TRANSVERSALITY FOR HOMOLOGY MANIFOLDS

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Abstract. Transversality phenomena are studied for homology manifolds. For homology manifolds X, Y and Z, with Z embedded in Y with a neighborhood $\nu(Z)$ which has a given bundle structure, we define a map $f : X \to Y$ to be transverse to Z, if $f^{-1}(Z) = Z'$ is a homology manifold, the neighborhood $f^{-1}(\nu(Z))$ has a bundle structure given by $f^*\nu(Z)$ and f induces the bundle map. In the case where the range is a manifold an arbitrary map is s-cobordant to a transverse map if the submanifold is codimension one and (π, π) or codimension greater than two. Appropriate homology manifold versions of related splitting and embedding theorems are proved for homology manifolds. As a group, bordism of high dimensional homology manifolds has one copy of the bordism of topological manifolds for each possible index.

1. Introduction

We are interested in the following problem which makes sense in many different categories of spaces: If X, Y and Z are objects in a given category $\mathcal{C}$, $\nu(Z)$ is a $\mathcal{C}-$normal neighborhood of Z in Y and $f : X \to Y$ is a morphism in the category $\mathcal{C}$, when can f be replaced by a $\mathcal{C}$-transverse map? For example Thom's celebrated transversality theorem says that any map in the smooth category can be approximated by a transverse map. Similar theorems for the PL and TOP categories are due to Rourke and Sanderson [RS] and Kirby-Siebenmann [KS] respectively.

This paper explores this question in the category of homology manifolds. A homology manifold X of dimension n is a finite dimensional ANR with the local homology of a manifold, i.e. for any point $x \in X$, $H_*(X, X \setminus x) \simeq H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. Homology manifolds are an object of classical mathematical interest. They have been studied via sheaf theory and point set topology (see the work of Bing and his school) and more recently via controlled topology, [Q] and [BFMW].

The question of transversality for homology manifolds was first proposed by Quinn. Transversality for homology manifolds is seen to fail in general due to the exotic local nature of the structures of unresolvable homology manifolds. Given two exotic local structures U_n and V_k of different types and of dimensions n and k respectively, there does not generally exist a splitting of the local structures $U_n = V_k \times W_{n-k}$. This parallels the local splitting problem in the equivariant setting. For example, Browder- Livesay studied the obstruction to decomposing an involution $(\mathbb{R}^n, \Sigma)$ as $(\mathbb{R}, -) \times (\mathbb{R}^{n-1}, T)$ for any involution T. See [J1] for a study of equivariant transversality for PL locally linear actions of the group $\mathbb{Z}_2$. The failure of transversality for homology manifolds is noted in [We2].

This paper explores the question of transversality for homology manifolds. The paper begins with some results related to transversality. Homology manifolds versions of the (π, π) and Browder splitting theorems and the Browder-Casson-Haefliger-Sullivan-Wall embedding theorems are proved.

Theorem 1.1. *Given X and Y high dimensional Poincaré spaces so that Y is Poincaré embedded in X via a spherical fibration of dimension at least 2, then any homology manifold structure on X defines an obstruction in $L(Y)$ which vanishes if and only if the structure splits, i.e. has a representative which restricts to a structure on Y. If Y is codimension 1 in X and $X = V_1 \cup_Y V_2$ so that the inclusion $Y \subset V_1$ induces an isomorphism of fundamental groups, then any structure on X splits.*

See theorems **??** and **??** below.

Theorem 1.2. *A Poincaré embedding of high dimensional homology manifolds which is given by a spherical fibraton of dimenson at least 2 is s-cobordant to an embedding.*

See theorem **??** below.

Interestingly, although the proofs of the manifold versions of these results rely heavily on transversality and the Wall surgery exact sequence, the homology manifold versions are proven without transversality, using only the homology manifold surgery exact sequence. In the homology manifold setting these splitting and embedding results are used to prove transversality theorems.

Transversality for homology manifolds is defined for X, Y and Z homology manifolds with $Z \subset \nu(Z) \subset Y$ where $\nu(Z)$ is the total space of a topological bundle over Z as follows: $f : X \to Y$ is transverse to Z if $f^{-1}(Z)$ is a homology manifold with a neighborhood $f^{-1}(\nu(Z)) = f^*\nu(Z)$ so that f is the bundle map. The two main theorems about homology manifold transversality give information about the success of transversality in this setting.

Theorem 1.3. *Given $f : X \to Y$ with X, Y and Z high dimensional homology manifolds as above. If in addition Y and Z are manifolds, then f is s-cobordant to a transverse map if $Z \subset Y$ is codimension one with the added condition that $Y = Y_1 \cup_Z Y_2$ and $\pi_1(Z) \simeq \pi_1(Y_1)$ or if $Z \subset Y$ is codimension at least* 3.

See theorems **??** and **??** below.

As in Thom's work, transversality results are closely connected to the calculation of bordism. Let $\Omega_*^H(X)$ denote homology manifold bordism.

Theorem 1.4. *In dimensions ≥ 6 we have an isomorphism of Abelian groups*

$$\Omega_*^H(X) \simeq \Omega_*^{TOP}(X)[8\mathbb{Z} + 1].$$

See theorem **??** below. As an Abelian group $\Omega_*^{TOP}(X)[8\mathbb{Z}+1]$ is just $Maps(\mathbb{Z}, \Omega_*^{TOP}(X))$ where the $\mathbb{Z}$ corresponds to the Quinn index by the map $x \to 8x + 1$. The notation is meant to suggest an expected multiplicative structure, but unfortunately the given map does not yield a ring isomorphism.

We make use of the analysis of obstructions to Poincaré transversality and related transversality structures (see [HV] for an account) and the homology manifold surgery exact sequence of [BFMW] to prove these theorems. First we apply Poincaré transversality theorems to get a Poincaré space P as the Poincaré transverse inverse image. Then we would like to perform Browder splitting to complete the proof. There is a priori an obstruction to doing this. However an embedding trick allows us to find a different solution to the Poincaré transversality problem which has vanishing splitting obstruction.

The bordism theorem is proven by comparing homology manifold bordism to Poincaré bordism and bordism of topological manifolds. The proof uses a construction of 0 and 1 surgery for homology manifolds to reduce to the situation in which there exists an isomorphism of fundamental groups. The proof uses the homology manifold surgery exact sequence and a construction similar to that used in the proof of the transversality theorem.

2. Definitions and Notation

2.1. **Poincaré Duality Spaces.**

Poincaré spaces are most known for their use in surgery theory for the study of manifold structures. These spaces have also been studied as interesting in and of themselves. See for example the comprehensive book of Hausmann and Vogel, [HV]. We will use the definitions of Poincaré and normal spaces, called PD-spaces and Q-spaces respectively, which are given in [HV].

Transversality for Poincaré spaces is defined using a type of normal structure called a CD_q-structure.

Definition 2.1. *A pair of spaces (X, A) with A closed in X is called a CD_q-pair, if it has a CD_q-structure, i.e. a pair $(N_A, \partial N_A)$ of closed subspaces of X giving rise to a decomposition. $X = N_A \cup [(X \setminus N_A) \cup \partial N_A]$, $N_A \cap [(X \setminus N_A) \cup \partial N_A] = \partial N_A$ such that*
1) N_A is a neighborhood of A and the inclusion $A \subset N_A$ is a homotopy equivalence.
2) The homotopy fiber of the inclusion $\partial N_A \to N_A$ is S^{q-1}.
3) The inclusion $\partial N_A \subset (N_A \setminus A)$ is a homotopy equivalence.

Definition 2.2. *We say that a map between Poincaré spaces $P \xrightarrow{f} X$, with a CD_q-pair (X, A), is Poincaré transverse to A, if*
1) $(P, f^{-1}(A))$ admits a CD_q-structure $(f^{-1}(N_A), f^{-1}(\partial N_A))$, so that f induces a map of spherical fibrations.
2) The same is true for ∂P and the inclusion $f^{-1}(N_A) \cap \partial P \subset f^{-1}(N_A)$ induces a morphism of spherical fibrations.
3)The decompositions of P and ∂P given by the CD_q-structures are Poincaré decompositions.

We shall say that a pair of spaces (X, Y) is a (π, π) -pair if the inclusion $X \subset Y$ induces an isomorphism of fundamental groups. The celebrated theorem of [Wa], that $L(\pi, \pi) \simeq 0$, is then called the (π, π) -theorem. A consequence of this theorem is the following Poincaré transversality theorem, see e.g. [HV].

Theorem 2.3. *Poincaré transversality holds for a CD_1-pair (X, A) so that $X = X_1 \cup_A X_2$ and (X_1, A) is a (π, π) -pair.*

In a setting where Poincaré transversality holds, a consistent method for making maps transverse is called a transversality structure. Given a Poincaré space P and Spivak fibration η, with Thom space $T(\eta)$, an (extrinsic) transversality structure on P is a way of making simplices of $T(\eta)$ transverse to P. Note that for any PL manifold M, $M \to T(\eta)$ can be made Poincaré transverse to P, by using the transversality structure on η to make all simplices of M Poincaré transverse to P in a consistent way.

According to Levitt and Morgan [LM], a Spivak fibration η has a transversality structure if and for $\dim \eta \geq 3$ and P 4-connected only if, η has a PL reduction.

Their methods and TOP transversality [KS] yield the same theorem in the TOP category. We will be interested in the easier direction.

Theorem 2.4. *Given $\eta \to P$ a spherical fiber space, a TOP reduction of η defines a transversality structure for η.*

2.2. **Homology Manifolds.**

Definition 2.5. *Define a homology manifold of dimension n to be a finite dimensional absolute neighborhood retract (ANR) X so that for every $x \in X$ $H_*(X, X \setminus x) = H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$*

Let I(X) denote Quinn's integer obstruction to resolution for homology manifolds. [Q3]

Let $\mathbb{L}_*$ or just $\mathbb{L}$ denote the 4-periodic 0-connective spectrum whose homotopy groups are the surgery obstruction groups of Wall. This spectrum is also known as the quadratic L-theory spectrum. Let $\mathbb{L}^*$ denote the corresponding symmetric L-theory spectrum of Ranicki. If X is a homology n-manifold, it has a canonical L-theory orientation $[X]_{\mathbb{L}} \in H_n(X, \mathbb{L}^*)$ so that $H^0(X, \mathbb{L}_*) \xrightarrow{\cap [X]_{\mathbb{L}}} H_n(X, \mathbb{L}_*)$ is an isomorphism, see [R2]. Ferry and Pedersen, [FP] have used this result to show that the Spivak fibration of a homology manifold has a canonical TOP reduction, which we will call the Ferry-Pedersen reduction.

