EMBEDDINGS OF HOMOLOGY MANIFOLDS IN CODIMENSION ≥ 3

J. L. BRYANT AND W. MIO

1. Introduction

Among the many problems attendant to the discovery of exotic generalized manifolds [2, 3] is the "normal bundle" problem, that is, the classification of neighborhoods of generalized manifolds tamely embedded in generalized manifolds (with the disjoint disks property). In this paper we study the normal structure of tame embeddings of a closed generalized manifold X^n into topological manifolds V^{n+q}, $q \geq 3$. If the local index $\imath(X) \neq 1$, then the codimension is necessarily ≥ 3 (see, e.g., Proposition 5.4 below). The main result is an extension to ENR homology manifolds of the classification of neighborhoods of locally flat embeddings of topological manifolds obtained by Rourke and Sanderson in [10]. We show that for $q \geq 3$ and $n+q \geq 5$, germs of tame manifold q-neighborhoods of X, or equivalently, controlled homeomorphism classes of $(q-1)$-spherical manifold approximate fibrations over X are in one-to-one correspondence with $[X, B\boldsymbol{T}op_q]$, where $B\boldsymbol{T}op_q$ is the classifying space for stable topological q-microbundle pairs [10]. Manifold approximate fibrations over topological manifolds have been studied by Hughes, Taylor and Williams in [5]. Our approach is to reduce the study of q-neighborhoods of X to the classification of q-neighborhoods of a (stable) regular neighborhood of X in euclidean space; this is done using the splitting theorem for manifold approximate fibrations proved in section 4. As applications, we obtain an analogue of Browder's theorem on smoothings and triangulations of Poincaré embeddings (Theorem 5.1) and of the Casson-Haefliger-Sullivan-Wall embedding theorem (Corollary 5.2) for generalized manifolds. These were also obtained independently by Johnston in [6].

2. Preliminaries

A *generalized n-manifold* is a locally compact euclidean neighborhood retract (ENR) X such that for each $x \in X$,

$$H_k(X, X \setminus \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

A compact generalized n-manifold X is *orientable* if there is a class $\xi \in H_n(X; \mathbb{Z})$ such that the inclusion $H_n(X; \mathbb{Z}) \to H_k(X, X \setminus \{x\}; \mathbb{Z})$ sends ξ to a generator of $H_k(X, X \setminus \{x\}; \mathbb{Z})$ for every $x \in X$. A choice of ξ is an orientation for X.

A subset $X \subseteq Y$ is 1-*LCC* in Y if, for every $x \in X$ and neighborhood U of x in Y, there is a neighborhood V of $x \in Y$ such that the inclusion induced homomorphism $\pi_1(V \setminus X) \to \pi_1(U \setminus X)$ is trivial. In codimension ≥ 3, we also refer to 1-*LCC* ENR subsets as *tame* subsets.

Partially supported by NSF grant DMS-9626624.

Given a space X, a *manifold approximate fibration with fiber F* over X is an approximate fibration $p\colon M \to X$, where M is a topological manifold and the homotopy fiber of p is homotopy equivalent to F. (Equivalently, each $p^{-1}(x)$ has the shape of the space F.) A group G ($\pi_1(F)$ in our constructions) is K-flat if $\mathrm{Wh}(G \times \mathbb{Z}^k) = 0$, for every $k \geq 0$.

Let $p_i\colon M_i \to X$, $i \in \{0,1\}$, be continuous maps. A *controlled map* f^c from $(M_0 \to X)$ to $(M_1 \to X)$ is a proper map $f\colon M_0 \times [0,1) \to M_1 \times [0,1)$ such that the composition

$$M_0 \times [0,1) \xrightarrow{f} M_1 \times [0,1) \xrightarrow{\text{proj}} M_1 \xrightarrow{p_1} X$$

extends continuously to $M_0 \times [0,1]$ via p_0 on $M_0 \times \{1\}$. Similarly, controlled maps $f_0^c, f_1^c\colon M_0 \to M_1$ are *controlled homotopic* if there is a controlled map H^c from $M_0 \times I \to X$ to $M_1 \to X$ such that $H^c|M_0 \times \{0\} = f_0^c$ and $H^c|M_1 \times \{1\} = f_1^c$. Controlled homeomorphisms and controlled homotopy equivalences are defined in the obvious way.

Remark.

(i) This definition is similar to that given in [5], except that we do not require controlled maps and homeomorphisms to be level preserving. However, controlled maps $f\colon M_0 \times [0,1) \to M_1 \times [0,1)$ are controlled homotopic to level-preserving mappings through linear homotopies. Similarly, level-preserving controlled homotopic maps are controlled homotopic through level-preserving homotopies.

(ii) If M_0 and M_1 are closed manifolds, h is a controlled homeomorphism of the manifold approximate fibrations $p_i\colon M_i \to X$ with fiber F, $i \in \{0,1\}$, and $\pi_1(F)$ is K-flat, then an infinite sequence of applications of the thin h-cobordism theorem [8] allows us to assume that h preserves a sequence of levels converging to 1, that is, $h(M_0 \times \{t\}) = M_1 \times \{t\}$ for an infinite sequence of t's converging to 1.

