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Topology of homology manifolds.

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Homology manifolds are topological spaces with the local homology properties of manifolds. More formally, an n-dimensional homology manifold is a topological space X such that at each point $x\in X$,

$$H_r(X,X\smallsetminus\{x\})=\begin{cases}\mathbf{Z} & \text{if } r=n,\\ 0 & \text{if } r\neq n.\end{cases}$$

Homology manifolds are also known as generalized manifolds.

Topological manifolds are themselves examples of homology manifolds. A resolution (M,f) of a space X is defined to be an n-dimensional topological manifold M with a surjection $f\colon M\to X$ such that each inverse image $f^{-1}(x)$ $(x\in X)$ is contractible in each of its neighbourhoods in M, in which case X is an n-dimensional homology manifold. Does every homology manifold admit a resolution? The paper under review constructs compact ANR homology manifolds in dimensions ≥ 5 which do not admit a resolution. Nonresolvable homology manifolds are called exotic. The exotic homology manifolds are important new objects in high-dimensional manifold theory. Their construction may be compared to the observational discovery of a new planet whose existence was predicted by calculations. In order to properly appreciate the significance of the exotic homology manifolds, it is best to trace the history of homology manifolds.

Homology manifolds were introduced in 1933 independently by Čech and Lefschetz. In the first instance, homology manifolds were artefacts of algebraic topology, serving to clarify the homological properties of manifolds. Much of the algebraic topology of manifolds works just as well for homology manifolds. In particular, the cohomology and homology of a compact oriented n-dimensional ANR homology manifold X are related by Poincaré duality isomorphisms $H^{n-*}(X)\cong H_*(X)$. Homology manifolds also became artefacts of geometric topology. The decomposition theory of manifolds pioneered in the 1920's and 1930's by R. L. Moore and G. T. Whyburn was refashioned in terms of homology manifolds by R. Wilder in the 1930's and 1940's, and carried forward by R. H. Bing in the 1950's and 1960's. All this led to the use of homology manifolds in the geometric study of the connectivity and separation properties of manifolds.

Homology manifolds play an important role in the recognition problem for topological manifolds: formulate a general position property on a homology manifold X which is necessary and sufficient for X to be a topological manifold. The beautiful paper by J. W. Cannon "The recognition problem: what is a topological manifold?" [Bull. Amer. Math. Soc. **84** (1978), no. 5, 832–866; MR **58** #13043] was written at a previous turning point in the history of the problem, shortly after the work of Cannon and R. D. Edwards on the double suspension of homology spheres. The paper included the formulation of the "disjoint disk property", and the statement of a theorem of Edwards that for $n \geq 5$ a resolvable n-dimensional ANR homology manifold with this property is an n-dimensional topological manifold. A proof of the theorem may be found in the book of R. J. Daverman [*Decompositions of manifolds*, Academic Press, Orlando, FL, 1986; MR 88a:57001]. The theorem reduces the recognition problem for topological manifolds to the problem of characterizing the ANR homology manifolds which admit resolutions. With hindsight, it is clear that this was the best result geometric topology could achieve without algebraic surgery theory.

The Browder-Novikov-Sullivan-Wall surgery theory was developed in the 1960's, solving the homotopy-theoretic version of the recognition problem: when is a space with n-dimensional Poincaré duality homotopy equivalent to an n-dimensional manifold? For $n \geq 5$ an n-dimensional Poincaré duality space X is homotopy equivalent to an n-dimensional topological manifold if and only if the Spivak normal fibration of X admits a topological reduction with 0 surgery obstruction. Since homology manifolds are Poincaré duality spaces it is natural to ask if they are homotopy equivalent to topological manifolds. D. E. Galewski and R. J. Stern [Invent. Math. **39** (1977), no. 3, 277–292; MR **56** #3847] showed that the Spivak normal fibration ν_X of a polyhedral homology manifold X admits a canonical topological reduction with 0 surgery obstruction, so that if $\dim(X) \geq 5$ then X is homotopy equivalent to a topological manifold. The triangulability of X was an essential ingredient of the method. The results of Cannon and Edwards allowed Galewski and Stern [Ann. of Math. (2) **111** (1980), no. 1, 1–34; MR 81f:57012] to prove that in fact every polyhedral homology manifold of dimension ≥ 5 admits a resolution by a (triangulated) topological manifold. With hindsight, it is clear that this was the best result algebraic topology could achieve without geometric surgery theory.

Further progress depended on the development of the controlled topology pioneered in the 1970's by T. A. Chapman, Ferry, and F. Quinn. Controlled topology grew out of the use of noncompact manifolds in Novikov's proof of the topological invariance of rational Pontryagin classes and in the Kirby-Siebenmann structure theory of compact topological manifolds. The methods of combinatorial topology can be extended to non-triangulable compact spaces by the systematic use of metric conditions to limit the size of allowed operations. From this point of view a resolution is a sufficiently controlled homotopy equivalence.