Definition 2.6. *Given a Poincaré space P, define $S^H(P)$ to be the (possibly empty) set of simple homotopy equivalences $X \to P$ for X a homology manifold up to s-cobordism.*

This is the definition used by Bryant, Ferry, Mio and Weinberger, for which they have proven a surgery exact sequence for homology manifolds, [BFMW]. In this sequence the homology manifold normal invariants are given by $H_n(P, \mathbb{L})$ where H_* denotes locally finite homology. Similarly there exists a surgery exact sequence for n-ads of homology manifolds.

3. Topology of Homology Manifolds

Many important splitting and embedding theorems for manifolds were proven using manifold surgery theory. In this section we use the homology manifold surgery exact sequence to prove similar theorems for homology manifolds. These splitting and embedding theorems will play a central role in proving the transversality theorems to follow.

Theorem 3.1. **((π, π) -splitting)** *Given a Poincaré space P, $\dim P = n \geq 6$ and a Poincaré decomposition of $P = P_1 \cup P_2$ where $P_1 \cap P_2 = P_0$ and (P_1, P_0) is (π, π) i.e. $\pi_1(P_0) \simeq \pi_1(P_1)$. Any simple homotopy equivalence $X \xrightarrow{f} P$ from a homology manifold X to P is s-cobordant to a simple homotopy equivalence $X' \xrightarrow{g} P$ which restricts to a simple homotopy equivalence $g^{-1}(P_0) \xrightarrow{g} P_0$.*

Proof: Consider $X \xrightarrow{f} P$ as an element of $S^H(P)$. In particular the Ferry-Pedersen TOP reduction of X gives a TOP reduction of P. P_1 has a TOP reduction, by restriction of the TOP reduction of P, and it is (π, π) , so there exists a homology manifold structure on P_1. We would like a homology manifold structure on the Poincaré space $P \times I$, which restricts to a structure on $P_1 \times 1$ and agrees with the

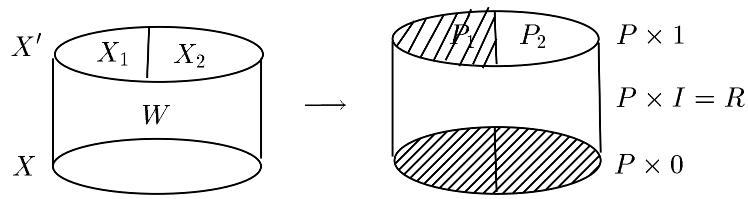

FIGURE 1. A homology manifold structure on $R = P \times I$ rel $P \times 0 \amalg P_1 \times 1$.

given structure on $P \times 0$. For a given structure in $S^H(P_1)$, consider the homology manifold surgery exact sequence for $P \times I$, rel $P_1 \times 1 \amalg P \times 0$, which we shall denote by R.

$S^H(R) \to H_{n+1}(P \times I, P_2; \mathbb{L}) \to L_{n+1}(P \times I, P_2) \to BS^H(R) \to H_n(P \times I, P_2; \mathbb{L}) \to L_n(P \times I, P_2)$

Notice that $L_*(P \times I, P_2) \simeq L_*(P_1, P_0) \simeq *$. So we have an isomorphism $BS^H(R) \simeq H_n(P \times I, P_2; \mathbb{L})$. But $H_n(P \times I, P_2; \mathbb{L}) \simeq H_n(P_1, P_0; \mathbb{L}) \simeq S^H(P_1)$. By naturality of the surgery exact sequence under inclusion we have the following commutative diagram, where the vertical maps are the sum of the maps induced by inclusion.

$$\begin{array}{ccc} S^H(P) \times S^H(P_1) & \longrightarrow & H_n(P, \mathbb{L}) \times H_n(P_1, P_0; \mathbb{L}) \\ \downarrow & & \downarrow \\ BS^H(R) & \longrightarrow & H_n(P \times I, P_2; \mathbb{L}) \end{array}$$

By commutativity of the diagram, the map $i : S^H(P_1) \to BS^H(R)$ induced by inclusion is the above isomorphism. We can choose a structure on P_1 which together with f in $S^H(P)$ gives a vanishing total surgery obstruction of R. Thus we have $W \to P \times I$ a simple homotopy equivalence, i.e. W is an s-cobordism from $X_1 \cup_{X_0} X_2$ to X' where $X' \to P$ is s-cobordant to the original structure $X \to P$. $\square$

This is called codimension one splitting. We say that the given homotopy equivalence "splits", i.e. restricts to a homotopy equivalence (over P_1 and hence) over P_0. A corollary of the proof which we will need later is that the splitting map $S^H(P) \to S^H(P_1)$ extends to a map of surgery exact sequences. We also have a relative version of this theorem, whose proof uses the relative version of the surgery exact sequence for homology manifolds.

Theorem 3.2. *(Browder Splitting) Let $f : X' \to X$ denote a simple homotopy equivalence where X' is a homology manifold of dimension $n \geq 6$ and X is a Poincaré space of dimension n. If Y is a Poincaré space of dimension $y \geq 6$ and $\nu(Y)$ is a fiber bundle over Y with fiber D^c, $c \geq 3$ so that $X = \nu(Y) \cup_{\partial \nu(Y)} V$ then there is a well defined obstruction $\sigma(f) \in L_y(Y)$ which depends only on the s-cobordism class of f so that f splits over Y i.e. f is s-cobordant to a map which is transverse to Y and restricts to a homotopy equivalence over Y if and only if $\sigma(f)$ vanishes.*

Proof: Apply (π, π) splitting to f and $\nu(Y)$ to get a structure $g : X'' \to X$ which is s-cobordant to f and so that $g|N$ where $N = g^{-1}(\nu(Y))$ is a simple homotopy equivalence. Call this s-cobordism W. Recall from above that there is a well-defined

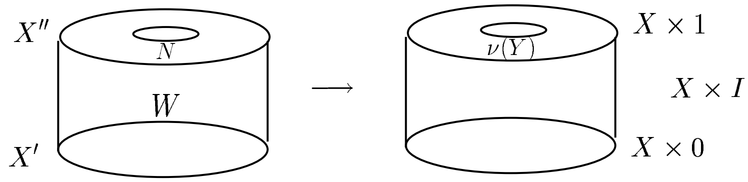

FIGURE 2. We perform (π, π) splitting on f to get a structure on $\nu(Y)$.

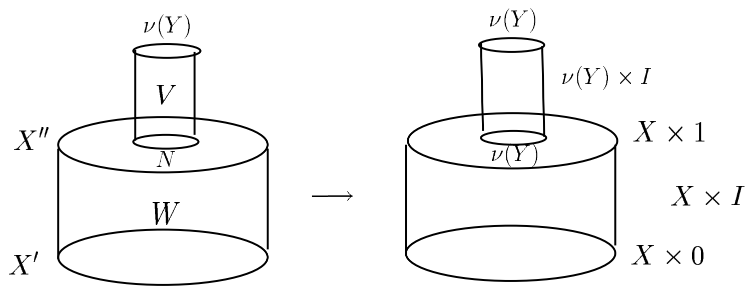

FIGURE 3. Glue together the two s-cobordisms V and W to get an s-cobordism from f to a new structure which contains an embedded $\nu(Y')$.

map to $S^H(\nu(Y))$ call the image of f under this map ρ. Further consider the map $S^H(\nu(Y)) \to H_n(\nu(Y), \mathbb{L})$, which is an isomorphism, because $\nu(Y)$ is (π, π) $(c \geq 3.)$ Let γ denote the image of ρ under this map. Further let α denote the image of γ under Ranicki's Thom isomorphism $H_n(\nu(Y), \mathbb{L}) \simeq H_y(Y, \mathbb{L})$, [R2]. Define $\sigma(f)$ to be the surgery obstruction of α in $L_y(Y)$.

Assume that $\sigma(f)$ vanishes. Then α is normally cobordant to a structure on Y, i.e. an element of $S^H(Y)$. Call this β. We have the following commutative diagram where the horizontal maps are transfers [R2].

$$\begin{array}{ccc} S^H(Y) & \longrightarrow & S^H(\nu(Y)) \\ \downarrow & & \downarrow \\ H_y(Y, \mathbb{L}) & \longrightarrow & H_n(\nu(Y), \mathbb{L}) \end{array}$$

Thus if β maps to α maps to γ, then also β maps to ρ maps to γ, i.e. ρ is in the image of the transfer map $S^H(Y) \to S^H(\nu(Y))$. Equivalently ρ is s-cobordant to a bundle map $h : \nu(Y') \to \nu(Y)$ for some structure $k : Y' \to Y$ representing β so that $\nu(Y') = k^*\nu(Y)$. Call this s-cobordism V. Gluing the two s-cobordisms together results in an s-cobordism from f to $\tilde{g} : \bar{X} \to X$ so that $\bar{X} = X'' \setminus N \cup_{\partial N} \partial_2 V \cup \nu(Y')$ where $\partial_2 V = \partial V \setminus (\nu(Y') \cup N)$ and $\tilde{g}|Y' \to Y$ is the simple homotopy equivalence k as desired.

Conversely assume that g is a transverse split map s-cobordant to f. Then the image of g in $S^H(\nu(Y))$ under codimension one (π, π) splitting is just $g|\nu(Y')$, which by transversality comes from $g|Y'$ which represents an element of $S^H(Y)$ by

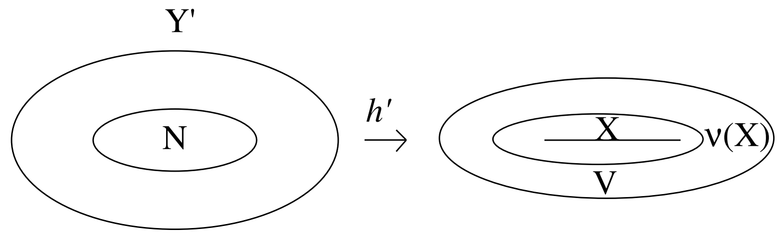

FIGURE 4. Apply the codimension one (π, π) splitting theorem to h to get h'.

the split assumption. By commutativity of the above diagram $\sigma(g)$ is the surgery obstruction corresponding to $g|Y'$ in $H_y(Y, \mathbb{L})$ which vanishes. □

We also have a relative version of the Browder splitting theorem, in which we assume that the given simple homotopy equivalence is already split along ∂Y. This translates into the appropriate hypothesis for a relative codimension one (π, π) splitting theorem. The relevant surgery obstruction group remains $L(Y)$, because all surgery is done relative to ∂Y. Thus the remainder of the proof goes through as before.