(iii) When we speak of controlled equivalences $f^c\colon M_0 \to M_1$ without specifying the control map p_0 on the domain, it is assumed that $p_0 = \lim\limits_{t \to 1} p_1 \circ f_t$ and that the limit exists, where $f_t(x) = f(x,t)$.

Although part of our discussion could be carried out in greater generality, unless stated otherwise, we assume that manifold approximate fibrations $p\colon M \to X$ have fiber S^{q-1}, $q \geq 3$, that the total space M is a closed manifold, and that the base space X is a closed ENR homology manifold.

Let $p\colon M \to X$ be a manifold approximate fibration. A *controlled structure* on p is a controlled homotopy equivalence $f^c\colon N \to M$, where $N \to X$ is a manifold approximate fibration. The *controlled structure set* of p, $\mathcal{S}_c(p)$, is the collection of all controlled homeomorphism classes of controlled structures on $p\colon M \to X$. For computational purposes, we next identify $\mathcal{S}_c(p)$ with a certain bounded structure set, in the sense of Ferry and Pedersen [4].

Given $p\colon M \to X$, assume that X is tamely embedded in S^N, N large, and that X is given the induced metric. Let $O(X)$ denote the open cone on X, and let $\wp\colon M \times [0,\infty) \to O(X)$ be defined by

$$\wp(m,t) = \begin{cases} (p(m), t), & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

where 0 denotes the cone point. We wish to identify $\mathcal{S}_c(p)$ and $\mathcal{S}_b^{>0}(\wp)$, where $\mathcal{S}_b^{>0}(\wp)$ is the bounded structure set of $\wp\colon M \times [0,\infty) \to O(X)$ away from zero (see [4] for more details).

Let $\psi\colon [0,1) \to [0,\infty)$ be a homeomorphism. A radial reparametrization (via ψ) of a controlled structure f^c on $p\colon M \to X$ may fail to yield a bounded structure (away from 0) on $\wp$, if convergence near X is too slow. This can be corrected with a suitable radial contraction of the given structure, as follows.

Let $\phi\colon [0,1) \to [0,1)$ be a homeomorphism such that $\phi(x) \leq x$, for every x, and let f^c be a controlled structure on $p\colon M \to X$ represented, say, by the level-preserving map $f\colon M_0 \times [0,1) \to M \times [0,1)$. The *$\phi$-contraction* of f^c is the controlled structure represented by the composition

$$M_0 \times [0,1) \xrightarrow{\mathrm{id}\times\phi^{-1}} M_0 \times [0,1) \xrightarrow{f} M \times [0,1) \xrightarrow{\mathrm{id}\times\phi} M \times [0,1).$$

If the contraction is fine enough, then its radial reparametrization under ψ gives a bounded structure on $\wp\colon M \times [0,\infty) \to O(X)$ away from 0. Appropriate restrictions on admissible functions ϕ guarantee that controlled homeomorphic structures are mapped to equivalent bounded structures, thus defining a map $\mathcal{S}_c(p) \to \mathcal{S}_b^{>0}(\wp)$. Conversely, if $f_0\colon N_0 \to M \times [0,\infty)$ represents a bounded structure away from zero, then N_0 has a simply-connected, tame end with respect to the control map $\rho : N_0 \to X$ given by the composition

$$N_0 \xrightarrow{f_0} M \times [0,\infty) \xrightarrow{\mathrm{proj}} M \xrightarrow{p} X.$$

By the end theorem [8] we can assume that, in a neighborhood of the end, $N_0 = N \times [0,1)$, and that the maps $\rho_t\colon N \to X$ given by $\rho_t(x) = \rho(x,t)$ converge to a spherical manifold approximate fibration $\rho_1\colon N \to X$, as $t \to 1$. Moreover, $\rho_1\colon N \to X$ is controlled homotopy equivalent to $p\colon M \to X$, under the map induced by f_0. This establishes a one-to-one correspondence

$$\mathcal{S}_c\begin{pmatrix} M \\ \downarrow \\ X \end{pmatrix} \longrightarrow \mathcal{S}_b^{>0}\begin{pmatrix} M \times [0,\infty) \\ \downarrow \\ O(X) \end{pmatrix}$$

between controlled and bounded structure sets.

3. Local structure

Let X^n be a closed oriented generalized n-manifold and V^{n+q} a topological manifold, $q \geq 3$. If X is 1-*LCC* in V and $q \geq 3$, then X has a mapping cylinder neighborhood $E = C_p$, where $p\colon \partial E \to X$ is a manifold approximate fibration with homotopy fiber S^{q-1} [8, 12]. Moreover, this spherical manifold approximate fibration structure is well defined up to controlled homeomorphisms over X. Conversely, by Proposition 3.1 below, any such spherical manifold approximate fibration arises as the normal structure of a tame embedding of X. This result is a consequence of Edwards-Quinn's characterization of manifolds, and is proven in [8] for polyhedral homology manifolds.

