RSE LECTURE

SIGNATURE OF 4-MANIFOLDS

PAST, PRESENT & FUTURE

SIGNATURE OF 4-MANIFOLDS

QUADRATIC FORM OVER $\mathbb{R}$ (NON-DEGENERATE)

$$\sum_{1}^{p} x_i^2 - \sum_{1}^{q} y_j^2$$ (SYLVESTER)

σ = SIGNATURE = $p - q$

X COMPACT ORIENTED 4-DIMENSIONAL MANIFOLD

ON $H_2(X, \mathbb{R})$ INTERSECTION FORM

ON $H^2(X, \mathbb{R})$ CUP-PRODUCT

$$H^2 \otimes H^2 \rightarrow H^4 \rightarrow \mathbb{R}$$

DIFF. FORMS $\alpha, \beta \rightarrow \alpha \wedge \beta \rightarrow \int \alpha \wedge \beta$

NOTE FOR dim 2 FORM IS SKEW-SYMMETRIC

NO ANALOGOUS INVARIANT

SIGNATURE OF 4-MANIFOLD $\sigma(X)$

FIRST NOTED BY HERMANN WEYL (1923)

[EXTENDS TO ALL DIMENSIONS $4n$] MULTIPLICATIVE

FOR SIMPLICITY WILL FOCUS ON DIMENSION 4

NOTE $\sigma(\bar{X}) = -\sigma(X)$ $\bar{X}$ IS X WITH OPPOSITE ORIENTATION

HODGE SIGNATURE THEOREM

X COMPLEX ALGEBRAIC SURFACE

REAL DIM = 4 . NATURAL ORIENTATION

HODGE NUMERS $h^{p,q}$ $(= \dim H^{p,q})$ $h^{p,q} = h^{q,p}$

$\sum_{p+q=n} h^{p,q} = \dim H^n$ $[H^{p,0} = \text{HOLOMORPHIC } p\text{-FORMS}]$

BETTI NUMBER $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$

HODGE THM $b_2^+ = 2h^{2,0} + 1$ $(= 2p_g + 1)$

(1933) $b_2^- = h^{1,1} - 1$

$\Rightarrow$ ONLY ONE + IN ALGEBRAIC CYCLES

A, B ALG. CURVES ON X A^2 OR $B^2 \geq 0$

THEN $(xA + yB)^2$ IS INDEFINITE

$* \quad \boxed{(A.B)^2 \geq A^2.B^2}$

HODGE PROOF BY ANALYSIS (HARMONIC FORMS)

(PURELY ALG. PROOF OF * BY MUMFORD)

Ex. CP_2 WITH n POINTS BLOWN UP

$b_2^+ = 1$ $\quad b_2^- = (n-1)$

APPLICATION : ISO-PERIMETRIC INEQUALITY

MINKOWSKI MIXED VOLUMES

A, B 2 CONVEX SETS IN (REAL) PLANE

CONSIDER SET $xA + yB$ OF ALL POINTS

$xa + yb \quad a \in A, b \in B$

$(x \geq 0, y \geq b)$

DENOTE AREA OF A BY $\|A\|^2$

$$\|xA + yB\|^2 = x^2 \|A\|^2 + 2xy \underbrace{(A \cdot B)}_{\text{MINKOWSKI MIXED AREA}} + y^2 \|B\|^2$$

ALEXANDROV- FENCHEL INEQUALITY

[ALSO IN n-DIMENSION]

$$(A \cdot B)^2 \geq \|A\|^2 . \|B\|^2$$

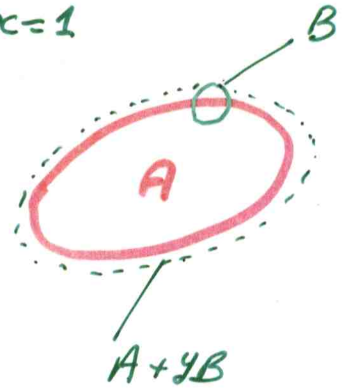
1) HODGE $\Rightarrow$ A-F

2) A·F $\Rightarrow$ ISO·PER

TAKE B = CIRCLE y SMALL, $x = 1$

$2 A \cdot B$ = LENGTH L OF ∂A

A·F $\Rightarrow \quad \frac{L^2}{4} \geq \|A\|^2 . \pi$



PROOF OF HODGE ⟹ A-F (B. TESSIER)

