Edinburgh Lectures on Geometry, Analysis and Physics

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Contents

Preface

These lecture notes are based on a set of six lectures that I gave in Edinburgh in 2008/2009 and they cover some topics in the interface between Geometry and Physics. They involve some unsolved problems and conjectures and I hope they may stimulate readers to investigate them.

I am very grateful to Thomas Köppe for writing up and polishing the lectures, turning them into intelligible text, while keeping their informal nature. This involved a substantial effort at times in competition with the demands of a Ph.D thesis. Unusually for such lecture notes I found little to alter in them.

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LECTURE SERIES 1

From Euclidean 3-space to complex matrices

December 8 and 15, 2008

1.1 Introduction

We will formulate an elementary conjecture for n distinct points in $\mathbb{R}^3$, which is unsolved for $n \geq 5$, and for which we have computer evidence for $n \leq 30$. The conjecture would have been understood 200 years ago (by Gauss). What is the future for this conjecture?

- A counter-example may be found for large n.
- Someone (perhaps from the audience?) gives a proof.
- It remains a conjecture for 300 years (like Fermat).

To formulate the conjecture, we recall some basic concepts from Euclidean and hyperbolic geometry and from Special Relativity.

1.2 Euclidean geometry and projective space

The two-dimensional sphere $S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\}$ is "the same as" the complex projective line $\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$, on which we have homogeneous coordinates $[u_1 : u_2]$. Stereographic projection through a "north pole" $N \in S^2$ identifies $S^2 \setminus \{N\}$ with $\mathbb{C}$, and it extends to an identification of S^2 with $\mathbb{CP}^1$ by sending N to ∞.

Exercise 1.2.1. Suppose we have two stereographic projections from two "north poles" N and N'. Show that these give a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ which is a *complex linear transformation*

$$u' = \frac{au + b}{cu + d}\ , \text{ where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

Hint: Start by considering stereographic projection from S^1 to $\mathbb{R}$ first.

1.3 From points to polynomials

We will now associate to each set of n distinct points in $\mathbb{R}^3$ a set of n complex polynomials (defined up to scaling).

The case $n = 2$. Given two points $x_1, x_2 \in \mathbb{R}^3$ with $x_1 \neq x_2$, define

$$f(x_1, x_2) := \frac{x_2 - x_1}{\|x_2 - x_1\|} \in S^2\ ,$$

which gives a unit vector in the direction from x_1 to x_2. Under the identification $S^2 \cong \mathbb{CP}^1$, f associates to each pair (x_1, x_2) a point in $\mathbb{CP}^1$. Exchanging x_1 and x_2 is just the antipodal map $x \mapsto -x$ on S^2.

The general case. Given n (ordered) points $x_1, \ldots, x_n \in \mathbb{R}^3$, we obtain $n(n-1)$ points in $\mathbb{CP}^1$ by defining

$$u_{ij} := \frac{x_j - x_i}{\|x_j - x_i\|} \in S^2 \cong \mathbb{CP}^1 \quad \text{for all } i \neq j. \tag{1.1}$$

For each $i = 1, \ldots, n$ we define a polynomial $\beta_i \in \mathbb{C}[z]$ with roots u_{ij} $(j \neq i)$:

$$\beta_i(z) = \prod_{j \neq i} (z - u_{ij}) \tag{1.2}$$

The polynomials β_i are determined by their roots up to scaling. We make the convention that if for some j we have $u_{ij} = \infty$, then we omit the j^{th} factor, so that β_i drops one degree. In fact, a more invariant picture arises if instead we consider the associated *homogeneous* polynomials $B_i \in \mathbb{C}[Z_0, Z_1]$ given by $B_i(Z_0, Z_1) = \prod_j \bigl(V_{ij} Z_0 - U_{ij} Z_1\bigr)$, where $[U_{ij} : V_{ij}] = [u_{ij} : 1]$, so $\beta_i(z) = B_i(z, 1)$.

We are now ready to state the simplest version of the conjecture:

Conjecture 1.3.1 (Euclidean conjecture)**.** *For all sets $(x_1, \dots, x_n) \subset \mathbb{R}^3$ of n distinct points, the n polynomials $\beta_1(z), \dots, \beta_n(z)$ are linearly independent over $\mathbb{C}$.*

Remark 1.3.2. The condition of linear independence of the polynomials β_i is independent of the choice of stereographic projection in Equation 1.1 by Exercise 1.2.1.

