Mathematisches Forschungsinstitut Oberwolfach

Report No. 29/2003

Mini-Workshop: Exotic Homology Manifolds

June 29th – July 5th, 2003

Homology manifolds were developed in the first half of the 20th century to give a precise setting for Poincaré's ideas on duality. Major results in the second half of the century came from two different areas. Methods from the point-set tradition were used to study homology manifolds obtained by dividing genuine manifolds by families of contractible subsets. "Exotic" homology manifolds are ones that cannot be obtained in this way, and these have been investigated using algebraic and geometric methods.

The Mini-Workshop brought together experts from both point-set and algebraic areas, along with new Ph.D.'s and experts in related areas. This was the first time this was done in a meeting focused only on homology manifolds. The 17 participants had 14 formal lectures and a problem session. There was a particular focus on the proof, 10 years ago, of the existence of exotic homology manifolds. This gave experts in each area an the opportunity to learn more about details coming from the other area. There had also been concerns about the correctness of one of the lemmas, and this was discussed in detail. One of the high points of the conference was the discovery of a short and beautiful new proof of this lemma. Extensive discussions of examples and problems have undoubtedly helped prepare for future progress in the field.

The organizers plan to publish a proceedings for the meeting, including an article on the history of the subject and a problem list.

There was also a wonderful interaction with the Mini-Workshop "Henri Poincaré and topology" which was held in the same week. There was a joint discussion on the early history of manifolds, and both groups offered evening lectures on topics of interest to the other. Several of the daytime history lectures also drew large numbers of homology manifold participants.

Abstracts

DDP Resolutions of *ANR* Homology Manifolds

John L. Bryant
(joint work with S. Ferry, W. Mio, and S. Weinberger)

We outline a proof of the following:

Theorem. If X is a compact ANR homology n-manifold, $n \geqslant 6$, there is a cell-like map $\phi\colon Y \to X$, where Y is an ANR homology n-manifold with the disjoint disks property. The space Y is the inverse limit of a sequence of patch space approximations $X_1, X_2, \ldots$ to X obtained by deforming the natural homotopy equivalence from each X_{i+1} to X_i (through X) first to a finer equivalence over X_{i-1}, using controlled surgery, and then to a UV^1 map. Iterating this construction gives a sequence of maps of Y to X that converge to a cell-like map.

Designer homology manifolds

Robert J. Daverman

The talk reviewed techniques, developed in collaborations with J. W. Cannon and J. J. Walsh, for producing cell-like maps from a given n-manifold onto a homology n-manifold X with various properties. Particular applications were outlined, including the controls (from the collaboration with Cannon) leading to a cell-like map from the n-sphere onto such an X, where no point of X has simply connected complement yet the product of X with the reals is a manifold. Finally, also discussed were more general methods (from the collaboration with Walsh) leading to another cell-like map from the n-sphere onto another homology n-manifold X containing no embedded 2-disks.

Infinite dimensional homology manifolds

Alexander N. Dranishnikov

Infinite dimensional homology manifolds exist in dimension 5 and higher. All known infinite dimensional homology manifolds are constructed as cell-like images of manifolds. A map $f : X \to Y$ is called *cell-like* if all point pre-images $f^{-1}(x)$ are Čech contractible. Every cell-like map of a closed manifold produce a homology manifold. If it raises the dimension then it produces an infinite dimensional homology manifold. Dimension raising cell-like maps were constructed by means of Edwards Resolution Theorem of infinite dimensional compacta with finite cohomological dimension. Such compacta were constructed first by the author for cohomological dimension 3 by means of K-theory. Using the Sullivan Conjecture (= Miller's theorem) Dydak and Walsh constructed such compacta with cohomological dimension 2. Kozlowsky and Walsh proved that cell-like maps of 3-dimensional manifold cannot raise the dimension. Whether there are dimension raising cell-like maps of 4-manifolds is an open question.

Every finite n-dimensional homology manifold admits a degree one map onto S^n. Infinite dimensional homology manifolds in dimension n constructed by the above procedure have nowhere dense infinite dimensional singular sets. Therefore they also admit a degree one map onto the n-sphere. We prove the following

THEOREM 1. *There are infinite dimensional homology manifolds in dimension 7 that do not admit a map of degree one onto* S^7.