1) APPROXIMATE A, B BY CONVEX POLYGONS WITH RATIONAL COORDINATES

2) RESCALE TO MAKE INTEGER COORDS

3) INTERPRET AS <u>NEWTON</u> POLYGONS OF POLYNOMIALS A, B IN 2 COMPLEX VARIABLES

4) A, B ⟶ ALG. CURVES ON TOROIDAL SURFACE (CLOSURE OF $C^* \times C^*$ ORBIT IN CP_N)

5) APPLY <u>HODGE</u> TO THESE CURVES

6) INTERPRET DEGREES OF CURVES AS COMPLEX KAHLER VOLUMES

7) USE <u>ARCHIMEDES</u> THEOREM TO RELATE COMPLEX (SYMPLECTIC) VOLUMES TO REAL EUCLIDEAN AREAS

⟹ A-F !!!

NOTE CIRCULARITY OF METHOD

ANALYTIC PROBLEM

APPROX BY COMBINATORIAL

CONVERT TO ALG. GEOMETRY

USE HODGE (BASED ON ANALYSIS!)

(BUT COULD USE MUMFORD - PURE A·G)

HIRZEBRUCH RIEMANN-ROCH THEOREM

(AS INDEX THEOREM)

$\chi_q = \Sigma(-1)^p h^{p,q}$ CAN BE EXPRESSED IN TERMS OF

CHERN NUMBERS (TOP. INVARIANTS)

FOR ALG. SURFACE

EULER NUMBER $\Sigma(-1)^n b_n = c_2$

$$\sigma = \frac{c_1^2 - 2c_2}{3} = \frac{p_1}{3}$$ (PONTRJAGIN CLASS)

ROHLIN, THOM (COBORDISM)

ARITHMETIC GENUS

$$\chi_0 = \Sigma(-1)^p h^{p,0} = \frac{c_1^2 + c_2}{12}$$

G-SIGNATURE

G FINITE GROUP ACTING ON X

ACTS ON $H^*(M, \mathbb{R})$ PRESERVING QUAD. FORM

DEFINE G-SIGNATURE OF X AS CHARACTER OF G (FUNCTION ON G)

$\sigma(X, g) = \sum$ OVER FIXED-POINTS OF g

ASSUME ONLY ISOLATED (FINITE) FIXED POINTS P_j WITH ROTATION ANGLES α_j, β_j. THEN

$$\boxed{\sigma(X,g) = \sum_j -\cot\frac{\alpha_j}{2} \cdot \cot\frac{\beta_j}{2}}$$

A FIXED SURFACE Y_k CONTRIBUTES $\boxed{\dfrac{Y_k^2}{\sin^2 \theta_k/2}}$

WHERE θ_k IS NORMAL ROTATION ANGLE

NOTE X/G IS RATIONAL HOMOLOGY MANIFOLD

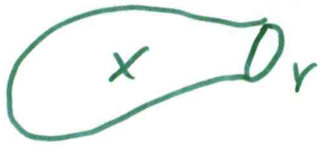
$\sigma(X/G)$ IS DEFINED & IS GIVEN BY

$$\sigma(X/G) = \frac{1}{|G|}\sigma(X) + \frac{1}{|G|}\sum_{g \neq 1} \sigma(X, g)$$

(SO SINGULAR POINTS GIVE CONTRIBUTION)
OF X/G

MANIFOLDS WITH BOUNDARY

X^4 BOUNDARY Y^3



$\sigma(X)$ CAN STILL BE DEFINED
(USE COMPACTLY SUPPORTED COH ON $X-Y$
& IGNORE DEGENERATE PART)

<u>NOTE</u> X RIEMANNIAN METRIC THEN
CAN DEFINE PONTRJAGIN 4-FORM p_1
FOR CLOSED X $\int_X p_1 = p_1(X)$ NUMBER

TAKE METRIC ON X PRODUCT NEAR Y

THEN (APS)

$$\boxed{\sigma(X) = \frac{1}{3}\int_X p - \eta(0)}$$

[cf. GAUSS-BONNET]

