Epsilon Surgery Theory

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1. Introduction

In classical surgery theory one begins with a Poincaré duality space X and a normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{\tilde{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

The problem is to vary (M, f) by a normal cobordism to obtain a homotopy equivalence $f' : M' \to X$.

It is desirable to have an epsilon or controlled version of surgery theory. Thus, X comes equipped with a reference map to a metric space K, and the aim is to produce a homotopy equivalence $f' : M' \to X$, which is small measured in K. The existence of such an f' implies that X is a small Poincaré duality space in the sense that cells are close to their dual cells, measured in K. That X be a small Poincaré duality space must therefore be part of the original data.

For many applications the most interesting question is whether such a map f' exists with arbitrarily small control in K. In this case, X would have to be an ϵ-Poincaré duality space for all $\epsilon > 0$. Unfortunately, there are technical difficulties in defining and dealing with such ϵ-Poincaré duality spaces. Our approach is to work instead with a bounded surgery theory. Our bounded surgery theory generalizes epsilon surgery theory in the same sense that simple proper surgery theory (as developed in [39, 41, 18, 26]) generalizes classical compact surgery theory. The fact that bounded categories are categories avoids many technical difficulties.

Consider a classical surgery problem as above. Cross with the real line and look for an infinite simple homotopy equivalence $f' : M' \to X \times \mathbb{R}$. Such a manifold M' has the form $N' \times \mathbb{R}$ for some closed manifold N', so $f' : N' \to X$ solves the classical surgery problem. This means that the two-ended simple surgery theory is as good for applications as the compact theory. The two-ended theory is more general, though, since it applies to any two-ended manifold with fundamental group equal to the fundamental group of each end.

This is our approach to epsilon surgery theory and its generalization. We consider surgery problems parameterized over $K \times \mathbb{R}$, where $K \times \mathbb{R}$ is given a metric so that $K \times \{t\}$ becomes t times as big as $K \times \{1\}$ for $t > 1$. Call this space $O(K_+)$. (This description is not quite accurate. See §2 for precise definitions and details).[1]

[1]It is easy to see that if Z is a Poincaré duality space with a map $Z \to K$ such that Z has ϵ-Poincaré duality for all $\epsilon > 0$ when measured in K (after subdivision), e.g. a homology manifold, then $Z \times \mathbb{R}$ is an $O(K_+)$-bounded Poincaré complex. The converse

We let X be a complex with bounded Poincaré duality over this control space. Given a proper normal map

$$\begin{array}{ccccc} \nu_M & \xrightarrow{\tilde{f}} & \xi & & \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & X & \longrightarrow & O(K_+) \end{array}$$

we study the problem of producing a proper normal cobordism to $f' : M \to X$ such that f' is a bounded simple homotopy equivalence measured in $O(K_+)$. The obstruction groups obtained are the desired obstruction groups for epsilon surgery. This is our codification of the idea that an ϵ-Poincaré space (for all $\epsilon > 0$) is a sequence of smaller and smaller Poincaré duality spaces joined by smaller and smaller Poincaré cobordisms.

Thus, in case X is a homology manifold (homology manifolds are naturally epsilon Poincaré for all epsilon) with a reference map $\varphi : X \to K$, we replace an epsilon surgery problem

$$\begin{array}{ccccc} \nu_M & \xrightarrow{\tilde{f}} & \xi & & \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & X & \longrightarrow & K \end{array}$$

by the bounded surgery problem

$$\begin{array}{ccccc} \nu_M \times 1 & \xrightarrow{\tilde{f}} & \xi \times 1 & & \\ \downarrow & & \downarrow & & \\ M \times \mathbb{R} & \xrightarrow{f} & X \times \mathbb{R} & \longrightarrow & O(K_+) \end{array}$$

If $f \times id$ is properly normally cobordant to a bounded simple equivalence $f' : M \to X \times \mathbb{R}$, we split M' near the end to obtain a sequence of more and more controlled solutions to the original problem. Our approach generalizes this sort of epsilon surgery in case other data happen to be available. For an application in which a parameterization over an open cone appears naturally, see Theorem 18.1, which is *not* a bounded translation of an epsilon problem. The other applications in §18 do, however, illustrate how one passes between the bounded and epsilon worlds.

Naturally, we require hypothesis in addition to the general situation described above. In the first part of the paper, our main hypotheses are that the control map $X \to K$ have constant coefficients in the sense that it "looks like" a product on π_1, and that K be a finite complex. This restricts

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(while true) will not concern us here.

applicability in, say, the case of group actions, to semifree group actions. In §14 we show how to extend the theory to treat a more general equivariant case, but for readability, we have chosen to give most details in the special case of constant coefficients.

Some surprising phenomena come up. The method allows studying many more objects than exist in the compact world. An n-manifold parameterized by $\mathbb{R}^{n+k}$ is an object of dimension $-k$, so we have objects that in some sense correspond to negative dimensional manifolds. This leads to nonconnective surgery spectra. The necessary algebra for our theory has been developed in [23, 27, 9].

Needless to say, we have benefited from discussions with many colleagues. We should like particularly to mention Anderson, Hambleton, Hughes, Lück, Munkholm, Quinn, Ranicki, Taylor, Weibel, Weinberger, and Williams. In fact, related theories have been developed by Hughes, Taylor and Williams, [14], Madsen and Rothenberg [17], and Weinberger [44].

Finally we want to acknowledge that the very stimulating atmosphere of the SFB at Göttingen has had a major effect on the developments of this paper. The first author would also like to thank Odense University for its support in the fall of 1987.

2. Algebraic preliminaries

Let M be a metric space, and let R be a ring with anti-involution. For definiteness, the reader should keep in mind the model case in which M is the infinite open cone $O(K)$ on a complex $K \subset S^n \subset \mathbb{R}^{n+1}$ and $R = \mathbb{Z}\pi$, with π a finitely presented group. The category $\mathcal{C}_M(R)$ is defined as follows:

Definition 2.1. An object A of $\mathcal{C}_M(R)$ is a collection of finitely generated free right R-modules A_x, one for each $x \in M$, such that for each ball $C \subset M$ of finite radius, only finitely many A_x, $x \in C$, are nonzero. A morphism $\varphi : A \to B$ is a collection of morphisms $\varphi^x_y : A_x \to B_y$ such that there exists $k = k(\varphi)$ such that $\varphi^x_y = 0$ for $d(x, y) > k$.

The composition of $\varphi : A \to B$ and $\psi : B \to C$ is given by $(\psi \circ \varphi)^x_y = \sum_{z \in M} \psi^z_y \varphi^x_z$. The composition $(\psi \circ \varphi)$ satisfies the local finiteness and boundedness conditions whenever ψ and φ do.

Definition 2.2. The dual of an object A of $\mathcal{C}_M(R)$ is the object A^* with $(A^*)_x = A^*_x = \mathrm{Hom}_R(A_x, R)$ for each $x \in M$. A^*_x is naturally a left R-module, which we convert to a right R-module by means of the anti-involution. If $\phi : A \to B$ is a morphism, then $\phi^* : B^* \to A^*$ and $(\phi^*)^x_y(h) = h \circ \phi^y_x$, where $h : B_x \to R$ and $\phi^y_x : A_y \to B_x$. ϕ^* is bounded whenever ϕ is. Again, ϕ^* is naturally a left module homomorphism which induces a homomorphism of right modules $B^* \to A^*$ via the anti-involution.

Definition 2.3. There are functors $\oplus$ and Π from $\mathcal{C}_M(R)$ to $\mathrm{Mod}(R)$, the category of free modules over R. $\oplus A = \bigoplus_{x \in M} A_x$ and $\Pi A = \prod_{x \in M} A_x$. Notice that $\Pi A^* = (\oplus A)^*$.

Definition 2.4. Consider a map $p : X \to M$

(i) The map $p : X \to M$ is *eventually continuous* if there exist k and a covering $\{U_\alpha\}$ of X, such that the diameter of $p(U_\alpha)$ is less than k.

(ii) A *bounded CW complex* over M is a pair (X, p) consisting of a CW complex X and an eventually continuous map $p : X \to M$ such that there exists k such that $\mathrm{diam}(p(C)) < k$ for each cell C of X. (X, p) is called *proper* if the closure of $p^{-1}(D)$ is compact for each compact $D \subset M$. We consider (X, p_1) and (X, p_2) to be the same, if there exists k so that $d(p_1(x), p_2(x)) < k$ for all x.

Remark 2.5. We do not require the control map p to be continuous in the above definition. It is, however, often the case that p may be chosen to be continuous. This is the case if the metric space is "boundedly highly connected" in an appropriate sense. See Definition 5.2.

Definition 2.6. Consider a bounded CW complex (X, p)

(i) The bounded CW complex (X, p) is (-1)-*connected* if there is $k \in \mathbb{R}_+$ so that for each point $m \in M$, there is a point $x \in X$ such that $d(p(x), m) < k$.

(ii) (X, p) is 0-*connected* if for every $d > 0$ there exist $k = k(d)$ so that if $x, y \in X$ and $d(p(x), p(y)) \leq d$, then x and y may be joined by a path in X whose image in M has diameter $< k(d)$. Notice that we have set up our definitions so that 0-connected does not imply -1-connected.

Definition 2.7. Let $p : X \to M$ be 0-connected, but not necessarily (-1)-connected.

(i) (X, p) has *trivial bounded fundamental group* if for each $d > 0$, there exist $k = k(d)$ so that for every loop $\alpha : S^1 \to X$ with $diam(p \circ \alpha(S^1)) < d$, there is a map $\bar{\alpha} : D^2 \to X$ so that the diameter of $p \circ \bar{\alpha}(D^2)$ is smaller than k.

(ii) (X, p) has *bounded fundamental group* π if there is a π-cover $\widetilde{X}$ so that $\widetilde{X} \to M$ has trivial bounded fundamental group.

If X is a CW complex, we will denote the cellular chains of $\widetilde{X}$ by $C_\#(X)$, considered as a chain complex of free right $\mathbb{Z}\pi$-modules. When $p : X \to M$ is a proper bounded CW complex with bounded fundamental group, we can consider $C_\#(X)$ to be a chain complex in $\mathcal{C}_M(\mathbb{Z}\pi)$ as follows: For each cell $C \in X$, choose a point $c \in C$ and let $D_\#(X)_y$ be the free submodule of $C_\#(X)$ generated by cells for which $p(c) = y$. The boundary map is bounded, since cells have a fixed maximal size. We will denote the cellular chains of $\widetilde{X}$ by $D_\#(X)$ when we consider them as a chain complex in $\mathcal{C}_M(\mathbb{Z}\pi)$ and by $C_\#(X)$ when we consider them as an ordinary chain complex of $\mathbb{Z}\pi$ modules. We will denote $D_\#(X)^*$ by $D^\#(X)$. If $(X, \partial X)$ is a bounded CW pair, $D_\#(X, \partial X)$ denotes the relative cellular chain complex regarded as a chain complex in $\mathcal{C}_M(\mathbb{Z}\pi)$.

Lemma 2.8. *When $p : X \to M$ is a proper bounded CW complex with bounded fundamental group, we have the following formulas*

(i) $\oplus D_\#(X) = C_\#(X)$
(ii) $\oplus D^\#(X) = C^\#_{cs}(X)$ *(≡ cochains with compact support)*
(iii) $\Pi D_\#(X) = C^{lf}_\#(X)$ *(≡ locally finite chains)*
(iv) $\Pi D^\#(X) = C^\#(X)$.

Proof. Statement (i) is clear, and (iv) follows from the formula $\Pi(A^*) = (\oplus A)^*$. Statements (ii) and (iii) follow easily from the fact that p is proper. In case p is not proper this suggests an extension of the concepts of homology with locally finite chains and cohomology with compact support to concepts requiring compactness or locally finiteness only in a designated direction. □

We recall the *open cone construction* from [27]. If K is a subset of $S^n \subset \mathbb{R}^{n+1}$, we define $O(K)$ to be the metric space $O(K) = \{t \cdot x | 0 \leq t, x \in K\} \subset \mathbb{R}^{n+1}$. Here, $O(K)$ inherits a metric from $\mathbb{R}^{n+1}$. We think of this as a 1-parameter family of metrics on K, in which distance grows larger with increasing t. We state the main result of [27]:

Theorem 2.9. *Let K be of the homotopy type of a finite complex. The K-theory of the categories $\mathcal{C}_{O(K)}(R)$ is given by*

$$K_*(\mathcal{C}_{O(K)}(R)) = KR_{*-1}(K)$$

where KR is the nonconnective homology theory associated to the algebraic K-theory of the ring R.

We refer the reader to [27] for further facts about the O construction and the K-theory of $\mathcal{C}_{O(K)}(R)$.

Definition 2.10. Let R be a ring with unit. We denote $\mathrm{Coker}(K_*(\mathcal{C}_M(\mathbb{Z})) \to K_*(\mathcal{C}_M(R)))$ by $\widetilde{K}_*(\mathcal{C}_M(R))$. If R is a group ring, $R = \mathbb{Z}\pi$, we denote $\widetilde{K}_1(\mathcal{C}_M(R))/\pm\pi$ by $Wh_M(\pi)$.

Definition 2.11. We define

(i) a metric space M is *allowable* if there exist a bounded finite-dimensional simplicial complex K and a map $p : K \to M$ which is -1, 0- and 1-connected.

(ii) a map $f : X \to Y$ between metric spaces is *eventually Lipschitz* if the inverse image of every bounded set is bounded and if there are numbers $k > 0$ and $d > 0$ so that $d(f(x), f(x')) \leq max(d, kd(x, x'))$ for all $x, x' \in X$. We say that X and Y are *eventually Lipschitz equivalent* if there exist a number $M > 0$ and eventual Lipschitz maps $f : X \to Y$, $g : Y \to X$ so that $d(f \circ g, id) < M$ and $d(g \circ f, id) < M$.

In connection with our study of the resolution problem, we will want to apply this theory to open cones on compact finite-dimensional ANR's, so we prove that such spaces are allowable.

Proposition 2.12. *If $X \subset S^n$ is a compact* ANR*, then $O(X)$ is an allowable metric space.*

Proof. Let $r_t : U \to X$ be a homotopy from a neighborhood U of X in S^m to X such that $r_0 = id$, $r_1 : U \to X$ is a retraction, and $r_t|X = id$ for all t. Let $X = \bigcap_{i=1}^{\infty} P_i$, where $P_1 \supset P_2 \supset \ldots$ are finite polyhedra and $r_t(P_{i+1}) \subset P_i$ for all $t \in [0,1]$. We may assume that $P_n \subset N_{\frac{1}{n}}(X)$. Form the telescope $K = \bigcup_{n=1}^{\infty} O(P_n) \cap B(n, 0)$, where $B(n, 0)$ is the closed ball of radius n around $0 \in \mathbb{R}^{m+1}$. The map r_1 induces a map $\bar{r} : K \to O(X)$ which is -1, 0 and 1-connected. □

Remark 2.13. Notice the following

(i) If $M = O(X)$ is the open cone on a compact ANR and $p : Z \to M$ is a finite-dimensional bounded CW complex, then p may be chosen to be continuous. The argument is an induction over the skeleta of Z starting with p restricted to the 0-skeleton.

(ii) Assume that K is a connected PL complex. Then we have

$$\widetilde{K}_1(\mathcal{C}_{O(K)}(R)) = K_1(\mathcal{C}_{O(K)}(R)) \text{ and } Wh_{O(K)}(\pi) = \widetilde{K}_1(\mathcal{C}_{O(K)}(\mathbb{Z}\pi)),$$

except when K is empty, in which case $O(K) = pt$. The argument is an Eilenberg swindle.

Definition 2.14. Let M be a metric space and let $p : X \to M$ and $q : Y \to M$ be maps. A map $f : X \to Y$ is said to be *bounded over M* (or simply *bounded*) if there is a number $k > 0$ so that $d(q \circ f(x), p(x)) < k$ for all $x \in X$. We say that f is a *bounded homotopy equivalence* if there exist $g : Y \to X$ and homotopies $h : f \circ g \sim id$, $l : g \circ f \sim id$ so that f, g, h and l are bounded.

Theorem 2.15. *Let M be an allowable metric space and let $p_X : X \to M$, $p_Y : Y \to M$ be proper bounded finite-dimensional CW complexes, both -1 and 0-connected with bounded fundamental group π. If $f : X \to Y$ is a cellular map such that:*

(i) $f \circ p_Y = p_X$.
(ii) f *induces a* π_1*-isomorphism.*
(iii) f *induces a (bounded) homotopy equivalence of chain complexes in* $\mathcal{C}_M(\mathbb{Z}\pi)$, $f_\# : D_\#(X) \to D_\#(Y)$.

Then f *is a bounded homotopy equivalence.*

Proof. This is proved in [2] for the case of a continuous control map. An alternative approach that works in our generality is to replace X and Y by proper regular neighborhoods $N(X) \subset N(Y)$ in some high-dimensional euclidean space and then apply the bounded h-cobordism theorem 2.17. The problem with torsion is solved by crossing with S^1, thus killing the torsion, thus getting a bounded homotopy equivalence of $X \times S^1 \to Y \times S^1$, and hence $X \to Y$ since X is bounded homotopy to $X \times \mathbb{R}$ which is the cyclic cover of $X \times S^1$. □

Remark 2.16. The theorem above plays a rôle in our theory which is analogous to the Whitehead theorem's rôle in standard surgery theory.

Finally we note that we have the bounded analogue of the s-cobordism theorem.

Theorem 2.17. (Bounded s-cobordism theorem). *Assume* $W \to M$ *is a* -1 *and* 0*-connected manifold with bounded fundamental group* π *such that the boundary of* W *has 2 components* $\partial_0 W \subset W$ *and* $\partial_1 W \subset W$*, such that the inclusions are bounded homotopy equivalences. The torsion* τ *of* W *is defined by the torsion of the contractible chain complex* $D_\#(W, \partial_0 W) \in Wh_M(\pi)$ *see* [37]. *For* $\dim(W) > 5$ *we have that* W *is isomorphic to* $\partial_0 W \times I$ *if and only if* $\tau = 0$.

Proof. The s-cobordism theorem in [25] is only stated for the parameter space $\mathbb{R}^n$, and in that context a bounded *and* thin h-cobordism theorem is proved. As far as the bounded s-cobordism theorem is concerned, the arguments only use that the h-cobordism is -1 and 0-connected with bounded fundamental group. □

3. Bounded Poincaré complexes

Given a bounded CW complex $p : X \to M$ with bounded fundamental group π, an element $[y] \in H_n^{lf}(X, \mathbb{Z})$ induces a cap-product $y \cap - : D^\#(X) \to D_{n-\#}(X)$. Here, $y \cap -$ is a homomorphism of chain complexes in $\mathcal{C}_M(\mathbb{Z})$ and is well-defined up to chain homotopy. The formula defining $y \cap -$ is the usual one using the Alexander-Whitney diagonal approximation. The size estimate on $y \cap -$ follows from the fact that the diagonal approximation takes the generator $c \in (D_n(X))_m$ to a sum $\sum c_i \otimes c_i'$ where $c_i \in (D_\#(X))_{m_i}$,

$c_i' \in (D_\#(X))_{m_i'}$, and $d(p(m_i), p(m_i')) \le 2k$, where k is the bound on the diameter of the cells of X as measured in M.