Theorem 3.3. *(Browder-Casson-Haefliger-Sullivan-Wall Embedding) Given X a homology manifold Poincaré embedded in a homology manifold Y of dimension $n \geq 6$ by a homotopy equivalence $h : Y \to \nu(X) \cup_{\partial\nu(X)} V$ where $\partial\nu(X)$ is a spherical fibration of dimension at least 2, with mapping cylinder $\nu(X)$, then X embeds in Y'', which is s-cobordant to Y. (If Y is a manifold, then X embeds in Y itself.)*

Proof: Apply the codimension one (π, π) splitting theorem to h to get $h' : Y' \to \nu(X) \cup_{\partial\nu(X)} V$ so that $(h')^{-1}(\nu(X)) = N$ a homology manifold and $h'|N : N \to \nu(X)$ is a homotopy equivalence. Given that h is only a homotopy equivalence and not a simple homotopy equivalence, the codimension one (π, π) splitting theorem gives Y' only h-cobordant to Y. However we may assume that Y' is s-cobordant to Y by gluing on an h-cobordism of appropriate torsion. We may do this on V (away from N,) because the high codimension gives $\pi_1(Y) \simeq \pi_1(V)$. Let f denote $h'|N$. We are interested in the controlled surgery exact sequence for $\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix}$. Notice that $\nu(X)$ is a controlled Poincaré complex over X, because X is a homology manifold and the fiber of $\nu(X)$ over X is a Poincaré space.

Let $S^{c,H}(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix})$ denote the set of controlled homology manifold structures on the controlled Poincaré complex $(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix})$. This surgery exact sequence is studied in [F]. The special case we use here also follows from the bounded surgery exact sequence for homology manifolds as studied in [BFMW]. In particular, we need to apply the techniques discussed there to generalize the surgery exact sequence from X a finite polyhedron to X a general finite dimensional ANR.

Claim: The map $S^{c,H}(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix}) \to S^H(\nu(X))$ induced by forgetting control is surjective.

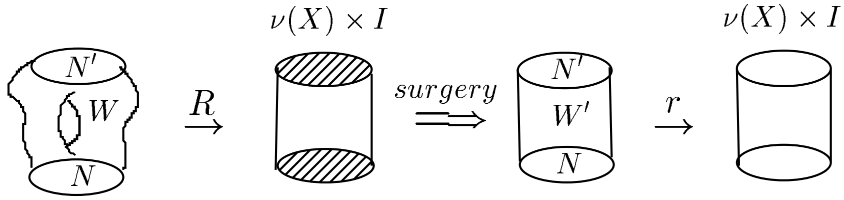

FIGURE 5. W is the trace of surgery on f not rel ∂. W' is the result of surgery on W rel $\nu(X)\times 0,1$.

Proof of Claim: Consider $f : N \to \nu(X)$ as a normal invariant. The controlled surgery exact sequence for $S^{c,H}(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix})$ is given by

$$\cdots \to S^{c,H}(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix}) \to NI(\nu(X)) \to L^c(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix}) \to \cdots .$$

By our dimension assumption the fiber of $(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix})$ is (π,π) and by the controlled (π,π) theorem $L^c(\begin{smallmatrix}\nu(X)\\ \downarrow\\ X\end{smallmatrix}) = 0$. Thus $f : N \to \nu(X)$ can be surgered to a controlled homotopy equivalence, $\tilde{f} : N' \to \nu(X)$. Let $R : W \to \nu(X)\times I$ denote the trace of this surgery. We can do surgery on $R : W \to \nu(X)\times I$ rel $\nu(X)\times 0,1$ to get a homotopy equivalence, because the relevant surgery obstruction group is $L_{n+1}(\nu(X), \partial\nu(X)) = 0$. This surgery results in $r : W' \to \nu(X)\times I$ an h-cobordism between $\tilde{f}$ and f, i.e. $\tilde{f}$ the desired controlled structure which represents the same element of $S^H(\nu(X))$ as f. Our claim is proved.

Given $\tilde{f}|\partial N' = g$ an element of $S^{c,H}(\begin{smallmatrix}\partial\nu(X)\\ \downarrow\\ X\end{smallmatrix})$, we will now construct an element of $S^H(\nu(X))$ in which X embeds. Let $H : \partial_2 W' \to \partial\nu(X)\times I$ denote the h- cobordism $r|\partial_2 W'$ where $\partial_2 W' = \partial W' \backslash (N' \cup N)$. Let $k = p \circ g$ denote the composition of g and $p : \partial\nu(X) \to X$ the projection map. Since g is a controlled homotopy equivalence, H extends to a homotopy equivalence $\bar{H} : Y_k \to \nu(X)$ where $Y_k = \partial_2 W'/\sim$ for $n_1 \sim n_2$ if $k(n_1) = k(n_2)$. Y_k is a homology manifold, because g is a controlled homotopy equivalence.

The map $\bar{H} : Y_k \to \nu(X)$ defines an element of $S^H(\nu(X) \text{ rel } \partial)$ in which X is embedded. We will show that this implies that X can be embedded in any element of $S^H(\nu(X) \text{ rel } \partial)$, i.e. X can be embedded in N above.

Consider the surgery exact sequence for $\nu(X)$ rel ∂.

$$L_{n+1}(X) \to S^H(\nu(X) \text{ rel } \partial) \to H_n(\nu(X), \mathbb{L}) \to L_n(\nu(X)).$$

Recall that for the homology manifold surgery exact sequence $NI(\nu(X))$ are naturally given by controlled surgery obstructions. (Even in the manifold case, we may still consider $NI(\nu(X))$ as contained in the group of controlled surgery obstructions.)

Let $\alpha \in H_n(\nu(X), \mathbb{L}) \simeq H_n(X, \mathbb{L})$ denote the difference of the images of $\bar{H}$ and f there. In the controlled surgery exact sequence for $p : \partial\nu(X) \to X$ we have

$$H_n(X, \mathbb{L}) \to S^{c,H}(\begin{smallmatrix}\partial\nu(X)\\ \downarrow\\ X\end{smallmatrix}) \to H_{n-1}(\partial\nu(X), \mathbb{L}) \to H_{n-1}(X, \mathbb{L}).$$

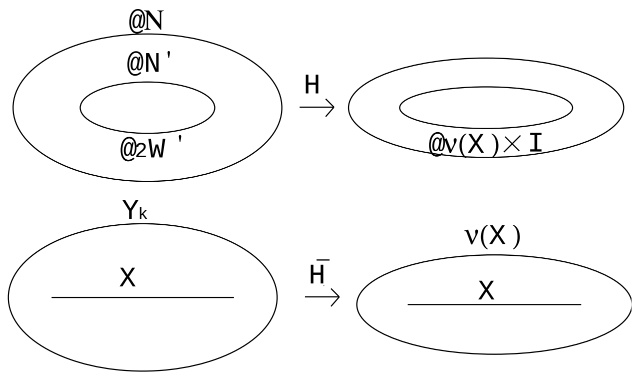

FIGURE 6. The space $\partial_2 W'$ and the map H are used to define $\bar{H} : Y_k \to \nu(X)$.

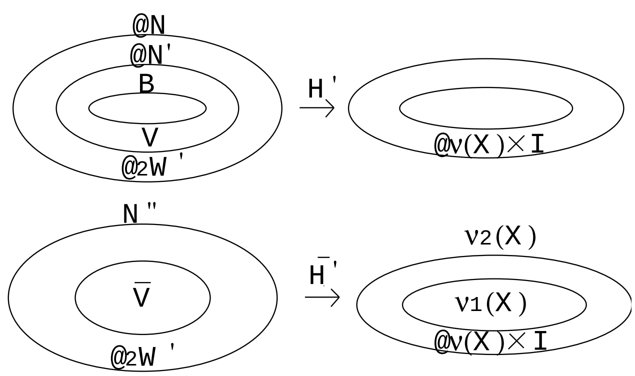

FIGURE 7. The space $\partial_2 W' \cup V$ and the map H' are used to define $N'' = \partial_2 W' \cup V/\sim$ and $\bar{H}'$.

Take $g : \partial N' \to \partial\nu(X)$ and do a Wall realization $\alpha : V \to \partial\nu(X) \times I$ with controlled surgery obstruction α to get $g' : B \to \partial\nu(X)$ another controlled structure. Now consider the homotopy equivalence $H' : \partial_2 W' \cup V \to \partial\nu(X) \times I$ so $H'|\partial_2 W' = H$ and $H'|V = \alpha$.

Now $N'' = \partial_2 W' \cup V/\sim$ where $\sim$ is given by $n_1 \sim n_2$ if $p(g'(n_1)) = p(g'(n_2))$, is a homology manifold and $\bar{H}' : N'' \to \nu(X)$ gives a homology manifold structure on $\nu(X)$ rel ∂. To see that the image of $\bar{H}'$ in $H_n(X, \mathbb{L})$ agrees with that of f i.e. that it differs from the normal invariant for $\bar{H}$ by α, compare the controlled surgery obstructions for H and H' in $H_n(X, \mathbb{L})$ and observe that they differ by α.

Now we have $\bar{H}' : N'' \to \nu(X)$ an element of $S^H(\nu(X) \text{ rel } \partial)$ which has the same image in $H_n(X, \mathbb{L})$ as f does. Thus we can act on $\bar{H}'$ by an element $\sigma \in L_{n+1}(X)$ to get f. Note that by construction $\bar{H}'$ maps the manifold $\bar{V} = V/\sim$ to the subset $\nu_1(X) = \partial\nu(X) \times [0, \frac{1}{2}]/\sim$ and maps $\partial_2 W'$ to the complement $\partial\nu(X) \times [\frac{1}{2}, 1]$.