Proposition 3.1. *If $p : \partial E \to X$ is a manifold approximate fibration with homotopy fiber S^{q-1}, $q \geq 3$, then the mapping cylinder $E = C_p$ of p is a topological manifold, provided that $n + q \geq 5$. Furthermore, X is tamely embedded in E as the zero section.*

Classifying manifold neighborhoods of X is, therefore, equivalent to classifying spherical manifold approximate fibrations over X, up to controlled homeomorphisms. We first address this problem within a fixed controlled homotopy type over X.

Let $p\colon \partial E \to X$ be a manifold approximate fibration with homotopy fiber S^{q-1}, and let $p_1\colon \partial E_1 \to X$ be a manifold approximate fibration controlled equivalent to p via a level-preserving controlled homotopy equivalence

$$\begin{array}{ccc} \partial E_1 \times [0,1) & \xrightarrow{\psi} & \partial E \times [0,1)\,. \\ & & \big\downarrow{\scriptstyle p} \\ & & X \end{array}$$

The map ψ induces a homotopy equivalence $\tilde{\psi}\colon (E_1, \partial E_1) \to (E, \partial E)$, where E_1 and E are the mapping cylinders of p_1 and p, respectively.

Let $\eta(\tilde{\psi}) \in [E; G/Top] \cong [X; G/Top] \cong H_n(X; G/Top)$ be the normal invariant of $\tilde{\psi}$. Normal invariants of mapping cylinders induce a map

$$\rho\colon \mathcal{S}_c \begin{pmatrix} \partial E \\ \downarrow \\ X \end{pmatrix} \longrightarrow H_n(X; G/Top)$$

given by $\rho([\,\psi^c\,]) = \eta(\tilde{\psi})$, for any controlled structure ψ^c.

Proposition 3.2. *ρ is a bijection.*

Proof. The result follows from a comparison of the controlled surgery exact sequence of $p\colon \partial E \to X$ with the G/Top-homology Gysin sequence of $p\colon \partial E \to X$. It is a consequence of the 5-lemma applied to the commutative diagram

$$\begin{array}{ccccccc} H_{n+q}(X;\mathbb{L}) & \longrightarrow & \mathcal{S}_c\begin{pmatrix} \partial E \\ \downarrow \\ X \end{pmatrix} & \longrightarrow & H_{n+q-1}(\partial E; G/Top) & \longrightarrow & H_{n+q-1}(X;\mathbb{L}) \\ \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle\cong} \\ H_{n+q}(E; G/Top) & \to & H_n(X; G/Top) & \to & H_{n+q-1}(\partial E; G/Top) & \to & H_{n+q-1}(E; G/Top) \end{array}$$

The first row is the bounded surgery sequence away from 0 of $\partial E \times [0,\infty) \to O(X)$, under the identification of structure sets described in section 2. The second row is the G/Top-homology exact sequence of the pair $(E, \partial E)$ with the term $H_{n+q}(E, \partial E; G/Top)$ identified with $[E, G/Top] \cong [X, G/Top] \cong H_n(X; G/Top)$, via Poincaré duality. For $k \geq 1$, the isomorphism $H_{n+k}(X;\mathbb{L}) \cong H_{n+k}(X; G/Top)$ follows from the Atiyah-Hirzebruch spectral sequence. □

We now consider the general classification problem. Let $\mathcal{N}_q(X)$ be the collection of germs of tame codimension q manifold neighborhoods of X. Two embeddings $\iota_k\colon X^n \to V_k^{n+q}$, $k \in \{1,2\}$, represent the same element of $\mathcal{N}_q(X)$, if there are neighborhoods N_k of X in V_k, and a homeomorphism $h\colon N_1 \to N_2$ such that $h \circ \iota_1 = \iota_2$. Our previous discussion shows that $\mathcal{N}_q(X)$ is in 1–1 correspondence with controlled homeomorphism classes of $(q-1)$-spherical manifold approximate fibrations over X.

Let $B\boldsymbol{T}op_{q+k,k}$ denote the classifying space for topological microbundle pairs $\epsilon^k \subseteq \zeta^{k+q}$, where ϵ_k denotes the trivial microbundle of rank k. In [10], Rourke and

Sanderson showed that if M is a topological manifold, there is a bijection

$$\gamma\colon \mathcal{N}_q(M) \to [M, B\boldsymbol{T}op_q],$$

where $B\boldsymbol{T}op_q = \lim_{k\to\infty} B\boldsymbol{T}op_{q+k,k}$. To the embedding $M \subseteq V$, they associate the pair $\tau_M \oplus \nu_M \subseteq \tau_V|_M \oplus \nu_M$, which represents the "formal" normal bundle of M in V. Here, ν_M is a stable inverse to τ_M.

A closed generalized manifold X has a canonical (up to controlled homeomorphisms) stable normal spherical manifold approximate fibration structure on the Spivak normal fibration determined by neighborhoods of embeddings of X in large euclidean spaces; we shall denote it ν_X. The uniqueness of this stable structure follows from the relative end theorem and the thin h-cobordism theorem [8] applied to a concordance between any two embeddings of X in $\mathbb{R}^N$, N large. On the other hand, Ferry and Pedersen have shown that the Spivak normal fibration of an ENR homology manifold has a canonical *Top* reduction ν_X^{fp} [4]. Notice that, if the local index of $X \neq 1$, we cannot have a simultaneous geometric realization of both structures since the index is multiplicative. In other words, we cannot have a *Top*-bundle over X whose total space is a manifold. Our approach is to use both ν_X and ν_X^{fp} to reduce the study of q-neighborhoods of X to the study of q-neighborhoods of a (stable) neighborhood of X in euclidean space. The fact that there is a bijective correspondence between these is stated as Corollary 3.4.