Example ($n = 3$)**.** Suppose x_1, x_2, x_3 are distinct points in $\mathbb{R}^3$. They are automatically co-planar, so that $x_1, x_2, x_3 \in \mathbb{R}^2 \subset \mathbb{R}^3$. So the points u_{ij} lie in some great circle $S^1 \subset S^2 \cong \mathbb{CP}^1$.

We can choose the north pole N for the stereographic projection in Equation 1.1 either such that all u_{ij} lie in the equator, in which case $|u_{ij}| = 1$ and $u_{ji} = -u_{ij}$, or such that all u_{ij} lie on a meridian, in which case $u_{ij} \in \mathbb{RP}^1$ and $u_{ji} = -1/u_{ij}$.

Let us stick with the first convention, so that all u_{ij} lie on the equator and we have $|u_{ij}| = 1$ and $u_{ji} = -u_{ij}$. This defines three quadratics

$$\begin{aligned}
\beta_1(z) &= (z - u_{12})(z - u_{13}) = (z - u_{12})(z - u_{13}) \\
\beta_2(z) &= (z - u_{21})(z - u_{23}) = (z + u_{12})(z - u_{23}) \\
\beta_3(z) &= (z - u_{31})(z - u_{32}) = (z + u_{13})(z + u_{23})
\end{aligned}$$

In this case we can prove Conjecture 1.3.1 in two ways:

- By geometric methods: Represent quadratics by lines in a plane, then linear dependence of the β_i is the same as concurrence.

- By algebraic methods: Compute the determinant of the (3×3)-matrix of coefficients of the β_i and show that it has non-vanishing determinant.

For the case $n = 4$, there exists a proof using computer algebra. For $n \geq 5$, no proof is known, even for *co-planar* points (i.e. *real* polynomials). A proof will be rewarded with a bottle of champagne or equivalent. The easiest point of departure is to consider four points in a plane.

1.4 Some physics: hyperbolic geometry

Consider again the 2-sphere $S^2 \subset \mathbb{R}^3$, and add a fourth variable t (for "time"):

$$x^2 + y^2 + z^2 - R^2t^2 = 0 \tag{1.3}$$

This is the *metric* of Minkowski space-time. Here R is the speed of light, and Equation (1.3) defines a *light cone*. Our original 2-sphere is the base of the light cone, the "celestial sphere" of an observer.

The (proper, orthochronous) Lorentz group $SO^+(3,1)$ acts on $S^2 \cong \mathbb{CP}^1$ as a group of complex projective transformations $SL(2;\mathbb{C})/\pm 1 = PSL(2;\mathbb{C})$.

The Euclidean version of this picture is the following: The rotation group of $\mathbb{R}^3$, $SO(3)$, acts as $SU(2)/\pm 1 =: PSU(2) \cong PU(2)$ on $S^2 \cong \mathbb{CP}^1$ preserving the metric given by Equation (1.3) ("rigid motion"). We can also see this as the projectivisation of the action of $SU(2)$ or $U(2)$ on $\mathbb{C}^2$, and the projectivisation map

$$SU(2) \twoheadrightarrow PSU(2) \cong SO(3)$$

is a *double cover*. This map is the restriction to the maximal compact subgroup of the double cover $SL(2;\mathbb{C}) \twoheadrightarrow PSL(2;\mathbb{C}) \cong SO^+(3,1)$.

We have two different representations of $SL(2;\mathbb{C}) \cong \widetilde{SO^+}(3,1)$ (double cover): It acts on real 4-dimensional space-time $\mathbb{R}^{3,1}$ by proper, orthochronous Lorentz transformations, and it acts on complex 2-dimensional space $\mathbb{C}^2$ (whose elements we call *spinors*). The fundamental link between these two representations is via *projective* spinors: A (projectivised) point in $(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \cong \mathbb{CP}^1$ corresponds to a point on the base of the light cone, S^2.

Consider the hyperboloid given by $x^2 + y^2 + z^2 - R^2t^2 = -m^2$. Denote the interior of the base of the light cone by H_m. The metric induced on H_m has constant negative curvature, and indeed it turns H_m into a model of hyperbolic 3-space with curvature $-1/m^2$.

The Lorentz group $SO^+(3,1)$ acts transitively on hyperbolic 3-space H^3 by isometries, and it acts by $SL(2;\mathbb{C})/\pm 1$ on the 2-sphere at infinity.