Our notational conventions in the following definition are based on [43, pp. 21–22]

Definition 3.1. Let $p : X \to M$ be a proper bounded CW complex with bounded fundamental group π, and let $\widetilde{X} \to X$ be an orientation double covering. Then X is a *bounded n-dimensional Poincaré duality space* if there is an element $[X] \in H_n^{lf}(X;\mathbb{Z})$, such that $[X] \cap - : D^\#(X) \to D_{n-\#}(X)$ is a bounded homotopy equivalence of chain complexes. Here, $\mathbb{Z}$ is made into a left $\mathbb{Z}\pi$ module using the antiinvolution on $\mathbb{Z}\pi$. X is a *simple* bounded Poincaré duality space if the torsion of $[X] \cap -$ is trivial in $Wh_M(\pi)$. If $p : X \to M$ is a disjoint union of spaces satisfying this condition, we shall also call X a Poincaré space. Notice that X may have infinitely many components, but the properness of $p : X \to M$ ensures that locally there are only finitely many components.

Definition 3.2. Let $p : (X, \partial X) \to M$ be a proper bounded CW pair so that X has bounded fundamental group π. The pair $(X, \partial X)$ is *an n-dimensional bounded Poincaré duality pair* if ∂X is an $(n-1)$-dimensional bounded Poincaré complex with orientation double covering the pullback of the orientation double covering on X and if there is an element $[X] \in H_n^{lf}(X, \partial X;\mathbb{Z})$, such that $[X] \cap - : D^\#(X) \to D_{n-\#}(X, \partial X)$ is a bounded homotopy equivalence of chain complexes. $(X, \partial X)$ is a *simple* bounded Poincaré duality space, if the torsion of $[X] \cap -$ is trivial in $Wh_M(\pi)$.

Remark 3.3. We note that $[X] \cap -$ is independent up to chain homotopy of the choice of representative chain for $[X]$. This is true since any other choice is of the form $X+\delta z$ and $(X+\delta z)\cap y - x\cap y = \delta z \cap y = \delta(z\cap y) - z\cap \delta y$, so $z \cap -$ is a chain homotopy between the two maps $X\cap$ and $(X+\delta z)\cap$.

Example 3.4. If X happens to be a manifold with a bounded handle decomposition, the usual proof of Poincaré duality produces a bounded Poincaré structure on X.

Definition 3.5. Let $\phi : (W, \partial W) \to (X, \partial X)$ be a map of bounded Poincaré duality pairs such that $\phi_*([W]) = [X]$. We define $K^\#(W, \partial W)$ to be the algebraic mapping cone of $\phi^* : D^\#(X, \partial X) \to D^\#(W, \partial W)$. We define $K_\#(W, \partial W)$ to be the dual of $K^\#(W, \partial W)$. We have short exact sequences of chain complexes

$$0 \longrightarrow D^\#(X, \partial X) \xrightarrow{\phi^*} D^\#(W, \partial W) \longrightarrow K^\#(W, \partial W) \longrightarrow 0$$

$$0 \longrightarrow K_\#(W, \partial W) \longrightarrow D_\#(W, \partial W) \xrightarrow{\phi^*} D_\#(X, \partial X) \longrightarrow 0.$$

As in classical surgery theory we have the following:

Lemma 3.6. *Let* $\phi : (W, \partial W) \to (X, \partial X)$ *be a map of bounded Poincaré duality pairs such that* $\phi_*([W]) = [X]$. *Then cap product with* $[W]$ *and* $[X]$ *induces a bounded chain homotopy equivalence from* $K_\#(W, \partial W)$ *to* $K^{n-\#}(W)$.

Remark 3.7. We have

(i) This definition of $K_\#(W, \partial W)$ gives $K_\#(W, \partial W)$ the same indexing as the kernel complex in [4] and [43]. Except for a shift in the index and changes in signs, $K_\#(W, \partial W)$ is just the algebraic mapping cone of $D_\#(W, \partial W) \to D_\#(X, \partial X)$.

(ii) Parameterizing an open manifold by the identity, our constructions give a simple proof of Poincaré duality on open manifolds from homology with locally finite coefficients to standard cohomology, or from cohomology with compact supports to standard homology. One applies Π and $\oplus$ respectively to the chain homotopy equivalence $[X] \cap -$.

4. Spivak normal fibre space

In this section we construct the Spivak normal fibre space of a bounded Poincaré duality space. Since bounded Poincaré complexes are certainly open Poincaré complexes in the sense of Taylor [41], we could simply refer to [41], but for the readers' convenience we give the existence proof:

Construct a proper embedding of $X \subset \mathbb{R}^n$, $n - \dim X \geq 3$. Let W a regular neighborhood of X and $r : W \to X$ a retraction. $W \to M$ has a bounded fundamental group, and we can triangulate sufficiently finely to get a bounded CW structure on W. Let F be the homotopy fibre of the map $\partial W \to X$.

Lemma 4.1. *The fibre* F *is homotopy equivalent to a sphere of dimension* $n - \dim X - 1$.

Proof. By the codimension 3 condition, F is simply connected. It is clear that F is also the fibre of the pullback to the universal cover of X, so consider the relative fibration $(*, F) \to (\widetilde{W}, \partial\widetilde{W}) \to \widetilde{X}$. We have at $\mathbb{Z}\pi$-module chain level homotopy equivalence $D_\#(W, \partial W) \cong D^{n-\#}(W) \cong D^{n-\#}(X) \cong D_{\dim X - n + \#}(X)$, so applying $\oplus$ we get $C_\#(W, \partial W) \cong C_{\# + \dim X - n}(X)$.

Therefore $H_*(\widetilde{W}, \partial\widetilde{W}) = H_{*+\dim X - n}(\widetilde{X})$. The usual spectral sequence argument shows that

$$H_i(*, F) = \begin{cases} 0 & i < n - \dim X \\ \mathbb{Z} & i = n - \dim X \\ 0 & i > n - \dim X \end{cases}$$

and F is thus a sphere, since $\pi_1 F = 0$. Considering $W \subset \mathbb{R}^n \subset \mathbb{R}^n_+ = S^n$ and collapsing everything outside W produces a spherical Thom class. □

The proof of uniqueness of the Spivak normal fibre space is also standard, and is left to the reader.

5. Surgery below the middle dimension

Our definition of "bounded surgery problem" is a straightforward translation of Wall's "surgery problem" [43, p. 9] into bounded topology.

Definition 5.1. Let X^n be a bounded Poincaré duality space over a metric space M and let ν be a (TOP, PL or O) bundle over X. A *bounded surgery problem* is a triple (W^n, ϕ, F) where $\phi : W \to X$ is a proper map from an n-manifold W to X such that $\phi_*([W]) = [X]$ and F is a stable trivialization of $\tau_W \oplus \phi^*\nu$. Two problems (W, ϕ, F) and $(\overline{W}, \bar{\phi}, \bar{F})$ are *equivalent* if there exist an $(n+1)$-dimensional manifold P with $\partial P = W \coprod \overline{W}$, a proper map $\Phi : P \to X$ extending ϕ and $\bar{\phi}$, and a stable trivialization of $\tau_P \oplus \Phi^*\nu$ extending F and $\bar{F}$. See [43, p. 9] for further details.

We will use the notation $\begin{smallmatrix} W & \xrightarrow{\phi} & X \\ & & \downarrow \\ & & M \end{smallmatrix}$ to denote a bounded surgery problem. When M is understood, we will shorten the notation to $\phi : W \to X$ or even to ϕ. In all cases, the bundle information is included as part of the data. Our theorem on surgery below the middle dimension and its proof are parallel to Theorem 1.2 on p. 11 of [43]. In order to state the theorem, we need a definition.

Definition 5.2. If $p : X \to M$ is a control map, we will say that $f : Y \to X$ is *boundedly k-connected over M* if for every $c > 0$ there is a number $d > 0$ so that for each $-1 \leq l < k$ and map $\alpha : S^l \to Y$ with extension $\beta : D^{l+1} \to X$ of $f \circ \alpha$ with $\mathrm{diam}(p \circ \beta(D^{l+1})) \leq c$, there exist a map $\gamma : D^{l+1} \to Y$ and a homotopy $h : D^{l+1} \to X$ with $h_0 = f \circ \gamma$, $h_1 = \beta$, and $\mathrm{diam}(p \circ h(D^{l+1} \times I)) \leq d$.

Note that if X is a bounded CW complex over M, then $X^{(k)} \to X$ is boundedly k-connected. The notion of boundedly k-connected *over M* differs from the notion of bounded connectivity of the *control map* discussed in §2. In particular, there is a dimension shift which is analogous to the dimension shift one normally encounters in discussing the connectivity of the *space* X as compared to the connectivity of the *map* $X \to *$.

Here is our theorem on surgery below the middle dimension.

Theorem 5.3. *Let $(X^n, \partial X)$ be a bounded Poincaré duality space over M, $n \geq 6$, or $n \geq 5$ if ∂X is empty. Consider a bounded surgery problem $\phi : (W, \partial W) \to (X^n, \partial X)$. Then $\phi : (W, \partial W) \to (X^n, \partial X)$ is equivalent to a bounded surgery problem $\bar{\phi} : (\overline{W}, \partial\overline{W}) \to (X^n, \partial X)$ such that $\bar{\phi}$ is boundedly $\left[\frac{n}{2}\right]$-connected over M and $\bar{\phi}| : \partial\overline{W} \to \partial X$ is boundedly $\left[\frac{n-1}{2}\right]$-connected.*

Proof. We start by considering the case in which $\partial X = \emptyset$. Triangulate W so that the diameters $p \circ \phi(\sigma)$, σ a simplex of W, are bounded. Replacing X by the mapping cylinder of ϕ, we can assume that $W \subset X$.

We inductively define a bordism $U^{(i)}$, $-1 \leq i \leq \left[\frac{n+1}{2}\right]$ and maps $\Phi^{(i)} : U^{(i)} \to W \cup X^{(i)}$, so that $\partial U^{(i)} = W \coprod \overline{W}^{(i)}$ and so that $\Phi^{(i)}$ is a bounded homotopy equivalence. We begin by setting $U^{(-1)} = W \times I$, and letting $\Phi^{(i)} \to X$ be $\phi \circ \text{proj}$. Let $U^{(0)}$ be obtained from $U^{(-1)}$ by adding a disjoint $(n+1)$-ball corresponding to each 0-cell of $X - W$. The map $\Phi^{(0)}$ is constructed by collapsing each new ball to a point and sending the point to the corresponding 0-cell of $X - W$.

Assume that $\Phi^{(i)} : U^{(i)} \to X$ has been constructed in such a way that $U^{(i)}$ is an abstract regular neighborhood of a complex consisting of W together with cells in dimensions $\leq i$ corresponding to the cells of $X - W$ in those dimensions. Assume further that $\Phi^{(i)}$ is the composition of the regular neighborhood collapse with a map which takes cells to corresponding cells. Each $(i+1)$-cell of $X - W$ induces an attaching map $S^i \to U^{(i)}$. If $2i+1 \leq n$, general position allows us to move this map off of the underlying complex and approximate the attaching map by an embedding $S^i \to \overline{W}^{(i)}$. The bundle information tells us how to thicken this embedding to an embedding of of $S^i \times D^{n-i}$ and attach $(i+1)$-handles to $U^{(i)}$, forming $U^{(i+1)}$. We extend $\Phi^{(i)}$ to $\Phi^{(i+1)}$ in the obvious manner. This process terminates with the construction of $U^{\left[\frac{n+1}{2}\right]}$. Turning $U^{\left[\frac{n+1}{2}\right]}$ upside down, we see that $U^{\left[\frac{n+1}{2}\right]}$ is obtained from $\overline{W}^{\left[\frac{n+1}{2}\right]}$ by attaching handles of index $> \left[\frac{n+1}{2}\right]$. Thus, the composite map $\overline{W}^{\left[\frac{n+1}{2}\right]} \to X$ is boundedly $\left[\frac{n}{2}\right]$-connected over M.

In case $\partial X \neq \emptyset$, the argument is similar. We first construct U over the boundary (and, therefore, over a collar neighborhood of the boundary) and then construct U over the interior. □

Remark 5.4. Notice the following

(i) The direct manipulation of cells and handles has replaced the usual appeals to homotopy theory and the Hurewicz-Namioka Theorem.

This is a general technique for adapting arguments from ordinary algebraic topology to the bounded category.

(ii) The construction in the proof yields somewhat more – we wind up with $(\overline{W}, \partial\overline{W}) \subset (X, \partial X)$. When $n = 2k+1$, $\overline{W}$ and X are equal through the k-skeleton. When $n = 2k$, $\partial\overline{W}$ is equal to ∂X through the $(k-1)$-skeleton and $\overline{W}$ contains every k-cell of $X - \partial X$. Since $\overline{W} \to X$ is k-connected, every k-cell in ∂X is homotopic rel boundary to a map into W. By attaching a $k+2$-cell to this homotopy along a face, we can guarantee that for every k-cell in ∂X there is a $k+1$-cell in X so that half of the boundary of the $k+1$-cell maps homeomorphically onto the k-cell and the other half maps into W.

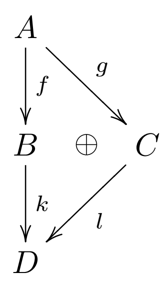
6. Controlled cell-trading

In this section we prove bounded versions of Whitehead's cell-trading lemma. There are algebraic and geometric versions of this lemma. We will need to use both in this paper. These operations apply equally well to cells in a bounded CW complex and to handles in a bounded handle decomposition. We will use cell terminology throughout, except for the term "handle addition."

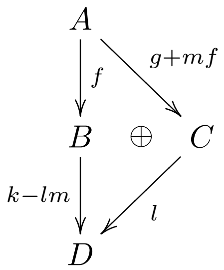
We need some notation. If we have a sequence

$$A \xrightarrow{(f,g)} B \oplus C \xrightarrow{k+l} D$$

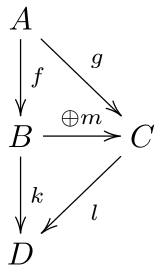
of objects and morphisms, we can represent it pictorially as:



Performing the elementary operation – sliding handles corresponding to basis elements in B over handles in C – corresponding to a bounded homomorphism $m : B \to C$ results in the diagram



We will write this operation schematically as

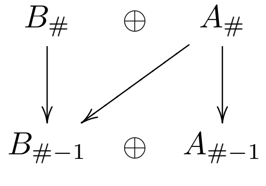


and call it *adding the B-cells to C via m.* When the sequence $A \to B \oplus C \to D$ is a part of a cellular $\mathbb{Z}\pi$-chain complex, this operation is realized geometrically by handle-addition by taking each generator x in B and sliding it across $m(x)$. Changing the attaching maps of the cells this way clearly has the effect described above on the cellular chains.

The next construction is *cancellation of cells.* If a portion of a chain complex looks like $\dots A \to B \oplus C \to D \oplus C' \dots$ and the composite $C \to B \oplus C \to D \oplus C' \to C'$ is an isomorphism sending generators to generators, then the chain complex is bounded chain homotopy equivalent to $\dots A \to B \to D \dots$. This has a geometric counterpart in cancellation of n- and $(n+1)$-cells. Note that it is not sufficient that the map $C \to C'$ be an isomorphism. It must send generators to generators. The complementary process of changing $\dots A_n \xrightarrow{\partial} A_{n-1} \to \dots$ to $\dots A_n \oplus D \xrightarrow{\partial \oplus 1} A_{n-1} \oplus D$ is called *introducing cancelling $n-1$ and n cells.*

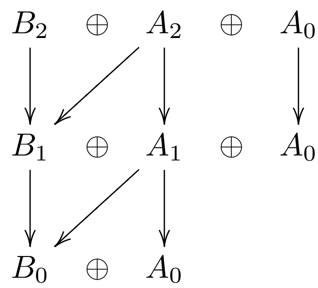
Here is our algebraic cell-trading lemma. This process involves introducing cells, adding cells, and cancelling cells, and results in n-cells being "traded for" $(n+2)$-cells.

Lemma 6.1. *Suppose given a bounded chain complex decomposed as modules as $B_\# \oplus A_\#$ for which the boundary map has the form*

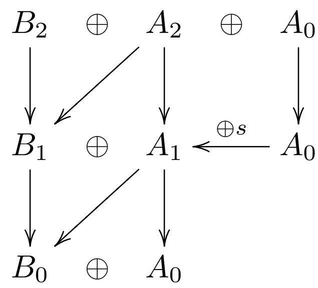


If there is a bounded chain homotopy s, with $(s|B_\#) = 0$, from the identity to a morphism which is 0 on $A_\#$ for $\# < k$, then $B_\# \oplus A_\#$ is boundedly chain-homotopy equivalent to $B_\# \oplus A'_\#$ where $A'_\# = 0$ for $\# < k$ and $A'_\# = A_\#$ for $\# \geq k+2$.

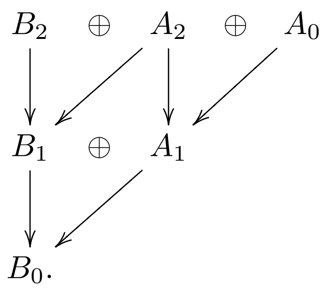
Proof. First introduce cancelling 1- and 2-cells corresponding to A_0 to obtain



Now add the new A_0-cells in dimension 1 to A_1 via s to obtain



The lower map from A_0 to A_0 is the identity, so the lower A_0 modules may be canceled to obtain



Repeat this process, and define $A'_\#$ so that $B_\# \oplus A'_\#$ is the resulting chain complex. □

Lemma 6.2. *Let $X \to M$ be a bounded CW complex which is 0-connected with bounded fundamental group π, so that the cellular chain complex over $\mathbb{Z}\pi$ is decomposed as (based) modules $B_\# \oplus A_\#$ for which the boundary map* *has the form*

$$\begin{array}{ccc} B_{\#} & \oplus & A_{\#} \\ \downarrow & \swarrow & \downarrow \\ B_{\#-1} & \oplus & A_{\#-1}. \end{array}$$

If there is a bounded chain homotopy s *with* $(s|B_{\#}) = 0$*, from the identity to a morphism which is* 0 *on* $A_{\#}$ *for* $\# < k$*, and if* $A_{\#} = 0$ *for* $\# \leq 2$*, then* X *may be changed by a bounded simple homotopy equivalence to* X'*, so that the cellular chains have the form* $B_{\#} \oplus A'_{\#}$ *where* $A'_{\#} = 0$ *for* $\# < k$ *and* $A'_{\#} = A_{\#}$ *for* $\# \geq k+2$*.*

Proof. Perform the same operations as above, but do them geometrically, using handle additions, rather than algebraically. □

In the above lemma, if $k > \dim(A_{\#} \oplus B_{\#})$, the cellular chain complex becomes $B_{\#}$ in low dimensions together with modules in some pair of adjacent high dimensions with ∂ an isomorphism between them. The hypothesis that $A_{\#} = 0$ for $\# \leq 2$ is necessary to avoid π_1 problems.

