CHAPTER 21

Homology Manifolds

Shmuel Weinberger*

Department of Mathematics, University of Chicago, IL 60637, Chicago, USA

Contents

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Generalized manifolds, for the purpose of this article, are finite-dimensional metric ANR homology manifolds, i.e., spaces X with $H_n(X, X - x) = \mathbb{Z}$ and $H_i(X, X - x) = 0$ for $i \neq n$, for all points $x \in X$. (See [5] for a weaker sounding equivalent condition.) These spaces arise naturally in a number of contexts, and, moreover, their study is intimately connected to fundamental problems in geometric topology and beyond.

There have been a number of expositions of the central ideas of this subject. I will just mention [19,20,11,9,30,44,52,58]. For the most part, I will try not to repeat the ideas discussed in these sources. I do this in an attempt to encourage readers to study these other sources whose authors (aside from that of the last cited reference) tend to have rather complementary perspectives.

1. Examples

Although the definition of a (topological) manifold is simple and natural, its very nature makes it hard to check directly: one is required to construct coordinate charts. A related, but seemingly different, complaint is that the admission fee for working in the topological category is quite steep: smooth and PL manifolds have pleasant tubular and regular neighborhood theories, Morse and handlebody theories, etc. In Top, it turns out that, after a great deal of effort, these tools do exist, except in dimension four – and, furthermore, in all dimensions, the structure of the theory is simpler than either Diff or PL. (See, e.g., [41,51,57].) Our thesis is that there is a natural class of spaces that are specified by possessing certain topological properties, and that for this class, while the admission price is yet steeper, the structure of the theory is even simpler.

But I am running ahead of myself. Let me first describe some places where homology manifolds arise naturally.

(A) Suppose one has a polyhedron and would like to decide if it is a manifold. What can one do? Since calculating homology and cohomology, etc. are algorithmic, one can check Poincaré duality. However, the polyhedron might accidentally satisfy PD, without being a manifold, like the letter T does. The next idea is to use Poincaré duality for open sets, and observe that the open set around the "join" point of the T does not satisfy duality. So, we are led to call a space a "generalized manifold" or "homology manifold" if every open subset satisfies proper Poincaré duality. A little sheaf theoretic argument shows that one only need look at open regular neighborhoods of simplices.

This condition is equivalent to the even easier to motivate requirement that for each $p \in X$, the local homology groups $H_*(X, X - p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$ are same as those for a manifold (e.g., those of Euclidean space).

Beyond this, it is hard to tell. For instance, given any homology sphere Σ, the open cone $c\Sigma$ (or its suspension $\Sigma\Sigma$!) is a homology manifold. It is easy to see that if Σ is nonsimply-connected, then this cannot be a manifold: the cone point has no simply-connected (small) deleted neighborhoods. If Σ is simply-connected, then $c\Sigma$ is a manifold: except for the cones of counterexamples to the 3-dimensional Poincaré conjecture, this follows from the truth of the Poincaré conjecture: such a Σ is a sphere. (The missing case follows from [36].)

On the other hand, there is no algorithm to determine if a homology n-sphere is simply-connected for n greater than four,[1] so one cannot algorithmically determine manifoldness among polyhedra. It would seem that is would only get worse when the singularities become nonisolated. In the next section we will see that that is not true. The singularities of a PL homology manifold P, from the topological point of view, are always isolated (theorem of Cannon and Edwards). However, it certainly reassured early topologists to know that all of their homotopical theorems remained true for homology manifolds.

Based on the observation of Kervaire that all PL homology spheres bound PL contractible manifolds (except in dimension 3, where Rochlin's theorem obstructs this), Cohen and Sullivan showed that the obstruction to finding a PL manifold V with a PL map $V \to P$ with contractible point inverses was an element of $H_{n-4}(P; \Theta_3)$,[2] where Θ_3 is the group of homology spheres up to homology h-cobordism. (Indeed, after one knows about Kirby–Siebenmann theory and the Cannon–Edwards theorem, it is not hard to use this analysis to figure out which topological manifolds are homeomorphic to polyhedra (as opposed to PL manifolds): see [38,43] for an approach not using the latter input.) The V produced will be homeomorphic to P iff P is a manifold.

(B) The topological version of such maps will play a central role in our discussion, and were studied by the great Bing school of geometric topologists over the course of several decades. Their point of view was typically opposite. Rather than starting with the homology manifold, one starts with a manifold V (usually it is Euclidean space) and then crushes various subsets to points to produce a new space, which we will not denote by P as it is almost never a polyhedron, but which we will denote as X.

Here is a well-known and important example. Consider a nullhomotopic embedding of a solid torus in itself as depicted in Figure 1.