This manifold is constructed as the image of a cell-like map $p : S^7 \to X$ that has a nontrivial kernel for complex K-homology with Z/p coefficients. This homology manifold was used by Ferry, Weinberger and the author do disprove some coarse conjectures due to Gromov and Roe motivated by the Novikov Higher Signature Conjecture. Using similar construction Steve Ferry and the author constructed examples of topologically different closed manifolds M_1 and M_2 and cell-like maps $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ with the same image. Thus, in the contrast to the case of finite dimensional homology manifolds (Quinn's Uniqueness of Resolution Theorem) in the realm of infinite dimensional homology manifold there is no uniqueness of resolution.

Constructing compact spaces of finite integral dimension and infinite covering dimension

Jerzy Dydak

The talk reviewed two constructions:

(a) (A.Dranishnikov) X such that $\dim_{\mathbb{Z}}(X) = 3$ and $\dim(X) = \infty$.

(b) (J.Dydak and J.Walsh) X such that $\dim_{\mathbb{Z}}(X) = 2$ and $\dim(X) = \infty$.

The main application of those two constructions for our conference is in creating an exotic homology 5-manifold of infinite covering dimension. Construction (a) uses existence of a generalized homology vanishing on $K(\mathbb{Z}, 3)$. Construction (b) uses a truncated cohomology represented by loop spaces of S^3. The Sullivan Conjecture (a theorem of H.Miller) shows that this particular cohomology vanishes on $K(\mathbb{Z}, 2)$.

The Approximation Theorem

Robert D. Edwards

The talk discussed various issues related to the following "classical" result. It is stated here for the empty-boundary case.

Theorem (rde, 1977) Suppose that $f\colon M \to X$ is a cell-like surjection (:= each $f^{-1}(x)$ is a cell-like compactum) from a topological manifold-without-boundary M to a finite dimensional space X, and suppose that $\dim M \geqslant 5$. Then f is ABH (:= approximable by homeomorphisms) $\iff$ X has the DDP (= disjoint discs property := any two maps from a 2-disc to X are arbitrarily closely approximable by maps having disjoint images).

It is known that any finite dimensional image of a closed manifold under a cell-like-map, such as X above, must in fact be an ENR homology manifold having the same dimension as M.

Bizarre spaces whose product with a line is a manifold

Denise M. Halverson

The disjoint homotopies property, a general position property first proposed by Bob Edwards, has proven highly effective in detecting non-manifold spaces of dimension $n \geqslant 4$ whose product with a line is a manifold. Such spaces are called *codimension one manifold factors*. A space X has the *disjoint homotopies property* (DHP) if for any pair of path homotopies $f, g : D \times I \to X$ where $D = I = [0, 1]$, there are approximations f', g' :

$D \times I \to X$ such that $f'(D \times \{t\}) \cap g'(D \times \{t\}) = \emptyset$ for all $t \in I$. If X is an ANR with DHP, then $X \times \mathbf{R}$ has the disjoint disks property (DDP). Hence resolvable generalized n-manifolds with DHP, are codimension one manifold factors.

Spaces that have DHP include resolvable generalized n-manifolds, $n \geqslant 4$, that have the plentiful 2-manifolds property. A space X has the *plentiful 2-manifolds property* ($P2MP$) if any path $\alpha : D \to X$ can be approximated by a path $\alpha' : D \to N \subset X$ where N is a 2-manifold embedded in X. Examples of spaces that have $P2MP$ include resolvable generalized manifolds of dimension $n \geqslant 4$ that arise from a nested defining sequence and carefully constructed k-ghastly spaces for $2 < k < n$. A space is said to be *k-ghastly* if it contains no embedded k-cells but does contain embedded $(k-1)$-cells. Hence these spaces are codimension one manifold factors.

Recently it has also been shown that there are also 2-ghastly spaces that have DHP. The 2-ghastly spaces clearly do not satisfy $P2MP$. It has been shown previously that such spaces can be constructed to be codimension one manifold factors. Thus, the fact that 2-ghastly spaces have DHP demonstrates that DHP is a fairly effective general position property in characterizing codimension one manifold factors of dimension $n \geqslant 4$.