7. The bounded π-π Theorem

As in [43, Chapter 9], the π-π Theorem is the key theorem in setting up a geometric version of bounded surgery theory.

Theorem 7.1. *(Bounded* π-π *Theorem) Let* $(X^n, \partial X)$*,* $n \geq 6$*, be a bounded Poincaré duality space over an allowable control space* M*. Consider a bounded surgery problem*

$$\begin{array}{ccc} (W, \partial W) & \xrightarrow{\phi} & (X^n, \partial X) \\ & & \downarrow{\scriptstyle p} \\ & & M \end{array}$$

with bundle information assumed as part of the notation.

If both $p : X \to M$ *and* $p| : \partial X \to M$ *are* (-1)-*, and* 0-*connected and if the inclusion* $\partial X \to X$ *induces an isomorphism of bounded fundamental groups* π*, then we may do surgery to obtain a bounded normal bordism from* $(W, \partial W) \to (X, \partial X)$ *to* $(W', \partial W') \to (X, \partial X)$*, where the second map is a bounded simple homotopy equivalence of pairs.*

Proof. We begin with the case $n = 2k$. By Theorem 5.3 we may do surgery below the middle dimension. We obtain a surgery problem $W' \xrightarrow{\phi'} X$ so that ϕ' is an inclusion which is the identity through dimension k.

This means that cancelling cells in $K_\#(W', \partial W')$ yields a complex which is 0 through dimension $k-1$. Abusing the notation, we will assume that the chain complex $K_\#(W', \partial W')$ is 0 for $\# \le k-1$. The generators of $K_{k-1}(W)$ correspond to k-cells in $\partial X - W$. Cancelling these against the k-cells described in Remark 5.4(ii), and leaving out the primes for notational convenience, we have

$$\begin{aligned} K_\#(W, \partial W) &= 0 \qquad \# \le k-1 \\ K_\#(W) &= 0 \qquad \# \le k-1. \end{aligned}$$

Since

$$K^{n-\#}(W, \partial W) \simeq K_\#(W)$$

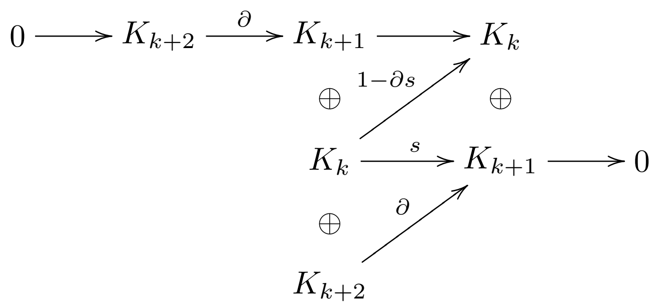
there is a bounded algebraic homotopy σ on $K^\#(W, \partial W)$ satisfying $\sigma\delta + \delta\sigma = 1$ for $\# \ge k+1$. Taking duals as in Definition 2.2, there is an algebraic homotopy s on $K_\#(W, \partial W)$ such that $s\partial + \partial s = 1$ for $\# \ge k+1$. Since $K_\# = K_\#(W, \partial W)$ is 0 in high dimensions, the "cell trading" procedure may be applied upside down, so that the $K_\#$ is changed to

$$0 \to K'_{k+2} \xrightarrow{\partial} K'{}_{k+1} \xrightarrow{\partial} K_k \to 0$$

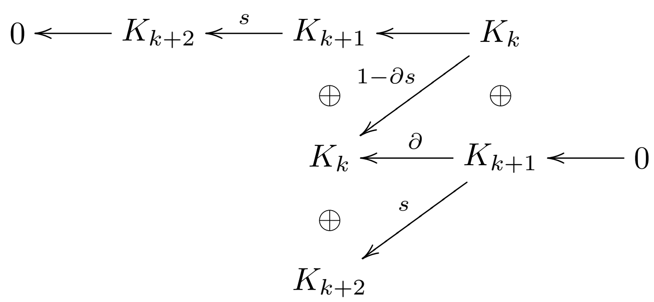
together with a homotopy s so that $s\partial + \partial s = 1$ except at degree k. We leave out the primes for notational convenience. Corresponding to each generator of K_{k+2} (and at a point near where the generator sits in the control space) we introduce a pair of cancelling $(k-1)$- and k-handles and excise the interior of the $(k-1)$-handle from $(W, \partial W)$. The chain complex for this modified W is

$$0 \longrightarrow K_{k+2} \longrightarrow K_{k+1} \longrightarrow \begin{matrix} K_k \\ \oplus \\ K_{k+1} \end{matrix} \longrightarrow 0$$

All generators of $K_k \oplus K_{k+1}$ are represented by discs. We may represent any linear combination of these discs by an embedded disc, and these embedded discs may be assumed to be disjoint by the usual piping argument. See [43, p. 39]. This uses the surjective part of the π-π condition. We do surgery on the following elements: For each generator x of K_k, we do surgery on $(x - \partial sx, sx)$ and for each generator y of K_{k+2}, we do surgery on $(0, \partial y)$. This time, we can think of the process as introducing pairs of cancelling k- and $(k+1)$-handles, performing handle additions with the k-handles, and then excising the k-handles from $(W, \partial W)$. The resulting chain complex is:



which is easily seen to be contractible, the contraction being



Dualizing, we see that after surgery, $K_{\#}(W, \partial W)$ is boundedly chain contractible. Poincaré duality shows that $K_{\#}(W)$ is boundedly chain contractible. Together, these imply the bounded chain contractibility of $K_{\#}(\partial W)$. Using Theorem 2.15 now shows that $\partial X \to \partial W$ and $X \to W$ are bounded homotopy equivalences. This is where the hypothesis of allowability and the full π-π condition are used. An easy argument composing deformations in the mapping cylinder of $(W, \partial W) \to (X, \partial X)$ completes the proof that $(W, \partial W) \to (X, \partial X)$ is a controlled homotopy equivalence.

Having obtained a homotopy equivalence of pairs, we can vary by an h-cobordism of pairs to obtain a simple homotopy equivalence of pairs. The argument for this is easy, using only standard facts about torsion of h-cobordisms and the fact that if $\phi : (M^n, \partial M) \to (N^n, \partial N)$ is a homotopy equivalence of pairs, then $\tau(f|\partial M) = \tau(f) + (-1)^n \tau(f)^*$. This is a straightforward consequence of the *simplicity* of Poincaré duality at the chain level. This completes the even-dimensional case.

To obtain the π-π-Theorem in the odd-dimensional case we resort to a trick.

(i) Cross with S^1 to get back to an even dimension and do the simple surgery.
(ii) Go to the cyclic cover and use the simplicity of the above homotopy equivalence to split and obtain a homotopy equivalence of the ends. See Theorem 7.2 below.

(iii) Vary by an h-cobordism of pairs to get a simple homotopy equivalence of pairs.

This completes the proof. □

In the above, we used the following splitting theorem from [9], which is essentially a bounded version of Quinn's End Theorem [29, 30]. In section 12 we give a proof based on the algebraic theory of surgery [36].

Theorem 7.2. *Let $(X^n, \partial X)$, $n \geq 6$, be a bounded Poincaré duality space over M. If both $p : X \to M$ and $p| : \partial X \to M$ are (-1), and 0-connected and if $\phi : (W^n, \partial W) \to (X^n, \partial X) \times \mathbb{R}$ is a bounded simple homotopy equivalence of pairs over $M \times \mathbb{R}$, where M is allowable, then $(W^n, \partial W) \cong (N^n, \partial N) \times \mathbb{R}$ and ϕ is boundedly homotopic to $\phi' \times id$, where $\phi' : (N^n, \partial N) \to (X^n, \partial X)$ is a bounded homotopy equivalence over M.*

8. Manifold 1-skeleton

Our construction of bounded surgery groups is modeled on Ch. IX in Wall [43]. An essential ingredient there is Wall's Lemma 2.8, which says that Poincaré duality spaces have manifold 1-skeleta.

In this and the following section we specialize to allowable metric spaces.

Proposition 8.1. *Suppose that M is an allowable metric space. Given a finitely presented group π and an integer $n \geq 4$, there exists a (-1)- and 0-connected n-dimensional manifold $W \to M$ with bounded fundamental group equal to π.*

Proof. Let N be a compact 4-manifold with fundamental group π, and let W be a regular neighborhood of a proper embedding of the 2-skeleton of M (or rather of a 0- and 1-connected PL-complex mapping to M see Definition 2.11) in $\mathbb{R}^5$. Now $\partial W \times N \to M$ satisfies the conditions except for dimension.

By general position once again, we may embed the 2-skeleton of $\partial W \times N$ properly in $\mathbb{R}^{n+1}$. The boundary of a regular neighborhood mapped to M now satisfies the conditions. □

Proposition 8.2. *A bounded Poincaré duality space $X \to M$ of dimension ≥ 5, with bounded fundamental group π has a manifold 1-skeleton, i.e., there exist a manifold with boundary $(W, \partial W) \to M$ and a bounded homotopy equivalence $X \sim W \cup_{\partial W} Y$ where Y is obtained from ∂W by attaching cells of dimension 2 and higher, and $(Y, \partial W)$ is a bounded Poincaré pair.*

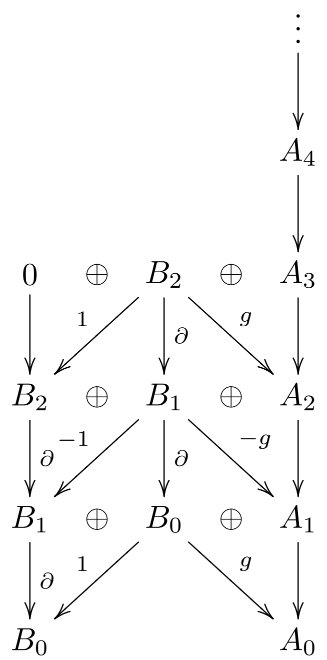
Proof of proposition. Our proof, which is modeled on Wall's, consists of changing the CW structure on X to make it similar to the CW structure of the dual chains in high dimensions. We then exploit the fact that the boundary map from the $(n-1)$- to the n-cells in the dual complex has a very special form.

Let $A_\# = D^{n-\#}(X)$, a chain complex in $\mathcal{C}_M(\mathbb{Z}\pi)$. Denote $D_\#(X)$ by $B_\#$. Poincaré duality gives a homotopy equivalence

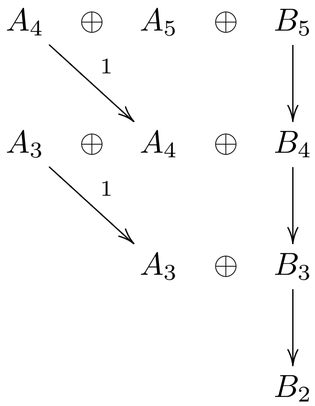
$$f : A_\# \to B_\#, \qquad g : B_\# \to A_\#.$$

We start by constructing an algebraic model for the new cell structure on X. The complex we construct will be equal to $B_\#$ for $\# = 0, 1, 2$ and $A_\#$ for $\# \geq 5$.

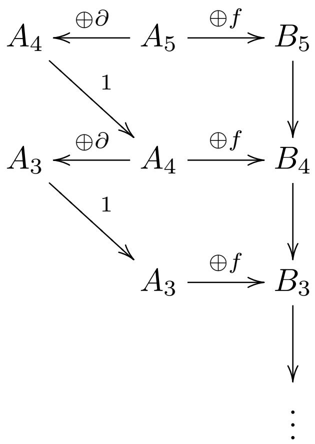
Consider the 2-skeleton of $B_\#$ and the mapping cylinder chain complex of $g|B_\#^{(2)} : B_\#^{(2)} \to A$.



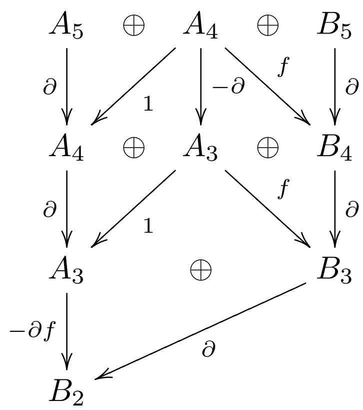
This is boundedly chain homotopy equivalent to $A_\#$. Since g is a homotopy equivalence, we can trade cells to get a new chain complex which looks like $B_\#$ through dimension 2. Calling the resulting chain complex $A_\#$ again, we begin to work geometrically. Introducing cancelling cells, we get a space bounded homotopy equivalent to X with chain complex changed as follows:



Adding A_n-cells to A_{n-1} via ∂ for $n \geq 4$ and A_n-cells to B_n via f for $n \geq 3$ results in the chain complex

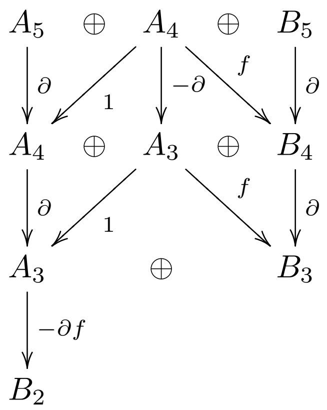


which gives the following boundary maps:

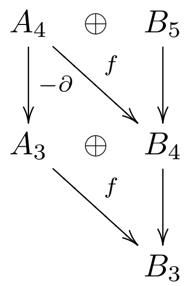


Note that the map from $A_n \to B_{n-1}$ is trivial for a nontrivial reason, adding A_n to B_n via f makes it $-\partial f$ but then adding A_n to A_{n-1} via ∂ makes it $-\partial f + f\partial = 0$.

Finally, we add B_n to A_n via $-g$. A short computation shows that this changes the boundary map $B_3 \to B_2$ to 0 (since $g : B_2 \to A_2$ is the identity) and it changes nothing else (once again $B_n \to A_{n-1}$ is 0 for a nontrivial reason). We now have



The chain complex



is contractible, since it is the mapping cone of a homotopy equivalence. (Remember that $A_\# = B_\#$ in low dimensions). We may now trade up this subchain complex geometrically to above the dimension of X. The n-skeleton of the resulting CW complex has cellular chains equal to $A_\#$ and the map to $D_\#(X)$ is the given map. It follows from Theorem 2.15 that this is a bounded homotopy equivalence, since it induces a homotopy equivalence on the cellular chains.

The proof is now finished by observing that the algebraic boundary map from n- to $(n-1)$-cells is the dual of the boundary map $D_1(X) \to D_0(X)$, and that each 1-cell hits $(\ell_0 \pm g \cdot f_0)$ where ℓ_0 and f_0 are 0-cells and $g \in \pi$. This means that if we attach the n-cells so that the map to the $(n-1)$-skeleton mod $(n-2)$-skeleton reflects the algebra completely, then the n-cells have patches on the boundary mapped homeomorphically to their images, and the complement goes to the $(n-2)$ skeleton. These patches are paired off 2 and 2, so the n-cells are identified via the $(n-1)$-cells to build a manifold W. This is the desired manifold 1-skeleton.

The point is that the fundamental class is the sum of all the n-handles suitably modified by multiplication by group elements. This is seen from the special nature of the boundary map. Using this, let W be the union of slightly shrunken n-cells with their identifications along $(n-1)$-cells. In the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_\# W & \hookrightarrow & D_\# X & \longrightarrow & D_\#(\overline{X-W}, \partial W) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle [X]\cap -} & & \uparrow & & \\ 0 & \longrightarrow & D^\#(W, \partial W) & \hookrightarrow & D^\#(X) & \longrightarrow & D^\#(\overline{X-W}) & \longrightarrow & 0 \end{array}$$

$[X]\cap$ restricted to $D^\#(W, \partial W)$ is seen to be an isomorphism, and it follows from arguments similar to above (rolling up in the mapping cones) that $[X] \cap -$ induces a homotopy equivalence of $D_\#(\overline{X-W}) \leftarrow D^\#(\overline{X-W})$, so we have split off a Poincaré duality complex. The reader is referred to [42, Corollary 2.3.2] for further details. □

We finally need the following:

Lemma 8.3. *Let $X \xrightarrow{p} M$ be a bounded Poincaré duality complex, and let $W \xrightarrow{f} X$ be a degree 1 normal map of a manifold W to X. This means that $W \to X \to M$ is proper, there is a bundle ξ over X with a framing of $\tau_W \oplus f'\xi$, and that the fundamental class $[W] \in H_n^{\ell f}(W; \mathbb{Z})$ is sent to $[X] \in H_n^{\ell f}(X; \mathbb{Z})$. Then there is a normal bordism of $W \xrightarrow{f} X$ to a manifold (which we will again call W) so that f is a homeomorphism over the 1-skeleton of X found in Proposition* 8.2, *i.e., so that $f|f^{-1}$ (regular neighborhood of 1-skeleton) is a homeomorphism.*

Proof. We start with $F = f \circ \mathrm{proj} : W \times I \to X$ and make $F|W \times 1$ transverse to the barycenter of each n-cell of X. Since f is degree 1, the inverse image of this point counted with signs must be 1. On pairs of points of opposite sign we attach a 1-handle to $W \times 1$, and extend F over the resulting bordism, sending the core of the 1-handle to the point and the normal bundle of the core to normal bundle of the point. The restriction of F to the new boundary has 2 fewer double points. Continuing this process, $F|$(new boundary) becomes a homeomorphism over the 0-handles of our 1-skeleton.

Now consider the 1-handles. Assuming $F|$ transverse to the core of a 1-handle, the inverse image must be a union of finitely many S^1's and one interval. After a homotopy the interval may be assumed to map homeomorphically onto its image, and we only need to eliminate the S^1's. But each S^1 maps to the interior of an interval so this map is homotopy trivial. Attaching a 2-handle to each S^1 in the induced framing, extending the map to the core D^2 by the null homotopy, and extending to the normal direction by the framing removes S^1 from the inverse image of the 1-handle. Doing this to all of the 1-handles completes the process. □

9. The surgery groups

Our surgery groups are bordism groups patterned on Wall's Chapter 9. As usual, we restrict ourselves to allowable metric spaces. Following [43, p. 86], we define an "n-dimensional unrestricted object" to consist of:

(i) A bounded Poincaré pair, i.e. a pair (Y, X) with control map to M, such that each component has bounded fundamental group and is a bounded Poincaré duality complex in the sense of §3.
(ii) A proper map $\varphi : (W, \partial W) \to (Y, X)$ of pairs of degree 1, where W is a manifold and $\varphi| : \partial W \to X$ is a simple homotopy equivalence, simplicity being measured in $Wh_M(\pi_1(X))$.
(iii) A stable framing F of $\tau_W \oplus \varphi^*(\tau)$.
(iv) A map $\omega : Y \to K$, where K is a pointed CW complex which is fixed with a fixed pointed double cover $\widehat{K}$. It is required that the pullback of this double cover to Y be the orientation covering.

