrow L_{n-1}^{\delta'', \epsilon''}(A) \oplus L_{n-1}^{\delta'', \epsilon''}(B)$ is defined. If $\delta \geq \lambda_n \delta''$, $N' \supset N^{\lambda_n \delta''}$, and $W = (N')^{\kappa_n \delta}$, then the relax-control image of the kernel of i_* in $L_{n-1}^{\{A \cup W, B \cup W\}, \delta}(W)$ is contained in the image of ∂ associated with the two closed subsets $A \cup N'$, $B \cup N'$:

$$\begin{array}{ccc} L_{n-1}^{\{A, B\}, \delta', \epsilon'}(N) & \xrightarrow{i_*} & L_{n-1}^{\delta'', \epsilon''}(A) \oplus L_{n-1}^{\delta'', \epsilon''}(B) \\ \downarrow & & \\ L_n^\delta(X) & \xrightarrow{\partial} & L_{n-1}^{\{A \cup W, B \cup W\}, \delta}(W) \end{array}$$

Proof (1) Take an element $x = [Q, \psi] \in L_n^{\{A, B\}, \delta, \epsilon}(N)$. The image $i_*(x)$ is a pair $([c_A], -[c_B])$ where c_A and c_B are free quadratic Poincaré complexes on p_A and p_B that are both homotopy equivalent to (Q, ψ) , and hence $[c_A] = [c_B] \in L_n^{\delta'', \epsilon''}(X)$. Therefore, $j_* i_*(x) = [c_A] - [c_B] = 0$.

(2) First, temporarily use the constant λ_n posited in the splitting lemma. Take an element $x = ([C_A, \psi_A], [C_B, \psi_B]) \in L_n^{\delta', \epsilon'}(A) \oplus L_n^{\delta', \epsilon'}(B)$ and assume $j_*(x) = 0$. There exists a null-cobordism $(f : C_A \oplus C_B \rightarrow D, (\delta\psi, \psi_A \oplus -\psi_B))$. Its boundary is already split according to A and B , so use the relative splitting to this null-cobordism to get cobordisms of radius $\lambda_n \delta''$:

$$\begin{aligned} (f_A : (C_A, 1) \oplus Q &\rightarrow (D_A, 1), (\delta\psi_A, \psi_A \oplus -\bar{\psi})) \text{ on } p_{A \cup W}, \\ (f_B : (C_B, 1) \oplus Q &\rightarrow (D_B, 1), (\delta\psi_B, \psi_B \oplus -\bar{\psi})) \text{ on } p_{B \cup W}. \end{aligned}$$

By the Poincaré duality $D_A^{n+1-*} \simeq \mathcal{C}(f_A)$, we have

$$[D_A] - [C_A] - [Q] = [\mathcal{C}(f_A)] = [D_A^{n+1-*}] = 0 \in \tilde{K}_0^{n+1, 4\lambda_n \delta''}(A \cup W),$$

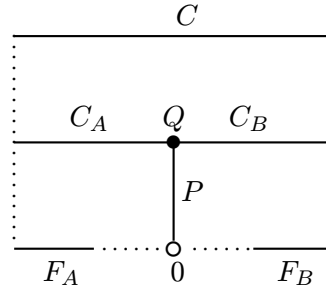
and, hence, we have $[Q] = 0$ in $\tilde{K}_0^{n, 36\lambda_n \delta''}(A \cup W)$. See §3 of [14]. Similarly $[Q] = 0$ in $\tilde{K}_0^{n, 36\lambda_n \delta''}(B \cup W)$. Thus we obtain an element $[Q, \bar{\psi}]$ of $L_n^{\{A \cup W, B \cup W\}, 36\lambda_n \delta''}(W)$. Replace λ_n by something bigger (at least $36\lambda_n$) so that its image via i_* in $L_n^{\alpha\delta}(A \cup W) \oplus L_n^{\alpha\delta}(B \cup W)$ is equal to $([C_A, \psi_A], [C_B, \psi_B])$ whenever $\delta \geq \lambda_n \delta''$.

(3) If we start with an element $x = ([C_A, \psi_A], [C_B, \psi_B])$, then $j_*(x)$ is represented by $(C_A, \psi_A) \oplus (C_B, \psi_B)$ which is already split according to A and B . Therefore $\partial j_*(x) = 0$.