Since $\pi_1(\nu(X) \text{ rel } \partial) \simeq \pi_1(\partial\nu(X) \times I \text{ rel } \partial) \simeq \pi_1(X)$, we may represent σ by an element of $L_{n+1}(\partial\nu(X) \times I \text{ rel } \partial)$, $s : K \to (\partial\nu(X) \times [\frac{1}{2}, 1]) \times I$ so that $s|s^{-1}(\partial\nu(X) \times [\frac{1}{2}, 1] \times 0) = \bar{H}'|\partial_2 W'$ and $s|s^{-1}(\partial\nu(X) \times [\frac{1}{2}, 1] \times 1)$ gives a new structure on $\partial\nu(X) \times [\frac{1}{2}, 1]$ rel ∂.

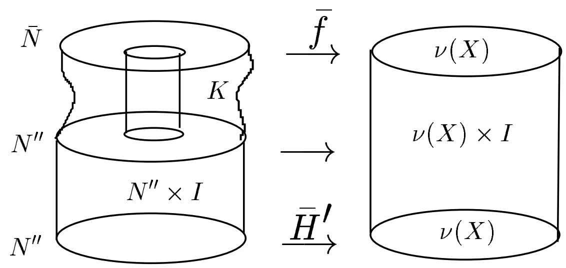

FIGURE 8. Glue the Wall realization K onto $N'' \times I$ away from the neighborhood $\bar{V}$ of X to get a structure $\bar{f} : \bar{N} \to \nu(X)$ which is h-cobordant to f.

Gluing K to $N'' \times I$ gives a Wall realization of σ for $S(\nu(X) \text{ rel } \partial)$ with $\bar{H}'$ at one end and a new structure $\bar{f}$ at the other end. Thus $\bar{f}$ is the same element of $S^H(\nu(X) \text{ rel } \partial)$ as f, i.e. $\bar{f}$ is h-cobordant to f. Let τ denote the torsion of this h-cobordism. Say $\bar{f} : \bar{N} = \bar{V} \cup C \to \nu(X)$ and let $\bar{K}$ denote an h-cobordism of torsion τ from C to C'. By gluing $\bar{K}$ to $\bar{N} \times I$ we get an h-cobordism from $\bar{f}$ to f', where $f' : \bar{N}' \to \nu(X)$ is s-cobordant to $f : N \to \nu(X)$. this s-cobordism together with the oe from Y' to Y' gives an s-cobordism from a new homology manifold Y'' to Y. Since by construction X is embedded in $\bar{V}$ which is a submanifold of $\bar{N}'$ we have X embedded in $\bar{N}' \subset Y''$ as desired.

Note that if Y were a manifold, we could do everything in the manifold category so that in the end we would have Y'' manifold s-cobordant to Y and hence by the manifold s-cobordism theorem $Y'' = Y$. This requires the following extra step.

Claim: If W is a manifold and $g : \partial_1 W \to \partial\nu(X)$ is a controlled homotopy equivalence over X then for $k = p \circ g$ the space $W_k = W/\sim$ where $w_1 \sim w_2$ if $k(w_1) = k(w_2)$ is a manifold.

The claim can be used to show that Y_k and N'' are both manifolds. If we use the manifold surgery exact sequence instead of the homology manifold surgery exact sequence the other constructions obviously remain in the manifold category. Note that this does not preclude us from considering $NI(\nu(X))$ as controlled surgery obstructions, we just don't get all possible controlled surgery obstructions this way.

Proof of claim: To see that W_k is a manifold use the fact that, by Edwards [E] and Quinn [Q3], a DDP homology manifold with manifold points is a manifold. Clearly W_k has manifold points and is a homology manifold (because g is a controlled homotopy equivalence.) To see that W_k has DDP, consider two disks in D_1 and D_2 in W_k. Because the homotopy fiber of $k : \partial_1 W \to X$ is highly connected, X is 1-l.c.c. embedded in W_k, and we can move each disk off X by a small move. If the disks are contained in $W_k \setminus X$ then we can make them disjoint using DDP for the manifold W.

□

4. Manifold Range

In this section we will study homology manifold transversality for the following special problem: Given X a homology manifold, M a manifold, a submanifold $N \subset M$, and a map $f : X \to M$ when can f be made transverse to N?

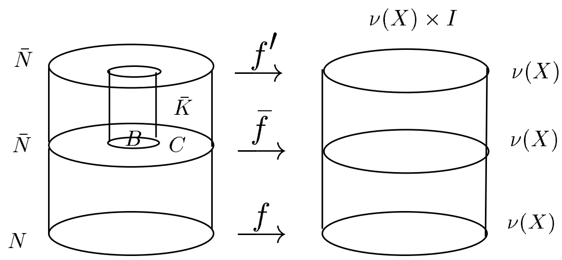

FIGURE 9. Glue the h-cobordism $\bar{K}$ of torsion τ to $\bar{N}$ to get a map f' s-cobordant to f.

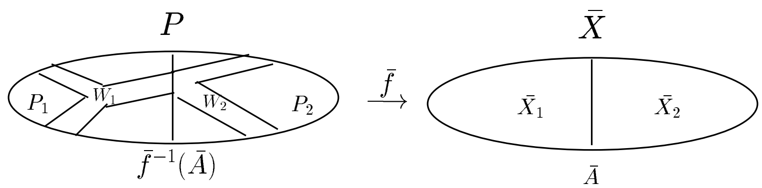

FIGURE 10. The manifold two-skeleton of P is the union of the manifold two-skeleta for P_1 and P_2 along the manifold two-skeleton for P_0.

4.1. **Codimension One** (π, π)**.** The following variation of Poincaré transversality theorem **??** will be needed in this section.

Corollary 4.1. *If (X, A) is a CD_1-pair so that $X = X_1 | \cup_A X_2$ and (X_1, A) is (π, π) , then it is possible to make f transverse to A with the added conclusion that $f^{-1}(A) \subset f^{-1}(X_1)$ induces an isomorphism on fundamental groups.*

Proof: The proof of **??** proceeds by approximating f by a Serre fibration $P \xrightarrow{\bar{f}} (\bar{X}, \bar{A}) \xrightarrow{\hat{f}} (X, A)$. Then the homotopy equivalence $\bar{f}$ is made Poincaré transverse to $\bar{A}$. Thus we may assume that we are working with the homotopy equivalence $\bar{f}$. We would like to do Poincaré surgery on $\bar{f}$ to achieve an isomorphism of fundamental groups. If we were working with manifolds this would be the usual handle trading argument in the first few steps of (π, π) codimension one splitting, see for example [B1]. We will need the following lemma to reduce to the manifold case.

Lemma 4.2. *If a Poincaré space P has Poincaré decomposition $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = P_0$ then the P_i have manifold two skeleta W_i for $i = 0, 1$ and 2 so that $W = W_1 \cup W_2$ with $W_1 \cap W_2 = W_0$ is the manifold two skeleton of P.*

This lemma is a corollary of the existence of a manifold two-skeleton for a Poincaré space together with the relative version, 2.15 and 2.20 of [HV].

We now return to the proof of the theorem. Poincaré transversality gives $\bar{f}^{-1}(A) = P_0$ and $P = P_1 \cup_{P_0} P_2$. Applying the lemma we get a manifold two skeleton $W = W_1 \cup_{W_0} W_2$. Now we can repeat the handle trading argument for the manifold case on the map $\bar{f}|W$. Changing the fundamental group as desired involves doing surgery on embedded S^1 or D^2 representatives. Thus we can assume that these representatives lie within the manifold two skeleta and do surgery there. This results in a homotopy of $\bar{f}|W$ which changes the fundamental group of $(\bar{f}|W)^{-1}(\bar{X}_1, \bar{A})$ while preserving transversality. Because everything is taking place on the interior of W we can extend this homotopy by the identity to all of P to achieve the desired result for the Poincaré spaces. $\square$

Let X be a compact oriented homology manifold of dimension n and M and N compact oriented manifolds so that N is a codimension one submanifold of M which divides it into two pieces M_1 and M_2, N has a neighborhood $N \times I$ in M, and the inclusion of N into M_1 is an isomorphism on fundamental groups.

Theorem 4.3. *Given X, M and N as above, and an arbitrary map $X \xrightarrow{f} M$ there exists an s-cobordism of homology manifolds W from X to X' and a map $W \to M$ such that $W|X = f$ and $W|X' = g$, so that $g^{-1}(N) = Y$ is a homology manifold with neighborhood $Y \times I$ in X' so that $g^{-1}(N \times I) = Y \times I$ and g preserves the product structure.*

Remark 4.4. *A relative version of this theorem follows easily from relative versions of Poincaré transversality and codimension one (π, π) splitting.*

Proof By **??** we can assume that f is Poincaré transverse to N say with $f^{-1}(N) = P$ and $f^{-1}(M_1) = P_1$. By **??** we can assume that the inclusion $P \subset P_1$ induces an isomorphism on fundamental groups.

Now apply **??** to the identity map $id_X : X \to P_1 \cup_{P_0} P_2$. This results in $X' = X_1 \cup_{X_0} X_2 \to P_1 \cup_{P_0} P_2$ s-cobordant to id_X, where $k|X_i$ is a homotopy equivalence for each i. This achieves our main result, $X' \xrightarrow{k} X \xrightarrow{f} M$ is s-cobordant to $X \xrightarrow{f} M$ and has $f^{-1}(N) = X_0$ a homology manifold. To achieve the desired $X_0 \times I$ neighborhood of X_0 we must slightly alter X'. Note that X' is s-cobordant to $X'' = X_1 \cup_{X_0} X_0 \times I \cup_{X_0} X_2$ and there is an s-cobordism of maps from $f \circ k : X' \to M$ to $g : X'' \to M_1 \cup_N N \times I \cup_N M_2 = M$ so that g is given by $f \circ k$ on $X_1 \cup X_2$ and by $(f \circ k|N) \times id$ on $N \times I$. $\square$

4.2. **Codimension ≥ 3.** Let X be a compact oriented homology manifold of dimension n, M and N compact oriented manifolds so that N is a submanifold of codimension q, where $q \geq 3$ and $n-q \geq 6$ with $\nu(N)$ a bundle normal neighborhood of N in M.