Bordism of these objects is defined similarly. See the reference above for details. We denote the bordism group of unrestricted objects by $L^1_{n,M}(K)$. Note that, as in Wall, (Y, X) is allowed to vary along with $(W, \partial W)$. There is a natural group structure on $L^1_{n,M}(K)$ with the empty set as the 0 element and the sum represented by disjoint union. As is usual in bordism theories, the groups are functorial in K, in the sense that a pointed map from K_1 to K_2 which is covered by a pointed map of double covers induces a group homomorphism. Note that Y may have infinitely many components, but that since $Y \to M$ is proper, there are only finitely many components locally.

Still following Wall, we define "restricted objects" by requiring that $X \to M$ be (-1)- and 0-connected and have bounded fundamental group and that the map $Y \to K$ induce a π_1-isomorphism. We define $L^2_{n,M}(K)$ to be the set of restricted bordism classes of restricted objects, i.e., we require objects as well as bordisms to be (-1)- and 0-connected and to have the same bounded fundamental group as K. See [43, pp. 86–88]. Note that $L^2_{n,M}(K)$ is only a set – we have no zero object and no sum, since the empty set is not allowed and since disjoint union destroys 0- connectedness.

The π-π Theorem shows, just as in the classical case, that we may do surgery with a fixed restricted target if and only if the invariant in $L^2_{n,M}(K)$ vanishes ([43, Theorem 9.3]).

Theorem 9.1. *Let $\phi : (W, \partial W) \to (Y, X)$ be a restricted surgery problem, i.e., a (-1)- and 0-connected surgery problem with bounded fundamental group π and reference map $Y \to K$ inducing an isomorphism of fundamental groups and pulling back the orientation double cover of Y. We assume $n = dim(X) \geq 5$. Then there is a normal cobordism rel ∂W of W to a bounded* *homotopy equivalence if and only if the equivalence class of ϕ in $L^2_{n,M}(K)$ vanishes.*

We also have the analogue of [43, Theorem 9.4].

Theorem 9.2. *The natural map*

$$L^2_{n,M}(K) \longrightarrow L^1_{n,M}(K)$$

is a bijection if $n \geq 5$ and K has a finite 2-skeleton.

Proof. By Proposition 8.1, there exists $W \to M$ so that the fundamental group is bounded $\pi_1 K$ and W is (-1)-connected. The existence of a map $W \to K$ inducing an isomorphism on π_1 is assured by the construction of W. The surgery problem $V \xrightarrow{f} X$ is equivalent to $V \amalg W \xrightarrow{f \amalg 1} X \amalg W$, since crossing with I may be considered a bordism (one is allowed to forget components that are homotopy equivalences).

Each component of X has a manifold 1-skeleton, and after a bordism of V we may assume f to be a homeomorphism of these 1-skeleta, by Lemma 8.3, so the stage is set for simultaneous surgery. Attach a 1-handle from every 0-handle of X to a 0-handle of W so that the image of this 1-handle in M is small.

Now mimic this construction in the domain by attaching a 1-handle from V to W. For each 1-handle of X we get a path from a point in W to a point in W through the one handles we attached above. Join up these points in W so that the loop created maps trivially to K. This is possible, since W has bounded π_1 isomorphic to $\pi_1 K$. Now attach 2-handles simultaneously in the domain and range to kill these loops. Since every 0- and 1-handle of X has been equated with some element of W, the result is a target which is (-1)-connected and which has bounded fundamental group equal to $\pi_1 K$ induced by the map to K. We have thus constructed a bordism from any object to a restricted object. Injectivity is proved using the same argument on the bordism. □

The group $L^1_{n,M}(K)$ and the set $L^2_{n,M}(K)$ can thus be given the common name $L_{n,M}(K)$, or better $L^s_{n,M}(K)$ because we require simple homotopy equivalence. Naturally, the definition may be varied by only requiring homotopy equivalence. These groups are denoted $L^h_{n,M}(K)$. We may, of course, also specify a $*$-invariant subgroup G of $Wh_M(\pi)$ defining $L^G_{n,M}(K)$. We shall sometimes suppress the upper index.

It is very important to notice that the functor $L_{n,M}(K)$, as a functor in M, does not send restricted objects to restricted objects, and that it is only for restricted objects that the invariant measures whether or not one may do surgery to obtain a bounded homotopy equivalence. We discuss this in the following:

Example 9.3. Consider the inclusion $L \subset L_+$, where L_+ is L union a disjoint basepoint. Assume that a restricted surgery problem has an obstruction in $L_{n,O(L)}(K)$ which vanishes in $L_{n,O(L_+)}(K)$. To understand this geometrically, we first have to replace the image in $L_{n,O(L_+)}(K)$ by a restricted object. The image is not (-1)-connected, but this may be corrected by doing simultaneous surgery – adding a tail to the surgery problem. This doesn't change the surgery problem away from a compact subset of the target. This means that if the surgery obstruction vanishes in the new problem, then it is possible to solve the original surgery problem "near infinity."

Similarly, if one considers the map from L_+ to L sending the extra point to some point of L, the induced map from $L_{n,O(L_+)}(K)$ to $L_{n,O(L)}(K)$ will hit an element which is not 0-connected, so some simultaneous surgery has to be done before the vanishing of this invariant implies that one can surger the source to a homotopy equivalence. Again, it is possible to do this via a small modification of the original problem, only changing the target along a ray out to infinity, rather than by using the more general construction of the proof of Theorem 9.2.

Notice that an analogous phenomenon occurs in classical compact surgery. A surgery problem comes equipped with a reference map, usually to $B\pi$ where π is the fundamental group of the target. Given a group homomorphism $\pi \to \rho$, the vanishing of the image $L^h_n(\mathbb{Z}\pi) \to L^h_n(\mathbb{Z}\rho)$ means that after simultaneous surgery on source and target to make the fundamental groups equal to ρ, one can do surgery to obtain a homotopy equivalence.

Theorem 9.4. *A map $K_1 \to K_2$ which induces an isomorphism of fundamental groups and which is covered by a map of based covering spaces induces an isomorphism $L_{n,M}(K_1)$ to $L_{n,M}(K_2)$ for $n \geq 5$.*

Proof. Given $K_1 \to K_2$ inducing an isomorphism on π_1, etc., we get a long exact sequence of bordism groups, as in any bordism theory. The relative groups are 0 by the π-π Theorem. □

As noted by Quinn [32], the proof of Theorem 9.5 of [43] can be used to prove the following:

Theorem 9.5. *Given a (-1)- and 0-connected manifold $V^{(n-1)}$ with bounded $\pi_1 = \pi$ and an element of $\alpha \in L_{n,M}(\pi)$, $n \geq 6$, then α may be represented with $V \times I$ as target.*

Proof. We may assume that α is realized by $\phi : (W, \partial W) \to (Y, X)$ where X and Y are (-1)- and 0-connected with bounded fundamental group π, and $\partial W \to X$ is a homotopy equivalence. We may glue ∂W to X by a mapping cylinder and assume that $\partial W \to Y$ is the identity. There is a bordism from $\partial W \to \partial W$ to $V \to V$ by equating 0- and 1-handles. Attaching this to ∂W, we may change the representative above to be of the form $\phi : (W, V) \to (X', V)$, where $\phi | V = id$. By the π-π Theorem, there is a bordism $(P, \partial P)$ *not* rel V from ϕ to $\phi' : (W', V) \to (X', V)$ which is a homotopy equivalence of pairs. $(P, \partial P)$ may be reinterpreted to be a bordism from ϕ to a surgery problem with $V \times I$ as target. □

Finally, we should mention that there is an important variation of the theory where we only ask for conditions to be satisfied near ∞. Thus, we only ask for bounded homotopy equivalence and Poincaré duality over $O(K) = \{t \cdot x | t \in [0, \infty), x \in K\}$ for t large. The groups $L^{s,\infty}_{n,O(K)}(\mathbb{Z}\pi)$ are defined as in the beginning of this section, but everything is only done near ∞. Spaces and maps only have to be defined near ∞, as well. It is shown in [9] that $\mathcal{C}_{O(K),\infty}(\mathcal{A}) \to \mathcal{C}_{O(K_+)}(\mathcal{A})$ is a homotopy equivalence, so Whitehead torsion makes sense in this framework. We obtain the following quite useful:

Theorem 9.6.

$$L^{s}_{n,O(K_+)}(\mathbb{Z}\pi) \to L^{s,\infty}_{n,O(K)}(\mathbb{Z}\pi)$$

is an isomorphism if $n \geq 5$.

Proof. The map is the forgetful map taking a problem over $O(K_+)$ to its germ near infinity over $O(K)$.

For simplicity, assume first that $n \gg dim(K)$. In this case we can form a manifold P with a map to $O(K_+)$ as follows. Embed K in S^{n-5} and let P' be the boundary of a regular neighborhood of K in S^{n-5}. Form P by gluing together the part of $O(P')$ outside the unit sphere and a copy of $(-\infty, 1] \times P'$. Map $[0,1] \times P'$ to the part of $O(K)$ inside the unit sphere and send $(-\infty, 0] \times P'$ to $O(+)$. Form a manifold V^{n-1} by taking the product of P with a closed manifold Q^4 which has fundamental group π. Note that it is reasonable to talk about "levels" in V, just as in $O(K_+)$.

To show that the forgetful map is onto, we first use a version of Theorem 9.5 near infinity to show that we can represent a given $\alpha \in L^{s,\infty}_{n,O(K)}(\mathbb{Z}\pi)$ by a map $\phi : (W, \partial W) \to (V \times I, \partial(V \times I))$ such that $\phi | \partial W$ is a simple homotopy equivalence near infinity. Using the simplicity, we can split ϕ over the part of V at level T for some large T, obtaining $\phi| : (W_0, \partial W_0) \to (V \times I, \partial(V \times I)) \cap (\mathit{level}\ T)$ with $\phi | \partial W_0$ a homotopy equivalence. Adding a copy of $(W_0, \partial W_0) \times (-\infty, T]$ to the part of W outside of level T and mapping to $V \times I$ in the obvious way produces an element of $L^{s}_{n,O(K_+)}(\mathbb{Z}\pi)$ whose image in $L^{s,\infty}_{n,O(K)}(\mathbb{Z}\pi)$ is α.

To show that the map is monic, let $\alpha \in L^s_{n,O(K_+)}(\mathbb{Z}\pi)$ be an element which becomes trivial in $L^{s,\infty}_{n,O(K)}(\mathbb{Z}\pi)$. Representing α by a map $\phi : (W, \partial W) \to (V \times I, \partial(V \times I))$ as above, ϕ is bordant rel ∂ to a map $\phi' : (W', \partial W') \to (V \times I, \partial(V \times I))$ which is a simple homotopy equivalence near infinity. Using the simplicity, we can split the homotopy equivalence over some level T. Expanding a collar around this level in the domain and sliding the remainder of the manifold down towards the cone point and out the "tail", $O(+)$, produces a bordism from ϕ' to a simple homotopy equivalence. Note that we actually *gain* control through this sliding process and that the bordism from ϕ to ϕ' need only be defined on a neighborhood of infinity for this process to succeed, since an easy transversality argument allows us to construct a bordism from $(W, \partial W)$ to a manifold which equals $(W', \partial W')$ near infinity in $O(K)$ and $(W, \partial W)$ over $O(+)$.

The general case, where we do not have $n \gg dim(K)$ is similar. The manifold target is less homogeneous, though, so we represent the problem with manifold target $(V \times I, \partial(V \times I))$, but here V has no good homogeneity properties, split at countably many levels going out the tail $O(+)$, and use the compact π-π-Theorem between the splittings to solve the surgery problem. Alternatively, we could use the periodicity result of §12 to deduce the low-dimensional case from the high-dimensional case treated above. □

Remark 9.7. Of course, we are not limited to defining absolute surgery groups. The same definition may be varied as in pp. 91–93 of [43] to define relative, or even n-ad, surgery groups.

10. Ranicki-Rothenberg sequences, and $L^{-\infty}$

In this section we study the properties of $L^s_{n,M}(\mathbb{Z}\pi)$ in the special case where $M = O(K)$.

Proposition 10.1. *Assume $n \geq 5$. There is a long exact Ranicki-Rothenberg sequence*

$$\to L^s_{n,M}(\mathbb{Z}\pi) \to L^h_{n,M}(\mathbb{Z}\pi) \to \widehat{H}^n(\mathbb{Z}_2\,; \widetilde{K}_1(\mathcal{C}_M(\mathbb{Z}))) \to$$

Proof. The proof is formal, the sequence is a bordism long exact sequence where the Tate cohomology groups are identified with the relative bordism groups of surgery problems with simple boundaries (see [26]). □

Note that $O(\Sigma K) = O(K) \times \mathbb{R}$ *as a metric space.* This leads to the following useful proposition.

Proposition 10.2. *Assume $n \geq 5$. Crossing with $\mathbb{R}$ produces an isomorphism*

$$L^h_{n,O(K)}(\mathbb{Z}\pi) \to L^s_{n+1,O(\Sigma K)}(\mathbb{Z}\pi).$$

Proof. First, consider the case where $K = \emptyset$. Then $O(K) = pt$ and $O(\Sigma K) = \mathbb{R}$. To see that the map is monic, let $\phi : (W, \partial W) \to (X, \partial X)$ be a compact surgery problem. Crossing with $\mathbb{R}$ gives $\phi \times id : (W, \partial W) \times \mathbb{R} \to (X, \partial X) \times \mathbb{R}$. If $\phi \times id$ represents 0 in $L^s_{n+1,R}(\mathbb{Z}\pi)$, then $\phi \times id$ is normally bordant rel ∂W to a simple homotopy equivalence $\phi' \times id : (W', \partial W') \times \mathbb{R} \to (X, \partial X) \times \mathbb{R}$. Since the homotopy equivalence is simple, W' has trivial end obstruction, so $W' = M' \times \mathbb{R}$ and by transversality we get W bordant rel ∂W to $M' \simeq X$, showing that ϕ represents 0 in $L^h_{n,pt}(\mathbb{Z}\pi)$.

To see that the map is an epimorphism, represent $\alpha \in L^s_{n+1,R}(\mathbb{Z}\pi)$ by a problem $\phi : (W, \partial W) \to ((V \times I) \times \mathbb{R}, \partial(V \times I) \times \mathbb{R})$ with V^{n-1} a closed manifold with fundamental group π. Since the homotopy equivalence on the boundary is simple, we can split the homotopy equivalence over $(V \times \partial I) \times \{T\}$ and continue the splitting over $(V \times I) \times \{T\}$ by transversality, getting a compact surgery problem $(M, \partial M) \to ((V \times I), \partial(V \times I))$. Letting a collar around M grow in both directions gives a bordism from the original problem to $(M, \partial M) \times \mathbb{R} \to ((V \times I), \partial(V \times I)) \times \mathbb{R}$.

In case $K \neq \emptyset$, the argument is similar. One does the same things boundedly over $O(K)$. □

Definition 10.3. By the groups $L^{2-k}_{n-k}(\mathbb{Z}\pi)$ we shall mean $L^s_{n-k}(\mathbb{Z}\pi)$ when $k = 0$, $L^h_{n-k}(\mathbb{Z}\pi)$ when $k = 1$, $L^p_{n-k}(\mathbb{Z}\pi)$ when $k = 2$, and the negative L-groups of [34, 35] for $k > 2$. It is well known that these groups are 4-periodic in the lower index.

Theorem 10.4. *When* $n \geq 5$

$$L^s_{n,R^k}(\mathbb{Z}\pi) \cong L^{2-k}_{n-k}(\mathbb{Z}\pi)$$
$$L^h_{n,R^{k-1}}(\mathbb{Z}\pi) \cong L^{2-k}_{n-k}(\mathbb{Z}\pi).$$

Proof. First, we consider the case $n \geq k + 5$. The general case will follow from 4-periodicity which is proved algebraically in §12. We use induction on the Ranicki-Rothenberg exact sequence.

We have an algebraically defined inclusion $L^{-i}_n(\mathbb{Z}\pi) \subset L^s_{n+i+2}(\mathbb{Z}(\pi \times \mathbb{Z}^{i+2}))$. There is also a map $L^s_{n+i+2}(\mathbb{Z}(\pi \times \mathbb{Z}^{i+2})) \to L^s_{n+i+2,\mathbb{R}^{i+2}}(\mathbb{Z}\pi)$, which is defined geometrically by taking cyclic covers. Combining these maps, we get a map of exact sequences:

$$\begin{array}{ccccc}
L^{-i+1}_n(\mathbb{Z}\pi) & \longrightarrow & L^{-i}_n(\mathbb{Z}\pi) & \longrightarrow & \widehat{H}^n(\mathbb{Z}_2, K_{-i}(\mathbb{Z}\pi)) \\
\downarrow & & \downarrow & & \downarrow \\
L^s_{n+i+1,\mathbb{R}^{i+1}}(\mathbb{Z}\pi) & \longrightarrow & L^h_{n+i+1,\mathbb{R}^{i+1}}(\mathbb{Z}\pi) & \longrightarrow & \widehat{H}^n(\mathbb{Z}_2, K_{-i}(\mathbb{Z}\pi)).
\end{array}$$

Here, we are using that $K_1(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathbb{Z}\pi)) \cong K_{-i}(\mathbb{Z}\pi)$ ([23]). Combining with the isomorphisms

$$L^h_{i,\mathbb{R}^n}(\mathbb{Z}\pi) \cong L^s_{i+1,\mathbb{R}^{n+1}}(\mathbb{Z}\pi)$$

this inductively proves that

$$L^{-i}_n(\mathbb{Z}\pi) \cong L^s_{n+i+2,\mathbb{R}^{i+2}}(\mathbb{Z}\pi).$$

□

Note that this proves that the L-groups are 4-periodic, at least when $K = S^i$, i.e. $O(K) = \mathbb{R}^{i+1}$ and $n \geq i+6$. We shall now investigate $L^s_{n,O(K)}(\mathbb{Z}\pi)$ as a functor of K, from the category of finite complexes and Lipschitz morphisms.

Theorem 10.5. *$L^s_{n,O(K)}(\mathbb{Z}\pi)$ is homotopy invariant, $n \geq 5$.*

Proof. We have to show that

$$L^s_{n,O(K)}(\mathbb{Z}\pi) \to L^s_{n,O(K\times I)}(\mathbb{Z}\pi)$$

is an isomorphism. By functoriality, it is a split monomorphism. To see that it is onto, we note that a homotopy can always be viewed as an unrestricted bordism. Thus a surgery problem parameterized by $O(K \times I)$ becomes bordant, and hence equivalent in the unrestricted bordism group of §9, to the induced problem parameterized by projecting to one end. □

Continuing to investigate $L^s_{n,O(K)}(\mathbb{Z}\pi)$ as a functor in K, we define

$$L^{-\infty}_{n,O(K)}(\mathbb{Z}\pi) = \lim_i L^s_{n+i,O(\Sigma^i K)}(\mathbb{Z}\pi)$$

where the maps are given by crossing with the reals.

Theorem 10.6. *$L^{-\infty}_{n,O(K)}(\mathbb{Z}\pi)$ is a reduced homology theory in the variable K.*

Remark 10.7. This is a geometric version of Theorem 3.4 of the thesis of Yamasaki [45]. There, $L^{-\infty}(\mathbb{Z}\pi)$ is defined abstractly as a spectrum.