We can re-embed a whole series of such solid tori in one another. Their intersection is an interesting subset, called Wh, after J.H.C. Whitehead who first considered its complement in the 3-sphere: it is nullhomotopic in any small neighborhood of itself, because the tori are nullhomotopic in one another. This implies that $\mathbb{R}^3/Wh$ is a homology manifold. It is certainly not a manifold, indeed the fundamental group of any small deleted neighborhood of Wh in $\mathbb{R}^3$ is infinitely generated and thus the same is true for the singular point of the constructed homology manifold!

Amazingly, according to Shapiro and Andrews–Rubin, the product $\mathbb{R} \times \mathbb{R}^3/Wh$ is $\mathbb{R}^4$! (The first published manifold factor is the more complicated example [4].) I should remark that the analysis of these "Bing doublings" of knots and associated decomposition spaces is at the heart of Freedman's proof of the four-dimensional Poincaré conjecture [36].

Other explicit decompositions gave rise to a space with no manifold points that is a manifold factor [53], or even one all of whose points have only nonsimply-connected deleted

[1] It is still unknown (but unlikely) whether recognition of the 4-sphere is possible. This is because one cannot produce arbitrary finitely-presented perfect groups with vanishing H_2 as the fundamental group of a homology 4-sphere [39]. The known general constructions require "balanced presentations" which are vanishingly rare among the outputs of machines for producing groups and presentations with logical complexity.

[2] We will always denote Borel–Moore, or locally finite, homology by the unadorned symbol H – "usual homology" will be viewed as homology with compact supports.

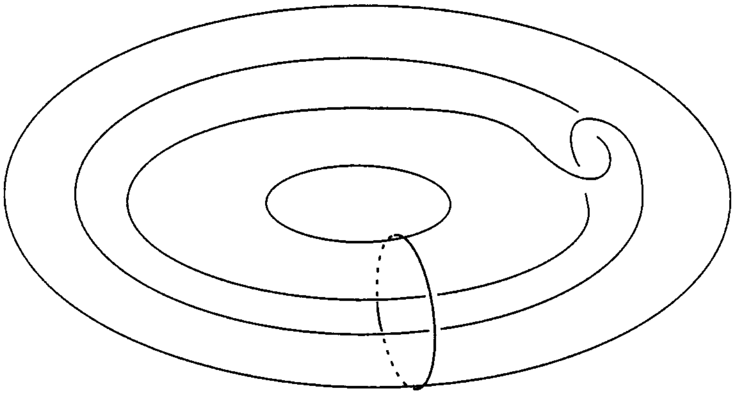

Fig. 1. First stage of the construction of the Whitehead continuum.

neighborhoods [13]. We shall discuss some of the other pathologies[3] below. I should also point out that in the early days of decomposition space theory, one produced, by carefully interlacing shrinkings around fixed-point sets, the first nonlinear group actions on Euclidean spaces (see [3,46]).

We note that this construction of homology manifolds produces them together with manifolds that very closely resemble them: We have a map $\varphi : M \to X$ which is a *hereditary homotopy equivalence*, i.e., we have that $\varphi|_{\varphi^{-1}(\mathcal{O})} : \varphi^{-1}(\mathcal{O}) \to \mathcal{O}$ is a (proper) homotopy equivalence for every open set $\mathcal{O} \subset X$. Such maps are called *CE maps*. M is said to *resolve* X, and φ is the *resolution*.

(C) A place where homology manifolds entered mathematics outside of pure topology was in the course of the proof of the following theorem:

THEOREM ([37]). *If $n > 3$, there are finitely many homeomorphism classes of Riemannian manifolds with any given upper bound on diameter and lower bounds on curvature and and volume.*

The point is that one shows that, if as one takes an infinite sequence of such manifolds, one can always extract a Gromov–Hausdorff convergent subsequence. A bit of work shows that in this case the limit is necessarily a homology manifold. Uniqueness of "approximate resolutions" (see Section 3) shows that all but finitely many manifolds in this subsequence are homeomorphic to each other, which proves the theorem.

For more information, see [47].

(D) Yet another place where homology manifolds arise naturally is in the theory of group actions on manifolds. I have already mentioned the examples of Bing and others of nonlinear actions on Euclidean spaces that stem from nonlocally flat fixed-point sets.

[3] Here pathology is not intended to be pejorative. I am speaking lovingly as would an immunologist. Indeed, the Bing school did develop many "splicing" techniques to create designer pathologies of various sorts. In studying their work, one can come to the point of view that the true subject here is the examples, not the theory built up surrounding them that I am focusing on in this survey.

In general, one can often deduce that a fixed-point set or the orbit space of the principal orbit type of a group action on a manifold is a homology manifold.[4] In my talk [58] I gave examples of results in fixed-point theory that would be obstructed if one only allowed manifold fixed-point sets, but become uniformly true when one allows (nonresolvable) homology manifolds as fixed sets.