(4) Temporarily set the constant λ_n to be the one posited in the splitting lemma. Take an element $[C, \psi]$ in $L_n^{\delta', \epsilon'}(X)$ such that $\partial[C, \psi] = 0$. (C, ψ) splits into two adjacent pairs:

$$a = (f_A : Q \rightarrow (C_A, 1), (\delta\psi_A, -\bar{\psi})) \quad \text{and} \quad b = (f_B : Q \rightarrow (C_B, 1), (\delta\psi_B, \bar{\psi}))$$

such that $[Q, \bar{\psi}] = 0$ in $L_{n-1}^{\{A \cup W, B \cup W\}, \delta'', \epsilon''}(W)$. Take a δ'' null-cobordism on p_W $p = (g : Q \rightarrow P, (\delta\bar{\psi}, \bar{\psi}))$ such that the reduced projective class of P is zero on $p_{A \cup W}$ and also on $p_{B \cup W}$. $C_A \cup_Q P$ is chain equivalent to an n -dimensional free complex F_A on p_A , and $C_B \cup_Q P$ is chain equivalent to an n -dimensional free complex F_B on p_B . Use these to fill in the bottom squares with cobordisms:



Replacing λ_n with something larger if necessary, we obtain free quadratic Poincaré complexes on $p_{A \cup W}$ and $p_{B \cup W}$ whose sum is $\lambda_n \delta''$ cobordant to (C, ψ) .

(5) If we start with an element $x = [C, \psi]$ in $L_n^{\delta, \epsilon}(X)$, then $j_*(x)$ is represented by the projective piece $(Q, \bar{\psi})$ obtained by splitting, and the null-cobordisms required to show $i_* \partial(x) = 0$ are easily constructed from the split pieces.

(6) Take an element $[Q, \psi]$ of $L_{n-1}^{\{A,B\},\delta',\epsilon'}(N)$. Then (Q, ψ) is homotopy equivalent to a free quadratic Poincaré complex $((C_A, 1), \psi_A)$ on p_A and also to a free quadratic Poincaré complex $((C_B, 1), \psi_B)$ on p_B . If $i_*[Q, \psi] = 0$, then these are both null-cobordant; there are quadratic Poincaré pairs

$$\begin{aligned} (f_A : C_A \rightarrow D_A, (\delta\psi_A, \psi_A)) & \quad \text{on } p_A, \text{ and} \\ (f_B : C_B \rightarrow D_B, (\delta\psi_B, \psi_B)) & \quad \text{on } p_B. \end{aligned}$$

Use the homotopy equivalence $(C_A, \psi_A) \simeq (C_B, \psi_B)$ to replace the boundary of the latter by (C_A, ψ_B) , and glue them together to get an element $[D, \delta\psi]$ of $L_n^\delta(X)$ for some $\delta > 0$. Note that $(D, \delta\psi)$ has a splitting into two pairs with the common boundary piece equal to (Q, ψ) , so we have $\partial[D, \delta\psi] = [Q, \psi]$ in $L_{n-1}^{\{A \cup W, B \cup W\}, \delta}(W)$. \square

8 A special case

In this section we treat the case when there are no controlled K -theoretic difficulties.

First assume that X is a finite polyhedron. We fix its triangulation. Under this assumption we can simplify the Mayer-Vietoris type sequence of the previous section at least for sufficiently small ϵ 's and δ 's. X is equipped with a deformation $\{f_t : X \rightarrow X\}$ called 'rectification' ([5]) which deforms sufficiently small neighborhoods of the i -skeleton $X^{(i)}$ into $X^{(i)}$ such that f_t 's are uniformly Lipschitz. This can be used to rectify the enlargement of the relevant subsets at the expense of enlargement of ϵ 's and δ 's. We thank Frank Quinn for showing us his description of *uniformly continuous CW complexes* which are designed for taking care of these situations in a more general setting.

Next let us assume that X is a finite polyhedron and that the control map $p_X : M \rightarrow X$ is a fibration with path-connected fiber F such that

$$Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$$

for every $k \geq 0$. The condition on the fundamental group is satisfied if $\pi_1(F) \cong \mathbb{Z}^l$ for some $l \geq 0$. If we study the proofs of 8.1 and 8.2 of [14] carefully, we obtain the following.