To prove the theorem we need the following:

Lemma 10.8. *Let $C(L)$ be the cone on L. Then $L^s_{n,O(CL)}(\mathbb{Z}\pi) = 0$.*

Proof. First, note that there is an isometry $O(C(L)) \cong O(L) \times [0,\infty)$. Let $\alpha \in L^s_{n,O(CL)}(\mathbb{Z}\pi)$ be represented by a surgery problem:

$$\begin{array}{ccc} (W,\partial W)) & \longrightarrow & (X,\partial X) \\ & & \Big\downarrow p \\ & & O(L)\times[0,\infty) \end{array}$$

This is the boundary of:

$$\begin{array}{ccc} (W,\partial W) & \longrightarrow & (X,\partial X) \\ & & \downarrow \\ & & O(L)\times[0,\infty) \end{array}$$

where $q(x,t) = p(x) + t$. This shows that $\alpha = 0$. □

Proof of theorem. To show that the functor is half exact in the variable K, consider a cofibration

$$L \subset K \to K \cup CL.$$

Applying O we get:

$$O(L) \to O(K) \to O(K \cup C(L)) \cong O(K) \cup_{O(L)} O(L) \times [0,\infty).$$

The composite is the trivial map, since it factors through $O(CL)$, so consider a surgery problem:

$$\begin{array}{ccc} (W,\partial W) & \longrightarrow & (X,\partial X) \\ & & \downarrow \\ & & O(K) \end{array}$$

which goes to 0 in $L^{-\infty}_{n,O(K\cup C(L))}(\mathbb{Z}\pi)$. As usual, we may assume that X is a manifold and that ϕ is (-1)- and 0-connected. The vanishing of $[\phi]$ over $O(K\cup C(L)) \cong O(K)\cup_{O(L)} O(L)\times[0,\infty)$ means that after simultaneous surgery on the domain and range to obtain a (-1)- and 0-connected $\phi' : W' \to X'$, ϕ' is bordant to a bounded homotopy equivalence. Clearly, we can do the simultaneous surgery in such a way that the parts of W' and X' over $O(L)\times[1,\infty)$ are products.

If ϕ' is bordant to a bounded *simple* homotopy equivalence ϕ'', we can split the bordism over $O(L)\times\{T\}$ for some large T using controlled splitting over the boundary and at ϕ'' [9]. Projecting back to $O(K)$, the split bordism becomes an unrestricted bordism from the original problem to a bounded homotopy equivalence together with the inverse image of $O(L)\times\{T\}$, showing that the original problem was in the image of $L^{-\infty}_{n,O(L)}(\mathbb{Z}\pi)$.

If ϕ' is bordant to a bounded *non-simple* homotopy equivalence, we cross with $\mathbb{R}$ to kill the torsion and proceed as above. This torsion problem is the reason that we have to stabilize to obtain a homology theory. In fact, this argument shows that $L^{s}_{n,O(K)}(\mathbb{Z}\pi)$ is *not* half exact.

To finish the proof, note that we have already shown that there is an isomorphism $L^h_{n,O(K)}(\mathbb{Z}\pi) \cong L^s_{n+1,O(\Sigma K)}(\mathbb{Z}\pi)$. Since $L^{-\infty}_{n,O(S^k)}(\mathbb{Z}\pi)$ is naturally 4-periodic, it follows from half exactness that $L^{-\infty}_{n,O(-)}(\mathbb{Z}\pi)$ is a 4-periodic homology theory. □

Remark 10.9. We get a *periodic* homology theory and thus not a *connective* homology theory. A geometric interpretation of this is that a (-1)-connected $M^k \to \mathbb{R}^{m+k}$ behaves like a $-m$-dimensional manifold.

11. The surgery exact sequence

In this section we consider the following: Let M be an allowable metric space and $X \to M$ a 0- and -1-connected bounded Poincaré duality space with bounded fundamental group $= \pi$, and assume there is a given lift of

$$\begin{array}{ccc} & & B\,\mathrm{CAT} \\ & & \downarrow \\ X & \longrightarrow & BG \end{array}$$

where CAT = TOP, PL or O. We define the bounded structure set $\mathcal{S}_b(X)$ as usual in surgery theory: An element consists of a manifold W and a bounded homotopy equivalence

$$\begin{array}{ccc} W & \simeq & X \\ & & \downarrow \\ & & M \end{array}$$

two such being equivalent if there is a homeomorphism $h : W_1 \to W_2$ such that the diagram

$$\begin{array}{ccc} W_1 & & \\ & \searrow & \\ \downarrow h & & X \\ & \nearrow & \\ W_2 & & \end{array}$$

is bounded homotopy commutative. As usual we get a surgery exact sequence.

Theorem 11.1. *For $n \geq 5$ there is an exact sequence of surgery*

$$\begin{aligned} \cdots \to \mathcal{S}_b(X \times I, \delta(X \times I)) \to [\Sigma X, F/\,\mathrm{CAT}] \to \\ \to L^s_{n+1,M}(\mathbb{Z}\pi) \to \mathcal{S}_b(X) \longrightarrow [X, F/\,\mathrm{CAT}] \to L^s_{n,M}(\mathbb{Z}\pi) \end{aligned}$$

and relative versions hereof.

Proof. The proof is standard as in [40]. Given a lift of $X \to BF$ to $B\,\mathrm{CAT}$ there is a surgery problem obtained by transversality. If the obstruction to doing surgery vanishes, we obtain a bounded homotopy equivalence. Given two elements in the structure set, they determine two lifts to $B\,\mathrm{CAT}$. If the lifts are fibre homotopic we obtain a normal cobordism by transversality as in standard surgery theory. □

Given a finitely dominated Poincaré complex X, we obtain a surgery exact sequence for $X \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ as follows. Cross X with S^1 to obtain a finite Poincaré complex over $\mathbb{R}$, pass to the cyclic cover, and cross with $\mathbb{R}^{k-1}$ to obtain a bounded Poincaré model for $X \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$. A radial homeomorphism $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ which is Lipschitz (but whose inverse is not necessarily Lipschitz) induces a map of the surgery exact sequences, the point being that if a homotopy is bounded with respect to a map p to $\mathbb{R}^k$, it is certainly also bounded with respect to $f \cdot p$.

Theorem 11.2. *Reparameterization by a radial homeomorphism which is Lipschitz induces the identity on the surgery exact sequence.*

Proof. Since f is a radial homeomorphism, $\bar{x} + t \cdot f(\bar{x})$ is a homotopy of f through Lipschitz maps to the identity, and it follows easily that f induces the identity of L-groups and normal invariants. To see that f induces the identity on the structure set we need the result of Chapman [7] that a map from a manifold to $\mathbb{R}^k$ which has the bounded homotopy lifting property can be boundedly approximated by a map with the epsilon homotopy lifting property for all epsilon. If $W \longrightarrow X \times \mathbb{R}^k$ is a bounded equivalence, then the composition $W \longrightarrow \mathbb{R}^k$ is boundedly approximated by an approximate fibration. There is then an approximate fiber homotopy equivalence $W \longrightarrow X \times \mathbb{R}^k$ boundedly homotopic to the original map. This approximate fiber homotopy equivalence remains bounded under arbitrary radial reparameterization. □

12. The algebraic surgery theory

The algebraic surgery theory has been developed over a number of years by Ranicki see [34, 35] . The extension to additive categories with involution has also been developed by A. Ranicki (see [36]). This section depends strongly on [36]. Here are some of the basic definitions:

An involution on an additive category $\mathcal{A}$ is a contravariant functor

$$* : \mathcal{A} \to \mathcal{A};\ A \to A^*$$

$$(f : A \to B) \to (f^* : B^* \to A^*)$$

together with a natural equivalence

$$e : id_{\mathcal{A}} \longrightarrow **$$

such that the coherence condition

$$e(A^*) = (e(A)^{-1})^* : A^* \longrightarrow A^{***}$$

is satisfied.

Example 12.1. Let R be an associative ring with an antiinvolution, $R = \mathbb{Z}G$ with $\overline{\Sigma n_g g} = \Sigma w(g) n_g g^{-1}$, for example. Consider the category of finitely-generated projective R-modules. Then duality induces an involution on the category.

Of more interest to us is the following:

Example 12.2. Let $\mathcal{R}$ be the category of finitely generated free R-modules with involution as above. We get an induced involution on $\mathcal{C}_M(\mathcal{R})$ for M a metric space by the prescription $(A^*)_x = (A_x)^*$.

Ranicki has shown that his theory of algebraic surgery extends to additive categories with involution, so, in particular, he has defined $L^t_n(\mathcal{C}_M(\mathbb{Z}\pi))$, where the decoration t is h or corresponds to any involution-invariant subgroup of $Wh_M(\pi)$, as is usual in L-theory. In this setup L^p is the composite of L^h with idempotent completion of the additive category. To be able to treat the simple L-groups L^s corresponding to the 0-subgroup of $Wh_M(\pi)$ one needs a system of stable isomorphisms of the objects so that composites that are automorphisms have trivial torsion. This is obtained from an Eilenberg swindle on the *objects* in case M is unbounded (and as usual a specific choice of basis in case M is bounded). In this section we prove that these algebraically defined L-groups are the obstruction groups for bounded surgery problems in the case where there is no boundary or that there is a homotopy equivalence on the boundary. First recall from [36, p. 169] the basic definitions.

Let $\mathcal{A}$ be an additive category with involution. A sequence of objects and morphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact if g is split by a morphism h such that $(f, h) : A \oplus C \to B$ is an isomorphism. Let ϵ denote ± 1. An ϵ-quadratic form in $\mathcal{A}$ is an equivalence class of maps $\psi : A \to A^*$ two such being equivalent if they differ by a morphism of the form $\phi - \epsilon\phi^*$. It is nonsingular if $\psi + \epsilon\psi^*$ (which only depends on the equivalence class) is an isomorphism. A Lagrangian in a non-singular form (A, ψ) is a morphism $i : B \to A$ such that $\psi \cdot i = 0$ and $0 \to B \to A \to B^* \to 0$ is split exact. Ranicki then proves that a non-singular ϵ quadratic form is equivalent to the hyperbolic form $(B \oplus B^*, \{\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\})$ if and only if it admits a Lagrangian. The even L-groups are now defined as the Grothendieck construction on isomorphism classes of non-singular quadratic forms with $\epsilon = 1$ in dimensions $\equiv 0(4)$ and $\epsilon = -1$ when the dimension is $\equiv 2(4)$.

To define the odd L-groups one needs formations. A nonsingular ϵ-quadratic formation in $\mathcal{A}$, (A, ψ, F, G) is a non-singular ϵ-quadratic form (A, ψ) together with an ordered pair of Lagrangians F and G. (H_ϵ, P, P^*) is considered a trivial formation where H_ϵ is the hyperbolic form on $P \oplus P^*$. With the obvious notion of isomorphism Ranicki defines the odd L-groups to be the Grothendieck construction on isomorphism classes of formations modulo trivial formations and the relation $(A, \psi; F, G) + (A, \psi; G, H) = (A, \psi; F, H)$, with $\epsilon = 1$ in dimensions $\equiv 3(4)$ and $\epsilon = -1$ in dimensions $\equiv 1(4)$. Given this we now proceed along the lines of Wall's original method.

Theorem 12.3. *Consider a bounded surgery problem*

$$\begin{array}{ccc} (M^n, \partial M) & \longrightarrow & (X, \partial X) \\ & & \downarrow \\ & & Z \end{array}$$

where $\partial M \to \partial X$ is a bounded simple homotopy equivalence, X is 0 and -1-connected with bounded fundamental group π, and $n \geq 5$. Then one can do surgery rel boundary to produce a bounded simple homotopy equivalence if and only if an invariant in

$$L_n^s(\mathcal{C}_Z(\mathbb{Z}\pi))$$

vanishes. Moreover every element of $L_n^s(\mathcal{C}_Z(\mathbb{Z}\pi))$ is realized as the obstruction on a surgery problem with target $N \times I$ and homotopy equivalence on the boundary for an arbitrary $n-1$-dimensional manifold $N \to Z$ which is -1 and 0-connected with bounded fundamental group π.

Proof. First consider the even-dimensional case. We proceed as in §7 and obtain a highly connected surgery problem. We obtain a chain complex homotopy equivalent to $K_\#(M)$ which is concentrated in 3 dimensions:

$$0 \to K_{k+2} \to K_{k+1} \to K_k \to 0$$

and a contracting homotopy s (except in dimension k) which is obtained from Poincaré duality. Introducing cancelling $k+1$ and $k+2$ handles, we may change this to

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{k+2} & \longrightarrow & K_{k+1} & \longrightarrow & K_k \\ & & & & \oplus & & \oplus \\ & & & & K_{k+2} & \xrightarrow{1} & K_{k+2} \end{array}$$

after adding $k+1$-handles to K_{k+2} along s we may cancel the K_{k+2}-handles to shorten this chain complex to a 2-term chain complex which we write $0 \to K'_{k+1} \to K'_k \to 0$. Abusing notation we omit the primes. Notice that all generators of K_k are still represented by immersed spheres. For each generator of K_{k+1} we do a trivial surgery to get a chain complex

$$0 \longrightarrow K_{k+1} \longrightarrow \begin{matrix} K_k \\ \oplus \\ K_{k+1} \\ \oplus \\ K_{k+1} \end{matrix}$$

We still have the contraction s in dimension $k+1$, so we may add handles along s and cancel K_{k+1} to obtain a chain-complex concentrated in one degree. Recall there is a similar need to do trivial surgeries in compact surgery theory because the homology modules are only stably free. Denote the remaining module by A. Poincare duality produces an isomorphism $\phi : A \to A^*$ which determines the intersections of *different generators* i. e. $\phi(e_i)(e_j)$ determines the intersections of e_i and e_j when e_i and e_j are different. Now total order the basis and define a map $\nu : A \to A^*$ so that $\nu(e_i)(e_j)$ is 0 when $i > j$ and the intersection counted with sign in $\mathbb{Z}\pi$ when $i \leq j$. By symmetrization $\nu + \epsilon\nu^* = \phi$, hence an isomorphism. This represents the surgery obstruction. If this obstruction is zero, [36, Proposition 2.6] shows us how to find a Lagrangian, and doing surgery on this Lagrangian will produce a homotopy equivalence. More specifically [36, Proposition 2.6] tells us that after stabilization with a hyperbolic form, we may find a Lagrangian. Using -1-connectedness we may do trivial surgeries at points chosen such that this hyperbolic form is added to A. Once we have a Lagrangian each basis element in the Lagrangian is a linear combination of generators in A, so we find representations by immersed spheres by tubing up the generators in A. This uses 0-connectedness. Using the assumption that we have bounded fundamental group π we may do the Whitney tricks to cancel double points so that the geometric intersections correspond to the algebraic intersections meaning that the generators of the Lagrangian are represented by framed, imbedded spheres. After surgery on these spheres an easy calculation shows that the new $K_{\#}(M)$ is contractible.

To see the obstruction is well defined it suffices to show that the obstruction is zero on a boundary, but doing surgery on the bounding manifold to make it highly connected will produce a Lagrangian as in classical surgery surgery theory. Plumbing shows that all algebraically defined surgery obstructions are realized by some surgery problem with boundary. This is done as follows: Let (A, ν) be an element of L_n, $n \geq 5$. Choose a -1, 0 connected $2k-1$-dimensional manifold $N \to Z$, with bounded fundamental group π, and trivially imbedded $k-1$-spheres corresponding to the generators of A.

Consider

$$(\cup S^{k-1}) \times I \subset N \times I$$

Piping against the boundary $N \times 1$ we change these imbeddings to immersions having selfintersection form defined by ν. Do surgery on the spheres imbedded in $M \times 1$, the non-singularity of ν implies that the trace of this surgery $W \to N \times I$ is a homotopy equivalence on the boundary and realizes the given surgery obstruction (A, ν)

In odd dimensions we can dodge the problem by crossing with the reals at both the manifold and parameter space level. Now we have a surgery problem parameterized by $O(\Sigma K)$. Since we now have an even-dimensional problem, we can translate to algebra as above, and use the algebraic fact that

$$L^s_{n+1}(\mathcal{C}_{O(\Sigma K)}(\mathbb{Z}\pi)) \cong L^h_n(\mathcal{C}_{O(K)}(\mathbb{Z}\pi))$$

[37] to finish off the proof by an application of Theorem 7.2. This gives the L^h result, but not the L^s result. An argument that solves the odd-dimensional case directly was shown to us by A. Ranicki. It goes as follows:

Doing surgery below the mid-dimension and furthermore proceeding as above we may obtain a length 2 chain complex

$$0 \to K_{k+1} \to K_k \to 0.$$

Now do surgeries on embedded $S^k \times D^{k+1}$'s in such a fashion that, denoting the trace of the surgery by W, the chain complexes $K_\#(W, M)$, $K_\#(W)$ and $K_\#(W, M')$ are homotopy equivalent to chain complexes which are zero except in dimension $k+1$. One way to do this could be to do surgeries to all the generators of K_k. Denote the resulting manifold by M'. The surgery obstruction is now defined to be the following formation

$$(K_{k+1}(W, M) \oplus K_{k+1}(W, M'), K_{k+1}(W, M), K_{k+1}(W))$$

where the first Lagrangian is the inclusion on the first factor, and the second Lagrangian is induced by the pair of inclusions. Poincaré duality shows that these are indeed Lagrangians. We need to see this is a well-defined element in the odd L-group. First, the choice of imbeddings of $S^k \times D^{k+1}$ may be changed by a regular homotopy, but that changes the formation by an isomorphism. Next we need to compare the effect of choosing a different set of spheres. Let W_1 and W_2 be two traces satisfying the conditions above. Let W_{12} denote the result of surgery by both sets of spheres. After attaching the first set of handles the second set is attached by homotopically trivial spheres so after a regular homotopy we have that W_{12} is W_1 with further trivial surgeries done. It is easy to see that trivial surgeries do not change the equivalence class of the formation, so W_1 and W_{12} define equivalent formations. Similarly W_2 and W_{12} define equivalent formations and we are done. We need to see this is a normal cobordism invariant, but given 2 highly connected normally cobordant surgery problems, we may do surgery on the normal cobordism and cancel handles as in the even dimensional case to obtain a normal cobordism which is just a trace of surgeries as described. This means we have a well defined element in the L-group. If $K_{\#}(M)$ is contractible we may choose to do no surgeries and thus get the 0-formation which does represent 0 in the L-group. Since the operations that are allowed on Lagrangians in the odd L-groups [36] can be mimicked geometrically, using the -1, 0-connectedness, and bounded fundamental group assumptions we see that surgery can be done if and only if the element is 0 in the L-group. Showing all algebraically defined elements are realized geometrically is done by plumbing: Given a non-singular formation we may think of it as $(H \oplus H^*, H, K)$, using the first Lagrangian to identify the form with a hyperbolic form. Start out with a -1, 0-connected $2k$-manifold $N \to Z$, with bounded fundamental group π and do trivial surgeries to a set of generators corresponding to generators of H. In the resulting manifold M' we have the kernel $K_k = H \oplus H^*$. Now do surgeries to spheres corresponding to generators of K to obtain a homotopy equivalence again. The union of the traces of these surgeries along M', $W \to N \times I$ will have surgery obstruction given by $(H \oplus H^*, H, K)$. □

Corollary 12.4. *The groups $L_{n,O(K)}(\mathbb{Z}\pi)$ are 4-periodic for $n \geq 5$. The isomorphism is given by multiplication by $\mathbb{C}\mathbb{P}^2$.*

Proof. First, note that it suffices to prove periodicity in L^h, since the Ranicki-Rothenberg sequence and the 5 Lemma then give L^s periodicity. (Multiplication by $\mathbb{C}\mathbb{P}^2$ is an isomorphism on Tate cohomology since $\mathbb{C}\mathbb{P}^2$ has odd Euler characteristic.) The L^h groups are 4-periodic because the algebraically defined groups only depend on n mod 4. To see that the isomorphism is given by multiplication by $\mathbb{C}\mathbb{P}^2$, one has to go through steps analogous to the compact proof [43, Theorem 9.9, p. 96]. □

Remark 12.5. As remarked above, it would be nicer to have a direct description of a map from the geometrically defined bordism groups to the algebraically defined bordism groups. Note, however, that the identification of $L^s_{n,O(S^i)}(\mathbb{Z}\pi)$ with $L^{1-i}_{n-i-1}(\mathbb{Z}\pi)$ is independent of the algebra of this section.