WHERE $\eta(0)$ IS A SPECTRAL INVARIANT OF
THE RIEMANNIAN MANIFOLD Y
LET A BE SELFADJOINT DIFF. OPERATOR ON
ALL EVEN FORMS BY

$$A(\varphi) = (-1)^p (*d - d*) \qquad \varphi \in \Omega^{2p}$$

PUT $$\eta(s) = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \operatorname{spec}(A)}} \operatorname{sign}\lambda \, . \, |\lambda|^{-s}$$ (HOLOMORPHIC FOR $\operatorname{Re}(s) > -\frac{1}{2}$)

HILBERT MODULAR SURFACES

RECALL $\Gamma = SL(2,\mathbb{Z})$ ACTS ON UPPER-HALF PLANE H AND QUOTIENT IS ALG CURVE H/Γ WITH INTERIOR SINGULAR POINTS AND CUSPS AT ∞

SIMILAR STORY FOR ALGEBRAIC SURFACES ARISING FROM REAL QUADRATIC FIELDS

$\mathcal{O}$ RING OF INTEGERS

$SL(2,\mathcal{O})$ ACTS ON $H \times H$ WITH QUOTIENT HILBERT MODULAR SURFACE X. THIS HAS INTERIOR ISOLATED SINGS. AND CUSPS AT ∞.

HOW TO COMPUTE $\sigma(X)$? HIRZEBRUCH

USE APS SIGNATURE THEOREM FOR INTERIOR SINGS. ALSO USE BOUNDARY NEAR CUSPS.

⇒ HIRZEBRUCH FORMULA

AT CUSP $\eta(0) = L(0)$

$L(s)$ SHIMIZU L-FUNCTION
(CLASSICAL NUMBER THEORY).
PROVIDED MOTIVATION FOR GENERAL APS
THEOREM.

BOUNDARY Y AT CUSP IS $S^1 \times S^1$
BUNDLE OVER S^1 WITH MONODROMY
GIVEN BY AN ELEMENT OF $SL(2,\mathbb{Z})$
(WITH REAL EIGENVALUES).

HIRZEBRUCH USES RESOLUTION OF CUSP
SINGULARITIES, BEAUTIFUL RELATION TO
PERIODIC CONTINUED FRACTIONS

[INTERIOR SINGS. ALSO RESOLVED USING
CONTINUED FRACTIONS]

FUTURE - SPECULATION

KALUZA-KLEIN 5TH DIMENSION

IGNORE TIME STATIC SITUATION

⇒ 4 DIM. RIEMANNIAN GEOMETRY

MODELS (CLASSICAL) OF NUCLEI ?

PROTON / NEUTRON

2 BASIC INTEGER INVARIANTS

ELECTRIC CHARGE

BARYON NUMBER (COUNTING PROTONS + NEUTRONS)

RELATED TO BASIC TOP. INVARIANTS

EULER NUMBER χ

SIGNATURE σ

??

BOTH GIVEN BY INTEGRATING

DENSITIES

WHAT CLASS OF RIEMANNIAN MANIFOLDS ?

RIEMANN CURVATURE RICCI + WEYL

RICCI = SCALAR + (TRACE-FREE)

WEYL $W = W^+ + W^-$ CONFORMAL INV

SELF-DUAL ANTI-SELF-DUAL

4-MANIFOLD WITH $W^- = 0$ IS

SELF-DUAL

HAS PENROSE TWISTOR SPACE

(INVOLVES HOLOMORPHIC FUNCTIONS)

$$\sigma(X) = \frac{1}{12\pi^2}\left\{ \|W_+\|^2 - \|W_-\|^2 \right\}$$

≥ 0 FOR SELF-DUAL X

PERHAPS σ = BARYON NUMBER ?

NEED <u>NON-COMPACT</u> MODELS

CANDIDATE FOR PROTON ?

AFFINE CUBIC SURFACE

$$x^2 - 2y^2 = 1$$

HAS $\sigma = 1$

AND HAS SD METRIC : <u>HYPERKAHLER</u>

HOLONOMY SU(2)

AH MANIFOLD

HAS SU(2) - SYMMETRY

CONTAINS MINIMAL 2-SPHERE

(CORE OF PROTON ?)

WORK IN PROGRESS ! ?