Proposition 8.1 *Let p_X be as above and $n \geq 0$ be an integer. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu \leq 1$ which depend on X and n such that the*

relax-control maps

$$\begin{aligned}\tilde{K}_0^{n,\epsilon}(S; p_S, \mathbb{Z}) &\longrightarrow \tilde{K}_0^{n,\epsilon'}(S; p_S, \mathbb{Z}) \\ Wh^{n,\epsilon}(S; p_S, \mathbb{Z}) &\longrightarrow Wh^{n,\epsilon'}(S; p_S, \mathbb{Z})\end{aligned}$$

are zero maps for any subpolyhedron S , any $\epsilon' \leq \epsilon_0$ and any $\epsilon \leq \mu\epsilon'$.

This means that there is a homomorphism functorial with respect to relaxation of control

$$L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\mathcal{F} \cup \{S\},\delta',\epsilon'}(S; p_X, \mathbb{Z})$$

for any family \mathcal{F} of subpolyhedra of X containing S , if $\delta' \leq \epsilon_0$, $\delta \leq \mu\delta'$, and $\epsilon \leq \mu\epsilon'$. Compose this with the homomorphism

$$\iota_S : L_n^{\mathcal{F} \cup \{S\},\delta',\epsilon'}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z})$$

to get a homomorphism

$$\iota : L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \longrightarrow L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z}).$$

A stable inverse τ functorial with respect to relaxation of control can be defined by $\tau([C, \psi]) = [(C, 1), \psi]$, and we have a commutative diagram:

$$\begin{array}{ccc} L_n^{\delta,\epsilon}(S; p_X, \mathbb{Z}) & \xrightarrow{\tau} & L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \\ \downarrow & & \parallel \\ L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z}) & \xleftarrow{\iota} & L_n^{\mathcal{F},\delta,\epsilon}(S; p_X, \mathbb{Z}) \\ \parallel & & \downarrow \\ L_n^{\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z}) & \xrightarrow{\tau} & L_n^{\mathcal{F},\alpha\delta',\alpha\epsilon'}(S; p_X, \mathbb{Z}) \end{array}$$

Thus the Mayer-Vietoris type sequence is stably exact when we replace the controlled projective L -group terms with appropriate controlled L -groups.

Furthermore, since p_X is a fibration, we have a stability for controlled L -groups:

Proposition 8.2 ([5], Theorem 1) *Let $n \geq 0$. Suppose Y is a finite polyhedron and $p_Y : M \rightarrow Y$ is a fibration. Then there exist constants $\delta_0 > 0$ and $K > 1$, which depends on the integer n and Y , such that the relax-control map $L_n^{\delta',\epsilon'}(Y; p_Y, R) \rightarrow L_n^{\delta,\epsilon}(Y; p_Y, R)$ is an isomorphism if $\delta_0 \geq \delta \geq K\epsilon$, $\delta_0 \geq \delta' \geq K\epsilon'$, $\delta \geq \delta'$, and $\epsilon \geq \epsilon'$.*

Now let us denote these isomorphic groups $L_n^{\delta,\epsilon}(Y; p_Y, R)$ ($\delta_0 \geq \delta$, $\delta \geq \kappa\epsilon$) by $L_n(Y; p_Y, R)$. When the coefficient ring R is \mathbb{Z} , we omit \mathbb{Z} and use the notation $L_n(Y; p_Y)$.

Theorem 8.3 *Let $p_X : M \rightarrow X$ be a fibration over a finite polyhedron X . Then $L_n(X; p_X, R)$ is 4-periodic: $L_n(X; p_X, R) \cong L_{n+4}(X; p_X, R)$ ($n \geq 0$).*

Proof The proof of the 4-periodicity of $L_n(\mathbb{A})$ of an additive category with involution given in [10] adapts well to the controlled setting. \square

We have a Mayer-Vietoris exact sequence for L_n with coefficient ring \mathbb{Z} .

Theorem 8.4 *Let X be a finite polyhedron and suppose that $p_X : M \rightarrow X$ is a fibration with path-connected fiber F such that $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. If X is the union of two subpolyhedra A and B , then there is a long exact sequence*

$$\begin{aligned} \dots \xrightarrow{\partial} L_n(A \cap B; p_{A \cap B}) \xrightarrow{i_*} L_n(A; p_A) \oplus L_n(B; p_B) \xrightarrow{j_*} L_n(X; p_X) \\ \xrightarrow{\partial} L_{n-1}(A \cap B; p_{A \cap B}) \xrightarrow{i_*} \dots \xrightarrow{j_*} L_0(X; p_X) . \end{aligned}$$

Proof The exactness at the term $L_2(A; p_A) \oplus L_2(B; p_B)$ and at the terms to the left of it follows immediately from the stably-exact sequence. The exactness at other terms follows from the 4-periodicity. \square

