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★**Geometric topology: localization, periodicity and Galois symmetry.**

The 1970 MIT notes.

Edited and with a preface by Andrew Ranicki.

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In 1970, Sullivan circulated a set of notes, the "MIT notes", introducing localization and completion of topological spaces to homotopy theory, and other important concepts and constructions that have had a major influence on the development of topology. A version of the notes appeared as "Genetics of homotopy theory and the Adams conjecture" [Ann. of Math. (2) **100** (1974), 1–79; MR0442930 (56 #1305)]. Although it has been a long time since 1970, their publication now is more than an historical exercise. The development of these ideas, by Bousfield, Kan, Dwyer, and Dror-Farjoun for spaces and spectra, by H. R. Miller, and Carlsson for equivariant notions, and by Ranicki for surgery, reveal the fecundity of Sullivan's ideas, but not necessarily their motivation. The notes themselves, together with the reprint of Sullivan's paper "Galois symmetry in manifold theory at the primes" from the Proceedings of the 1970 ICM [in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, 169–175, Gauthier-Villars, Paris, 1971; MR0451254 (56 #9541)], and a lengthy personal Postscript provide the rich context in which the author developed these ideas.

C. T. C. Wall wrote of the MIT notes, "it is difficult to summarise Sullivan's work so briefly: the full philosophical exposition in (the notes) should be read." The exposition in the notes focuses on epistemological questions—in particular, what is the underlying algebraic nature of a manifold, and how can we know it? If we think of a manifold as a homotopy type together with its tangent bundle and associated sphere bundle, then the classification problem for manifolds takes us through characterization of a homotopy type, and determination of the associated sphere bundle. Since a sphere bundle associated to a geometric bundle on a manifold enjoys extra structure, the stable type is detected in the image of the J-homomorphism in stable homotopy (where the stable stems are the homotopy groups of the classifying space of the direct limit of the groups of homotopy equivalences of S^n). An important ingredient in understanding the image of J is the Adams conjecture, solved around 1968 by D. Quillen [Ann. of

Math. (2) **90** (1969), 205–295; MR0258031 (41 #2678)], and later by Becker and Gottlieb, and by Sullivan in these notes.

It was evident early on in the development of homotopy theory that problems sometimes simplified one prime at a time, and this motto applied to manifolds as well. Localization of a space at a prime p (or set of primes) associates to a space X a new space X_p and a mapping $X \to X_p$ which induces the homomorphism $\pi_i(X) \to \pi_i(X) \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the subring of $\mathbb{Q}$ with denominators relatively prime to p. The existence of such a space and mapping follows from a modification of the Postnikov tower of X. Another operation on a ring is completion, evidently useful in topology after Atiyah's computation of $K(BG)$. For groups, the profinite completion is the inverse limit of quotients with order a power of the prime p. It follows that $\widehat{G}_p$ is a totally disconnected compact topological group. To complete a space consider that the category of all mappings $X \to F$ with F satisfying $\pi_i(F)$ is finite for all i. The functor $\widehat{X}(Y) = \varprojlim [Y, F]$ is Brown representable by a homotopy type $\widehat{X}$, the completion of X.

The basic properties of localization and completion of spaces are proved in the notes, and related via arithmetic squares. A homotopy type is the pullback of the completion localized over the rationals, and the localization over the rationals completed formally over all primes. In the case of manifolds, as explained in the ICM paper, a manifold is reconstructed from a profinite completion and a rational space, fibred over an adelic completion. The gain from this viewpoint is a "pattern of symmetry" not directly visible at the manifold level. The main tool used in revealing the symmetry is Sullivan's Inertia Theorem, which explains when a direct limit of compatible automorphisms of a sequence of mappings extends to an automorphism of compatible spherical fibrations over the spaces in the sequence, and hence to a fibre homotopy preserving map of stable bundles.

The principal example is the case of $B_\infty = B\mathrm{U}$, the direct limit of the classifying spaces of the finite unitary groups $B\mathrm{U}_n$, which in turn are direct limits of Grassmann manifolds, $B\mathrm{U}_n = \lim_{k\to\infty} G_{n,k}\mathbb{C}$. The Grassmann manifolds are also algebraic varieties and so enjoy extra structure that is inherited by $B\mathrm{U}$. In particular, étale covers of a variety by finite covering spaces together with an automorphism of the variety can be passed to a limit and induce a mapping on the completion of the variety which is an inverse limit of the nerve of the cover. Since the Grassmann manifolds are defined as varieties over the integers, there is an action of the Galois group $\mathrm{Gal}(\tilde{\mathbb{Q}}/\mathbb{Q})$, where $\tilde{\mathbb{Q}}$ is the field of algebraic numbers.

The induced action of $\mathrm{Gal}(\tilde{\mathbb{Q}}/\mathbb{Q})$ on K-theory can be analyzed by considering the picture on completions and rationalizations over the p-primary bits and this leads to a proof of the Adams conjecture. The development of this symmetry on piecewise linear bundles is carried out in Chapter 6 with rich consequences for geometric topology.

Sullivan's use of localization and completion to fracture a homotopy type into manageable parts gave a strong impetus to the development of rational homotopy theory using new tools initiated in the notes. Also, the analysis of the picture above for Grassmannians in the real case led to a fixed point conjecture whose subsequent solution by Miller [Ann. of Math. (2) **120** (1984), no. 1, 39–87; MR0750716 (85i:55012); correction, Ann. of Math. (2) **121** (1985), no. 3, 605–609; MR0794376 (87k:55020)] has led to new methods in homotopy theory. The editor, A. K. Bousfield, and G. Mislin have provided an expansive and valuable bibliography of papers related to the notes.

My old copy of the MIT notes included a photo of the author, barely recognizable after so much photocopying. The photos in the new book of the author and his children together with the Postscript give a rare insight into the development of deep mathematical ideas. The notes remain worth reading for the boldness of their ideas, the clear mastery of available structure they command, and the fresh picture they provide for geometric topology. The editor must be thanked for making the notes available to another generation of topologists.

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