We begin by defining $\gamma\colon \mathcal{N}_q(X) \to [X, B\boldsymbol{T}op_q]$, for any closed generalized manifold X. Let $X^n \subseteq V^{n+q}$ be a tame embedding, $q \geq 3$, and let $E = C_p$ be a mapping cylinder neighborhood of X with projection $p\colon E \to X$. Let W be the total space of $\xi = p^*(\nu_X^{fp})$. Since ν_X has the same controlled homotopy type as ν_X^{fp}, and $\xi|_X = \nu_X^{fp}$, the splitting theorem proved in the next section shows that ξ is controlled homeomorphic to an approximate fibration that restricts to ν_X over X. Moreover, the mapping cylinder N of the projection of ν_X is embedded in W as a locally flat submanifold.

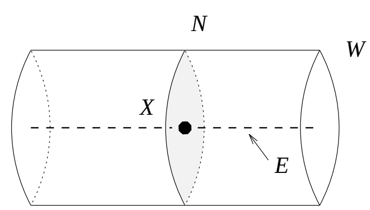

FIGURE 3.1

A relative version of this construction shows that any two such splittings are concordant. Therefore, the assignment $X \subseteq V \mapsto \tau_N|_X \oplus \nu_N|_X \subseteq \tau_W|_X \oplus \nu_N|_X$ induces a classifying map $\gamma\colon \mathcal{N}_q(X) \to [X, B\boldsymbol{T}op_q]$, which coincides with the Rourke-Sanderson map when X is a topological manifold.

Theorem 3.3. *Let X^n be a closed generalized manifold. The map $\gamma\colon \mathcal{N}_q(X) \to [X, B\boldsymbol{T}op_q]$ is a bijection, provided that $q \geq 3$ and $n+q \geq 5$.*

Using the same notation, define $\wp\colon \mathcal{N}_q(X) \to \mathcal{N}_q(N)$ by associating to $X \subseteq V$ the q-neighborhood $N \subseteq W$.

Corollary 3.4. *$\wp\colon \mathcal{N}_q(X) \to \mathcal{N}_q(N)$ is a bijection.*

Proof. Neighborhoods of N are classified by $\gamma_1\colon \mathcal{N}_q(N) \to [N, B\boldsymbol{T}op_q]$, where $\gamma_1([X \subseteq V])$ is represented by the microbundle pair $\tau_N \oplus \nu_N \subseteq \tau_W|_N \oplus \nu_N$ [10]. Therefore, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{N}_q(X) & \xrightarrow{\wp} & \mathcal{N}_q(N) \\
{\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_1} \\
[X, B\boldsymbol{T}op_q] & \xrightarrow{\cong} & [N, B\boldsymbol{T}op_q].
\end{array}$$

The result follows from Theorem 3.3. □

Proof of Theorem 3.3.

γ is injective. Let $p_i\colon \partial E_i \to X$, $i \in \{0,1\}$, be $(q-1)$-spherical manifold approximate fibrations such that $\gamma(X \subseteq E_0) = \gamma(X \subseteq E_1)$. Adding subscripts to the notation introduced above (see figure 3.1), Theorem 3.2b of [10] implies that W_0 and W_1 are equivalent neighborhoods of N, i.e., there is a homeomorphism $H\colon W_1 \to W_0$ inducing the identity on N.

Analogous to $B\boldsymbol{T}op_q$, there is a classifying space $B\boldsymbol{G}_q$ for pairs of spherical fibrations [10]. Since such pairs split uniquely, $B\boldsymbol{G}_q$ is homotopy equivalent to the classifying space for spherical fibrations BG_q. Since controlled homotopy classes of approximate fibrations are in one-to-one correspondence with fiber homotopy classes of spherical fibrations under the path fibration map [5], the fact that the image of $\gamma(X \subseteq E_0)$ and $\gamma(X \subseteq E_1)$ are the same under the forgetful map $B\boldsymbol{T}op_q \to B\boldsymbol{G}_q$ implies that H induces a controlled homotopy equivalence $f^c\colon \partial E_1 \to \partial E_0$ over X, which gives a homotopy equivalence $\tilde{f}\colon E_1 \to E_0$ of mapping cylinders. By Proposition 3.2, to establish the injectivity of γ, it suffices to show that the normal invariant of $\tilde{f}$ is trivial.

Since $\tilde{f}$ restricts to the identity on X, and W_i is the total space of the bundle over E_i obtained as the pull-back of ν_X^{fp} under the projection $p_i\colon E_i \to X$, there is a bundle map $F\colon W_1 \to W_0$

$$\begin{array}{ccc}
W_1 & \xrightarrow{F} & W_0 \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\
E_1 & \xrightarrow{\tilde{f}} & E_0.
\end{array}$$

covering $\tilde{f}\colon E_1 \to E_0$. Therefore, the normal invariants $\eta(F) \in [W_0, G/Top] \cong [X, G/Top]$ and $\eta(\tilde{f}) \in [E_0, G/Top] \cong [X, G/Top]$ are the same. Since F can be assumed to be homotopic to H as maps of pairs, it follows that $\eta(\tilde{f}) = \eta(F) = 0$.