Recall that there is a functor $\mathbb{L}(-)$ from spaces to Ω -spectra such that $\pi_n(\mathbb{L}(M)) = L_n(\mathbb{Z}[\pi_1(M)])$ constructed geometrically by Quinn [6], and algebraically by Ranicki [11]. Blockwise application of \mathbb{L} to p_X produces a generalized homology group $H_n(X; \mathbb{L}(p_X))$ [7]. There is a map $A : H_n(X; \mathbb{L}(p_X)) \rightarrow L_n(X; p_X)$ called the assembly map. See [16] for the $\mathbb{L}^{-\infty}$ -analogue, involving the lower L -groups of [12].

Theorem 8.5 *Let X be a finite polyhedron and suppose that $p_X : M \rightarrow X$ is a fibration with path-connected fiber F such that $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. Then the assembly map $A : H_n(X; \mathbb{L}(p_X)) \rightarrow L_n(X; p_X)$ is an isomorphism.*

Proof We actually prove the isomorphism $A : H_n(S; \mathbb{L}(p_S)) \rightarrow L_n(S; p_S)$ for all the subpolyhedra S of X by induction on the number of simplices.

When S consists of a single point v , then the both sides are $L_n(\mathbb{Z}[\pi_1(p_X^{-1}(v))])$ and A is the identity map.

Suppose S consists of $k > 1$ simplices and assume by induction that the assembly map is an isomorphism for all subpolyhedra consisting of less number

of simplices. Pick a simplex Δ which is not a face of other simplices and let $A = \Delta$ and $B = S - \text{interior}(\Delta)$. Since A contracts to a point v , it can be easily shown that $H_n(A; \mathbb{L}(p_A))$ and $L_n(A; p_A)$ are both $L_n(\mathbb{Z}[\pi_1(p_X^{-1}(v))])$, and the assembly map $A : H_n(A; \mathbb{L}(p_A)) \rightarrow L_n(A; p_A)$ is an isomorphism. By induction hypothesis the assembly maps for B and $A \cap B$ are both isomorphisms. We can conclude that the assembly map for S is an isomorphism by an application of 5-lemma to the ladder made up of the Mayer-Vietoris sequences for $H_*(-)$ and $L_*(-)$. \square

Remark If F is simply-connected, then $Wh(\pi_1(F) \times \mathbb{Z}^k) = Wh(\mathbb{Z}^k) = 0$ for every $k \geq 0$ by the celebrated result of Bass, Heller and Swan. In this case $H_n(X; \mathbb{L}(p_X))$ is isomorphic to the generalized homology group $H_n(X; \mathbb{L})$ where \mathbb{L} is the 4-periodic simply-connected surgery spectrum with $\pi_n(\mathbb{L}) = L_n(\mathbb{Z}[\{1\}])$ and we have an assembly isomorphism

$$A : H_n(X; \mathbb{L}) \cong L_n(X; p_X) .$$

This is the controlled surgery obstruction group which appears in the controlled surgery exact sequence of [4] (as required for the surgery classification of exotic homology manifolds in [1]). There the control map is not assumed to be a fibration. We believe that most of the arguments in this paper work in a more general situation.

As an application of 8.4, we consider the \mathbb{Z} -coefficient controlled L -group of $p_X \times 1 : M \times S^1 \rightarrow X \times S^1$.

Corollary 8.6 *Let $n \geq 0$, and let X and $p_X : M \rightarrow X$ be as in 8.4. Then there is a split short exact sequence :*

$$0 \rightarrow L_n(X; p_X) \xrightarrow{i_*} L_n(X \times S^1; p_X \times 1) \xrightarrow{B} L_{n-1}(X; p_X) \rightarrow 0 .$$

Proof Split the circle $S^1 = \partial([-1, 1] \times [-1, 1]) \subset \mathbb{R}^2$ into two pieces

$$S^1_+ = \{(x, y) \in S^1 | y \geq 0\} \quad \text{and} \quad S^1_- = \{(x, y) \in S^1 | y \leq 0\} ,$$

with intersection $\{p = (1, 0), q = (-1, 0)\}$. Let ∂ be the connecting homomorphism in the Mayer-Vietoris sequence 8.4 corresponding to this splitting, and consider the composite

$$\begin{aligned} B : L_n(X \times S^1; p_X \times 1) &\xrightarrow{\partial} L_{n-1}(X \times \{p\}; p_X \times 1) \oplus L_{n-1}(X \times \{q\}; p_X \times 1) \\ &\xrightarrow{\text{projection}} L_{n-1}(X \times \{p\}; p_X \times 1) \cong L_{n-1}(X; p_X) . \end{aligned}$$