Theorem 4.5. *Given X, M and N as above and a map $f : X \longrightarrow M$, there exists a homology manifold s-cobordism W from X to X' and a map $H : W \longrightarrow M$ so that $H|X = f$ and $H|X' = g$ such that $g^{-1}(N) = Y$ is a homology manifold, the neighborhood $g^{-1}(\nu(N))$ has a bundle structure given by $\nu(Y) = g^*\nu(N)$ and g respects the bundle structure.*

Proof: Because X is a homology manifold, its Spivak bundle has a canonical TOP reduction ξ, given by $E(\xi) \xrightarrow{p} X$, called the Ferry-Pedersen reduction, [FP]. The map $E(\xi) \to X \to M$ given by the projection p composed with f is a map of manifolds ($E(\xi)$ is a manifold neighborhood of X), so we can use manifold transversality to make $f \circ p$ transverse to N. Now we have $(f \circ p)^{-1}(N) = B \subset E(\xi)$. By **??** a TOP reduction on X gives an extrinsic transversality structure on ξ. We can use this transversality structure to make the manifold $B \subset E(\xi)$ Poincaré transverse to X. The transversality structure gives a way to make the manifold B Poincaré transverse to X. Call the new Poincaré transverse manifold $B' \subset E(\xi)$. Then $B' \cap X = P$ is a Poincaré space, Poincaré embedded in X via

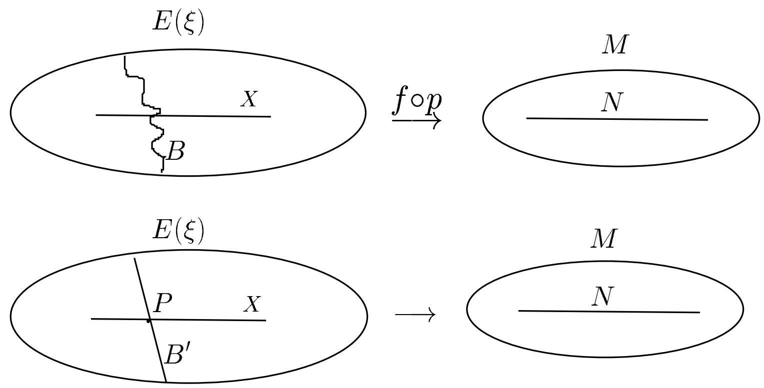

FIGURE 11. A transversality structure on ξ gives Poincaré transversality for $f : X \to M$.

$h : X \to V \cup \nu(P)$. Now if $\bar{h} : V \cup \nu(P) \to X$ denotes the homotopy inverse of h, then $X \xrightarrow{h} V \cup \nu(P) \xrightarrow{\bar{h}} X \xrightarrow{f} M$ is homotopic to our original map and has P as Poincaré transverse inverse image of N, i.e. $\nu(P) = (f \circ \bar{h})^* \nu(N)$. Note that since $\nu(N)$ has a TOP reduction the pull-back, $\nu(P)$ comes with a TOP reduction.

Consider h as an element of $H_n(V \cup \nu(P), \mathbb{L})$. Map via the restriction map to $H_n(\nu(P), \mathbb{L})$. Then via Ranicki's Thom isomorphism to $H_{n-q}(P, \mathbb{L})$, [R2]. We would like to put a homology manifold structure on P with this normal invariant. Unfortunately there is an obstruction to doing this in $L_{n-q}(P)$. Call this obstruction σ. It is the obstruction to Browder splitting given in theorem **??** above. This obstruction is possibly nontrivial with this particular h and P. The following lemma, whose proof we defer briefly, allows us to switch to a different Browder splitting problem which does have a vanishing obstruction.

Lemma 4.6. *Given a Poincaré space $(P, \partial P)$ of dimension n, a normal k-disk bundle $\nu(P)$ ($k \geq 3$) and a surgery obstruction $\sigma \in L_n(P)$, we can construct a Poincaré space P' so that $\partial P' = \partial P$, and a map $f : P' \to \nu(P)$ so that f is a Poincaré embedding with $\nu(P') = f^* \nu(P)$ and so that the surgery obstruction of $pr \circ f : P' \to P$ is σ.*

Since the result gives P' Poincaré embedded in $\nu(P)$, we actually have P' Poincaré embedded in $V \cup \nu(P)$ which we denote by $\bar{k} : V' \cup \nu(P') \to V \cup \nu(P)$. The surgery obstruction of $\bar{k}|P'$ is σ. Denote the homotopy inverse of $\bar{k}$ by $k : V \cup \nu(P) \to V' \cup \nu(P')$. Now the composition of maps $X \xrightarrow{h} V \cup \nu(P) \xrightarrow{k} V' \cup \nu(P') \xrightarrow{\bar{k}} V \cup \nu(P) \xrightarrow{\bar{h}} X \xrightarrow{f} M$ is homotopic to our original map and it has P' as Poincaré transverse inverse image of N, i.e. $\nu(P') = (f \circ \bar{h} \circ \bar{k})^* \nu(N)$.

Now we follow the same procedure as above to get the Browder splitting obstruction of $k \circ h$. Begin with the image of $k \circ h$ in $H_n(V' \cup \nu(P'), \mathbb{L})$. Map to $H_n(\nu(P'), \mathbb{L})$ by restriction, then to $H_{n-q}(P', \mathbb{L})$ by Ranicki's Thom isomorphism. Finally consider the image α in $L_{n-q}(P')$. Consider the following commutative

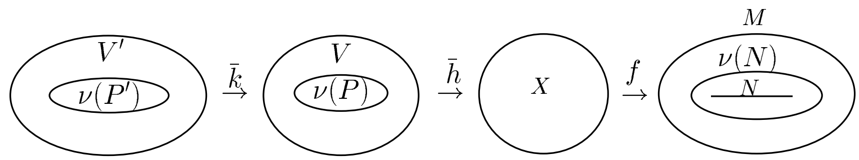

FIGURE 12. P' is the Poincaré transverse inverse image of N.

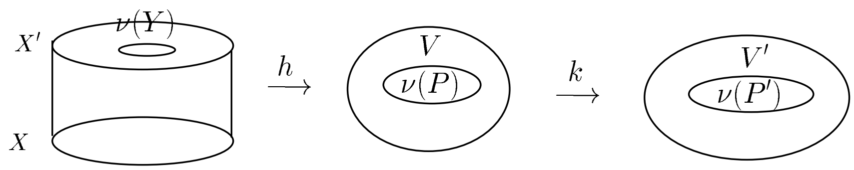

FIGURE 13. The result of Browder splitting on $k \circ h$.

diagram:

$$\begin{array}{ccccccc}
H_n(V' \cup \nu(P'), \mathbb{L}) & \longrightarrow & H_n(\nu(P'), \mathbb{L}) & \longrightarrow & H_{n-q}(P', \mathbb{L}) & \longrightarrow & L_{n-q}(P') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n(V \cup \nu(P), \mathbb{L}) & \longrightarrow & H_n(\nu(P), \mathbb{L}) & \longrightarrow & H_{n-q}(P, \mathbb{L}) & \longrightarrow & L_{n-q}(P)
\end{array}$$

The horizontal maps are as described above and the vertical maps are induced by $\bar{k}$. The diagram commutes by naturality of the surgery exact sequence and transversality of $\bar{k}$. The map induced on the surgery obstruction groups takes α to $\alpha + \sigma = \sigma$. Thus we have that the Browder splitting obstruction α in $L_{n-q}(P')$ vanishes. This means that there exists a transverse map $g' : X' \to V' \cup \nu(P')$ which is s-cobordant say via $\bar{H} : W \to V' \cup \nu(P')$ to $k \circ h : X \to V' \cup \nu(P')$. Let $(g')^{-1}(P') = Y$ and denote $(g')^{-1}(\nu(P'))$ by $\nu(Y)$.

Now $g = f \circ \bar{h} \circ \bar{k} \circ g'$ is the desired map homology manifold transverse to N. $f \circ \bar{h} \circ \bar{k} \circ \bar{H}$ gives an s-cobordism from g to $f \circ \bar{h} \circ \bar{k} \circ k \circ h$. Putting this together with the homotopy from $f \circ \bar{h} \circ \bar{k} \circ k \circ h$ to f yields the desired s-cobordism from f to g. $\square$

Proof of Lemma: Let D denote the boundary of the manifold two-skeleton of P, where $P = B \cup_D C$. Note that the surgery group for (P rel ∂P) corresponds to the fundamental group of P, and we can take a manifold two-skeleton for P which is disjoint from ∂P. In any case, the fundamental group of D is the same as that of P and the dimension of D is one less than that of P. Thus $L_n(P) \simeq L_n(D)$ allows us to consider the given obstruction σ as an element of $L_n(D)$ upon which we may perform Wall realization. Let $\sigma : W \to D \times I$ denote the result of the Wall realization.

Let $\nu(D)$ denote the restriction of $\nu(P)$ to D. Pull back this bundle via σ to W, call the result V. Note that $\partial(\nu(D) \times I) = \nu(D) \times \{0,1\} \cup \partial\nu(D) \times I$. For doing surgery on $V \to \nu(D) \times I$ rel $\nu(D) \times \{0,1\}$ the relevant surgery group is $L_{n+k}(\nu(D) \times I, \partial\nu(D) \times I) \simeq L_{n+k}(\nu(D), \partial\nu(D)) \simeq 0$, because $k \geq 3$ insures that this will be (π, π) . Surgery on $V \to \nu(D) \times I$ results in $V' \to \nu(D) \times I$ a simple homotopy equivalence. In particular V' is an s-cobordism with one end

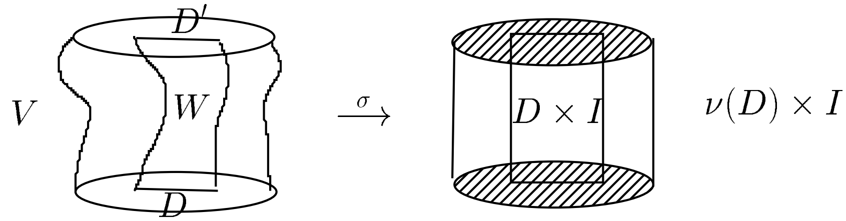

FIGURE 14. Pull back the bundle $\nu(D) \times I$ to W and surgery this to a homotopy equivalence rel $\nu(D) \times 0, 1$.