1.5 The hyperbolic conjecture.

Given n distinct, ordered points in H^3, define the point u_{ij} as the intersection of the oriented geodesic joining x_i to x_j with the S^2 at infinity. We define n polynomials $\beta_1, \ldots, \beta_n$, where β_i has roots u_{ij}, as before in Equation (1.2) (but note that in hyperbolic space we no longer have a notion of "antipodal points"). This brings us to the second, stronger version of the conjecture:

Conjecture 1.5.1 (Hyperbolic conjecture)**.** *For all sets $(x_1, \ldots, x_n) \subset H^3$ of n distinct points, the n polynomials $\beta_1(z), \ldots, \beta_n(z)$ are linearly independent over $\mathbb{C}$.*

Remarks 1.5.2.

- There is good numerical evidence for the hyperbolic conjecture.
- The conjecture uses only the intrinsic geometry of hyperbolic 3-space, so it is invariant under the group of isometries (i.e. the Lorentz group).
- A model for H^3 is the open ball $B^3 \subset \mathbb{R}^3$. We can actually forget about the geometry of H^3 and just consider the points $x_1, \ldots, x_n$ to lie in $B^3 \subset \mathbb{R}^3$. Letting the radius of the ball B^3 grow (which is equivalent to letting the curvature of the hyperbolic space go to zero) exhibits the Euclidean conjecture as a limiting case of the hyperbolic conjecture.

Remark 1.5.3 (The ball of radius R)**.** As we said in Remark 1.5.2 (3), we can view the hyperbolic conjecture as a statement about points inside the unit ball B^3, and more generally inside any ball B^3_R of radius $R \geq 0$ – this corresponds to hyperbolic space of constant curvature $-1/R^2$.

We might expect that if the conjecture is false, then a counter-example would be given by a rather special configuration of the n points $x_1, \ldots, x_n$. The following example treats the most special configuration, namely the collinear one.

Example. Let $x_1, \ldots, x_n$ be collinear in B^3_R, and choose complex coordinates on the boundary S^2 such that all the roots of p_1 are at infinity, so that $p_1(z) = 1$. But then $p_2(z) = z$, $p_3(z) = z^2$, …, $p_n(z) = z^{n-1}$, and these are clearly linearly independent.

1.6 The Minkowski space conjecture

Consider two world lines ξ_1, ξ_2 in $\mathbb{R}^{3,1}$ representing world-like motion of two "stars". Consider the two points x_1, x_2 on ξ_1, ξ_2, respectively, representing events when an "observer" looks up into the sky and "sees" the other star on his celestial sphere, and denote the points on the respective celestial spheres $S^2 \cong \mathbb{CP}^1$ by u_{12} and u_{21}.

We make this more precise and more general:

Given n moving stars (i.e. non-intersecting world lines) $\xi_1, \ldots, \xi_n$ and n events $x_i \in \xi_i$, let u_{ij} be the point in the celestial sphere of x_i at which the past light cone at x_i intersects the world line ξ_j. In other words, x_i "sees" $n-1$ other stars at points u_{ij} in its own celestial sphere. Since in (flat) Minkowski space all celestial spheres can be identified by parallel translations, we may consider all the points u_{ij} to live in the space $\mathbb{CP}^1$.

Again we form the polynomials β_i from the roots u_{ij} as in Equation (1.2) and come to the third and strongest version of the conjecture.

Conjecture 1.6.1 (Minkowski space conjecture)**.** *Let $\xi_1, \ldots, \xi_n \subset \mathbb{R}^{3,1}$ be n non-intersecting world lines in Minkowski space and $\{x_1, \ldots, x_n\}$ a set of n (distinct) events such that $x_i \in \xi_i$ for all i. Then the polynomials $\beta_1(z), \ldots, \beta_n(z)$ are linearly independent over $\mathbb{C}$.*

Remarks 1.6.2.

1. Since the Lorentz group is essentially $SL(2;\mathbb{C})$, the Minkowski space conjecture is "physical", i.e. Lorentz-invariant.
2. If all stars emerge from a "big bang", i.e. if all world lines meet in a point in the past, then the Minkowski space conjecture reduces to the hyperbolic conjecture.
3. If stars are "static", the Minkowski conjecture reduces to the Euclidean conjecture.
4. The Minkowski conjecture is true for $n = 2$ ($u_{12} \neq u_{21}$). There is no other evidence!
5. See [2] and [4] for details.

Challenge. Prove or disprove the Minkowski space conjecture for $n = 3$.

Remarks.