Then ∂ can be identified with

$$(B, -B) : L_n(X \times S^1; p_X \times 1) \longrightarrow L_{n-1}(X; p_X) \oplus L_{n-1}(X; p_X) .$$

The map i_* is the map induced by the inclusion map:

$$L_n(X; p_X) \cong L_n(X \times \{p\}; p_X \times 1) \xrightarrow{i_*} L_n(X \times S^1; p_X \times 1) .$$

The exactness follows easily from the exactness of the Mayer-Vietoris sequence. A splitting of B can be constructed by gluing two product cobordisms. \square

Corollary 8.7 *Let T^n be the n -dimensional torus $S^1 \times \cdots \times S^1$. Then*

$$\begin{aligned} L_m(T^n; 1_{T^n}) &\cong \bigoplus_{r=0}^n \binom{n}{r} L_{m-r}(\mathbb{Z}) \\ &\cong L_m(\mathbb{Z}[\pi_1(T^n)]) \quad (m \geq n) . \end{aligned}$$

Proof Use 8.6 repeatedly to obtain $L_m(T^n; 1_{T^n}) \cong \bigoplus_{r=0}^n \binom{n}{r} L_{m-r}(\mathbb{Z})$. The isomorphism $\bigoplus_{r=0}^n \binom{n}{r} L_{m-r}(\mathbb{Z}) \cong L_m(\mathbb{Z}[\pi_1(T^n)])$ is the well-known computation obtained geometrically by Shaneson and Wall, and algebraically by Novikov and Ranicki. \square

9 Locally Finite Analogues

Up to this point, we considered only finitely generated modules and chain complexes. In this section we deal with infinitely generated objects; such objects arise naturally when we take the pullback of a finitely generated object via an infinite-sheeted covering map. We restrict ourselves to a very special case necessary for our application.

Definition 9.1 (Ranicki and Yamasaki [4, p.14]) Consider the product $M \times N$ of two spaces. A geometric module on $M \times N$ is said to be *M-finite* if, for any $y \in N$, there is a neighbourhood U of y in N such that $M \times U$ contains only finitely many basis elements; a projective module (A, p) on $M \times N$ is said to be *q-finite* if A is *M-finite*; a projective chain complex (C, p) on $M \times N$ is *M-finite* if each (C_r, p_r) is *M-finite*. [In [14], we used the terminology “*M-locally finite*”, but this does not sound right and we decided to use “*M-finite*” instead.] When M is compact, *M-finiteness* is equivalent to the ordinary locally-finiteness.

Consider a control map $p_X : M \rightarrow X$ to a metric space X , and let N be another metric space. Give the maximum metric to the product $X \times N$, and let us use the map

$$p_X \times 1_N : M \times N \longrightarrow X \times N,$$

as the control map for $M \times N$.

Definition 9.2 For $\delta \geq \epsilon > 0$, $Y \subset X$, and a family \mathcal{F} of subsets of X containing Y , define M -finite (δ, ϵ) -controlled L -groups $L_n^{M, \delta, \epsilon}(X \times N, Y \times N; p_X \times 1, R)$, and M -finite (δ, ϵ) -controlled projective L -groups $L_n^{M, \mathcal{F}, \delta, \epsilon}(Y \times N; p_X \times 1, R)$ by requiring that every chain complexes concerned are M -finite.

All the materials up to §7 are valid for M -finite analogues. In the previous section, there are several places where we assumed X to be a finite polyhedron, and they may not automatically generalize to the M -finite case.

The most striking phenomenon about M -finite objects is the following vanishing result on the half line.