γ is surjective. Let the microbundle pair $\epsilon^k \subseteq \zeta^{q+k}$ represent a given element $\alpha \in [X, B\boldsymbol{T}op_q]$. Let $p\colon \partial\mathcal{E} \to X$ be the $(q-1)$-spherical fibration underlying α, and let $\mathcal{E}$ be the mapping cylinder of p. Abusing notation, the natural projection $p\colon \mathcal{E} \to X$ gives $(\mathcal{E}, \partial\mathcal{E})$ the structure of an arbitrarily fine Poincaré space over X. Furthermore, as stable spherical fibrations

$$\nu_X^{fp} = \nu_{\mathcal{E}}^{sp}|_X \oplus \zeta, \tag{3.1}$$

where $\nu_{\mathcal{E}}^{sp}$ is the Spivak normal fibration of $\mathcal{E}$. Therefore, equation 3.1 determines a Top reduction of $\nu_{\mathcal{E}}^{sp}|_X$. Since $\mathcal{E}$ deformation retracts to X, we also obtain a Top

reduction of $\nu^{sp}_{\mathcal{E}}$. Let

$$\begin{array}{ccc} (M,\partial M) & \xrightarrow{\phi} & (\mathcal{E},\partial\mathcal{E}) \\ & & \big\downarrow{\scriptstyle p} \\ & & X \end{array}$$

be a surgery problem associated to this reduction. Crossing with $\mathbb{R}$, we obtain a bounded surgery problem

$$\begin{array}{ccc} (\tilde{M},\partial\tilde{M}) & \xrightarrow{\tilde{\phi}} & (\mathcal{E}\times\mathbb{R},\partial\mathcal{E}\times\mathbb{R}) \\ & & \big\downarrow{\scriptstyle p'} \\ & & O(X^+) \end{array}$$

where $X^+ = X \amalg \{a\}$ is a disjoint union, and in "polar coordinates" in $O(X^+)$,

$$p'(x,t) = \begin{cases} (x,t), & \text{if } t > 0; \\ (a,|t|), & \text{if } t \leq 0. \end{cases}$$

By the bounded $\pi-\pi$ theorem [4], we can assume that $\tilde{\phi}$ is a bounded homotopy equivalence. Now, we split this equivalence near ∞ to obtain a manifold approximate fibration over X. Since $\partial\tilde{M}$ has a tame end (near $+\infty$) with respect to the composition

$$\partial\tilde{M} \xrightarrow{\tilde{\phi}} \partial\mathcal{E}\times\mathbb{R} \xrightarrow{\text{proj}} \partial\mathcal{E} \xrightarrow{p} X,$$

we can assume by the end theorem that in a neighborhood of the end, $\partial\tilde{M} = \partial\tilde{E}_1 \times [0,1)$. For each $0 < t < 1$, let $p_t\colon \partial E_1 \to X$ be the composition

$$\partial E_1 \times \{t\} \xrightarrow{\tilde{\phi}} \partial\mathcal{E} \xrightarrow{\text{proj}} \mathcal{E} \xrightarrow{p} X.$$

Then, $p_1 = \lim\limits_{t\to 1} p_t\colon \partial E_1 \to X$ is a manifold approximate fibration such that the spherical fibrations underlying $\gamma(X \subseteq E_1)$ and α are the same.

The stability theorem for $\boldsymbol{G}_q/\boldsymbol{T}op_q$ [10], $q \geq 3$, gives a pull-back diagram

$$\begin{array}{ccc} B\boldsymbol{T}op_q & \longrightarrow & BTop \\ \big\downarrow & & \big\downarrow \\ B\boldsymbol{G}_q & \longrightarrow & BG. \end{array}$$

Hence, the difference between α and $\gamma(X \subseteq E_1)$ is stable, and defines an element $\beta \in [X, G/Top] \cong H_n(X; G/Top)$. Proposition 3.2 applied to $p_1\colon E_1 \to X$ gives a manifold approximate fibration $\partial E \to X$ and a controlled equivalence $\psi^c\colon \partial E \to \partial E_1$ such that $\rho([\psi^c]) = \beta$. Then, $\gamma(X \subseteq E) = \alpha$. This concludes the proof. $\square$

Corollary 3.5. *Let X^n be a closed generalized manifold. If $q \geq 3$ and $n+q \geq 5$, $(q-1)$-spherical manifold approximate fibrations over X are classified by $B\boldsymbol{T}op_q$, i.e., there is a one-to-one correspondence between controlled homeomorphism classes of $(q-1)$-spherical manifold approximate fibrations over X and $[X, B\boldsymbol{T}op_q]$.*

Since $B\boldsymbol{G}_q \to BG_q$ is a homotopy equivalence and $\boldsymbol{G}_q/\boldsymbol{T}op_q$ is stable when $q \geq 3$, the classification of spherical manifold approximate fibration structures obtained in Proposition 3.2 can be rephrased in terms of reductions of structural groups, as follows.