1. Conjecture 1.6.1 refers to world lines. These can be interpreted as world lines of particles or "stars" in uniform motion and this gives one version of the conjecture. A stronger version arises if we allow all "physical motion" (i.e. not exceeding the velocity of light). In [2] I produced what purported to

be an elementary counterexample for $n = 3$. However, on closer inspection this involves motion faster than light, so the general conjecture is still open.

2. It is even tempting to consider motion on a curved space-time background but since we now have to worry about parallel transport it is not clear how to formulate a conjecture.

1.7 The normalised determinant

We begin by recalling some basic results from linear algebra. Consider the decomposition

$$\mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^4 \cong \mathrm{Sym}^2\big(\mathbb{R}^2\big) \oplus \Lambda^2\big(\mathbb{R}^2\big) \cong \mathbb{R}^3 \oplus \mathbb{R}^1 \ .$$

We can view the sum on the right-hand side as the decomposition of real (2×2)-matrices into symmetric and skew-symmetric parts, and we may think of the symmetric part $\mathrm{Sym}^2\big(\mathbb{R}^2\big)$ as a space of symmetric polynomials (of degree 2) and of the alternating part $\Lambda^2\big(\mathbb{R}^2\big)$ as the "area" or "determinant". The linear group $GL(2;\mathbb{R})$ acts on both summands and preserves this decomposition, and it acts on the area by multiplication by the determinant. $SL(2;\mathbb{R})$ acts trivially on the $\mathbb{R}^1$-summand.

The complex analogue of this picture is the following: The group $SL(2;\mathbb{C})$ acts trivially on $\Lambda^2\big(\mathbb{C}^2\big) \cong \mathbb{C}$ and on $\Lambda^n(\mathbb{C}^n) \cong \mathbb{C}$. (Note: $\mathbb{C}^n \cong \mathrm{Sym}^{n-1}\big(\mathbb{C}^2\big)$.) The group action preserves the standard symplectic form on $\mathbb{C}^2$.

Now suppose we have n distinct points $x_1, \ldots, x_n$ inside a ball of radius R, and the numbers $u_{ij} \in \mathbb{CP}^1$ are defined as in Equation 1.1. Lift the u_{ij} to any $v_{ij} \in \mathbb{C}^2$, i.e. pick a vector $v_{ij} = (z_1, z_2)$ such that $z_1\big/z_2 = u_{ij}$. Using the standard symplectic form, we identify $\mathbb{C}^2$ with its dual $(\mathbb{C}^2)^\vee$, and using this identification we consider the v_{ij} as one-forms. Since $u_{ij} \neq u_{ji}$, $v_{ij} \wedge v_{ji} \neq 0$. Now fix the constant multiplier by setting

$$p_i = \prod_{j \neq i} v_{ij} \in \mathrm{Sym}^{n-1}\big((\mathbb{C}^2)^\vee\big) \cong (\mathbb{C}^n)^\vee \ ,$$

and define

$$D_R(x_1, \ldots, x_n) = \frac{p_1 \wedge p_2 \wedge \cdots \wedge p_n}{\prod_{i<j}\big(v_{ij} \wedge v_{ji}\big)} \ . \tag{1.4}$$

Remarks 1.7.1. Here the numerator is an element of $\Lambda^n(\mathbb{C}^n) \cong \mathbb{C}$, concretely given by the determinant of the $(n \times n)$-matrix of the coefficients of the polynomials p_i. The denominator is a product of elements of $\Lambda^2\big(\mathbb{C}^2\big) \cong \mathbb{C}$. Changing the choice of V_{ij} by a factor λ_{ij} multiplies both numerator and denominator by the same factor $\prod_{i\neq j}\lambda_{ij}$, so D_R depends only on the points $x_1,\ldots,x_n$. Permuting the points $x_1,\ldots,x_n$ produces the same sign change in numerator and denominator, so D_R is invariant under permutations.

Definition 1.7.2 (Normalised determinant)**.** For n distinct points $x_1,\ldots,x_n$ in $\mathbb{R}^3$ inside a ball of radius R, we define the *normalised determinant* D_R to be as in Equation (1.4). (This normalization gives $D_R = 1$ for collinear points.)

Computation of D_R. Given n distinct points $x_1,\ldots,x_n$ inside a ball of radius R, choose for each pair $i<j$ lifts v_{ij}, v_{ji} such that $V_{ij}\wedge V_{ji}=\omega_2$. Write each p_i in terms of the monomials $t_0^{n-1-i}t_1^i$, where $\{t_0,t_1\}$ is a basis for $\mathbb{C}^2$ satisfying $t_0\wedge t_1=\omega_2$. If we denote by P the $(n\times n)$-matrix whose (i,j)-entry is the jth coefficient of p_i, then $D_R(x_1,\ldots,x_n)=\det P$ (hence the name "normalised determinant").