The B.F.M.W.-construction of generalized manifolds

Friedrich Hegenbarth

The lecture reviewed the systematic construction of generalized manifolds of dimension greater or equal to six given by J.Bryant, S.Ferry, M.Mio and S.Weinberger (Ann. of Math. 143 (1996), 435-476). The generalized manifold to be constructed can be considered as a limit of well-controlled Poincaré complexes and maps. It begins with a topological closed n-manifold M together with an element u of the nth L-homology group of M. Depending on the element u the resulting generalized manifold X can be homotopy equivalent to M or of general type. If the fundamental group of M is such that the assembly map is injective one obtains generalized manifolds which do not have the homotopy type of any topological manifold. The main ingredient to construct the controlled sequence of Poincaré complexes is the controlled surgery sequence established by E.Pedersen, F.Quinn and A.Ranicki (see "Controlled surgery with trivial local fundamental groups", available under arXiv:math.GT/0111269v1). The construction consists in plugging into the manifold M a well-controlled realization V of the element u to obtain a complex X' with a controlled Poincaré structure. This construction is repeated infinitely many times with better and better control. The resulting limit space is a generalized n-manifold. To show this one proves that X is an ANR and that the boundary of a regular neighbourhood is an approximate fibration (see R.J.Daverman, L.Husch: Decompositions and approximate fibrations, Mich. Math. J. 31 (1984), 197-214). Then Quinn's resolution obstruction $i(X)$ (see F.Quinn: An obstruction to the resolution of homology manifolds, Mich. Math. J. 34 (1987), 285-292) depends on the choice of u so it can be chosen to be $1 + 8k$, k arbitrary. Hence this construction produces generalized manifolds which cannot be homeomorphic to topological manifolds.

Transversality Obstructions

Heather M. Johnston

The multiplicativity of the Quinn Index implies that transversality for homology manifolds fails in general. Let $\mathbb{R}^k_i$ denote a contractible homology manifold of Quinn Index i, and X_j a homology manifold of Quinn Index j. If $f : X \to \mathbb{R}^k_i$ is transverse to a point x, then $I(f^{-1}(x) \times \mathbb{R}^n_i) = I(X_j)$ implies $i|j$. The surgery exact sequence for homology manifolds of Bryant, Ferry, Mio and Weinberger can be used to obtain transversality for homology manifolds when possible. In joint work with Andrew Ranicki, we show that there are no obstructions to transversality for $f : X^n \to (M, N)$ where X^n is a homology manifold and (M, N) is a manifold pair such that N has locally trivial normal bundle neighbourhood $\nu(N) \subset M$ and $n - \mathrm{codim}(M, N) \geqslant 7$, i.e., any such f is s-cobordant to a map $g : X' \to M$ such that $g^{-1}(N) = Z$ is a homology manifold and $g|g^{-1}(\nu(N)) : \nu(Z) \to \nu(N)$ is a bundle map.

Given $f : X^n \to \mathbb{R}^k_i$, if $p = n - k \geqslant 7$, then there is no obstruction to Poincaré transversality. If f is assumed to be Poincaré transverse to a point x, i.e. $f = g \circ h$ for a Poincaré space P^p, a homotopy equivalence $h : X \to (P \times D^k_i) \cup V$ and $g : (P \times D^k_i) \cup V \to \mathbb{R}^n_i$ such that $g|P \times D^k_i$ is projection followed by inclusion, then there is an obstruction $\sigma(f) \in H_p(P; \mathbb{L}/i\mathbb{L})$ which vanishes iff f is ϵ-s-cobordant to a transverse map for any $\epsilon < \epsilon_0$ where ϵ_0 depends on h.

UV^k Maps on Homology Manifolds with the Disjoint Disks Property

Washington Mio

The study of control improvement for maps defined on homology manifolds with the disjoint disks property (DDP) is essential in the investigation of the (local) geometric topology of these spaces. In this talk, we outline a proof of the following result. Let B be a compact metric ANR and $n \geqslant 6$. Given $\epsilon > 0$, there is $\delta > 0$ such that if X^n is a compact ENR homology n-manifold with the DDP and $f \colon X \to B$ is a δ-UV^k map, then f is ϵ-homotopic to a UV^k map, provided that $0 < 2k + 3 \leqslant n$.