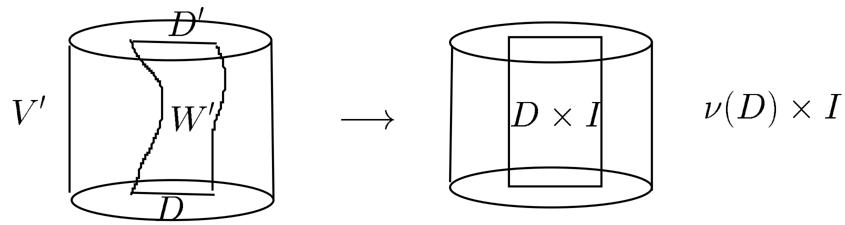

FIGURE 15. Apply manifold transversality to V' the result of surgery on V to get W' a Wall realization of σ which is embedded in $D \times I$.

$\nu(D)$, $V' = \nu(D) \times I$. Note that the transverse inverse image of $D \times I$ is now W'. By relative manifold transversality on the trace of the surgery we see that W' is normally cobordant rel ∂ to W, i.e. it is still a Wall realization of σ. This W' is embedded in $\nu(D) \times I$. The desired $P' = B \cup_D W' \cup_D C$ which is Poincaré embedded in $\nu(P) = \nu(P)|B \cup_{\nu(D)} \nu(D) \times I \cup_{\nu(D)} \nu(P)|C$. Because ∂P was contained entirely in C we have that $\partial P' = \partial P$ as will be useful for the relative version of this theorem. $\square$

Theorem 4.7. *Let $(X, \partial X)$ be a compact oriented homology manifold of dimension n, $(M, \partial M)$ and $(N, \partial N)$ compact oriented manifolds so that $(N, \partial N)$ is a codimension q submanifold of $(M, \partial M)$ where $q \geq 3$ and $n - q \geq 6$ with $\nu(N)$ a bundle normal neighborhood of N in M. Given a map of pairs $f : (X, \partial X) \longrightarrow (M \partial M)$ which is transverse to ∂N on ∂X so that $f^{-1}(\partial N) = Y_0$ and $\nu(Y_0) = f^* \nu(\partial N)$. There exists a homology manifold s-cobordism from $f : X \to M$ to $g : X' \to M$ so that g is transverse to N, $g^{-1}(\partial N) = Y_0$ and $g|Y_0 = f|Y_0$.*

The key difference in the proof of the relative version of the theorem is that we must take care to work rel Y_0 when changing the map to achieve Poincaré transverality. We construct a manifold neighborhood of X, $N(X)$ which contains a manifold neighborhood of Y_0, $N(Y_0)$ so that the map of manifolds given by the composition of the retraction $r : N(X) \to X$ and the map $f : X \to M$ is transverse to ∂N, with a neighborhood $\nu(N(Y_0) \subset N(X)) = (f \circ r)^* \nu(\partial N)$. We then use relative manifold transversality in making this map transverse to N with transverse inverse image a manifold B. Then when we use the manifold-type transversality that X has inside $N(X)$, because $r : N(X) \to X$ has a TOP reduction, we can make B transverse to X relative to $\nu(N(Y_0) \subset N(X))$. The result is a Poincaré

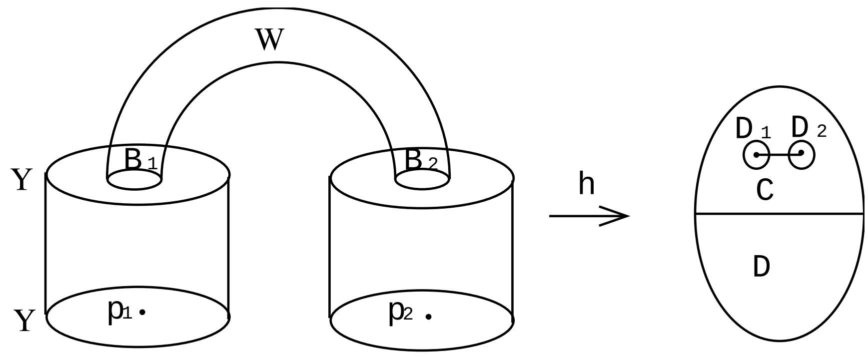

FIGURE 16. Gluing a one-handle onto $\bar{Y}$ which is s-cobordant to our original Y.

space P which is the Poincaré transverse image of N, so that $\partial P = Y_0$. Having done this the rest of the proof is as before using the surgery exact sequence for $(P, \partial P)$.

5. Bordism of Homology Manifolds

As manifold bordism was understood via manifold transversality, homology manifold bordism can be understood using the ideas of our transversality theorems. Let $\Omega_n^{SH}(X)$ denote the oriented bordism theory in the category of homology manifolds.

Theorem 5.1. $\Omega_n^{SH}(X) \simeq \Omega_n^{STOP}(X)[8\mathbb{Z}+1]$ *as an Abelian group for* $n \geq 6$.

This was asserted in [BFMW] for $X = pt$. It shows in particular that bordism of homology manifolds corresponds in high dimensions to a homology theory. It does not, however, give the ring structure from the ring structure on $\Omega_*^{STOP}(X)$. The appropriate normal bundle structure for homology manifolds, the Ferry-Pedersen bundle of a product of homology manifolds, is not in fact the product of their Ferry-Pedersen bundles. This suggests an unusual ring structure for homology manifold bordism.

Before proving the theorem we give two important lemmas and their proofs.

Lemma 5.2. *If Y is a homology manifold of dimension n, X is an arbitrary connected finite CW-complex then $f : Y \to X$ is bordant to a map $g : Y' \to X$ so that Y' is connected and g induces an isomorphism of fundamental groups.*

Proof of lemma: If Y is not connected consider an element α of $\pi_0(Y)$, i.e a pair of points p_1, p_2 in Y. We will show how to do geometric 0-surgery on α in Y. Recall that Y has a manifold two-skeleton, say this is given by C via a homotopy equivalence $h : Y \to C \cup D$ with homotopy inverse $g : C \cup D \to Y$. We will now work with $Y \xrightarrow{h} C \cup D \xrightarrow{g} Y \xrightarrow{f} X$, which is homotopy equivalent to f. Consider $h(\alpha(S^0))$ by a homotopy of h we can arrange that this lies in the manifold two-skeleton. Say the two points are $p_1', p_2' \in C$. Because C is a manifold, there exist neighborhoods D_i of p_i', where each D_i is homeomorphic to the standard disk D^n. Because n is large, $\pi_1(\partial D^n) \simeq \pi_1(D^n)$ and we can apply the (π, π) -splitting theorem **??** to h.

This results in $h' : \bar{Y} \to C \cup D$ s-cobordant to h. Now work with $\bar{Y} \xrightarrow{h'} C \cup D \xrightarrow{g} Y \xrightarrow{f} X$ which is bordant to the original map. By construction $(h')^{-1}(D_i) = B_i$ are contractible homology manifolds and each ∂B_i is a homology manifold homotopy equivalent to $\partial D^n = S^{n-1}$. We know from [BFMW] that $S^H(S^{n-1}) \simeq \mathbb{Z}$ which is given by the index. The index of ∂B_i agrees with the index of B_i which agrees with the index of Y, because the B_i are both subsets of $\bar{Y}$ and $\bar{Y}$ is s-cobordant to Y. Thus $h'|\partial B_i : \partial B_i \to S^{n-1}$ for $i = 1, 2$ give s-cobordant structures on S^{n-1}. Say $k : W \to S^{n-1}$ gives an s-cobordism of structures.

Take $\overset{\circ}{Y} = \bar{Y} \setminus (B_1 \cup B_2)$. Let $Y' = \overset{\circ}{Y} \cup W$ gluing W by the identity map along $\partial W = \partial B_1 \cup \partial B_2$. All that remains now is to extend $f \circ g \circ h' : \bar{Y} \to X$ to W. Since W is an s-cobordism we have $k : W \to S^{n-1} \times I$ a homotopy equivalence rel boundary. Define a map $r : S^{n-1} \times I \to C \cup_{h(p_1) \cup h(p_2)} [\frac{1}{4}, \frac{3}{4}]$ as follows: $r|S^{n-1} \times [0, \frac{1}{4}]$ is defined to be the composition of the cone map $S^{n-1} \times [0, \frac{1}{4}] \to S^{n-1} \times [0, \frac{1}{4}]/S^{n-1} \times \frac{1}{4} = D^n$ and the inclusion map $D_1 \subset C$ so that $r(S^{n-1} \times \frac{1}{4}) = h(p_1) = p_1'$. Similarly define $r|S^{n-1} \times [\frac{3}{4}, 1]$ so that $r(S^{n-1} \times \frac{3}{4}) = h(p_2) = p_2'$. Define $r|S^{n-1} \times [\frac{1}{4}, \frac{3}{4}]$ to be projection onto the second factor composed with the inclusion map. Recall that X was connected so we can define a path $\gamma : [\frac{1}{4}, \frac{3}{4}] \to X$ connecting $f \circ g \circ h'(p_1)$ and $f \circ g \circ h'(p_2)$. Now define a map $s : C \cup [\frac{1}{4}, \frac{3}{4}] \to X$ as follows. Define $s|C = f \circ g$ and $s|[\frac{1}{4}, \frac{3}{4}] = \gamma$. Define $f' : Y' \to X$ so that $f'|\overset{\circ}{Y} = f \circ g \circ h'|\overset{\circ}{Y}$ and $f'|W = s \circ r \circ k$.

We may now assume that Y is connected. If $f_* : \pi_1(Y) \to \pi_1(X)$ is not surjective. Consider $\gamma \in \pi_1(X)$ which is not in the image of f_*. Take any two points p_1 and p_2 in Y and consider $f(p_1)$ and $f(p_2)$ in X. Let q denote the base point of γ. Because X is connected, there exist paths β_i connecting $f(p_i)$ to q. Let γ' denote the path $\beta_1 * \gamma * \beta_2^{-1}$ which goes from $f(p_1)$ to q along β_1 then along γ, then back to $f(p_2)$ along β_2. By the above argument we may construct Y' by removing neighborhoods of p_1 and p_2 and gluing in an s-cobordism of homology manifolds W. Define the map $f' : Y' \to X$ so that $f'|\overset{\circ}{Y} = f|\overset{\circ}{Y}$ and $f'|W$ is defined similarly to the above, using the path γ'' from $f \circ g \circ h(p_1)$ to $f \circ g \circ h(p_2)$, where $\gamma'' = \alpha_1 * \gamma' * \alpha_2^{-1}$ and α_i are the paths connecting $f(p_i)$ and $f \circ g \circ h(p_i)$, induced by the homotopy $g \circ h \sim id_Y$. Now γ'' and hence γ is in the image of f'_*.