We now give a proof of theorem 7.2 based on the material in this section.

Proof of Theorem 7.2. Given the algebraic description of the surgery groups (in the case without boundary) we may establish the surgery exact sequences of the last section without reference to a [42, Chapter 9] type definition of the L-groups. First assume ∂X is empty. Consider the diagram of surgery exact sequences

$$\begin{array}{ccccccc} \cdots \longrightarrow & L_n^h(\mathcal{C}_M(\mathbb{Z}\pi)) & \longrightarrow & \mathcal{S}_b^h(X) & \longrightarrow & [X, F/\operatorname{TOP}] & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & L_{n+1}^s(\mathcal{C}_{M\times\mathbb{R}}(\mathbb{Z}\pi)) & \to & \mathcal{S}_b^s(X\times\mathbb{R}) & \to & [X\times\mathbb{R}, F/\operatorname{TOP}] & \to \cdots \end{array}$$

where the vertical maps are induced by crossing with $\mathbb{R}$. On the normal invariant we clearly get an isomorphism, and it is proved in [37] that

$$L^h(\mathcal{C}_M(\mathbb{Z}\pi)) \to L^s(\mathcal{C}_{M\times\mathbb{R}}(\mathbb{Z}\pi))$$

is an isomorphism (see also [6]) hence an element in the simple structure set parameterized over $M\times\mathbb{R}$ is the product with $\mathbb{R}$ with an element in $\mathcal{S}_b^h(X)$. The reader should note that, as usual in surgery theory, the surgery exact sequence is *not* a sequence of abelian groups and homomorphisms. The L-groups act on the structure set and exactness at the structure set means that two elements having the same normal invariant differ by an action of the L-group. It is however easy to see that a version of the 5-lemma sufficient for our purposes is valid, so we do get a 1-1 correspondence of structure sets. To get the splitting, we finally need to refer to the bounded s-cobordism theorem 2.17.

The relative case is treated by first splitting the boundary then working relative to the boundary. □

13. The annulus theorem, CE approximation, and triangulation

In this section we show how bounded surgery theory can be applied to give direct proofs of Kirby's annulus theorem and Siebenmann's CE approximation Theorem. We also take a look at triangulation theory through the lens of bounded topology.

Theorem 13.1. (Kirby [16]) *If C^n, $n \geq 5$, is a bicollared ball in $\mathbb{R}^n$ containing a bicollared ball D^n in its interior, then $C - \overset{\circ}{D}$ is homeomorphic to $S^{n-1}\times[0,1]$.*

Proof. By the generalized Schönflies Theorem [5], [19], there is a homeomorphism $h:\mathbb{R}^n\to\mathbb{R}^n$ with $h(D^n) = B^n$, where B^n is the standard ball. Now $h:\mathbb{R}^n\to\mathbb{R}^n$ is certainly a controlled homotopy equivalence, so h defines an element of the bounded structure set

$$\mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix}\mathbb{R}^n\\ \downarrow id\\ \mathbb{R}^n\end{pmatrix}.$$

The surgery exact sequence in this case is

$$\to L_{n,\mathbb{R}^{n+1}}(e) \to \mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix} \mathbb{R}^n \\ \downarrow id \\ \mathbb{R}^n \end{pmatrix} \to [\mathbb{R}^n, F/\,\mathrm{PL}] \to L_{n,\mathbb{R}^n}(e)$$

where there are no decorations on the L-groups because π_1 is trivial and all homotopy equivalences are therefore simple. This uses Bass-Heller-Swan and [24]. By Theorem 10.4, $L_{n,\mathbb{R}^{n+1}}(e) = L_1(e)$, which is zero by Kervaire-Milnor [4, p. 49]. The space $F/\,\mathrm{PL}$ is connected, so $[\mathbb{R}^n, F/\,\mathrm{PL}]$ is trivial.

Thus, $\mathcal{S}_b^{\mathrm{PL}}\left(\begin{smallmatrix} \mathbb{R}^n \\ \downarrow id \\ \mathbb{R}^n \end{smallmatrix}\right)$ is trivial, which means that there is a PL homeomorphism $k : \mathbb{R}^n \to \mathbb{R}^n$ which is boundedly close to h. Let $\mathbb{R}^n$ be compactified to D^n by adding a sphere at infinity. We use the notation LB^n to denote the ball of radius L centered at the origin in $\mathbb{R}^n$. Since $k^{-1} \circ h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism which extends to a homeomorphism $\overline{k^{-1} \circ h} : D^n \to D^n$, we see

(i) $D^n - L\overset{\circ}{B}{}^n$ is an annulus, so

$$\overline{k^{-1} \circ h}(D^n - L\overset{\circ}{B}{}^n) = D^n - (k^{-1} \circ h)(L\overset{\circ}{B}{}^n)$$

is an annulus for all L.

(ii) This implies that $(k^{-1} \circ h)(LB^n) - M\overset{\circ}{B}{}^n$ is an annulus for $L \gg M$, since the annulus $D^n - M\overset{\circ}{B}{}^n$ is $(k^{-1} \circ h)(LB^n) - M\overset{\circ}{B}{}^n$ plus the collar $D^n - (k^{-1} \circ h)(L\overset{\circ}{B}{}^n)$ and adding a collar on the boundary of a manifold leaves the homeomorphism type unchanged.

(iii) Applying k, we see that $h(LB^n) - k(M\overset{\circ}{B}{}^n)$ is an annulus for $L \gg M$.

(iv) Since k is PL, $k(M\overset{\circ}{B}{}^n) - \overset{\circ}{B}{}^n$ is an annulus for M large [38, p. 36] and the collaring trick shows that $h(LB^n) - \overset{\circ}{B}{}^n$ is an annulus for very large L.

(v) Applying the collaring trick yet again shows that $h(B^n) - \overset{\circ}{B}{}^n$ is an annulus.

□

Here is a geometric restatement of the surgery theory involved in this argument: Bundle theory over $\mathbb{R}^n$ is trivial, so a transversality argument shows that the bounded PL structure given by h is normally bordant to the identity. Repeated splitting shows that the obstruction to surgering this bordism to a bounded h-cobordism over $\mathbb{R}^n$ is the codimension n surgery obstruction over the transverse inverse images of points in $\mathbb{R}^n$. This uses the π-π theorem and periodicity, since we need to multiply the original problem by $\mathbb{C}\,\mathbb{P}^2$ to keep the dimensions from dropping below 5. All of the surgery groups that we use here are rel boundary, so the problem of transferring between the geometry and algebra alluded to in the last section does not arise in this connection. The codimension n surgery obstruction for surgering the bordism is an odd-dimensional simply connected (ordinary) surgery obstruction and is therefore zero. We can therefore surger to a bounded PL h-cobordism, at which point we can apply the bounded h-cobordism theorem over $\mathbb{R}^n$ (see [25]) to produce the bounded PL approximation k.

Corollary 13.2. (of the proof) *Orientation-preserving homeomorphisms of $\mathbb{R}^n$, $n \geq 5$ are stable.*

Proof. We assume that the reader is familiar with [16]. The homeomorphism k is stable because it is orientation-preserving and PL, while the homeomorphism $k^{-1} \circ h$ is stable because it is bounded. Compositions of stable homeomorphisms are stable, so $h = k \circ (k^{-1} \circ h)$ is stable. □

Remark 13.3. This is the lone surgical ingredient in the proof of the Kirby-Siebenmann Product Structure Theorem, which says that M^n has a PL structure if and only if $M \times \mathbb{R}^k$ has a PL structure for some k. See [15, p. 33]. We could also prove the product structure theorem directly using Theorem 7.2. The existence of handlebody decompositions for high-dimensional TOP manifolds is a direct consequence. See [15, pp. 104 ff.]. It also follows immediately by a general bundle theory argument [21] that M^n has a PL structure if and only if the stable tangent bundle of M has a PL reduction. Thus, triangulation is a lifting problem and the triangulation problem is reduced to determining the structure of TOP / PL.

The same lemma gives a proof of Siebenmann's CE approximation theorem.

Theorem 13.4. *If $n \geq 5$ and $f : M \to N$ is a* CE *map, then f is a uniform limit of homeomorphisms.*

Proof. Let $U \subset N$ be the interior of a bicollared ball in N. Then $f : f^{-1}(U) \to U \cong \mathbb{R}^n$ is a bounded structure on $\mathbb{R}^n$. The manifold $f^{-1}(U)$ is contractible, so by the Product Structure Theorem, $f^{-1}(U)$ has a PL structure and the argument above shows that there is a PL homeomorphism $k : f^{-1}(U) \to U$ approximating f so closely that the map $\bar{f} : M \to N$ defined by

$$\bar{f}(x) = \begin{cases} f(x) & x \notin f^{-1}(U) \\ k(x) & x \in f^{-1}(U) \end{cases}$$

is continuous. Performing similar modifications over all of the sets U in an open cover of N gives a homeomorphism homotopic to f. If the open sets U are taken to be small, the homeomorphism approximates f. □

We can approach Kirby-Siebenmann's triangulation theory similarly. A topological homeomorphism $h : V^{\mathrm{PL}} \to D^k \times \mathbb{R}^m$, $m+k = n \geq 5$, which is a

PL homeomorphism over a neighborhood of the boundary gives an element of $\mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix} D^k\times\mathbb{R}^m \\ \downarrow id \\ O(D^k\times S^{m-1})\end{pmatrix}$. The surgery exact sequence is

$$\ldots \to L_{n,O(D^k\times S^{m-1})}(e) \to \mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix} D^k\times\mathbb{R}^m \\ \downarrow id \\ O(D^k\times S^{m-1})\end{pmatrix} \to [D^k\times\mathbb{R}^m,\partial;F/\mathrm{PL}] \to L_{n,O(D^k\times S^{m-1})}(e).$$

By homotopy invariance, this is

$$\cdots \to L_{n,O(S^{m-1})}(e) \to \mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix} D^k\times\mathbb{R}^m \\ \downarrow id \\ O(D^k\times S^{m-1})\end{pmatrix} \to \pi_k(F/\mathrm{PL}) \to L_{n,O(S^{m-1})}(e)$$

which is

$$\pi_{k+1}(F/\mathrm{PL}) \to L_{k+1}(e) \to \mathcal{S}_b^{\mathrm{PL}}\begin{pmatrix} D^k\times\mathbb{R}^m \\ \downarrow id \\ O(D^k\times S^{m-1})\end{pmatrix} \to \pi_k(F/\mathrm{PL}) \to L_k(e).$$

The usual plumbing argument shows that the maps $\pi_k(F/\mathrm{PL}) \to L_k(e)$ are isomorphisms for $k \neq 4$, in which case Rochlin's Theorem shows that the map is multiplication by 2. This shows that such structures are trivial for $k \neq 3$ and allows the straightening of all but 3-handles. The one nontrivial structure on $D^3 \times \mathbb{R}^m$ comes from a homotopy equivalence

$$f : V \to D^k \times \mathbb{R}^m$$

which is a PL homeomorphism near the boundary and which is bounded over $O(D^k \times S^{m-1})$. Add a boundary to V to form $\bar{V}$ and extend the map. This uses Quinn's end theorem or our bounded modification thereof and requires $m+k \geq 6$. By the Generalized Poincaré Conjecture, $\bar{V}$ is a disk. The limiting map is CE and we approximate by a homeomorphism. Coning produces a TOP homeomorphism h proper homotopic to the original f. Comparing the bounded and proper surgery exact sequences shows that the bounded structure given by h is equivalent to the original f, so composing h with an appropriate PL homeomorphism which is the identity on the boundary gives a TOP homeomorphism boundedly close to f. Thus, the nonstraightenable "bounded homotopy handle" comes from a TOP homeomorphism, $\pi_3(\mathrm{TOP}/\mathrm{PL}) \cong \mathbb{Z}/2\mathbb{Z}$, and the development of the theory proceeds as in [15].

14. Extending the algebra

In this section we extend the bounded algebraic theory in two directions. First, we consider the equivariant case, i.e., we extend the theory to allow non-bounded fundamental groups coming from group action. Second, we introduce germ methods, which allow us to disregard what happens in a bounded neighborhood of a subset of the metric space.

In the following, suppose that M is a metric space with a group G acting by quasi-isometries.

Definition 14.1. An object of $\mathcal{C}_{M,G}(R)$ is a left RG-module A together with a set map $f : A \to F(M)$, where $F(M)$ is the finite subsets of M such that

(i) f is G-equivariant.
(ii) $A_x = \{a \in A \mid f(a) \subseteq \{x\}\}$ is a finitely generated free sub R-module.
(iii) As an R-module $A = \oplus_{x\in M} A_x$.
(iv) $f(a + b) \subseteq f(a) \cup f(b)$.
(v) The set $\{x \in M \mid A_x \neq 0\}$ is locally finite.

A morphism $\varphi : A \to B$ is a morphism of RG-modules so that there is a $k = k(\varphi)$ so that $\varphi^m_n : A_m \to B_n$ is 0 for $d(m, n) > k$.

Remark 14.2. In case G is the trivial group, $\mathcal{C}_{M,e}(R)$ and $\mathcal{C}_M(R)$ are identified by sending an object A in $\mathcal{C}_M(R)$ to $\oplus_{x\in M} A_x$ together with the map $f : \oplus_{x\in M} A_x \to F(M)$ picking out non-zero coefficients. Similarly when the action of G on M is trivial, the categories $\mathcal{C}_{M,G}(R)$ and $\mathcal{C}_M(RG)$ may be identified.

Definition 14.3. If R is a ring with involution, the category $\mathcal{C}_{M,G}(R)$ has an involution given by $A^* = \mathrm{Hom}^{lf}_R(A, R)$, the set of locally finite R-homomorphisms. We define $f^* : A^* \to F(M)$ by $f^*(\phi) = \{x \mid \phi(A_x) \neq 0\}$, which is finite by assumption.

Given a metric space M with an action by G and an equivariant submetric space $N \subset M$, let us denote the k-neighborhood of N by N^k. We shall now develop germ methods "away from N".

Definition 14.4. The category $\mathcal{C}^{>N}_{M,G}(R)$ has the same objects as $\mathcal{C}_{M,G}(R)$, but morphisms φ_1, $\varphi_2 : A \to B$ are identified if there exists k such that $\varphi^x_{1y} = \varphi^x_{2y}$ for $x \notin N^k$.

Using the methods of [27] , [33], and [6] we get the following:

Theorem 14.5. *Let $M \cup N \times [0, \infty)$ have the metric included from $M \times [0, \infty)$. The forgetful map (functor!)*

$$\begin{array}{ccc} \mathcal{C}_{M\cup N\times[0,\infty),G}(R) & \longrightarrow & \mathcal{C}^{>N\times[0,\infty)}_{M\cup N\times[0,\infty),G}(R) \\ & & \| \\ & & \mathcal{C}^{>N}_{M,G}(R) \end{array}$$

induces isomorphisms on algebraic K-theory and (if R is a ring with involution) on algebraic L-theory.

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathcal{U} = \mathcal{C}_{M \cup N \times [0,\infty),G}(R)$ with objects 0 except for a bounded neighborhood of $N \times [0,\infty)$. Then $\mathcal{U}$ is $\mathcal{A}$-filtered in the sense of Karoubi and the result follows from [27] since $\mathcal{A}$ has an obvious Eilenberg swindle making the K-theory trivial. $\square$

Arguing as above with Karoubi filtrations we get the following from [27], see also [6]:

Theorem 14.6. *Assume that M is a metric space with a group G acting by quasi-isometries, and let N be an invariant subspace. Form $M \cup N \times [0,\infty)$ with metric induced from $M \times \mathbb{R}$ and the induced G-action. Then the sequence of categories (with morphisms restricted to isomorphisms)*

$$\mathcal{C}_{N,G}(R) \to \mathcal{C}_{M,G}(R) \to \mathcal{C}_{M \cup N \times [0,\infty),G}(R)$$

induces a fibration of K-theory spectra, and hence a long exact sequence in K-theory.

In the important special case where the metric space M is $O(K)$ for some finite complex K with a cellular action on K, the combination of these two theorems allows the computation (in the sense of providing exact sequences) of $K_*(\mathcal{C}_{O(K),G}(R))$. Computations are further facilitated by the fact [13] that these functors are Mackey functors in the variable G.

When R is a ring with involution $\mathcal{C}_{M,G}(R)$ is a category with involution, so following Ranicki the algebraic L-theory is defined. There are exact sequences similar to the above sequences for computing L-theory. See Remark 19.4, and [6].

15. Extending the geometry

In this section, the basic setup is going to be a group G acting on a metric space M by quasi-isometries and freely, cellularly, on a bounded CW-complex X such that the reference map $p : X \to M$ is equivariant. We call this a *free bounded $G - CW$ complex.* The cellular chains take values in the category $C_{M,G}(\mathbb{Z})$ and will be denoted $D_\#(X)$. Thus, the basic point of view is equivariant instead of working with a fundamental group. We do however have to worry about interference from the fundamental group of X.

Let N be an equivariant subset of M. We shall use the following language:

Definition 15.1. Let $p : X \to M$ be a bounded G-CW complex. The term *away from N* means "when restricted to a subset of X whose complement under p maps to a bounded neighborhood of N." Similarly, *in a bounded neighborhood of N* means a subset of X mapping to a bounded neighborhood of N under p. Similarly,

(i) $p : X \to M$ is (-1)-*connected away from* N if there exists k so that for every point x in M except for a bounded neighborhood of N there exists $y \in X$ such that $d(x, p(y)) < k$.

(ii) The bounded CW complex (X, p) is (-1)-*connected away from* N if there are $k, l \in \mathbb{R}_+$ so that for each point $m \in M$ either there is a point $x \in X$ such that $d(p(x), m) < k$ or $d(m, N) < l$.