Squeezing

Erik Kjær Pedersen

Let K be a finite complex embedded in S^n for some large n. Choose a disjoint base point $+$ in S^n. The open cone $O(K_+)$ is the set $\{t \cdot x \in \mathbb{R}^{n+1} | x \in K_+\}$.

The bounded category parametrized by a metric space M with coefficients in a ring R, denoted $\mathcal{C}_M(R)$, has objects based R-modules with a proper map from the basis to M. Morphisms are required to be bounded in the sense that a morphism applied to a basis elements only involves basis elements within a k-ball. Here k only depends on the morphism, not the given basis element. The boundedness condition is only relevant when M is non-compact, but we always obtain a filtered category this way.

We show that $K_2(\mathcal{C}_{O(K_+)}(R))$ codifies a sequence of ever smaller automorphisms in $\mathcal{C}_K(R)$, thus providing a categorical basis for squeezing. In joint work with Quinn and Ranicki we obtain a similar result in L-theory when $R = \mathbb{Z}$ using the principle that "splitting implies squeezing". The restriction $R = \mathbb{Z}$ has recently been removed in joint work with M. Yamasaki.

History of manifolds and homology manifolds

Frank S. Quinn

This lecture described the main developments in the theory, from Poincaré to the present day.

The early period was from the initiation by Poincaré in 1895 to 1932 when the modern definition using coordinate charts was formulated by Whitehead and Veblen. This period was better described in the "Poincaré and topology" workshop lecture of E. Scholz.

Analysis of the duality structure was developed by Lefschetz, Alexander, Čech in the 30s and 40s, and brought to essential completion by Wilder in the 40s and early 50s, though an important simplification was contributed by Bredon in the early 60s. After this the main stream in the study of manifolds concerned properties of smooth manifolds, as exemplified by the work of Whitney.

Study of wild and point-set properties of manifolds continued in the 1950s-70s primarily in the "Bing school", focused on cell-like quotients of manifolds. Contributors included Bing, Bryant, Lacher, Cannon, Seebeck, Daverman, Walsh, and Edwards. There were also important contributions by Chernavski and Homma. A culmination was reached in Edwards' theorem on homeomorphism approximation of manifold resolutions of ANR homology manifolds with the "disjoint disks property" (DDP).

The next major ingredient was Quinn's use, in the early 1980s, of surgery to construct resolutions. One outcome was a characterization of topological manifolds in terms of homology manifolds, the resolution obstruction, and the DDP. The existence of nonresolvable homology manifolds was shown in the early 1990s by Bryant, Ferry, Mio, and Weinberger. This proof and its elaborations was a major topic at the Workshop.

The resolution obstruction

Andrew A. Ranicki

The algebraic theory of bounded surgery allows the Quinn obstruction to the resolution of a compact connected n-dimensional ANR homology manifold X

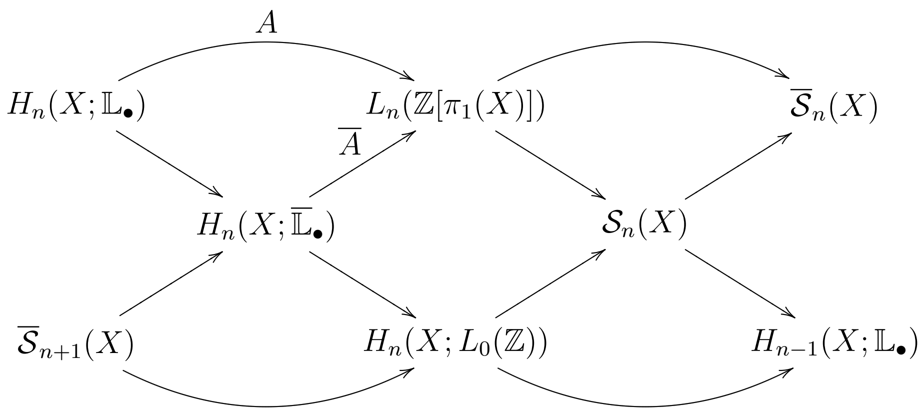
$$i(X) \in L_0(\mathbb{Z}) \;=\; \mathbb{Z}$$

to be interpreted as the $\mathbb{R}^n$-bounded surgery obstruction

$$i(X) \;=\; \sigma_*^b(f) \in L_n(\mathbb{C}_{\mathbb{R}^n}(\mathbb{Z})) \;=\; L_0(\mathbb{Z})$$

of an $\mathbb{R}^n$-bounded proper degree 1 normal map $f : X_1 \to \mathbb{R}^n$ on an open neighbourhood $X_1 \subset X$ of a point $x \in X$. See the book "Lower K- and L-theory" (Cambridge, 1992) for an exposition of bounded L-theory.