It only remains to show that we can kill $\ker f_*$ by a bordism of f. Let α denote an element of $\ker f_*$ we will construct $f' : Y' \to X$ bordant to f so that $\ker f'_* \simeq (\ker f_*)/(\alpha)$. Let $h : Y \to C \cup D$ denote the manifold two skeleton decomposition of Y as above. Let α' denote $h \circ \alpha$. By a homotopy of α' we may assume that it is in the manifold two skeleton and that it is embedded with a neighborhood $N(\alpha')$ of the form $S^1 \times D^{n-1}$. Note that homotoping α' does not change the homotopy class of $g \circ \alpha'$ where g is the homotopy inverse of h. By (π, π) splitting as above we get $h' : \bar{Y} \to C \cup D$ so that $(h')^{-1}(S^1 \times D^n) = B$ is a homology manifold and $h'|B : B \to S^1 \times D^n$ gives a homology manifold structure on $S^1 \times D^n$.

We are interested in gluing a homology manifold two-handle $D^2 \times S_i^{n-2}$ where S_i^{n-2} is a homology manifold of index $i = I(Y)$ homotopy equivalent to S^{n-2} onto $\overset{\circ}{Y} = \bar{Y} \setminus B$. Thus we must understand the structures on ∂B, i.e $S^H(S^1 \times S^{n-2})$. By the homology manifold analogue of Shaneson's Thesis or by a direct calculation, we get $S^H(S^1 \times S^{n-2}) \simeq S^H(S^{n-2}) \times S^H(S^{n-2} \times I, \text{ rel } \partial)$. $S^H(S^{n-2}) \simeq \mathbb{Z}$ detected by index, and $S^H(S^{n-2} \times I, \text{ rel } \partial) \simeq *$, by a quick calculation involving the [BFMW] surgery exact sequence. Thus since ∂B must have the same index as B which is the

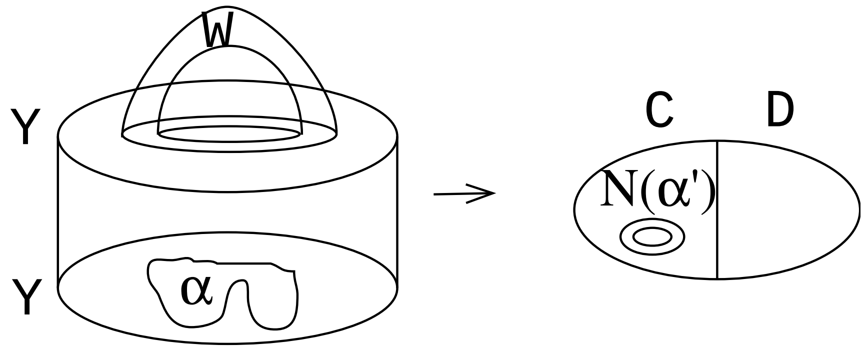

FIGURE 17. Gluing a two-handle onto $\bar{Y}$ s-cobordant to the original Y.

index of Y, we know that $h'|\partial B : \partial B \to S^1 \times S^{n-2}$ is s-cobordant to the structure given by $v : id_{S^1} \times u : S^1 \times S_i^{n-2} \to S^1 \times S^{n-2}$, where $u : S_i^{n-2} \to S^n$ is the non resolvable homology manifold structure of index i from [BFMW]. So there exists an s- cobordism from $h'|\partial B$ to v given by $W \to S^1 \times S^{n-2}$.

Our new space $Y' = \overset{\circ}{Y} \cup_{\partial B} W \cup_{S^1 \times S_i^{n-2}} D^2 \times S_i^{n-2}$. Again since W is an s-cobordism we have a homotopy equivalence rel boundary $r : W \to S^1 \times S^{n-2} \times I$. Define $f' : Y' \to X$ as follows: Define $f'|\overset{\circ}{Y}$ to be $f \circ g \circ h'|\overset{\circ}{Y}$. Define $f'|W$ to be $f \circ g \circ i \circ p \circ r$ where $p : S^1 \times S^{n-2} \times I \to S^1 \times (S^{n-2} \times I / S^{n-2} \times 1) = S^1 \times D^{n-1}$ is the projection map, and i is the inclusion $N(\alpha) \subset C$. Define $f'|D^2 \times S_i^{n-2}$ to be $H \circ p$ where p is projection onto the first factor and H is a null-homotopy of $f \circ g \circ \alpha' = f \circ g \circ h \circ \alpha \sim f \circ \alpha$. □

Lemma 5.3. *Given a manifold B and an integer k, we may find homology manifolds B', and B'' together with maps $b : B' \longrightarrow B$ and $\bar{b} : B'' \longrightarrow B'$ so that*
1) $Sp(B') = b^* Sp(B)$ where Sp denotes the Spivak fibration with its Ferry-Pedersen TOP-reduction.
2) $Ind(B') = 8k + 1$ where Ind denotes the Quinn index.
3) The composition of maps $b \circ \bar{b} : B'' \to B' \to B$ is normally cobordant to the identity map.

Proof of lemma: The construction is a variation on a construction found in [BFMW, section 7]. There the construction is performed on a torus, resulting in a homology manifold not homotopy equivalent to any manifold. We perform the construction on an arbitrary manifold with 1) 2) and 3) above as the result.

Slice B open along the boundary of a manifold two skeleton, ∂. So $B = C \cup_\partial D$. We first apply lemma 4.4 from [BFMW]. This will allow us to perform a small homotopy on $id_B : B \to B$ to get a new map $p_0 : B \to B$ which restricts to a UV^1 map on C, D and ∂. Because $p_0|\partial$ is a UV^1 map, its controlled surgery obstruction group $L^c \begin{pmatrix} \partial \\ \downarrow \\ B \end{pmatrix} \simeq H_n(B, \mathbb{L}) \simeq H_n(B, G/TOP) \times \mathbb{Z}$. Let σ denote the element of $L^c \begin{pmatrix} \partial \\ \downarrow \\ B \end{pmatrix}$ which corresponds to $Sp(B)$ and the desired index. Now by Wall realization we construct a normal invariant $\sigma : N \to \partial \times I$ with controlled surgery obstruction as desired, which is given by a controlled homotopy equivalence

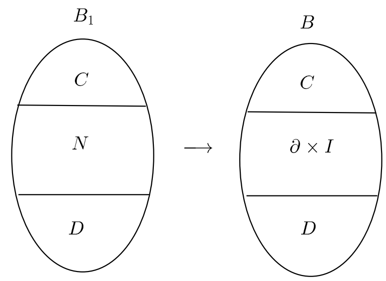

FIGURE 18. The map $b_1 : B_1 \to B$

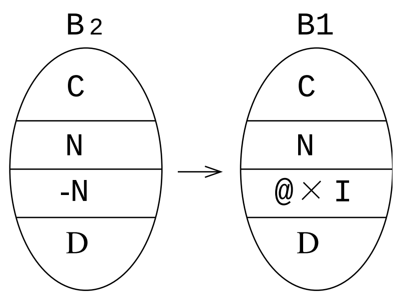

FIGURE 19. The map $b_2 : B_2 \to B_1$.

$k : \partial' \longrightarrow \partial$ on one end and by the identity on the other. Gluing C and D back onto N by the identity and by k respectively results in B_1 a Poincaré complex. We define a map $b_1 : B_1 \to B$ by $b_1|C \cup D = id$ and $b_1|N = \sigma$. By applying [BFMW, 4.4] we may assume b_1 is UV^1. A calculation of the total surgery obstruction of B_1 shows that it is homotopy equivalent to a homology manifold.

The rest of the construction is a limiting process in which the above type of construction is performed on finer and finer manifold two skeleta of B. We refer the reader to [BFMW] for the details of this stage. The construction yields a homology manifold B' and a homotopy equivalence $h : B' \to B_1$ so that the controlled surgery obstruction of h in $L^c\begin{pmatrix} B_1 \\ \downarrow b_1 \\ B \end{pmatrix}$ vanishes and the controlled surgery obstruction of $b = b_1 \circ h : B' \to B_1 \to B$ is given by σ in $L^c\begin{pmatrix} B \\ \downarrow p_0 \\ B \end{pmatrix} \simeq H_n(B, \mathbb{L}) \simeq [B, G/TOP] \times \mathbb{Z}$ which gives 1) and 2) as desired.

To see 3) consider $B_2 = C \cup_\partial N \cup_{\partial'} (-N) \cup_\partial D$. Define a map $b_2 : B_2 \to B_1$ so that $b_2|B_2 \setminus (-N) = id$ and $b_2|(-N) = -\sigma$. Now $B'' = B_2$ is a homology manifold. Consider the controlled surgery obstruction of b_2 in $L^c\begin{pmatrix} B_1 \\ \downarrow b_1 \\ B \end{pmatrix}$. This is $-\sigma$. Let $b' : B'' \to B'$ be given by $\bar{h} \circ b_2$ where $\bar{h}$ is the homotopy inverse of h. The composition $b \circ b' : B'' \to B' \to B$ has surgery obstruction $\sigma + (-\sigma) = 0$. Thus it is normally cobordant to the identity map and in particular it is bordant to the identity map as we will find useful in the proof of the theorem. $\square$

Proof of theorem: Assume that X is connected. The general case follows by applying the theorem to each of the connected components of X.

We will define a map $\Omega_n^{SH}(X) \to \Omega_n^{STOP}(X)[8\mathbb{Z}+1]$ as follows. Given $f : Y \to X$ in $\Omega_n^{SH}(X)$ consider $f_i : Y_i \to X$ where each Y_i is the union of the connected components of Y of index i, and $f_i = f|Y_i$. Note that, because the index of a homology manifold is bordism invariant, this separation according to index is well-defined.

For each $i \in 8\mathbb{Z}+1$, we will get an element of $\Omega_n^{STOP}(X)$ as follows. Let $n_i : M_i \to Y_i$ denote the normal invariant of Y_i corresponding to the Ferry-Pedersen TOP reduction of Y_i. We map f_i to the element $f_i \circ n_i : M_i \to Y_i \to X$ in $\Omega_n^{STOP}(X)$.