Proposition 9.3 Let $p_X : M \rightarrow X$ be a control map, N a metric space, and give $N \times [0, \infty)$ the maximum metric. For any $\epsilon > 0$ and $\delta \geq \epsilon$,

$$\begin{aligned} L_n^{M, \delta, \epsilon}(X \times N \times [0, \infty); p_X \times 1, R) &= 0, \\ \tilde{K}_0^{M, n, \epsilon}(X \times N \times [0, \infty); p_X \times 1, R) &= 0. \end{aligned}$$

Proof This is done using repeated shifts towards infinity and the ‘Eilenberg Swindle’. Let us consider the case of $L_n^{M, \delta, \epsilon}(X \times N \times [0, \infty); p_X \times 1, R)$. Let $J = [0, \infty)$ and define $T : M \times N \times J \rightarrow M \times N \times J$ by $T(x, x', t) = (x, x', t + \epsilon)$. Take an element $[c] \in L_n^{M, \delta, \epsilon}(X \times N \times J, p_X \times 1, R)$. It is zero, because there exist M -finite ϵ Poincaré cobordisms:

$$\begin{aligned} c &\sim c \oplus (T_{\#}(-c) \oplus T_{\#}^2(c)) \oplus (T_{\#}^3(-c) \oplus T_{\#}^4(c)) \oplus \dots \\ &= (c \oplus T_{\#}(-c)) \oplus (T_{\#}^2(c) \oplus T_{\#}^3(-c)) \oplus \dots \sim 0. \end{aligned}$$

The proof for controlled \tilde{K} is similar. See the appendix to [14]. □

Thus, the analogue of Mayer-Vietoris type sequence (7.3) for the control map $p_X \times 1 : M \times N \times \mathbb{R} \rightarrow X \times N \times \mathbb{R}$ with respect to the splitting $X \times N \times \mathbb{R} = X \times N \times (-\infty, 0] \cup X \times N \times [0, \infty)$ reduces to:

$$0 \longrightarrow L_n^{M, \delta, \epsilon}(X \times N \times \mathbb{R}; p_X \times 1, R) \xrightarrow{\partial} L_{n-1}^{M, p, \delta', \epsilon'}(X \times N \times I; p_X \times 1, R) \longrightarrow 0,$$

where $\delta' = \kappa_n \delta$, $\epsilon' = \kappa_n \epsilon$, $I = [-\kappa_n \delta, \kappa_n \delta]$, and the right hand side is the M -finite projective L -group $L_{n-1}^{M, \{\}, \delta', \epsilon'}(X \times N \times I; p_X \times 1, R)$ corresponding to the empty family $\mathcal{F} = \{\}$.

A diagram chase shows that there exists a well-defined homomorphism:

$$\beta : L_{n-1}^{M, p, \delta', \epsilon'}(X \times N \times I; p_X \times 1, R) \longrightarrow L_n^{M, \delta'', \epsilon''}(X \times N \times \mathbb{R}; p_X \times 1, R),$$

where $\gamma'' = \lambda_n \kappa_n \lambda_{n-1} \alpha \gamma'$ and $\epsilon'' = \lambda_n \kappa_n \lambda_{n-1} \alpha \epsilon'$. The homomorphisms ∂ and β are stable inverses of each other; the compositions are both relax-control maps.

Note that, for any $\delta \geq \epsilon > 0$, the retraction induces an isomorphism:

$$L_{n-1}^{M, p, \delta, \epsilon}(X \times N \times I; p_X \times 1, R) \cong L_{n-1}^{M, p, \delta, \epsilon}(X \times N \times \{0\}; p_X, R).$$

Thus, we have obtained:

Theorem 9.4 *Splitting along $X \times N \times \{0\}$ induces a stable isomorphism:*

$$\partial : L_n^{M, \delta, \epsilon}(X \times N \times \mathbb{R}; p_X \times 1, R) \longrightarrow L_{n-1}^{M, p, \delta', \epsilon'}(X \times N; p_X \times 1, R).$$

Now, as in the previous section, let us assume that X is a finite polyhedron and $p_X : M \rightarrow X$ is a fibration with a path-connected fiber F such that $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$.

The following is an M -finite analogue of 8.1.

Proposition 9.5 *Let p_X be as above and $n \geq 0$, $k \geq 0$ be integers. Then there exist numbers $\epsilon_0 > 0$ and $0 < \mu \leq 1$ which depend on X , n , and k such that the relax-control maps*

$$\begin{aligned} \tilde{K}_0^{M, n, \epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) &\longrightarrow \tilde{K}_0^{M, n, \epsilon'}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) \\ Wh^{M, n, \epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) &\longrightarrow Wh^{M, n, \epsilon'}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z}) \end{aligned}$$

is the zero map for any $\epsilon' \leq \epsilon_0$ and any $\epsilon \leq \mu \epsilon'$.

Proof First note that, since $X \times \mathbb{R}^k$ is not a finite polyhedron unless $k = 0$, the proof for 8.1 does not directly apply to the current situation.