A different sort of example is obtained by modding out $\mathbb{C}P^n$ by a decomposition of the sort discussed in (B), i.e., one which becomes "shrinkable" (the terminology will be explained in Section 2) after crossing with $\mathbb{R}$. Consider now, the circle bundle over this quotient space induced from the homotopy equivalence to $\mathbb{C}P^n$. The total space of this bundle is a manifold which the hypothesis on shrinkability identifies with S^{2n+1}. Rotating the fibers of this bundle gives a new circle action on that sphere. This action can easily be distinguished from the usual linear action on S^{2n+1} because its orbit space is not a manifold. On the other hand, the same shrinking fact easily yields the fact that for all finite subgroups $\mathbb{Z}_k \subset S^1$ the restricted actions are equivalent to the linear action. (To contrast, smoothly, one can show that a free action of S^1 on S^{2n+1} which is differentiably conjugate to a linear action for all of the finite subgroups, is in fact linear.)

2. Geometric methods

In this section, I would like to summarize some of the high points achieved by pure geometric reasoning, as distinguished from those to be discussed in Section 3 that require some algebraic organizational apparatus as well.

From the "classical period" I should probably mention the highlight theorems regarding dimension 2.

THEOREM (see [59]). *If X is a homology 2-manifold, then it is a 2-manifold.*

This was stated in a more quaint way in the old literature: e.g., a reasonable space which contains a circle and in which every circle separates the space into exactly 2 components is a 2-sphere (with a generalization to the surfaces possible, with care). Similarly, Montgomery had characterized 2-manifolds as homogeneous 2-dimensional ANR's during this classical period.

A result that seems less useful for the classification of manifolds, but is more in the spirit of later developments is the following result which paraphrases a result of R.L. Moore:

THEOREM. *Any CE map from a surface is a uniform limit of homeomorphisms.*

We say that uniform limits of homeomorphisms are *approximable by homeomorphisms*, and abbreviate this as ABH. (We leave it as an exercise to see that limits of homeomorphisms are CE.) Of course, we have seen above that this fails in dimension 3. Starting with Bing, a great deal of attention has been focused on the problem of seeing when a CE map is

[4] Actually, for some problems, one is also led by this route to consider mod p homology manifolds and other more exotic classes of spaces. For semifree circle actions the fixed-point set is a (possibly non-ANR) homology manifold.

ABH, and conversely, using funny decompositions to build designer homology manifolds with various properties. Almost all of this work rests on the fundamental:

BING SHRINKING CRITERION. A map $\varphi : X \to Y$ between compact metric spaces is ABH iff for every ε there is a self homeomorphism h of X which takes each point inverse of φ to a set of diameter $< \varepsilon$, and for which $d(\varphi, \varphi h) < \varepsilon$.

See [25,19] for an elegant (short) Baire category proof of this theorem. The decomposition of such an X into the preimages of φ is referred to as *shrinkable* when the requisite homeomorphisms h exist. Bing's criterion asserts that φ is ABH iff the decomposition is shrinkable.

It is quite instructive to convince oneself that the Whitehead decomposition is not shrinkable (although it is obvious that the quotient is not a manifold). The proof that after crossing with $\mathbb{R}$ the new decomposition is shrinkable is not at all hard (see [19, pp. 81–84]; it is all summarized in one picture on p. 83).

Besides its successes as a practical tool for deciding when various decompositions produce manifolds, one can wonder whether one is asking for too much in wanting the construction of the coordinate charts for our X to be close to the projection defining X. The following result of Siebenmann [54] shows that this is not at all the case:

THEOREM. *If M and X are manifolds of dimension > 4, then any CE $\varphi : M \to X$ is ABH.*

So shrinking the decomposition is equivalent to deciding manifoldness of the quotient. (Siebenmann's theorem is of significance for other reasons as well, as will be clear in the next section.) I should also mention that Armentrout had earlier proven the 3-dimensional version of this result assuming the Poincaré conjecture (indeed, it is equivalent to the Poincaré conjecture). The 4-dimensional case can be found in [49]. Siebenmann's motivation was the observation made by Sullivan that Novikov's proof of the topological invariance of rational Pontrjagin classes only used the hereditary homotopy equivalence property of homeomorphisms, not that they actually have unique point inverses.

The following result of Bob Edwards, conjectured by Jim Cannon following the verification of a number of important special cases (such as the double suspension problem), gives an extremely useful shrinkability criterion:

THEOREM ([25,19]). *If $\varphi : M \to X$ is a CE map, M a manifold of dimension > 4, and X is finite-dimensional (or ANR), then φ is ABH iff X has the DDP (disjoint disk property).*

DDP means that any two maps of D^2 into X can be arbitrarily closely approximated by maps whose images are disjoint from each other. It is quite easy to see that for, say, the cone on a nonsimply-connected homology sphere DDP fails: D^2's mapped in by radially extending nontrivial elements of π_1 cannot be moved disjoint from each other (indeed they cannot be moved disjoint from the cone point). On the other hand, once one crosses with $\mathbb{R}$, one can then slide in the $\mathbb{R}$ direction and make disks disjoint. Thus[5] one sees that:

[5] Modulo a little argument producing the resolution. However, aside from the 3-dimensional case, there is even a PL resolution, as we discussed in Section 1(A).