(iii) (X, p) is 0-*connected away from* N if for every $d > 0$ there exist k and l depending on d so that if $x, y \in X$ and $d(p(x), p(y)) \leq d$, then either x and y may be joined by a path in X whose image in M has diameter $< k(d)$ or $d(x, N) < l(d)$ or $d(y, N) < l(d)$. Notice that we have set up our definitions so that 0-connected does not imply (-1)-connected.

(iv) (X, p) is 1-*connected away from* N if for every $d > 0$, there exist $k = k(d)$ and $l = l(d)$ so that for every loop $\alpha : S^1 \to X$ with $d(\alpha(1), N) > l$ and $\operatorname{diam}(p \circ \alpha(S^1)) < d$, there is a map $\bar{\alpha} : D^2 \to X$ so that the diameter of $p \circ (D^2)$ is smaller than k. We also require $p : X \to M$ to be 0-connected away from N, but not (-1)-connected.

(v) (X, p) has *bounded fundamental group* π *away from* N if there exists a π-covering of X away from N which is 0- and 1-connected away from N. We do not require (X, p) to be (-1)-connected away from N.

Definition 15.2. A free bounded G-CW complex $X \to M$ is a *G-Poincaré duality complex away from* N if $X \to M$ is 0- and 1-connected away from N, and there is a class $[X] \in H^{lf}(X/G; C)$ so that a transfer of $[X]$ induces a bounded homotopy equivalence $[X] \cap - : D^{\#}(X) \to D_{\#}(X)$ as chain complexes in $\mathcal{C}^{>N}_{M,G}(\mathbb{Z})$.

Definition 15.3. A G-metric space M (i.e., a metric space M with a group G acting by quasi-isometries) is allowable if there exists a finite dimensional complex K with a free cellular G-action and a map $p : K \to M$ making K a free bounded (-1)-, 0- and 1- connected free bounded $G - CW$-complex.

With these definitions, the theory detailed in the preceding sections for the case of a boundedly constant fundamental group carries through, so we obtain the analogue of the main theorem of this paper, the surgery exact sequence.

Theorem 15.4. *Let $X \to M$ be a (-1), 0, and 1-connected n-dimensional G-Poincaré duality complex away from N. For $n \geq 5$, there is a surgery exact sequence*

$$\cdots \to \mathcal{S}_b((X \times I), \delta(X \times I), G)^{>N} \to [\Sigma X/G, F/\operatorname{CAT}]^{>N} \to$$
$$\to L^s_{n+1}(\mathcal{C}^{>N}_{M,G}(\mathbb{Z})) \to \mathcal{S}_b(X, G) \to [X, F/\operatorname{CAT}]^{>N} \to L^s_n(\mathcal{C}^{>N}_{M,G}(\mathbb{Z})).$$

Here simpleness is measured in $K_1(\mathcal{C}^{>N}_{M,G})/G$, $\mathcal{S}_b(X,G)^{>N}$ denotes bounded equivariant structures away from N, and $[X, F/\,\mathrm{CAT}]^{>N}$ denotes germs of homotopy classes of maps away from N.

Of course, we have L^h-groups as well as L^s-groups and these groups are connected via the usual Ranicki-Rothenberg exact sequence.

Proposition 15.5. *Assume $n \geq 5$. There is a long exact Ranicki-Rothenberg sequence*

$$\to L^s_n(\mathcal{C}^{>N}_{M,G}(\mathbb{Z})) \to L^h_n(\mathcal{C}^{>N}_{M,G}(\mathbb{Z})) \to \widehat{H}^n(\mathbb{Z}_2, \widetilde{K}_1(\mathcal{C}^{>N}_{M,G}(\mathbb{Z}))) \to .$$

16. Spectra and resolution of ANR homology manifolds

Following the tradition of Quinn, Ranicki, and Nicas, we spacify our bounded surgery groups, producing spectra such that the surgery groups are the homotopy groups of these spectra.

Theorem 16.1. *Let (M,G) be an allowable G-metric space. There is an infinite loop space $\mathbb{L}^s_{M,G}(\mathbb{Z})$ depending functorially on (M,G), such that*

$$\pi_i \mathbb{L}^s_{M,G}(\mathbb{Z}) = L^s_i(\mathcal{C}_{M,G}(\mathbb{Z})).$$

Proof. We construct a $\triangle$-set whose n-simplices are n-ads of (M,G)-surgery problems. The realization is an infinite loop space, as in the classical case. See [28] and Nicas [22] for details. $\square$

In the special case where G is the trivial group, i.e., the case of simply-connected bounded surgery, we can improve a bit on the situation, getting an analogue of the main theorem of Pedersen-Weibel [27].

Theorem 16.2. *The functor sending a finite complex K to $\mathbb{L}_{O(K)}(\mathbb{Z})$ sends cofibrations to fibrations.*

Proof. Let $L \hookrightarrow K \to K \cup CL$ be a cofibration. The composite $\mathbb{L}_{O(L)}(\mathbb{Z}) \to \mathbb{L}_{O(K)}(\mathbb{Z}) \to \mathbb{L}_{O(K\cup CL)}(\mathbb{Z})$ is the zero map, since it factors through $\mathbb{L}_{O(CL)}(\mathbb{Z})$, which is contractible. On homotopy groups we get an exact sequence (by Theorem 10.5) and thus the Five Lemma shows that $\mathbb{L}_{O(L)}(\mathbb{Z})$ is homotopy equivalent to the homotopy fibre of $\mathbb{L}_{O(K)}(\mathbb{Z}) \to \mathbb{L}_{O(K\cup CL)}(\mathbb{Z})$. $\square$

Following [27], this identifies the homology theory. Denoting the four-periodic simply connected surgery spectrum by $\mathbb{L}$ we have:

Theorem 16.3.

$$L_n(\mathcal{C}_{O(K)}(\mathbb{Z})) \cong h_{n-1}(K, \mathbb{L})$$

where $h(-,\mathbb{L})$ denotes the reduced homology theory associated with the spectrum $\mathbb{L}$.

Proof. It is shown in [27], Theorem 3.1 that the spectrum for the homology theory $L_n(\mathcal{C}_{O(K)})$ is given by the spectrum whose n'th space is $\mathbb{L}_{\mathbb{R}^n}(\mathbb{Z})$, but that is exactly four-periodic simply connected L-theory. □

Let k denote the (unreduced) homology theory with coefficients in connective simply connected $\mathbb{L}$-theory, and h the (unreduced) homology theory with coefficients in 4-periodic simply connected $\mathbb{L}$-theory. As always, there is a natural transformation from h to k, sending the periodic spectrum to the connective version. We define a natural transformation α from k to h as follows:

Let K be a finite complex, W a regular neighborhood of K. Now $k_*(K) \cong k_*(W) \cong k^{|W|-*}(W/\partial W) \cong k^0(\Sigma^{*-|W|}W/\partial W) = [\Sigma^{*-|W|}W/\partial W, F/\operatorname{TOP}]$. Consider $W \times [0,\infty) \to O(K)$, a simply connected bounded Poincaré duality complex away from 0, and with boundary. The normal invariant of $W \times [0,\infty)$ away from 0 relative to the boundary is given by $[\Sigma^{*-|W|}W/\partial W, F/\operatorname{TOP}]$ and the surgery exact sequence maps from there to $L_{(*-|W|)+|W|}(\mathcal{C}_{O(K)}(\mathbb{Z})) = h_*(K)$.

Theorem 16.4. *The composite of natural transformations* $k_* \xrightarrow{\alpha} h_* \to k_*$ *is an isomorphism.*

Proof. It is enough to verify this for spheres. What we need to prove is that the bounded structure space $\mathcal{S}_b\begin{pmatrix} S^n\times[0,\infty) \\ \downarrow \\ O(S^n)\end{pmatrix}^{>0}$ is contractible. The classical structure space of a sphere is a point (by the high-dimensional Poincaré conjecture). Crossing with $\mathbb{R}^k$ into bounded $\mathbb{L}$-theory is an isomorphism both on normal invariants and L-groups, so $[\Sigma^i(S^n \times \mathbb{R}^k); F/\operatorname{TOP}] \cong L(\mathcal{C}_{\mathbb{R}^k}(\mathbb{Z}))$. Away from 0, the Poincaré complex $S^n \times \mathbb{R}^k \to \mathbb{R}^k$ is $S^n \times S^{k-1} \times [0,\infty) \to O(S^{k-1})$, but that, on the other hand, is the image of $S^n \times S^{k-1} \times [0,\infty) \to O(S^n \times S^{k-1})$ away from 0 induced by the projection $S^n \times S^{k-1} \to S^{k-1}$. By naturality, we obtain that α is an isomorphism on one of the summands when applied to $S^n \times S^{k-1}$, but then by naturality it must be an isomorphism on spheres. □

The point of the proof above is to relate the obvious isomorphism for the case $S^n \times \mathbb{R}^k \to \mathbb{R}^k$ to the definition of α. This theorem can also be proved using Chapman and Ferry's α-approximation theorem [8].

We get a new proof of Quinn's obstruction to resolution.

Theorem 16.5. *Let X be an* ANR *homology manifold. Then there is an integral obstruction to producing a resolution of X.*

Proof. First, assume that X admits a TOP reduction of its Spivak normal fibre space. Theorem 16.6 below shows that such a reduction always exists.

Consider the bounded surgery exact sequence

$$\mathcal{S}_b\begin{pmatrix} X\times[0,\infty) \\ \downarrow \\ O(X)\end{pmatrix}^{>0} \to [X, F/\operatorname{TOP}] \to L_{|X|}(\mathcal{C}^{>0}_{O(X)}(\mathbb{Z})).$$

By the theorem above and Theorem 16.7 below,

$$L_{|X|}(\mathcal{C}^{>0}_{O(X)}(\mathbb{Z})) \cong [X, F/\operatorname{TOP}\times\mathbb{Z}],$$

and the map $[X, F/\operatorname{TOP}] \to [X, F/\operatorname{TOP}\times\mathbb{Z}]$ followed by the map to connective L-theory, i.e., to $[X, F/\operatorname{TOP}]$, is an isomorphism. Hence it is only the component in $F/\operatorname{TOP}\times\mathbb{Z}$ that X maps into which is the obstruction to the nonemptiness of the bounded structure set $\mathcal{S}_b\begin{pmatrix} X\times[0,\infty) \\ \downarrow \\ O(X)\end{pmatrix}^{>0}$. Assume that this integral obstruction vanishes. Choose an element in the bounded structure set

$$\begin{array}{ccc} \phi : W & \cong & X\times[0,\infty) \\ & & \downarrow \\ & & O(X). \end{array}$$

ϕ is a bounded homotopy equivalence away from $0 \in O(X)$. A neighborhood of infinity in W maps to X, and since $X\times[0,\infty) \to O(X)$ is the identity away from 0, and ϕ is a bounded homotopy equivalence, the end of W mapping to X is tame and simply connected, so we may add an end M to W and extend the map $W \to X$ to M. The map $M \to X$ is a resolution because it is an arbitrarily small homotopy equivalence. The theorem will now follow from: □

Theorem 16.6. *An* ANR *homology manifold X has a canonical* TOP *reduction.*

Preparing for the proof, first notice that by [1] $L_n(\mathcal{C}_{O(-)}(\mathbb{Z}))$ is a functor defined on compact subsets of S^N, N large, and all continuous maps, not only Lipschitz maps. We now have

Theorem 16.7. *$L_n(\mathcal{C}_{O(-)}(\mathbb{Z}))$ satisfies Milnor's wedge axiom.*

Proof. Consider $\bigvee X_\alpha \subset S^N$. We have

$$L_n(\mathcal{C}_{O(\bigvee X_\alpha)}(\mathbb{Z})) \cong L_n(\mathcal{C}^{>O(*)}_{O(\bigvee X_\alpha)}(\mathbb{Z})).$$

By suspension, we may assume n divisible by 4, so an element is given by a self-intersection form ν, which is bounded, so when we disregard a neighborhood of $O(*)$ we get a self-intersection form on each $L^{>O(*)}_{O(K_\alpha)}(\mathbb{Z})) \cong$

$L_n(\mathcal{C}_{O(X_\alpha)}(\mathbb{Z}))$. To combine an element in $\Pi_\alpha L_n(\mathcal{C}^{>O(*)}_{O(X_\alpha)}(\mathbb{Z}))$ to get an element in $L_n(\mathcal{C}_{O(\bigvee X_\alpha)}(\mathbb{Z}))$, all we need to do is reparameterize radially so that all components have the same bound. $\square$

This means that the identification of bounded L-theory over open cones with homology theory extends beyond finite complexes as a Steenrod homology theory, and that homology with locally finite coefficients may be defined as reduced homology of the one-point compactification.

Proof of Theorem 16.6. Cover X by open sets U_α so that the Spivak normal fibration restricted to U_α is trivial. On U_α we obviously have a TOP reduction, giving rise to a surgery exact sequence as above, denoting the *topological* boundary of U_α by ∂U_α,

$$\mathcal{S}_b\begin{pmatrix} U_\alpha\times[0,\infty) \\ \downarrow \\ O(\bar{U}_\alpha)\end{pmatrix}^{>\partial U_\alpha} \to [U_\alpha, F/\operatorname{TOP}] \xrightarrow{\phi} L(\mathcal{C}^{>O(\partial\bar{U}_\alpha)}_{O(\bar{U}_\alpha)}(\mathbb{Z})).$$

By Poincaré duality, ϕ may be identified with

$$[U_\alpha, F/\operatorname{TOP}] \to [U_\alpha, F/\operatorname{TOP}\times\mathbb{Z}],$$

so by changing the lift of U_α we may ensure that the surgery obstruction is just an integer, and assuming this integer vanishes, we can produce a resolution over U_α as above. This surgery exact sequence is natural with respect to restriction to smaller open sets, so the lifts combine to give a lift over the whole of X. $\square$

Remark 16.8. Strictly speaking, the argument above is flawed in that arbitrary wedges of polyhedra cannot be embedded in a single finite-dimensional sphere. This can be cured by using the unit sphere in a Hilbert space or, better, by embedding X isometrically into the bounded functions from X to $\mathbb{R}$ and taking a cone there.

Next, we want to understand assembly from the point of view of bounded surgery. Given a boundedly simply-connected surgery problem away from 0 parametrized by $O(K)$, the induced map from K to a point gives a surgery problem parametrized by $[0,\infty)$ away from 0. We can turn this problem into a simply-connected surgery problem by doing simultaneous surgery on source and target, giving the usual functorial property with respect to K, but avoiding that, we obtain a simple surgery problem (simplicity measured in $Wh(\pi_1(K))$) together with a reference map to K, in other words, an element in $L^s(\mathcal{C}^{>0}_{[0,\infty)}(\mathbb{Z}\pi_1(K))) \cong L^s(\mathcal{C}_{\mathbb{R}}(\mathbb{Z}\pi_1(K))) \cong L^h(\mathbb{Z}\pi_1(K))$. We claim this forget control map is the assembly map.

Theorem 16.9. *Let M be a manifold. Then the forgetful map*

$$F: L(\mathcal{C}^{>0}_{O(M)}(\mathbb{Z})) \to L^h(\mathbb{Z}(\pi_1(M)))$$

is the assembly map.

Proof. Consider the following diagram

$$\begin{array}{ccccc} \cdots \longrightarrow & [\Sigma^i M, F/\mathrm{TOP}] & \xrightarrow{\alpha} & L_{m+i+1}(\mathcal{C}_{O(M)}(\mathbb{Z})) \\ & \| & & \downarrow F \\ \cdots \longrightarrow & [\Sigma^i M, F/\mathrm{TOP}] & \xrightarrow{A} & L^h_{m+i}(\mathbb{Z}(\pi_1(M))) \end{array}$$

where the lower row is the classical surgery exact sequence with the assembly map. We have just proved that α is an isomorphism for $i > 0$ and the inclusion of a direct summand for $i = 0$. For $i > 0$, this identifies the map with assembly. Since the algebraically defined groups are 4-periodic, this also identifies F with the higher assembly maps when $i = 0$. □

This gives a curious relation between the resolution problem and the Novikov Conjecture.

Theorem 16.10. *Let M be a closed $K(\pi, 1)$-manifold such that the assembly map is an integral monomorphism. Then an* ANR *homology manifold X homotopy equivalent to M admits a resolution.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{S}_b\begin{pmatrix} X\times[0,\infty) \\ \downarrow \\ O(X)\end{pmatrix}^{>0} & \longrightarrow & [X, F/\mathrm{TOP}] & \xrightarrow{\alpha} & L_{m+1}(\mathcal{C}^{>0}_{O(X)}(\mathbb{Z})) \\ \downarrow & & \| & & \downarrow F \\ \mathcal{S}(X) & \longrightarrow & [X, F/\mathrm{TOP}] & \xrightarrow{A} & L^h_m(\mathbb{Z}\pi_1(X)). \end{array}$$

Since $\mathcal{S}(X)$ is nonempty, there is a $\sigma \in [X, F/\mathrm{TOP}]$ so that $A(\sigma) = 0$. But then $\alpha(\sigma) = 0$ and $\mathcal{S}_b\begin{pmatrix} X\times[0,\infty) \\ \downarrow \\ O(X)\end{pmatrix}^{>0}$ is nonempty, showing that X admits a resolution. □

Bob Daverman has pointed out that there is an easy geometric proof that there is no nonresolvable ANR homology manifold X homotopy equivalent to the n-torus. The universal cover of such an ANR homology manifold could be compactified by adding a sphere at infinity. Adding an external collar would then give an ANR homology manifold with both manifold points and points with neighborhoods from X, showing that the resolution obstruction for X was trivial. Many of the classes of groups for which the map α is known to be 1-1 admit similarly nice compactifications. In fact, Ferry and Weinberger have recently announced a proof of the Novikov conjecture for all Γ such that $K = K(\Gamma, 1)$ is a finite complex (*not* necessarily a manifold) and $\widetilde{K}$ admits a sufficiently nice compactification [11].

17. Geometric constructions

In this section we prepare for the applications in the next section by describing some geometric constructions. Consider a continuous proper map $X \to O(K)$.

Definition 17.1. The K-completion $\widehat{X}^K$, of X is defined as follows: As a set $\widehat{X}^K$ is the disjoint union of X and K. The open sets of $\widehat{X}^K$ have as a basis:

(i) all open sets of X
(ii) for every open set U of K, and every $k \in \mathbb{R}_+$, the set

$$\{p^{-1}\ (x,t) \in O(K) | x \in U, t > k\}\ \cup U.$$

This construction is sometimes called the tear drop construction. It is easy to see that $\widehat{X}^K$ is a compact metric space. This construction generalizes one-point compactification.