Let us consider the Whitehead group case first. Since the $k = 0$ case was already treated in 8.1, let us suppose $k > 0$. Let T^k denote the k -torus $(S^1)^k$, and define $p_X^{(k)} : M \times T^k \rightarrow X$ to be the following composite map:

$$M \times T^k \xrightarrow{\text{projection}} M \xrightarrow{p_X} X.$$

By the Mayer-Vietoris type sequence for controlled K -theory, the group

$$Wh^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$$

is stably isomorphic to

$$\tilde{K}_0^{M,n-1,\epsilon}(X \times \mathbb{R}^{k-1}; p_X \times 1, \mathbb{Z}),$$

which is also stably isomorphic to

$$Wh^{M \times S^1, n, \epsilon}(X \times \mathbb{R}^{k-1}; p'_X \times 1, \mathbb{Z}).$$

The last statement is a locally-finite analogue of 7.1 of [14]. The proof given there works equally well here. Therefore $Wh^{M,n,\epsilon}(X \times \mathbb{R}^k; p_X \times 1, \mathbb{Z})$ is stably isomorphic to

$$Wh^{M \times T^k, n, \epsilon}(X; p_X^{(k)}, \mathbb{Z}),$$

for which the stable vanishing is already known. This completes the Whitehead group case.

The \tilde{K}_0 case follows from the stable vanishing of

$$Wh^{M, n+1, \epsilon}(X \times \mathbb{R}^{k+1}; p_X \times 1, \mathbb{Z}).$$

□

From this we get:

Proposition 9.6 Assume that X is a finite polyhedron and $p_X : M \rightarrow X$ is a fibration with a path-connected fiber F such that $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$ for every $k \geq 0$. Splitting along $X \times \mathbb{R}^{m-1} \times \{0\}$ induces a stable isomorphism:

$$\partial : L_n^{M, \delta, \epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) \longrightarrow L_{n-1}^{M, \delta', \epsilon'}(X \times \mathbb{R}^{m-1}; p_X \times 1, \mathbb{Z}).$$

Corollary 9.7 Let X and p_X be as above, then stability holds for $L_n^{M, \delta, \epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z})$; i.e. it is isomorphic to the limit

$$L_n^M(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) = \lim_{0 < \epsilon \ll \delta \rightarrow 0} L_n^{M, \delta, \epsilon}(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z})$$

when $0 < \epsilon \ll \delta$ and δ is sufficiently small.

Proof By the 4-periodicity, we may assume that $n > m$. Then the proposition above gives a stable isomorphism with $L_{n-m}^{\delta, \epsilon}(X; p_X, \mathbb{Z})$, and the result follows. □

Corollary 9.8 Let X and p_X be as above, then splitting along $X \times \mathbb{R}^{m-1} \times \{0\}$ induces an isomorphism

$$\partial : L_n^M(X \times \mathbb{R}^m; p_X \times 1, \mathbb{Z}) \longrightarrow L_n^M(X \times \mathbb{R}^{m-1} \times \{0\}; p_X \times 1, \mathbb{Z}).$$

Proof Immediate from 9.6 and 9.7. □

10 Controlled surgery obstructions

We discuss the controlled surgery obstructions and an application. We only consider the identity control maps on polyhedra or on the products of polyhedra and \mathbb{R}^m . X -finiteness on $X \times \mathbb{R}^m$ is the same as the usual local finiteness, so we use the following notation throughout this section :

$$\begin{aligned} L_n^{lf, \delta, \epsilon}(X \times \mathbb{R}^m) &= L_n^{X, \delta, \epsilon}(X \times \mathbb{R}^m; 1_X \times 1, \mathbb{Z}) , \\ L_n^{lf}(X \times \mathbb{R}^m) &= L_n^X(X \times \mathbb{R}^m; 1_X \times 1, \mathbb{Z}) . \end{aligned}$$

We omit the decoration ‘ lf ’ when $m = 0$.

Let $(f, b) : M \rightarrow N$ be a degree 1 normal map between connected oriented closed PL manifolds of dimension n . Quadratic construction on this produces an element $\sigma_N(f, b) \in L_n^{\delta, \epsilon}(N)$ for arbitrarily small $\delta \gg \epsilon > 0$ [13]. By 8.2, this defines an element $\sigma_N(f, b) \in L_n^{lf}(N)$. This is the *controlled surgery obstruction* of (f, b) , and its image via the forget-control map

$$L_n(N) \rightarrow L_n(\{pt.\}; N \rightarrow \{pt.\}, \mathbb{Z}) = L_n(\mathbb{Z}[\pi_1(N)])$$

is the ordinary surgery obstruction $\sigma(f, b)$ of (f, b) . The controlled surgery obstruction $\sigma_N(f, b)$ vanishes, if (f, b) is normally bordant to a sufficiently small homotopy equivalence measured on N .