COROLLARY (Double suspension theorem). *The second suspension of any manifold homology sphere is a manifold (and hence the sphere).*

Thus, at least in high dimensions the problem of characterizing topological manifolds is reduced to the question of finding resolutions, to which we turn in the next section.

REMARK. I have given short shrift to a great deal of beautiful work that surrounds CE maps such as M. Brown's work on the Schoenflies theorem and taming theory for topological embeddings. For this, see, e.g., [20 and also 11,19].

3. Application of controlled topology

Controlled topology is the systematic reinvestigation of the classical problems of topology, such as the problem of putting boundaries on open manifolds (originally studied by Browder, Livesay, Levine, and Siebenmann), of making h-cobordisms into products (Smale, Barden, Mazur, Stallings), homotoping homotopy equivalences into homeomorphism (Kervaire, Milnor, Browder, Novikov, Sullivan, Wall), etc. with a view to keeping track of the size of the solutions of the problems.

Thus, one might want to know, for instance, how far a homotopy to a homeomorphism must move points during the course of the homotopy. Thought about this way, "controlling topology" should directly connect the type of geometric topology that has deep connections to algebraic K-theory and surgery (L-theory) to the type that studies whether maps are ABH, shrinkable, how embeddings might be tamed, etc., i.e., the type of question considered in the previous section. This is indeed the case.

Unfortunately for the novice there are many different types of control in the literature, each of which is best adapted to some problem or other: for instance, there are (epsilon) control, bounded control, and continuous control at infinity. Farrell and Jones [27] have considered "foliated control" to very good effect (see the chapter here on Topological Rigidity Theorems). We shall need "approximate" control in what follows, although, in some ways, it is the most difficult type of control of them all.

The most basic theorem in approximately controlled topology is the α-approximation theorem of Chapman and Ferry [16] (although this is an anachronistic description as it preceded the terminology by a good number of years[6]).

THEOREM. *For every open cover α of a manifold M there is a refinement β, such that* "β-controlled homotopy equivalences" *from a manifold N, $\varphi : N \to M$ are α-homotopic to homeomorphisms.*

A "β-controlled homotopy equivalence" φ, with homotopy inverse ψ, is one where the images in M of the paths followed by points during the course of the homotopies of $\varphi\psi$ and $\psi\varphi$ to the identities of M and N, respectively, always lie entirely in some element of

[6] It would be wrong for me not to mention here, though, the prescient paper [18] that did suggest some of the general context we now view as natural.

β. (Similarly for α-homotopic.) If M is compact, one can think of α and β as being small numbers measuring sizes.

Note that a CE map is a β-controlled homotopy equivalence for every β, so this theorem (essentially) includes Siebenmann's. Siebenmann's theorem is a "totally controlled" theorem, while the Chapman–Ferry theorem is the "approximately controlled" analogue. One approach to the α-approximation theorem is to obtain it from Siebenmann's theorem: show that for every α there is a β such that, for any γ, β-homotopy equivalences are α-homotopic to γ-homotopy equivalences. Letting γ get small very quickly enables one to check convergence of the sequence of maps to a continuous map, which is CE, and then by Siebenmann, ABH.

Moreover this latter approach, if done carefully, never uses the fact that the target is a manifold: one thus has a result to the effect that every "approximate resolution" is near a resolution; this refinement is necessary for the [37] result, discussed in Section 1.

Let us elucidate this point a bit. One of the important ideas in Quinn's landmark paper [48] was to liberate the control space from the problem being solved. That is one should introduce an auxiliary space P, just a finite-dimensional ANR, and require that our spaces M, N, X, whatever, all be equipped with maps to P. All of the covers, or distance measurements, will be made in P. (For the Chapman–Ferry theorem, the control map is $id: M \to M = P$.)

By allowing the control space to be an arbitrary homology manifold, one can consider the problem of homotoping a map that compares two different resolutions of the same X to be a (controlled) homeomorphism. After showing that controlled surgery theory exists, one can readily deduce from α-approximation that this is always possible (i.e., that a *uniqueness of resolution theorem* holds).[7]

Indeed, almost the same argument almost produces an existence of resolution theorem! (See [50,7].) The reason for this is that surgery theory works just as well to study the problem of when a Poincaré space is homotopy equivalent to a manifold as it does in deciding when a homotopy equivalence is homotopic to a homeomorphism (indeed the latter problem is studied as a special relative case of the former).