Corollary 3.6. *Let X be a closed generalized n-manifold, and let ξ be a $(q-1)$-spherical fibration over X. If $q \geq 3$ and $n+q \geq 5$, then manifold approximate fibrations over X fiber homotopy equivalent to ξ are in 1–1 correspondence with fiber homotopy classes of lifts to $B\boldsymbol{T}op_q$ of the map $X \to BG_q$ that classifies ξ.*

Remark. The classification of manifold approximate fibrations with spherical fibers obtained in Corollary 3.5, the H-space structure on $BTop$ induced by Whitney sums of bundles, and the stability theorem for $\boldsymbol{G}_q/\boldsymbol{T}op_q$, for $q \geq 3$, suggest possible definitions of *Whitney sums* and *pull-backs* of spherical manifold approximate fibrations over ENR homology manifolds. However, such definitions do not seem to be entirely satisfactory. For example, one would not obtain product formulae for characteristic classes, since 0-dimensional classes (like the local index of a homology manifold) are not visible to the operation in $BTop$ arising from Whitney sums of bundles. In order to account for these, it appears to be necessary to address the more general classification problem of tame neighborhoods of generalized manifolds in generalized manifolds with the disjoint disks property. We conjecture that these q-neighborhoods are classified by $B\boldsymbol{T}op_q \times \mathbb{Z}$. From the viewpoint of the techniques utilized in this paper, the main obstacles to completing such a study are the validity of the s-cobordism theorem and of the simply-connected end theorem for generalized manifolds with the disjoint disks property.

4. A splitting theorem

Let E^m be a compact topological m-manifold, and $p\colon M^{m+r} \to E$ be a manifold approximate fibration with homotopy fiber F, where F is a closed r-manifold. For the duration of this section we adopt the following notation: if $A \subset E$, then $\hat{A} = p^{-1}(A) \subset M$.

Let $X^n \subseteq E^m$ be a closed ENR homology manifold tamely embedded in E, and let $q\colon \hat{X} \to X$ denote the restriction of p to $\hat{X}$. In this generality, $\hat{X}$ is not necessarily an ANR, and q may not be an approximate fibration.

Definition 4.1. $p\colon M \to E$ is *split* along X, if $\hat{X}$ is a closed $(n+r)$-manifold tamely embedded in M, and $q = p|_{\hat{X}}\colon \hat{X} \to X$ is an approximate fibration.

Suppose $q_1\colon N \to X$ is an approximate fibration with homotopy fiber F, where N is a closed $(n+r)$-manifold

Theorem 4.2. *If $q_1\colon N \to X$ is fiberwise shape equivalent to $q\colon \hat{X} \to X$ over X, $m-n \geq 3$, $r \geq 3$, $n+r \geq 5$, and $\pi_1(F)$ is K-flat, then $p\colon M \to E$ is controlled homeomorphic to an approximate fibration $p_1\colon M \to E$, which is split along X and restricts to q_1 over X.*

Proof. Let V be a mapping cylinder neighborhood of X in E with mapping cylinder projection $\gamma\colon V \to X$. Then $\operatorname{int}\hat{V}$ has a tame end over ∂V (the control map being the composition of p with projection on a collar of ∂V to ∂V). Since $\pi_1(F) = 0$, $\hat{V}$ has a controlled collar at ∞ over ∂V [8]. Let U be a compact manifold in $\operatorname{int}\hat{V}$ containing $\hat{X}$ obtained by removing a small open collar from the end of $\hat{V}$. Then the inclusion $\hat{X} \subset U$ is a shape equivalence. The map $\gamma \circ p \colon U \setminus \hat{X} \to X$ also has a tame end, hence a controlled collar at ∞. Thus U has a "mapping-cylinder-like" structure over $\hat{X}$, controlled over X, in the sense that small neighborhoods of $\hat{X}$ in U can be isotopied arbitrarily close to $\hat{X}$ by isotopies that fix $\hat{X}$ and are controlled over X. (That is, $\hat{X}$ is a *tame FANR* in U over X.)

Since the manifold approximate fibration $q_1 \colon N \to X$ is fiberwise shape equivalent to $p \colon \hat{X} \to X$, there is a controlled homotopy equivalence $G^c \colon N \to U$ over X. Let $G \colon N \times [0,1) \to U \times [0,1)$ represent G^c. As in section 2, under an appropriate radial reparametrization, we may assume that G is a bounded homotopy equivalence over $O(X)$, the open cone on X. By the bounded analogue of the Casson-Haefliger-Sullivan-Wall embedding theorem [11], after a bounded homotopy over $O(X)$, we can assume that G is an embedding.