For any space X there is defined a commutative braid of exact sequences



with $\mathbb{L}_\bullet$ (resp. $\overline{\mathbb{L}}_\bullet$) the 1- (resp. 0-) connective quadratic $\mathbb{L}$-spectrum of $\mathbb{Z}$ such that

$$\mathbb{L}_0 \;\simeq\; G/TOP \text{ (resp. } \overline{\mathbb{L}}_0 \;\simeq\; L_0(\mathbb{Z}) \times G/TOP)$$

with A (resp. $\overline{A}$) the assembly map to the Wall surgery obstruction group of $\mathbb{Z}[\pi_1(X)]$. For a finite n-dimensional Poincaré complex X and $n \geqslant 5$ (resp. 6) the total surgery obstruction $s(X) \in \mathcal{S}_n(X)$ (resp. the 4-periodic version $\overline{s}(X) = [s(X)] \in \overline{\mathcal{S}}_n(X)$) is such that $s(X) = 0$ (resp. $\overline{s}(X) = 0$) if and only if X is homotopy equivalent to a compact n-dimensional topological manifold (resp. ANR homology manifold) M. For any such M $\mathcal{S}_{n+1}(M) = \mathcal{S}^{TOP}(M)$ (resp. $\overline{\mathcal{S}}_{n+1}(M) = \mathcal{S}^H(M)$) is the topological (resp. homology) manifold structure set. The total surgery obstruction of a homology manifold M is the image of the resolution obstruction

$$s(M) \;=\; [i(M)] \in \mathrm{im}(H_n(M; L_0(\mathbb{Z})) \to \mathcal{S}_n(M)) \;=\; \ker(\mathcal{S}_n(M) \to \overline{\mathcal{S}}_n(M)) \;.$$

See the book "Algebraic L-theory and topological manifolds" (Cambridge, 1992) for the exposition of the total surgery obstructions.

Homology 3-manifolds

DUSAN REPOVŠ

Dimension 3 is in many respects peculiar for generalized manifolds:

(1) This is the lowest dimension when genuine singularities appear;
(2) Unlike higher dimensions, generalized 3-manifolds cannot have "cone" singularities; and
(3) The unresolved status of the Poincaré conjecture represents a significant obstruction to recognizing topological 3-manifolds.

In this talk we centred the discussion on the history of the Recognition problem for 3-manifolds. According to Cannon's program, one should first address the Resolution problem for generalized 3-manifolds - given a generalized 3-manifold, blow it up via a cell-like map into a genuine 3-manifold. Initially, we looked at examples, of various degrees of sophistication, which can be constructed if fake cubes exist (Wilder, Brin, Brin-McMillan, Jakobsche, and Jakobsche-Repovš). Modulo the Poincaré conjecture, there are now several partial results concerning resolvability of generalized 3-manifolds X, with various conditions on X, most of the time on the dimension of the singular set $S(X)$ (Bryant-Lacher, Lacher-Repovš, Brin, Brin-McMillan, Thickstun, and Daverman-Thickstun). Moving on to the second (and final) step of Cannon's program, the General position problem - find simple "homotopical" properties which detect manifolds, we studied various appropriate 3-dimensional "analogues" of Cannon's Disjoint disks property (DDP) which - as Edwards has shown so dramatically, works perfectly in higher dimensions: we proposed the Dehn lemma property (DLP), the Map separation property (MSP), the Light map separation property ($LMSP$), the Spherical simplicial approximation property ($SSAP$), and the Relative simplicial approximation property ($RSAP$). They were introduced at various times by different people and they produced with different degree of success, "shrinking" theorems for the corresponding cell-like upper semicontinuous decompositions of topological 3-manifolds, which in turn were constructed as cell-like resolutions of given generalized 3-manifolds (Lacher-Repovš, Daverman-Repovš, and Daverman-Thickstun). Several open problems and conjectures regarding generalized 3-manifolds were also formulated.