THEOREM. *Let X be a connected finite dimensional ANR homology manifold of dimension* > 3, *then there is an integer* $i(X)$, *which is locally defined* (*i.e., can be determined by restriction to any open subset*) *and which* $= 1$ *iff X can be resolved.* $i(X)$ *has several properties*:

(i) $i(e) = 1$,
(ii) $i(X) \equiv 1 \bmod 8$,
(iii) $i(X \times Y) = i(X)i(Y)$,
(iv) i *is invariant under CE maps.*

Indeed, it is often best to think of $i(X)$ as being the 0th Pontrjagin class of X. That is, recall that according to Novikov the rational Pontrjagin classes of manifolds are topological invariants. There are several possible approaches to the problem of directly defining

[7] Actually Quinn does not discuss controlled surgery in [48], but rather controlled h-cobordism. As a result, he requires more input (an h-cobordism rather than a controlled homotopy equivalence) to begin looking for homeomorphisms. This input he gets from Edwards' theorem.

Pontrjagin classes in a topologically invariant fashion: most traditionally, one can appeal to the theory of topological microbundles and use the fact that BSTop and BSO have the same rational homotopy to transport these classes from $H^*(BSO)$ to $H^*(BSTop)$ and then to the manifold using the classifying map of the topological tangent microbundle.

However, there is another approach, which is more closely related to what Novikov did. (Indeed, the pedigree goes back to Thom–Milnor and Rochlin before him who used such ideas in producing a PL invariant L-class for rational homology manifolds.) This method uses signatures and signatures of submanifolds to produce a homology class.[8] For manifolds one sees that this class in the top dimension is the fundamental class.

For general homology manifolds one has enough duality around on enough subsets to be able to apply the same procedure. The orientation class property is not so clear on inspection in general, and indeed it turns out not to be true. One could detect it though if one had a signature-type invariant of Poincaré spaces, which for manifolds involved the whole L-class, for then using the Poincaré structure of the homology manifold we would be able to see an invariant whose values would reflect the 0th (cohomology) L-class.

Finding such invariants of Poincaré spaces is exactly what the Novikov higher signature conjecture is about. (See the proceedings [33] for some surveys on this problem.) For instance, if $f : X \to T^n$ is a map, then the image of the L-class of X in $H_*(T^n)$ is a homotopy invariant according to old theorems of Farrell–Hsiang, Kasparov, and most important for us here Lusztig [42]. The advantage of Lusztig's approach is that he goes far enough to find an explicit homotopy invariant of Poincaré spaces with free abelian fundamental group, which for manifolds is the push-forward of the homology L-class (while the others merely show that that push-forward is a homotopy invariant).

Now, if M is a manifold of dimension n, and, say, the one-dimensional cohomology generates $H^n(M)$ under cup product, then one knows that the value of Lusztig's invariant is automatically the generator of $H_0(T^n)$. The signature of the transverse image of a point is necessarily one, because we have a degree one map to T^n and the point inverse image is a 0-manifold. What is at issue is whether this is necessarily the case for homology manifolds, and the answer is that it is not! There are nonresolvable homology manifolds not homotopy equivalent to any manifold. If one adds on a normalization axiom regarding the value of $i(M)$ for homology manifolds with free abelian fundamental groups, then this characterizes $i(X)$ (with the other axioms[9]).

Another definition of $i(X)$ for a homotopy sphere is simply this: if M is a manifold homotopy equivalent to $S^n \times \mathbb{C}P^2$ with $p_1 = 3k$, which is an approximate fibration over X,[10] then $i(X) = k$. In fact, for any X one can determine $i(X)$ from Pontrjagin classes of manifold approximate fibrations over X.

Rather than go through all of the details and repeat arguments that already exist in the literature, let me give a second description of where the $\mathbb{Z}$ obstruction comes from, and why it should be the only obstruction. (I'll engage in high-level philosophy, rather than even in heuristics.)

8 More precisely, what is used is the theory of self-dual complexes of sheaves [14].

9 The normalization and product axioms actually follow from a calculation for a torus and multiplicativity of Lusztig's invariant.

10 This map is a generalized CE map: one wants the homotopy fiber of the restriction of the map to any open set to be independent of the open set – rather than be contractible, which is the CE condition.