Pushing the image of G toward $\hat{X}$ using the mapping-cylinder-like structure on U, we may also assume that $G(N \times \{t\}) \subset \hat{V}_t$, where $V_t \subset V$ is the part of the mapping cylinder having mapping cylinder parameter $\geq t$. Set $Z = G(N \times [0,1) \cup \hat{X} \times \{1\}) \subset U \times [0,1]$. Then $U \times [0,1] \setminus Z$ has a tame end over X, hence, a controlled collar over X. The closed region between a boundary of the end and $\partial U \times [0,1]$ is a thin h-cobordism of triples (over X), hence, a product. Thus, there is a controlled homeomorphism $\lambda \colon \partial U \times ([0,1], 0, 1) \times [0,1) \to U \times ([0,1], 0, 1) \setminus Z$ over X, with $\lambda | \partial U \times [0,1] \times [0,\delta] = \mathrm{id}$ for some $\delta > 0$.

It is not difficult to show that $\lambda(\partial U \times [0,1] \times \{t\})$ is an $\epsilon(t)$-thin h-cobordism over X, where $\epsilon(t) \to 0$ as $t \to 1$. Thus, an infinite sequence of applications of the thin h-cobordism theorem yields a homeomorphism $h \colon (U \setminus \hat{X}) \times [0,1] \to (U \times [0,1] \setminus Z)$, controlled over X. The composition $p \circ \mathrm{proj} \circ h^{-1} \colon (U \times [0,1] \setminus Z) \to V \setminus X$ extends to a map $H \colon U \times [0,1] \to V$. Since h is the identity in a neighborhood of ∂U, H can be used to deform p to an approximate fibration p_1 that agrees with p outside U and with q_1 on $N = G(N \times \{0\})$, and is controlled homeomorphic to p. ☐

5. Taming Poincaré embeddings

Let X^n be a closed generalized manifold and V^{n+q} a compact topological manifold. Following [1] (see also [7, 11]), we define a *Poincaré embedding* of X in V to be a triple $(\xi, (C, \partial \mathcal{E}), h)$ consisting of

(i) a $(q-1)$-spherical fibration ξ over X, with projection $p \colon \partial \mathcal{E} \to X$;
(ii) a finite Poincaré pair $(C, \partial \mathcal{E})$;
(iii) a (simple) homotopy equivalence $h \colon C \cup \mathcal{E} \to V$, where $\mathcal{E}$ is the mapping cylinder of p, and $C \cap \mathcal{E} = \partial \mathcal{E}$.

Remark.

1. If X is tamely embedded in V, let the spherical manifold approximate fibration $p \colon \partial E \to X$ represent the normal structure to X. Under the path fibration construction, this approximate fibration determines a spherical fibration over X which is controlled homotopy equivalent to $p \colon \partial E \to X$. Thus, underlying any tame embedding, there is a Poincaré embedding of X in V.
2. Lemma 11.1 of [11] shows that (ii) follows from (iii), when $q \geq 3$.

As in smoothing theory, we first consider reductions of the structural group of $p \colon \mathcal{E} \to X$ to $\boldsymbol{Top}_q$. Given a Poincaré embedding $h \colon C \underset{\partial \mathcal{E}}{\cup} \mathcal{E} \to V$, let f^s be the composition

$$X \xrightarrow{i} \mathcal{E} \subseteq C \underset{\partial\mathcal{E}}{\cup} \mathcal{E} \xrightarrow{h} V \xrightarrow{i} V \times \mathbb{R}^k,$$

k large. By general position, we can assume that f^s is a tame embedding. Since any other embedding homotopic to f^s is concordant to f^s, the stable controlled homeomorphism type of the normal spherical manifold approximate fibration is well-defined. Thus, associated to a Poincaré embedding $h\colon C \underset{\partial\mathcal{E}}{\cup} \mathcal{E} \to V$, there is a natural stable *Top*-reduction of $\partial\mathcal{E} \to X$. The stability theorem [10] for $\boldsymbol{G}_q/\boldsymbol{Top}_q$, $q \geq 3$, implies the same unstably, that is, associated to a Poincaré embedding, there is a canonical $\boldsymbol{Top}_q$-reduction of $\partial\mathcal{E} \to X$.

We now state the extension of Browder's *"Top-Hat"* theorem to embeddings of ENR homology manifolds into topological manifolds (see also [6]). For smooth or *PL* manifolds, a proof of this result is given in [11].

Theorem 5.1. *Let $(\xi, (C, \mathcal{E}), h)$ be a Poincaré embedding of a closed generalized manifold X^n into a compact topological manifold V^{n+q}, with $q \geq 3$ and $n + q \geq 5$. Then, there is a tame embedding of X in V inducing the given Poincaré embedding.*

Proof. The proof follows much the same line as the proof of Theorem 11.3 of [11]. Let $\phi^c\colon \partial E \to \partial\mathcal{E}$ represent the canonical Top_q-reduction of $\partial\mathcal{E} \to X$, where $\partial E \to X$ is a $(q-1)$-spherical manifold approximate fibration. Since ϕ^c induces a simple homotopy equivalence $\tilde{\phi}\colon E \to \mathcal{E}$, we may assume that $\mathcal{E} = E$ and $h\colon C \cup_{\partial E} E \to V$.