Properties of the normalised determinant.

1. $D_R(x_1,\ldots,x_n)$ is invariant under the $SL(2;\mathbb{R})$-action (i.e. the isometries of H^3_R) on the points $x_1,\ldots,x_n$, and it is continuous in $(x_1,\ldots,x_n)$.

2. The limit $D_\infty := \lim_{R\to\infty} D_R$ exists and is invariant under the group of Euclidean motions (translations and rotations of $\mathbb{R}^3$).

3. $D_R(x_1,\ldots,x_n)=1$ for collinear points.

4. $D_R\to\overline{D_R}$ under reflection of $\mathbb{R}^3$ (so D_R is real for coplanar points).

5. For $n=3$,
$$D_\infty = \frac{1}{2}\sum_{i=1}^{3}\cos^2\left(\tfrac{A_i}{2}\right) ,$$
where A_i are the angles of a triangle, varying between 1 for collinear and 9/8 for equilateral configurations. For $n\geq 4$, D_R is complex-valued in general.

6. D_∞ is scale-invariant: $D_\infty(\lambda x_1,\ldots,\lambda x_n)=D_\infty(x_1,\ldots,x_n)$ for $\lambda>0$.

7. In the hyperbolic case, $D_R(x_1,\ldots,x_n)\to D_R(x_1,\ldots,x_{n-1})$ as $|x_n|\to R$. (This generalises to the so-called "cluster decomposition": If the points $x_1,\ldots,x_n$ fall into two "clusters" at great distance, then D_R is approximately the product of the D_R's of the clusters.)

The formalism of the normalised determinant allows us to rephrase our conjectures, and assuming normalisation we can actually state stronger forms:

- The Euclidean conjecture 1.3.1. Weak form: $D_\infty \neq 0$. Strong form: $|D_\infty| \geq 1$ after normalisation.

- The hyperbolic conjecture 1.5.1. Weak form: $|D_R| \neq 0$. Strong form $|D_R| \geq 1$, after normalisation, with equality for collinear points.

- We also have a new conjecture, the *monotonicity conjecture*: $|D_R|$ increases with R (for fixed $x_1, \ldots, x_n$).

Remarks 1.7.3. $D_R(x) = D_{\lambda R}(\lambda x)$, so the hyperbolic conjecture is independent of R. So if it is true for finite R, then it is true for $R = \infty$.

The Minkowski space conjecture implies the hyperbolic conjecture: Shrink S^2_R to $S^2_{R'}$, where $R' = |x_n| = \max_i |x_i|$, then apply Property (7) inductively.

The normalised determinant D_R can be defined for points inside any ellipsoid S, in which case we denote it by D_S. This is because S can be changed into a standard sphere by affine linear transformations of R^3 (which preserve straight lines). We can reduce $x^2 + y^2 + z^2 = 1$ to $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by choice of $a, b, c \geq 1$.

Remark 1.7.4 (Ellipsoid version)**.** The Minkowski space conjecture can be stated in terms of ellipsoids: Suppose $S' \supseteq S$ are two ellipsoids in $\mathbb{R}^3$ containing n distinct points $(x_1, \ldots, x_n)$. Then

$$|D_{S'}(x_1, \ldots, x_n)| \geq |D_S(x_1, \ldots, x_n)| \ .$$

To see this, consider the situation where $S' \supseteq S$ are two light cones. Then $|D_{S'}| \geq |D_S|$. A physical interpretation is that if S' is the vacuum light cone and S the light cone in a medium, then $|D_{\text{med}}| \leq |D_{\text{vac}}|$.

1.8 Relation to analysis and physics

The Dirac equation. Let $s(x)$ be a spinor field in $\mathbb{R}^3$. The Dirac equation in vacuum is

$$Ds = \sum_{j=1}^{3} A_j \frac{\partial s}{\partial x_j} = 0 \ ,$$

where A_j are (2×2)-matrices, $A_j^2 = -1$, $A_iA_j = -A_jA_i = A_k$ (the Pauli matrices).

The point monopole. Given $(x_1, \ldots, x_n)$, consider these as locations of n Dirac monopoles and take the Dirac equation $Ds = 0$ in the background field. We need to impose suitable singular behaviour at $x_1, \ldots, x_n$ and decay at infinity.