A key property of surgery groups is their periodicity: $L_n(\square) \cong L_{n+4}(\square)$ whatever we put in the box. Now according to the α-approximation theorem there are no obstructions to making arbitrary controlled homotopy equivalences to M, controlled homotopic to homeomorphisms; in particular, there are no characteristic class obstructions. This means that the obstructions to getting the controlled homotopy equivalence must be rich enough to account for all the possible characteristic classes of all the maps that we might try to surger to a controlled equivalence. The aggregate of these is given by maps $[M : G/Top]$. By periodicity, we also see that the controlled surgery groups account for $[M \times D^4, \partial; G/Top]$ or D^{4k} for that matter. These sets of characteristic class sets are almost, but not quite, isomorphic: $\Omega^4(G/Top) \cong \mathbb{Z} \times G/Top$. The characteristic class theory is deficient in dimension 0; an integer which would be needed to have genuine periodicity – a periodicity which is forced by the structure of surgery theory to hold for the obstruction groups. This deficient $\mathbb{Z}$ is the $\mathbb{Z}$ that arises in the resolution theorem.

Also, since G/Top is a split summand of $\mathbb{Z} \times G/Top$ it is not surprising that one is not troubled in uniqueness theorems.

4. Toward a positive classification theory of homology manifolds

The previous section showed that many homology manifolds, namely those with $i(X) = 1$, are resolvable, and therefore one can recognize manifoldness for homology manifolds as being the same as DDP with $i(X) = 1$. It leaves open what one can say in general.

In [6,7] beginnings were made on this problem. In particular, the examples alluded to of homology manifolds not homotopy equivalent to resolvable ones were constructed. The main theorem of these papers can be phrased as:

THEOREM. *One can classify homology manifolds up to s-cobordism by means of surgery theory.*

This implies that for homology manifolds, one does have a fully periodic classification theory (above low dimensions). Let me state it carefully.

THEOREM. *If X is a homology manifold of dimension exceeding 5, then there is an isomorphism between the following two ("structure") sets: $S(X) \approx S(X \times D^4)$, where $S(X) = \{\varphi : Y \to X$, a simple homotopy equivalence, which is already a homeomorphism on any boundaries$\}$/s-cobordism and $S(X \times D^4)$ is the analogous set with $X \times D^4$ replacing X.*

Note the latter set is an abelian group (by exactly the same method one puts the group structure on homotopy groups: i.e., by "stacking")! Group structures were produced on topological manifold structure sets (see [55]), but periodicity fails as we discussed above.[11] Using homology manifold structure sets, one not only has a group structure, but much more: structure sets become covariant abelian group-valued functors: i.e., it is possible to

[11] Siebenmann observed that it was very close to being true.

push forward structures with respect to continuous maps between manifolds (or homology manifolds) whose dimensions differ by a multiple of four,[12] a process which one must admit is geometrically rather obscure.

Observe that if X is a manifold then the latter set is built up entirely out of manifolds (or at least resolvable homology manifolds, which are s-cobordant to manifolds by taking the mapping cylinder of a resolution map). Thus, homology manifolds fill in lacunae in the theory of manifolds. An important special case is:

$$S(S^n) \cong S(\mathbb{R}^n) \cong \mathbb{Z}.$$

The isomorphism is given by $(i(X) - 1)/8$. Thus, for instance, there are homotopy spheres other than the sphere in this category, but for each local index, it is unique up to s-cobordism, just as usual.

Obviously, one would want to know whether for local index other than 1 there are preferred local models which are, say, topologically homogeneous (for $i = 1$, these are the manifolds) and whether for these the s-cobordism theorem is true. I strongly believe this to be the case, but these problems have resisted a number of attempts.

To follow the outline suggested by Edwards' theorem, one might conjecture that the "good" models are determined by requiring the DDP. In [7] the construction produces DDP homology manifolds in every class of $S(X)$. In fact, in [8] we show that controlled surgery can be extended to DDP homology manifolds which implies, for instance:

COROLLARY. *Every homology manifold is resolvable by a DDP homology manifold.*

Let me briefly sketch the method by which these theorems are proven. All of the theorems are proven by (patching together suitable local versions of) the following argument: Suppose that we are given a Poincaré complex X which has vanishing total surgery obstruction, and we want to show it homotopy equivalent to a homology manifold. Then X has a normal invariant, i.e., there is a manifold M with a degree one normal map $M \to X$. Now, if the surgery obstruction of this vanished, then one could normally cobord this map to a homotopy equivalence from a manifold to X, and we would certainly be done. However, if the obstruction is nonzero, this is impossible, and we have to understand what it is that the condition of vanishing total surgery gives.

What it gives is this: the surgery obstruction must live in the image of $H_n(X; \mathbf{L}(e))$ under the assembly map. Now, this is the same thing as the controlled L-group $L^c(X \downarrow X)$, so we can act (in the sense of the action given in the surgery exact sequence, namely Wall realization) on the structures by this element on the structures of any manifold that UV^1 maps to X. The relevant manifold we choose is the boundary of a regular neighborhood of a 2-skeleton of M.