Let $g\colon V \to C \cup_{\partial E} E$ be a homotopy inverse to h. After removing a small open collar of ∂E from E, we can assume that g is transverse to ∂E. Let $A = g^{-1}(E)$. Then $g|A\colon (A, \partial A) \to (E, \partial E)$ is a degree 1 normal with normal invariant $\eta \in [E, G/Top] \cong [X, G/Top] \cong H_n(X; G/Top)$. Proposition 3.2 implies that there is a manifold approximate fibration $p_1\colon \partial E_1 \to X$ and a controlled homotopy equivalence $\psi^c\colon \partial E_1 \to \partial E$ such that $\eta(\tilde{\psi}) = \eta$. Hence, there is a normal bordism $F_1\colon (U, U_0) \to (E, \partial E)$ between $g\colon (A, \partial A) \to (E, \partial E)$ and $\tilde{\psi}\colon (E_1, \partial E_1) \to (E, \partial E)$. Identify $A \subset U$ with $A \times \{1\} \subset V \times \{1\} \subset V \times I$ to obtain $W = (V \times I) \cup_A U$. There is a degree-one normal map $F\colon (W, V \times \{0\}, \partial_+ W) \to (C \cup E \times I, C \cup E \times \{0\}, C \cup E \times \{1\})$ inducing h on $V \times \{0\}$ and $\tilde{\psi}$ on $E_1 \subset \partial_+ W$. By the $\pi - \pi$ theorem we can do surgery on W rel $V \times \{0\} \cup E_1$ to get an s-cobordism W' between $V \times \{0\}$ and V', with $E_2 \subset V'$. By the s-cobordism theorem $W' \cong V \times I$; hence, we get an embedding $f\colon X \subset E_1 \hookrightarrow V$ realizing the given Poincaré embedding. Notice that since $\partial E_1 \to X$ arises as the normal structure to X under the embedding f, $\partial E_1 \to X$ is controlled homeomorphic to $\partial E \to X$. □

Corollary 5.2. *Suppose X^n is a closed generalized n-manifold, V^{n+q} is a compact topological $(n+q)$-manifold, $(n+q) \geq 5$, $q \geq 3$, and $f\colon X \to V$ is a homotopy equivalence. Then f is homotopic to a tame embedding.*

Proof. Identical to the proof of Corollary 11.3.4 of [11]. □

Corollary 5.3. *Suppose that X is a closed generalized n-manifold, $n \geq 5$. Then there is a tame embedding of X into a topological manifold of dimension $n + 3$.*

Proof. By [4], the Spivak normal fibration of X admits a *Top*-reduction, which gives a degree-one normal map $f\colon M \to X$, where M is a topological n-manifold. By the $\pi - \pi$ theorem, we can do surgery on $f \times \mathrm{id}\colon M \times B^3 \to X \times B^3$, to get a (simple) homotopy equivalence $F\colon (V,\partial V) \to (X\times B^3, X\times S^2)$. Apply 5.2 to a homotopy inverse of F restricted to X. □

In contrast to 5.3 we have the following well-known fact.

Proposition 5.4. *If X is a closed generalized n-manifold and $\iota(X) \neq 1$, then there is no compact topological manifold $(V,\partial V)$ controlled homotopy equivalent to $(X\times B^2, X\times S^1)$ over X.*

Proof. If there were, the infinite cyclic cover U of ∂V corresponding to the $\mathbb{Z}$-factor of $\pi_1(\partial V)$ would have a tame end over X. The end theorem would then produce a completion $\overline{U}$ of the end over X, hence, a cell-like map $\partial\overline{U} \to X$, that is, a resolution of X. But this would imply that $\iota(X) = 1$ [9]. □

References

[1] W. Browder, *Embedding smooth manifolds*, Proc. Internat. Cong. Mathematicians, Moscow (1968), 712–719.

[2] J. Bryant, S. Ferry, W. Mio and S. Weinberger, *Topology of homology manifolds*, Bull. Amer. Math. Soc. **28** (1993), 324–328.

[3] ———, *Topology of homology manifolds*, Ann. of Math. **143** (1996), 435–467.

[4] S. C. Ferry and E. K. Pedersen, *Epsilon surgery theory*, in Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), 167–226, London Math. Soc. Lecture Ser., 227, Cambridge University Press, Cambridge, 1995.

[5] C. B. Hughes, L. R. Taylor and E. B. Williams, *Bundle theories for topological manifolds*, Trans. Amer. Math. Soc. **319** (1990), 1–65.

[6] H. Johnston, *Transversality for non-manifolds*, Thesis, The University of Chicago, 1996.

[7] N. Levitt, *On the structure of Poincaré duality spaces*, Topology **7** (1968), 369–388.

[8] F. Quinn, *Ends of maps I*, Ann. of Math. **110** (1979), 275–331.

[9] ———, *An obstruction to the resolution of homology manifolds*, Michigan Math. J. **301** (1987), 285–292.

[10] C. P. Rourke and B. J. Sanderson, *On topological neighborhoods*, Compositio Math. **22** (1970), 387–424.

[11] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, 1970.

[12] J. E. West, *Mapping Hilbert cube manifolds to ANR's: a solution to a conjecture of Borsuk*, Ann. of Math. **106** (1977), 1–18.

Department of Mathematics, Florida State University, Tallahassee, FL 32306

E-mail address, J. L. Bryant: `bryant@math.fsu.edu`

E-mail address, W. Mio: `mio@math.fsu.edu`