Squeezing in L-theory

Masayuki Yamasaki

Let X be a subcomplex of the standard N-simplex in $\mathbf{R}^{N+1}$ and let $n \geqslant 0$ be an integer. Then there exist constants $\epsilon_0 > 0$ and $\kappa > 1$ such that any R-coefficient quadratic Poincaré complex on a fibration $p : E \to X$ with radius $\epsilon \leq \epsilon_0$ is $\kappa\epsilon$-cobordant to an arbitrarily small quadratic Poincaré complex. Here R is a ring with involution. There is also a relative version of this, and we can conclude that, for all $\epsilon > 0$ and $\delta > 0$ satisfying $\kappa\epsilon \leq \delta \leq \epsilon_0$, the controlled L_n-groups $L_n(p; R, \epsilon, \delta)$ are all naturally isomorphic.

Edited by Frank Quinn and Andrew Ranicki

Participants

Prof. Dr. Markus Banagl
banagl@math.uc.edu
Dept. of Mathematical Sciences
University of Cincinnati
P.O.Box 210025
Cincinnati, OH 45221-0025 –USA

Prof. Dr. John L. Bryant
bryant@math.fsu.edu
Department of Mathematics
The Florida State University
Tallahassee, FL 32306-4510 – USA

Prof. Dr. Robert J. Daverman
daverman@math.utk.edu
Dept. of Mathematics
University of Tennessee at
Knoxville
121 Ayres Hall
Knoxville, TN 37996-1300 – USA

Prof. Dr. Alexander N. Dranish-nikov
dranish@math.ufl.edu
Dept. of Mathematics
University of Florida
358 Little Hall
P.O.Box 118105
Gainesville, FL 32611-8105 – USA

Prof. Dr. Jurek Dydak
dydak@math.utk.edu
Department of Mathematics
University of TEnnessee
Knoxville TN 37996-0001 – USA

Prof. Dr. Robert D. Edwards
rde@math.ucla.edu
Department of Mathematics
UCLA
Los Angeles, CA 90095-1555 – USA

Prof. Dr. Denise Halverson
deniseh@math.byu.edu
deniseh@math.utk.edu
Mathematics Department
Brigham Young University
274 TMCB
Provo UT 84602-6539 – USA

Dr. Friedrich Hegenbarth
friedrich.hegenbarth@mat.unimi.it
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
I-20133 Milano

Prof. Dr. Heather M. Johnston
johnston@vassar.edu
Department of Mathematics
Vassar College
Poughkeepsie NY 12604 – USA

Prof. Dr. Wolfgang Lück
lueck@math.uni-muenster.de
wolfgang.lueck@math.uni-muenster.de
Mathematisches Institut
Universität Münster
Einsteinstr. 62
D–48149 Münster

Prof. Dr. Washington Mio
mio@math.fsu.edu
Department of Mathematics
The Florida State University
Tallahassee, FL 32306-4510 – USA

Prof. Dr. Erik Kjaer Pedersen
erik@math.binghamton.edu
SFB 478
Geom. Strukturen in der Mathematik
Hittorfstr. 27
D–48149 Münster

Prof. Dr. Frank S. Quinn
quinn@math.vt.edu
Dept. of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0123 – USA

Prof. Dr. Andrew A. Ranicki
aar@maths.ed.ac.uk
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Building, Mayfield Road
GB-Edinburgh, EH9 3JZ

Prof. Dr. Dusan Repovs
dusan.repovs@uni-lj.si
Institute of Mathematics,
Physics and Mechanics
University of Ljubljana
P.O.Box 2964
1001 Ljubljana – Slovenia

Dr. Jonathan Woolf
jw301@cam.ac.uk
Christ's College
GB-Cambridge CB2 3BU

Prof. Dr. Masayuki Yamasaki
yamasaki@math.josai.ac.jp
Department of Mathematics
Josai University
1-1, Keyakidai Sakado
Saitama 350-0295 – Japan