Controlled topology was first applied to the resolution problem for homology manifolds by

Quinn [Invent. Math. **72** (1983), no. 2, 267–284; MR 85b:57023]. The paper had the following programme for proving that every ANR homology manifold X can be resolved: (1) view X as an X-controlled Poincaré duality space, meaning that the Poincaré duality chain equivalence $[X]\cap -\colon C(X)^{n-*}\to C(X)$ has "small point inverses" when measured in X, and (2) extend in that case Quinn's methods [Ann. of Math. (2) **110** (1979), no. 2, 275–331; MR 82k:57009] to develop just enough controlled surgery theory to prove that the controlled Poincaré space in (1) is controlled homotopy equivalent to a topological manifold, and hence resolvable. In Quinn's 1983 paper [op. cit.; MR 85b:57023] it was claimed that the programme works, and that every ANR homology manifold X of dimension ≥ 5 admits a resolution. Unfortunately, a potential obstruction in (2) had been overlooked. The result was withdrawn, and replaced by Quinn's 1987 study [Michigan Math. J. **34** (1987), no. 2, 285–291; MR 88j:57016]. That paper introduced an invariant $I(X)\in \mathbf{Z}$ (for connected X) such that $I(X)=0$ if and only if X admits a resolution. Roughly speaking, $8I(X)+1$ is the number of "points" in the generic "point inverse" of $[X]\cap -\colon C(X)^{n-*}\to C(X)$, with $I(X)=0$ for resolvable X by topological transversality. Either every X was resolvable or else the invariant was realized by an exotic X. The hunt was on.

The main new results in the paper under review are: (i) for every compact simply-connected n-dimensional topological manifold M with $n\geq 6$ and every integer $k\in\mathbf{Z}$ there exists a compact n-dimensional ANR homology manifold X homotopy equivalent to M with $I(X)=k$, and (ii) a non-simply-connected surgery classification theory for compact ANR homology manifolds of dimension ≥ 6, including analogues of the Sullivan-Wall surgery exact sequence and the reviewer's total surgery obstruction.

The simply-connected examples in (i) are obtained by an ingenious infinite process which constructs a sequence of ever more controlled Poincaré duality spaces, converging to a compact ANR homology manifold X in the homotopy type of M realizing the prescribed resolution obstruction $I(X)=k$. Unfortunately, Quinn did not work out all the controlled surgery theory necessary for the classification of homology manifolds. The surgery classification in (ii) uses the bounded surgery theory of Ferry and E. K. Pedersen [in *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, 167–226, Cambridge Univ. Press, Cambridge, 1995] instead. A controlled surgery problem determines a bounded surgery problem over the open cone. A solution of the bounded surgery problem determines a solution of the original controlled surgery problem by codimension-1 splitting, but it is not clear if the solution is sufficiently controlled for the application to (ii). The proof of such a result is to appear in a paper by Ferry and Pedersen ["Squeezing structures", to appear]. Until this paper or some appropriate substitute becomes available the surgery classification in (ii) must be regarded as somewhat provisional—although there is little doubt among the experts that it is correct.

Even though the exotic homology manifolds have now been hunted down, much remains to be done, apart from completing the surgery classification programme.

An n-dimensional topological manifold is homogeneous, since every point has a neighbourhood homeomorphic to the Euclidean space $\mathbf{R}^n$. Is there a corresponding homogeneity for exotic homology manifolds with the disjoint disk property? The paper formulates the intriguing conjecture that there exist spaces $\mathbf{R}^4_k$ ($k \in \mathbf{Z}$) such that every connected ANR homology manifold X_n, $n \geq 5$, with the disjoint disk property and $I(X) = k$ is locally homeomorphic to $\mathbf{R}^4_k \times \mathbf{R}^{n-4}$. The hunt is now on for $\mathbf{R}^4_k$!

Homology manifolds may well feature in work on the Borel conjecture for Poincaré duality groups, and the closely related Novikov conjecture on the homotopy invariance of the higher signatures. The classifying space of an n-dimensional Poincaré duality group π is an aspherical n-dimensional Poincaré duality space $B\pi$. The Borel conjecture for π is that $B\pi$ is homotopy equivalent to a compact n-dimensional topological manifold. The conjecture has been verified for a large class of Poincaré duality groups. The hunt is also on for aspherical nonresolvable homology manifolds, since their fundamental groups would be counterexamples to the Borel conjecture.

The report of Weinberger [in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, 637–647, Birkhäuser, Basel, 1995] explores the analogy between homology manifolds and orbifolds, pointing towards new applications of homology manifolds in the study of group actions.

In conclusion, it should be noted that the discovery of the exotic homology manifolds provides a remarkable example of collaboration between algebraic and geometric topologists. The reviewer hopes that collaborations of this type will lead to further progress into this fascinating realm of topology.

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