Theorem 17.2. *Let W_1 and W_2 be manifolds properly parameterized by $O(K)$, and assume that h is a bounded homotopy equivalence from W_1 to W_2. Then $\widehat{W}_1^K$ is a manifold if and only if $\widehat{W}_2^K$ is a manifold.*

Proof. Assume that $\widehat{W}_1^K$ is a manifold. It is easy to see that $\widehat{W}_2^K$ is an ANR homology manifold. Using the homotopy equivalence, the disjoint two-disc property is also carried over, so the result follows by the Manifold Recognition Theorem [31]. See [10] for a detailed proof of the disjoint two-disc property in the case where $K = S^1$. □

Remark 17.3. This result is very useful in proving the existence of group actions by varying the complement of the singular set. For studying group actions we also need the following:

Proposition 17.4. *Let G be a finite group with a stratum-preserving action on a finite complex K. Assume that W_1 and W_2 are manifolds parameterized by $O(K)$ on which G acts freely and compatibly with the action on K. Then W_1 is boundedly equivariantly homotopy equivalent to W_2 over $O(K)$ if and only if W_1/G is boundedly homotopy equivalent to W_2/G over $O(K/G)$.*

Proof. One way is trivial, so assume that $h : W_1/G \to W_2/G$ is a bounded homotopy equivalence. Certainly we get an equivariant homotopy equivalence $\tilde{h} : W_1 \to W_2$. The length of the path in the homotopy can be at most $|G|$ times the length of the path measured in $O(K/G)$, so we are done. □

18. More applications

We begin with an application to group actions.

Consider the standard $n+k$ sphere S^{n+k} with the standard $k-1$ subsphere $S^{k-1} \subset S^{n+k}$ so that $S^{n+k} = S^n * S^{k-1}$. Let G be a finite group and assume that G acts semifreely (topologically) on S^{n+k} fixing S^{k-1}. It was proved in [3] that G has to be a periodic group, since $(S^{n+k} - S^{k-1})/G$ is finitely dominated and $S^{n+k} - S^{k-1}$ has the homotopy type of a sphere. Hence, the homotopy type of $(S^{n+k} - S^{k-1})/G$ is given by G and a single k-invariant which is a unit in $\mathbb{Z}/|G|$. It was further proved that such actions exist if and only if a certain surgery problem has a solution, i.e., if and only if a certain Spivak normal bundle has a reduction and the resulting surgery problem can be solved.

This surgery program was completed in [12] and [20], and also the maps $L^h_n(\mathbb{Z}G) \to L^p_n(\mathbb{Z}G) \to L^{-1}_n(\mathbb{Z}G)$ were computed for the relevant groups, but the computation did not give the classification one usually obtains from surgery theory. It is the purpose of this section to show how ε-surgery can turn the computations of [12] and [20] into such a classification.

Let X be a Swan complex. The surgery exact sequence of §11 takes the following form:

$$\cdots \to L^s_{n+k+1,\mathbb{R}^k}(\mathbb{Z}G) \to \mathcal{S}_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \to [X, F/\operatorname{TOP}] \to L^s_{n+k,\mathbb{R}^k}(\mathbb{Z}G).$$

We proved in §10 that $L^s_{n+k,\mathbb{R}^k}(\mathbb{Z}G) = L^{2-k}_n(\mathbb{R}G)$, so we get a surgery exact sequence:

$$\cdots \to L^{1-k}_{n+1}(\mathbb{Z}G) \to \mathcal{S}_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \to [X, F/\operatorname{TOP}] \to L^{2-k}_n(\mathbb{Z}G) \to \cdots .$$

Comparing this with [12] and [20] we see that what is actually being computed is $\mathcal{S}_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix}$.

In [3] a map is defined:

$$\mathcal{S}_b\begin{pmatrix} X\times\mathbb{R}^k \\ \downarrow \\ \mathbb{R}^k \end{pmatrix} \xrightarrow{h} \left\{ \begin{matrix} \text{Conjugacy classes of semifree group ac-} \\ \text{tions of } G \text{ on } S^{n+k} \text{ fixing } S^{k+1} \text{ with } k\text{-} \\ \text{invariant of } S^{n+k} - S^{k-1}/G \text{ given by } X, \\ \text{so that } (S^{n+k} - S^{k-1})/G \simeq X. \end{matrix} \right\}$$

The construction in [3] starts with a compact manifold homotopy equivalent to $X \times T^n$, and then passes to the $\mathbb{Z}^n$ cover, but it is clear that the same construction gives a map as above. The right-hand side is a classification of semifree group actions, so we will be done once we prove:

Theorem 18.1. *h is an isomorphism.*

Proof. The map is well-defined because a bounded homeomorphism extends to a completion by the identity on the fixed sphere. To see that h is onto, consider a semifree action on S^{n+k} fixing S^{k-1} and let $W = S^{n-k} - S^{k-1}/G$. It is shown in [3] that W has a tame end at S^{k-1}. Killing the obstruction to completing the end by multiplying with a torus, we obtain that $W \times T^n \simeq W' \times \mathbb{R}^n$, where the radial directions in $\mathbb{R}^n$ point to the points of S^{n-1}. Going to the cyclic cover we have $W \simeq W \times \mathbb{R}^n \simeq \widetilde{W'} \times \mathbb{R}^n$ and $\widetilde{W'} \simeq X$ so we are done. That the map is monic follows from Theorem 11.2 that radial reparameterization induces the identity on the structure set. □

Remark 18.2. Note that this surgery theory is not restricted to the category of manifolds. All that is needed is that the objects be manifolds away from the singular set. It thus makes perfectly good sense to suspend group actions. Suspension is just crossing with the reals in the nonsingular part and suspending on the singular part, at least if one assumes nonempty singular sets. This means that questions such as the above may be treated in two stages:

(i) Suspend enough times that $K_{-i}(\mathbb{Z}[\pi]) = 0$, and apply $L^{-\infty}$.
(ii) Try to split off real factors to get back to the manifold situation.

As a second example, consider a closed PL manifold $M^n \subset S^{m-1} \subset \mathbb{R}^m$ which contains a simply-connected polyhedron Y. Let p:$M \to M/Y$ be the projection map. As in the proof of Theorem 9.6, form a two-ended manifold W which looks like $O(M)$ near $+\infty$ and like $M \times \mathbb{R}$ near $-\infty$ and parameterize W over $O(M_+/Y)$. The map $id : W \to W$ is a bounded structure on $W \to O(M_+/Y)$, so we have an exact surgery sequence:

$$\cdots \to L_{n+1,O(M_+/Y)}(\mathbb{Z}) \to \mathcal{S}_b \begin{pmatrix} W \\ \downarrow \\ O(M_+/Y) \end{pmatrix} \to [M, F/\operatorname{TOP}] \to L_{n,O(M_+/Y)}(\mathbb{Z})$$

where there are no decorations on the L's because of the simple connectivity. In this case $L^s_{n,O(M_+/Y)}(\mathbb{Z}) = L^{-\infty}_{n,O(M_+/Y)}(\mathbb{Z})$, so the obstructions lie in $h_n(M/Y; F/\operatorname{TOP})$. This is unreduced homology.

An element of $\mathcal{S}_b \begin{pmatrix} W \\ \downarrow \\ O(M_+/Y) \end{pmatrix}$ is an equivalence class of bounded homotopy equivalences $\phi : W' \to W$. Splitting such a ϕ over $M \times \{T\}$ for some large T produces a homotopy equivalence $\phi| : M' \to M$ which is arbitrarily small over M/Y. By the thin h-cobordism Theorem, this splitting is well-defined up to small homeomorphism over M/Y.

If $N \supset Y$ is a regular neighborhood of Y in M, the main theorem of [8] shows that $\phi|$ is close to a homeomorphism over $M - \operatorname{int}(N)$. Thus, M' is the union of a copy of $M - \text{int}(N)$ and a copy of $(\phi|)^{-1}(N) = N'$. Since $M - Y \cong M - N \cong M' - N'$, we see that M' is a compactification of $M - Y$ by a polyhedron homotopy equivalent to Y. If Y has codimension-three or greater in M, then a polyhedron Y' homotopy equivalent to Y and having the same dimension as Y embeds in N' and $M' - N' \cong M' - Y'$.

There is, of course, a related existence question. If M is an open manifold and we wish to compactify M by adding a complex K at ∞, we can proceed by constructing a nonmanifold "Poincaré completion" and then try to solve the resulting bounded surgery problem over the open cone on the one-point compactification of M. Note that the use of bounded surgery here is the reverse of the group actions application above. There, we started with a manifold and a control map and used our theory to vary the complement. Here, we control over the complement and allow the theory to construct the manifold completion. One interesting aspect of this theory is that, except for predicting the dimension of Y', it works well for Y of any codimension.

Another way of exploiting the same control map $M \to M/Y$ is to start with a homotopy equivalence $\phi : N \to M$ and try to solve the resulting controlled surgery problem over $O(M_+/Y)$, as above. As before, we encounter obstructions lying in $h_n(M/Y; F/\text{TOP})$. If we succeed in solving this surgery problem, we obtain a bordism from N to a manifold N' which is controlled homotopy equivalent to M over M/Y. As above, such a manifold splits into a copy of $M - Y$ and a polyhedron homotopy equivalent to Y. The bordism comes equipped with a degree one normal map to $M \times I$, so there is a further ordinary surgery obstruction to surgering the bordism to an s-cobordism from N' to N. Note that this is a nonsimply connected surgery obstruction, since M is not required to be simply connected. In the case $Y = pt$, the resulting exact sequence is the ordinary surgery exact sequence. In the general case, we have obtained a 2-stage obstruction to splitting N into a manifold homeomorphic to $M - Y$ and a complex homotopy equivalent to Y.

As a final example, consider a manifold M homotopy equivalent to the total space of a bundle (or quasifibration or approximate fibration) of manifolds:

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ M & \simeq & E \\ & & \downarrow \\ & & B \end{array}$$

We may ask whether M can be turned into a bundle of some sort over B. Assuming that the bundle splits at fundamental group level , we obtain a surgery problem

$$\begin{array}{ccc} M \times \mathbb{R} & \longrightarrow & E \times \mathbb{R} \\ & & \downarrow \\ & & O(B_+) \end{array}$$

with fundamental group $\pi = \pi_1(F)$. The obstruction lies in

$$L^s_{n,O(B_+)}(\mathbb{Z}[\pi_1 F]).$$

Assuming that the obstruction vanishes, we obtain a manifold N normally cobordant to M and a homotopy equivalence of N to E which is arbitrarily small when measured in B. But this means that $N \to B$ is an approximate fibration, so we have obtained a normal cobordism from M to a manifold which approximate fibers over B. As before, we now have an ordinary surgery obstruction to turning this cobordism into an s-cobordism. The result is an obstruction theory for homotoping a map to an approximate fibration. Note that the fact that $E \to B$ was a bundle was barely used. If $E \to B$ is any map from a manifold to a polyhedron which is a "trivial bundle on π_1," and $M \to E$ is a homotopy equivalence, then solving the same sequence of problems would produce a map $M \to B$ with the same "shape fiber structure" as $E \to B$.

19. A variant L-theory

It is sometimes a problem that the L-theory described in §9 is not a homology theory as a functor of the control space. This is unlike K-theory [27]. Inspired by discussions with Quinn, we give a a variant definition of L-theory which is (at least) a half exact functor in the control space. This, on the other hand, means that it cannot degenerate to usual L-theory when the control space is a point. The idea is to mix the torsion requirements. As in §9, our definition is modeled on [43, Ch. 9].

Given a space K, an object is a surgery problem

$$\begin{array}{ccc} (M, \partial M) & \longrightarrow & (X, \partial X) \\ & & \downarrow \\ & & O(K) \end{array}$$

where $(X, \partial X)$ is a bounded Poincaré pair with bounded fundamental group π and a specific CW structure. Thus, we have a specific simple type of $(X, \partial X)$ parameterized by $O(K)$, but we only require $(X, \partial X)$ to be a Poincaré pair. We do not require Poincaré torsion to vanish in $Wh_{O(K)}(\mathbb{Z}\pi)$. We assume $\partial M \to \partial X$ to be a simple homotopy equivalence.

Associated to such an object we have a Poincaré torsion $\tau(X)$. We use sign conventions for Poincaré torsion as in [23].

The usual equivalence of bordism is to say $(M_1, \partial M_1) \to (X_1, \partial X_1)$ is bordant to $(M_2, \partial M_2) \to (X_2, \partial X_2)$ if there is a triad surgery problem

$$(W, M_1, M_2) \to (Y, X_1, X_2).$$

We refine this relation by requiring that $\tau(Y, X_1) = 0$.

We claim that this refined type of bordism is an equivalence relation on the set of surgery problems. The condition $\tau(Y, X_1) = 0$ is equivalent to the condition $\tau(Y) = \tau(X_1) = \tau(X_2)$ (see e.g. [23]). We have $\tau(X \times I, X \times 0) = 0$ showing that an object is equivalent to itself. Symmetry follows from $\tau(Y, X_1) = \pm\tau(Y, X_2)$. Finally, if Y is a bordism from X_1 to X_2 and Z is a bordism from X_2 to X_3, then $\tau(Y \cup Z) = \tau(Y) + \tau(Z) - \tau(X_2)$ showing that $\tau(Y \cup Z) = \tau(X_1) = \tau(X_3)$.

All constructions involving simultaneous surgery as in §9 are allowed, since manifolds have trivial Poincaré torsion, and these are manifold constructions.

The Grothendieck construction on the set of surgery problems with fundamental group π, and only requiring homotopy equivalence of the Poincaré duality map, not simple homotopy equivalence, parameterized by $O(K)$, modulo the above equivalence relation, we shall denote by

$$L^{h,s}_{n,O(K)}(\mathbb{Z}\pi).$$

The basic idea is to require relations to be simpler than generators.

Theorem 19.1. *The functor $L^{h,s}_{n,O(K)}(\mathbb{Z}\pi)$, $n \geq 5$, is half exact in the variable K.*

Proof. In §9 we studied the functor $L^h_{O(K),n}(\mathbb{Z}\pi)$ as a functor in K. In trying to prove half exactness, there was a splitting obstruction, but this splitting obstruction must vanish because of the assumption of simpler relations. □

Let

$$\begin{aligned} H^h_n(\mathbb{Z}_2\,; K_1) &= \{\sigma \in K_1 \,|\, \sigma^* = (-1)^n \sigma\}\,, \\ H^s_n(\mathbb{Z}_2\,; K_1) &= \{\sigma \in K_1 \,|\, \sigma = \tau + (-1)^n \tau^*\} \end{aligned}$$

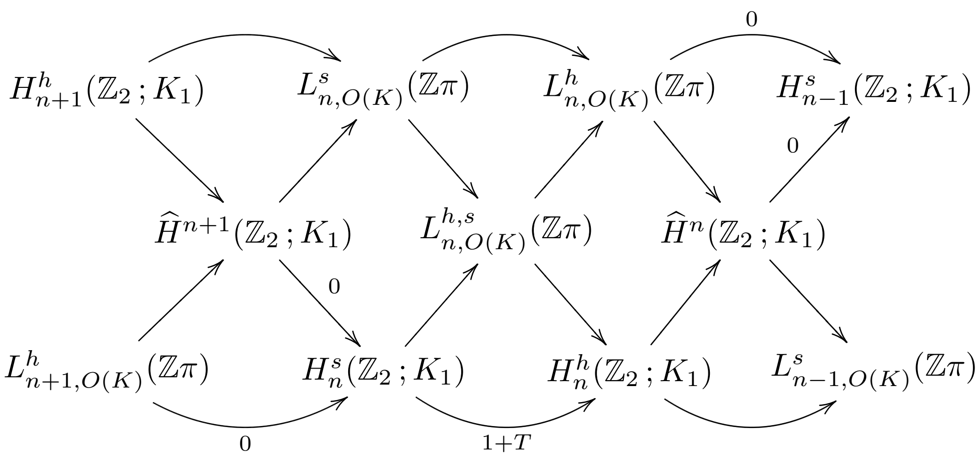
where $K_1 = K_1(\mathcal{C}_{O(K)}(\mathbb{Z}\pi))$.

Theorem 19.2. *For $n \geq 5$, there are exact sequences*

$$H^s_n(\mathbb{Z}_2\,; K_1) \to L^{h,s}_{n,O(K)}(\mathbb{Z}\pi) \to L^h_{n,O(K)}(\mathbb{Z}\pi) \to 0$$

$$0 \to L^s_{n,O(K)}(\mathbb{Z}\pi) \to L^{h,s}_{n,O(K)}(\mathbb{Z}\pi) \to H^h_n(\mathbb{Z}_2\,; K_1)$$

which together with the usual Ranicki-Rothenberg exact sequence fit into a commutative braid



Proof. In L^h we allow more relations than in $L^{h,s}$, so clearly there is an epimorphism. Similarly, in L^s we allow fewer generators than in $L^{h,s}$, so there is a monomorphism. The proof is now completed by a slight modification of the main argument in [26], realization of h-cobordisms, and the π-π theorem. □

Remark 19.3. The authors believe that the $L^{h,s}$ groups coincide with the diagonal L-groups as proposed by Quinn in various lectures. The notation is chosen to indicate that one may always define L-groups with two upper decorations instead of only one, corresponding to a $*$-invariant subgroup of the Whitehead group containing another $*$-invariant subgroup.

Remark 19.4. Ranicki has recently proved the existence of a useful exact sequence [37]. Given a cofibration $A \to X \to X \cup_A CA$ there is a long exact sequence:

$$\ldots \to L_n^h(\mathcal{C}_{O(A)}(R)) \to L_n^h(\mathcal{C}_{O(X)}(R)) \to L_n^K(\mathcal{C}_{O(X\cup_A CA)}(R)) \to \\ \to L_{n-1}^h(\mathcal{C}_{O(A)}(R)) \to \ldots$$

where $K = \mathrm{Im}(K_1(\mathcal{C}_{O(X)}(R)) \to K_1(\mathcal{C}_{O(X\cup_A CA)}(R)))$. This seems to be an adequate substitute for being a homology theory. See also the extensions of Ranicki's results given in [6, Section 4] which give a general result of the above mentioned type in the language of Karoubi–filtered categories.

20. Final Comments

Throughout this paper we have been working under the assumption of a constant fundamental group or a group action. This does exclude some examples one might want to study, for example

$$(S^7 \to CP(3)) \rightsquigarrow \{S^7 \times \mathbb{R} \to O(CP(3)_+))\}$$

as control map. In this example we have a locally constant fundamental group $\mathbb{Z}$ which is not globally constant. It is however possible to study questions of this type by the methods developed in this paper as follows: Cover $CP(3)$ by open sets U_α so that the restriction of the bundle to each U_α is trivial. A bounded surgery problem parameterized by $O(CP(3))$ with this fundamental group structure will now produce a surgery problem in each $L_n(\mathcal{C}^{\partial U_\alpha}_{\bar{U}_\alpha}(\mathbb{Z}[\mathbb{Z}]))$, and the original surgery problem can be solved if and only if all these surgery problems can be solved in a compatible way. But this can be investigated: If we can solve over $O(U_\alpha)$ and over $O(U_\beta)$ there will be an obstruction in $L_{n+1}(\mathcal{C}^{\partial(U_\alpha \cap U_\beta)}_{O(\bar{U}_\alpha \cap \bar{U}_\beta)}(\mathbb{Z}[\mathbb{Z}]))$. So the methods can in principle be made to work, if not in the most elegant way, for a locally constant system of fundamental groups in the stratified sense.

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