Similarly, if $(f, b) : V \rightarrow W$ is a degree 1 normal map between connected open oriented PL manifolds of dimension n , we obtain its *controlled surgery obstruction* $\sigma_W(f, b)$ in $L_n^{lf}(W)$.

Theorem 10.1 *Let X be a connected oriented closed PL manifold of dimension $4k$, and $f : V^n \rightarrow W^n = X \times \mathbb{R}^{n-4k}$ be a homeomorphism of open PL manifolds. Homotope f to produce a map $g : V \rightarrow W$ which is transverse regular to $X \times \{0\} \subset X \times \mathbb{R}^{n-4k}$. Then the PL submanifold $g^{-1}(X \times \{0\})$ of V and X have the same signature : $\sigma(g^{-1}(X \times \{0\})) = \sigma(X)$.*

Proof The homeomorphism f determines a degree 1 normal map F between V and W , and hence determines an element $\sigma_W(F) \in L_n^{lf}(X \times \mathbb{R}^{n-4k})$. Repeated application of splitting 9.8 induces an isomorphism:

$$\partial^{n-4k} : L_n^{lf}(X \times \mathbb{R}^{n-4k}) \rightarrow L_{4k}(X) .$$

The image of $\sigma_W(F)$ by this map is the controlled surgery obstruction $\sigma_X(g|, b)$ of the degree 1 normal PL map $(g|, b) : Y = g^{-1}(X \times \{0\}) \rightarrow X \times \{0\} = X$. Since f is a homeomorphism, F is normally bordant to arbitrarily small

homotopy equivalences. Therefore, $\sigma_W(F)$ is zero and hence $\sigma_X(g|, b)$ is zero. This means that the ordinary surgery obstruction $\sigma(g|, b)$ is also zero. The equality $\sigma(Y) = \sigma(X)$ follows from this. \square

Now we apply the above to prove the topological invariance of the rational Pontrjagin classes [3].

Theorem 10.2 (S. P. Novikov) *If $h : M^n \rightarrow N^n$ is a homeomorphism between oriented closed PL manifolds, then $h^*p_i(N) = p_i(M)$, where p_i are the rational Pontrjagin classes.*

Proof Recall that the rational Pontrjagin classes $p_*(N) \in H^{4*}(N; \mathbb{Q})$ determine and are determined by the \mathcal{L} -genus $\mathcal{L}_*(N) \in H^{4*}(N; \mathbb{Q})$, and that the degree $4k$ component $\mathcal{L}_k(N) \in H^{4k}(N; \mathbb{Q})$ of the \mathcal{L} -genus is characterized by the property $\langle \mathcal{L}_k(N), x \rangle = \sigma(X)$ for $x \in \text{im}([N, S^{n-4k}] \rightarrow H_{4k}(N; \mathbb{Q}))$, where $X^{4k} \subset N$ is the inverse image $f^{-1}(p)$ of some regular value $p \in S^{n-4k}$ of a map $f : N \rightarrow S^{n-4k}$ which represents the Poincaré dual of x and is PL transverse regular to p . Set $x' = (h^{-1})_*(x) \in H_{4k}(M; \mathbb{Q})$ and let us show that $\langle \mathcal{L}_k(M), x' \rangle = \langle \mathcal{L}_k(N), x \rangle$.

Since X is framed in N , it has an open PL neighborhood $W = X \times \mathbb{R}^{n-4k}$ in N . Let $V = h^{-1}(W) \subset M$, then h restricts to a homeomorphism $f : V \rightarrow W$. Homotope f to get a map g which is PL transverse regular to $X = X \times \{0\}$, and set Y to be the preimage $g^{-1}(X)$, then $\langle \mathcal{L}_k(M), x' \rangle = \sigma(Y)$ and this is equal to $\sigma(X) = \langle \mathcal{L}_k(N), x \rangle$ by 10.1. \square

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School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK

Department of Applied Science, Okayama University of Science, Okayama, Okayama 700-0005, Japan

Email: a.ranicki@ed.ac.uk masayuki@mdas.ous.ac.jp