This map is surjective by lemma **??**. Given $f : M \to X$ an element of $\Omega_n^{STOP}(X)$ and a given index i, this construction yields $m_i : M'_i \to M$ and $\bar{m}_i : M'' \to M'_i$. Notice that $\bar{m}_i$ is the normal invariant corresponding to the Ferry-Pedersen reduction of M'_i and that $m_i \circ \bar{m}_i : M'' \to M'_i \to M$ is bordant to $id : M \to M$. Thus to see that $f \circ m_i \circ \bar{m}_i$ and hence f is in the image we may apply the given map to $f \circ m_i$.

To see that this map is injective we need to see that if two homology manifolds have the same index and the maps $f : Y \to X$ and $g : Z \to X$ have the same image in $\Omega_n^{STOP}(X)$, then they are homology manifold bordant. We will first see that they are Poincaré bordant. By the above lemma **??** we may assume that Y and Z are connected and that f and g induce isomorphisms of fundamental groups.

Consider f and g as elements of Poincaré bordism of X. Note that we have the following exact sequence see [HV]:

$L_n(X) \to^{or} \Omega_n^{PD}(X) \to H_n(X, MSG) \to L_{n-1}(X)$.

Which corresponds to the sequence

$\Omega_{n+1}^{QP}(X \times BSG) \to \Omega_n^P(X \times BSG) \to \Omega_n^Q(X \times BSG) \to \Omega_n^{QP}(X \times BSG)$.

Claim the diagram commutes

$$\begin{array}{ccc}
\Omega_n^{SH}(X) & \longrightarrow & \Omega_n^{STOP}(X) \\
\big\downarrow & & \big\downarrow \\
{}^{or}\Omega_n^{PD}(X) & \longrightarrow & \Omega_n^Q(X \times BSG)
\end{array}$$

where the top horizontal map is given above and all other maps are the appropriate forgetful maps. This follows from the fact that $id \times j^Y : Y \to Y \times BSG$ and $(id \times j^Y) \circ n_i : M \to Y \to Y \times BSG$ are bordant as Q spaces, where j^Y is the classifying map of $Sp(Y)$. See [HV] for this fact and related definitions.

From the above exact sequence we see that if f and g have the same image in $\Omega_n^{STOP}(X)$, then the obstruction to them being Poincaré bordant is an element of $L_n(X)$. It is given by the difference of the surgery obstructions for f and g. Let $m : M \to Y$ and $n : N \to Z$ denote the normal invariants associated to the Ferry-Pedersen reductions of Y and Z. By assumption $f \circ m : M \to Y \to X$ and $g \circ n : N \to Z \to X$ are bordant as manifolds so we have $V \to X$ a bordism between them. Consider $id : Y \to Y$ and $m : M \to Y$. They bound as elements of Q-bordism. Therefore we get an element Q_Y of $\Omega_{n+1}^{QP}(Y)$ and a corresponding element of $L_n(Y)$. This element vanishes; it is given by comparing $id \circ m : M \to Y \to Y$ and $m \circ id : M \to M \to Y$ as elements of $L_n(Y)$. Thus id and m bound in $\Omega_n^P(Y)$. Denote their bordism by P_Y. Similarly $id : Z \to Z$ and $n : N \to Z$ are bordant by P_Z. Pasting these three bordisms together we get a Poincaré bordism from f to g, call this $k : W \to X$.

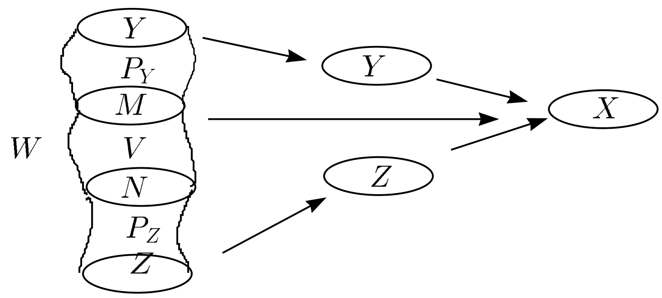

FIGURE 20. The Poincaré bordism from f to g.

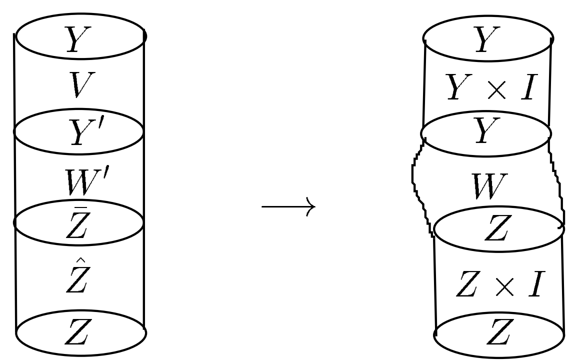

FIGURE 21. A homology manifold cobordism from Y to Z and a map to W which is the identity on the boundary.

Further we get a TOP reduction on W which restricts to the Ferry-Pedersen reductions on Y and Z as follows. We use the TOP-reductions of Y and Z to get TOP reductions of the Poincaré bordisms P_Y and P_Z respectively. Together with the TOP reduction of the manifold V this gives a TOP reduction of W as desired. Taking this TOP reduction together with the index i gives a homology manifold normal invariant on W which agrees with the given homology manifold normal invariants for Y and Z. By performing Poincaré surgery rel ∂ on $k : W \to X$ ($f : Y \to X$ and $g : Z \to X$ already induce isomorphisms of π_1,) we may assume that it is an isomorphism on fundamental groups.

We would like to put a homology manifold structure on W rel Z. Notice that since we have a homology manifold normal invariant of W which agrees with that for Y and Z, the only obstruction to this lives in $L_{n+1}(W, Y) \simeq *$.

So we have a structure $h : W' \to W$ so that $\partial W'$ gives $\alpha : Y' \to Y$ an arbitrary structure on Y and $\bar{Z} \to Z$ which is s-cobordant via $\hat{i} : \hat{Z} \to Z$ to the identity map $id_Z : Z \to Z$. Since this structure α on Y, has the same homology manifold normal invariant as the identity structure, it came from a Wall realization of Y. Thus there exists a Wall realization $r : V \to Y \times I$ so that $r|\partial V = \alpha \amalg id_Y$. The desired bordism from f to g is then given by $r \cup h \cup \hat{i} : V \cup_Y W' \cup_{\bar{Z}} \hat{Z} \to Y \times I \cup_Y W \cup_Z Z \times I$ composed with the projection to W and the given Poincaré bordism $k : W \to X$. $\square$

References

[AM] Anderson, D., Munkholm, H.: Boundedly controlled topology. Lecture Notes in Mathematics, vol.**1323**. Berlin-Heidelberg-New York: Springer-Verlag 1988

[B1] Browder, W.: Smooth structures on $M \times \mathbb{R}$. Proc. Cambridge Philos. Soc. **61**, 337-345 (1965)

[B2] Browder, W.: Surgery on simply connected manifolds. Berlin-Heidelberg-New York: Springer-Verlag 1972

[BM] Brumfiel, G. , Morgan, J.: Homotopy-theoretic consequences of N. Levitt's obstruction theory to transversality for spherical fibrations. Pacific J. Math., **67** (1976), 1-100.

[BFMW] Bryant, J., Ferry, S., Mio, W., Weinberger, S.: Topology of Homology Manifolds. Ann. of Math. (2), **143**, 435-467 (1996)

[C] Cannon, J.W.: The recognition problem: what is a topological manifold? Bull. Amer. Math. Soc., **84**, 832-866 (1978)

[E] Edwards, R.D.: The topology of manifolds and cell-like maps, Proc. Internat. Cong. Mathematicians, Helsinki, 1980.

[F] Ferry, S.: Controlled topology and the characterization of topological manifolds, CBMS Regional Conference Series in Mathematics, Amer. Math. Soc., To appear.

[FP] Ferry, S., Pedersen, E.K.: Epsilon surgery theory I. In: Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993) 167-226, London. Math. Soc. Lecture Note Ser. 227, Cambridge Univ. Press, Cambridge (1995)

[HV] Hausmann, J, Vogel, P.:Geometry on Poincaré Spaces. Princeton: Princeton University Press 1993.

[J1] Johnston, H.: Transversality for $\mathbb{Z}_2$-orbifolds. Preprint, 1996.

[J2] Jones, L.: Patch spaces: a geometric representation for Poincaré spaces. Ann. Math., **97** 306-343 (1973)

[KS] Kirby, R., Siebenmann, L.: Foundational essays on topological manifolds, smoothings, and triangulations. Princeton: Princeton University Press 1977

[LM] Levitt, N., Morgan, J.: Transversality structures and PL structures on spherical fibrations. Bull. Amer. Math. Soc., **78** 1065-1068 (1972)

[LR] Levitt, Ranicki: Intrinsic Transversality Structures, Pac. J. of Math. **129** (1987), 85-144.

[Q1] Quinn, F.: A geometric formulation of surgery. Thesis, Princeton University, 1969

[Q2] Quinn, F.: Ends of maps II. Invent. Math. **68**, 353-424 (1982)

[Q3] Quinn, F.: Resolutions of homology manifolds, and the topological characterization of manifolds. Invent. Math. **72**, 264-284 (1983); Corrigendum **85**(1986), 653

[Q4] Quinn, F.: Surgery on Poincaré and normal spaces, Bull. Amer. Math. Soc., 262-267 (1972)

[R1] Ranicki, A.: The total surgery obstruction. In: Proceedings of Symposium on Algebraic Topology, Aarhus, 1978, Dupont, J.L., Madsen, I.(ed), 275-236. Lecture Notes in Mathematics, vol.**763**. Berlin-Heidelberg-New York: Springer-Verlag 1979

[R2] Ranicki, A.: Algebraic L-theory and topological manifolds. Cambridge Tracts in Mathematics, 102. Cambrige Unviersity Press, Cambridge, 1992.

[RS] Rourke, C.P., Sanderson, B.J.: Block bundles I,II and III Ann. Math. **87**, 1-28, 256-278, 431-483 (1968)

[Sp] Spivak, M.: Spaces Satisfying Poincaré Duality,. Topology **6** (1969) 77-102.

[Wa] Wall, C.T.C.: Surgery on compact manifolds. New York: Academic Press 1971

[We1] Weinberger, S.: The topological classification of stratified spaces. Chicago: University of Chicago Press 1995.

[We2] Weinberger, S.: Non locally linear manifolds and orbifolds. Proc. ICM, Zurich 1994, Birkhauser, 1995.