This using the new homotopy equivalence from the other side of the Wall realization to the boundary we can glue on this contraption between the sides of M separated by the boundary of this regular neighborhood. What we then obtain is a space mapping to

[12] This is somewhat reminiscent of the kind of conditions that Atiyah and Hirzebruch imposed to get wrong-way functoriality in K-theory: a small spin and/or complexity condition and the assumption that the manifolds have dimensions of the same parity.

X, whose surgery obstruction is now zero. Since we used controlled Wall realization, the space has good controlled Poincaré duality over X (although it certainly is not a homology manifold). A little thought shows that surgery theory can be applied to spaces that have the mild singularities of this one, so we obtain a controlled Poincaré complex, controlled over X, homotopy equivalent to X.

This is progress. What we would like to have is a space which is controlled Poincaré when measured over *itself*, i.e., a homology manifold, rather than just over our initial X. To do this we iterate the construction, with some care, so that each stage becomes controlled over the previous one, so that the limit is controlled over itself. This completes the sketch.

None of these methods produce homeomorphisms or even CE maps out of a given initially chosen homology manifold. Such would be necessary for proving homogeneity of these spaces or the s-cobordism theorem and the other standard tools of topological topology for DDP homology manifolds. In [6] we conjectured that all these and more are true. If correct, the picture of DDP homology manifolds will be very similar to that of ordinary topological manifolds.

REMARK. Some "evidence by analogy", based on experience from the theory of orbifolds in particular, and stratified spaces in general, for this collection of conjectures is suggested in [58]; this analogy extends the analogy between taming theory and decomposition spaces that [11] and [20] discuss. For instance, there are homogeneity, s-cobordism, local contractibility of homeomorphism group theorems for classes of spaces described purely in terms of local homotopical properties.

Personally, the only conjecture from the list in [6] that I have any doubts at all about is the local contractibility of homeomorphism group conjecture, and even that one I think is likely.

It is probably also worth mentioning that the analogies between surgery theory and index theory of elliptic operators that are developed in the appendix to chapter nine of [57], as well as in a number of papers in [33], work a lot better if one includes homology manifolds in surgery theory. (It works even better if one allows generalized orbifolds, where the strata are only homology manifolds, to mimic equivariant indices of operators.)

REMARK. A start on another basic geometric problem for homology manifolds, namely transversality, was made by Johnston in her thesis [40]. She shows, that while transversality to sub-homology manifolds of homology manifolds is obstructed, there are many cases where maps out of homology manifolds into manifolds are unobstructed. Moreover, her work shows that there are maps that are homotopic to transverse maps, but not approximable by such.

While these phenomena might make one wonder about how seriously to take the idea that "what difference should it make whether $i = 1$?", note, however, that DDP is what is necessary for general position, which is more used in building homeomorphisms than is transversality, per se, and that (again relying on our stratified experience) in the case of orbifolds [58], there are obstructions to transversality as well, but that ultimately, despite this, one can succeed in in proving all of the theorems conjectured to hold for homology manifolds.[13] However, it *does* show that there are interesting phenomena left to be discovered about their geometric nature where homology manifolds do not too closely parallel manifolds.

5. Remarks on infinite dimensions

A great deal of work has been done on infinite-dimensional manifolds of various sorts, some by methods intrinsic to that subject, others adaptations of the ideas of Section 2. The result most in the spirit of that section is:

TORUNCYK'S THEOREM. *An ANR is a Hilbert cube manifold iff it has the disjoint k-disk property for all k.*

To emphasize again one of our themes, here is a theorem that deduces homogeneity from local homotopical properties. The main reason that Edwards can get by with 2-disks and Toruncyk requires all disks is that homology manifolds have local Poincaré duality, while in infinite dimensions there is no corresponding principle.

More in the spirit of Section 3, one can study homology n-manifolds that have infinite covering dimension. That such spaces exist at all is itself a remarkable result of Dranishnikov [22], who had shown that finite cohomological dimension is consistent with infinite covering dimension. His examples are cell-like images of finite-dimensional spaces (a theorem of Edwards forces this). He detects their infinite dimensionality by showing that they and their finite-dimensional "resolutions" have different K-theory.

(It is important here to use a nonconnective theory like K-theory because for ordinary homology and cohomology, the Vietoris–Begle theorem gives an equivalence, and one can deduce the case of any connective theory from this.)

Not so much is known about infinite-dimensional homology n-manifolds. It is quite easy to see that they can possess a dearth of finite-dimensional subspaces, and have other counterintuitive features. On the other hand, they do arise fairly naturally in some contexts: A theorem of Engel–Moore produces such examples as Gromov–Hausdorff limits of manifolds with some uniform contractibility function (class $\mathrm{LC}(\rho)$).

Subsequently, Ferry [29] established a simple homotopy theory for such limits, and showed that every compact homology n-manifold has at most finitely many resolutions [28]. This can be viewed as saying that only finitely many $\mathrm{LC}(\rho)$ manifolds can mutually degenerate in Gromov–Hausdorff space. Such theorems can be viewed as natural extensions of the theorem of [37] discussed above.

A really new wrinkle then arises: there are examples where there are distinct resolutions (always an odd number, because G/Top is Eilenberg–MacLane at 2, and is therefore essentially connective as far as convergence properties of spectral sequences are concerned). These were found by Dranishnikov and Ferry [23]. The same method also produces homology manifolds without finite-dimensional resolutions (by closed homology manifolds). The basic point behind the construction is that controlled surgery is a homology theory

[13] It did, however, interfere with with earlier attempts at classification results (notably the tour de force by [45]) which only succeeded for odd-order groups where transversality is shown to be possible.

closely related at odd primes to periodic K-theory. The difference between the Htheories of X and its resolving manifold gives a place for different resolutions to come from.

Finally, in [24], these methods are applied to produce interesting noncompact finite-dimensional Riemannian manifolds which contradict some of the standard "large scale" conjectures related to the Novikov conjecture. For instance, a uniformly contractible manifold is constructed there, which is not boundedly topologically rigid. (Universal covers of aspherical manifolds are always uniformly contractible, so the rigidity of such manifolds is an analogue of the Borel rigidity conjecture for aspherical manifolds – and its verification for many manifolds, such as Hadamard manifolds, is not that difficult and is an important tool in proving the Novikov conjecture for the compact aspherical manifolds they cover; see [35].) The homology manifold that arises in this example is a Gromov–Hausdorff limit of rescaled versions of the spheres of various radii and is a cell-like image of the sphere whose K-theory differs from that of the usual sphere.

While it is not yet clear whether the point-set geometric topology of these spaces has any features to recommend it, it seems that these spaces do arise from time to time, and do deserve study, at least to the extent of using them as control spaces, and maybe even to the point of studying them under some less restrictive equivalence relation that homeomorphism.

6. Some problems

This is not, by any means, intended to be a comprehensive list of problems. It is a list of questions some of which I think are actually important, and some others of which one can make progress on.

(A) Homogeneity (etc.). This was mention in Section 4. The problem is to show that there are nice homogeneous models for k-dimensional homology manifolds with given local index $= i$. "Manifolds" modelled on these should satisfy the s-cobordism theorem.

Presumably, the locally best examples will be characterized by satisfying the DDP, although it is conceivable that some other property will pick out the "best examples" and only later will it end up being equivalent to DDP.

In other words, one might find the good models somewhere, without being able to prove right away the analogue of the double suspension theorem or Edwards' theorem. Probably it would be a mistake to think of that as a failure.

(B) $\times\mathbb{R}$ problem. One does not know whether or not every decomposition of a manifold becomes shrinkable after crossing with a single copy of $\mathbb{R}$. Daverman has long ago shown that 2 copies are enough. Even a moron can show that 5 are enough.

(C) Systematic classification of nonlocally flat embeddings. Much of the theory discussed in Section 3 grew out of the problems of producing mapping cylinder structures for submanifolds, or taming them up to approximation. I think it might be interesting to study nonlocally flat embeddings with some given measure of nonlocal flatness, say in terms of a perfect group which local fundamental groups are given a map to. (This is a measure of nonlocal flatness because one knows that local flatness in codimension not two is the same as local simple connectivity.) For instance, it is not hard to see that while all embeddings of S^n in S^{n+c}, $c > 2$, can be approximated by locally flat ones (which are of course unique up to isotopy), they are not all isotopic if we require that the complement always have a map to, say, the dodecahedral group. (The circles occurring from double suspensions of different homology spheres are often not concordant in this sense.)

I am not sure whether every concordance class has a homogeneous representative (this could be especially interesting in codimension 3).

(D) Low dimensions. I have mentioned that the theory in dimension three has connections to the Poincaré conjecture. See [52] for more survey of dimension three. In dimension four, one does not even have a conjectured characterization of manifolds. This is clearly a matter of prime importance.

(E) Infinite dimensions. It would be good to have an understanding of which infinite dimensional homology n-manifolds can be resolved, and by how many manifolds. The point-set theory here seems to be wide open.

(F) Mod p-homology manifolds or $\mathbb{Q}$-homology manifolds. Here very little is known. If one starts by thinking about the polyhedral case, one sees immediate difficulties because the group of homology n-spheres up to homology h-cobordism is rather large and complicated (see [1] and [2]), unlike the case for integral homology spheres.

However, in transformation groups these spaces arise naturally, and it would be good to have at least obstructions to resolution for them and surgery up to s-cobordism.

Incidentally, for ANR $\mathbb{Q}$-homology manifolds, one can still define the local index. I very much doubt that $i(X) \equiv 1 \bmod 8$ still holds, but I haven't succeeded in producing examples.

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