We expect an n-dimensional space of solutions. Examine the asymptotic behaviour at infinity: Can we find our polynomials β_i in this (e.g. as a basis of the solutions)? Would this imply the Euclidean conjecture?

In the hyperbolic case, the asymptotic behaviour may be exponential decay, with polynomial angular dependence. Would this imply the radius-R conjecture for finite R?

The four-dimensional variant. Let M^4 be the Hawking-Gibbons 4-manifold, which has an action of $U(1)$. The quotient is $M^4/U(1) = \mathbb{R}^3$, and the $U(1)$-action has n fixed points, which determine n points $x_1, \ldots, x_n$ in $\mathbb{R}^3$.

Reinterpret on M^4: The solutions of the four-dimensional Dirac equation on M^4 inherit an action of $U(1)$. The invariant solutions on M^4 correspond to the singular solutions on $\mathbb{R}^3$. This disposes of the singular behaviour at x_i. We still require decay at infinity.

Next step: The Dirac equation is conformally invariant, so we can form the conformal compactification $\overline{M}$ (which has a mild singularity at infinity). This replaces asymptotic behaviour by local behaviour near infinity.

In the final step, we form the twistor space of $\overline{M}$ and use the complex methods of sheaf theory: Under the twistor transform, solutions of the Dirac equation correspond to sheaf cohomology. In particular, we expect a certain first cohomology to have dimension n.

To relate this to polynomials and our conjectures, we must use *real* numbers and positivity. This is close to (real) algebraic geometry.

Hyperbolic analogue. Four-manifold N^4 with special metric and $U(1)$-action with n fixed points and $N^4/U(1) = B^3$, the "inside" of S^2 in $\mathbb{R}^3$ with the hyperbolic metric. It admits a conformal compactification $\overline{N}$, on which we have a $U(1)$-action with n fixed points and a fixed $S^2 \subset \overline{N} \setminus N^4$. Twistor methods still apply to this case, but is it better than the Euclidean case? This leads to the theory of LeBrun manifolds. See Atiyah-Witten, which includes a problem about the existence of G_2 metrics on 7-manifolds which are $\mathbb{R}^3$-bundles over N^4 and generalise the cases $n = 0$ and $n = 1$.

Lie group generalisation. The Euclidean conjecture implies the existence of a continuous map

$$f_n\colon C_n\big(\mathbb{R}^3\big) \to GL(n;\mathbb{C})\big/(\mathbb{C}^\times)^n \to U(n)\big/T^n$$

compatible with the action of the symmetric group. Specifically, the value in $GL(n;\mathbb{C})$ is the matrix of coefficients of the polynomials p_i, and the quotient by $(\mathbb{C}^\times)^n$ accounts for the freedom of scale.

The configuration space can be described as follows.

$$C_n\big(\mathbb{R}^3\big) = \mathrm{Lie}\big(T^n\big) \otimes \mathbb{R}^3 \setminus \mathcal{S}\ ,$$

where the $\mathbb{R}^3$-factor contains the coordinates of the points, the factor $\mathrm{Lie}\big(T^n\big)$ accounts for the n points, and $\mathcal{S}$ is the union of codimension-3 linear subspaces $\mathcal{S}_\alpha$, where $\mathcal{S}_\alpha$ is the kernel of the linear map (the root map)

$$\alpha \otimes \mathrm{id}_{\mathbb{R}^3}\colon\ \mathrm{Lie}\big(T^n\big) \otimes \mathbb{R}^3 \to \mathbb{R}^3$$

extending the roots α of $U(n)$, accounting for the fact that the n points are required to be *distinct*. (The roots of $U(n)$ are formed by elements $x_i - x_j$. Note that $\mathrm{Lie}\big(T^n\big)$ is the Cartan subalgebra of $\mathfrak{u}(n)$.)

This leads us to a generalisation of our conjectures. Let G be a compact Lie group (e.g. $SO(n;\mathbb{R})$) and $G_\mathbb{C}$ its complexification (e.g. $SO(n;\mathbb{C})$). Let $T \leq G$ be a maximal torus with complexification $T_\mathbb{C}$, and let $W := N(T)\big/T$ be the Weyl group of G, which permutes the roots.

Conjecture 1.8.1 (Lie group conjecture)**.** *If G is a Lie group as above with rank n, then there exists a continuous map*

$$f_n\colon\ \mathrm{Lie}(T) \otimes \mathbb{R}^3 = \mathcal{S} \to \frac{G_\mathbb{C}}{T_\mathbb{C}} \to \frac{G}{T}$$

compatible with the action of the Weyl group W.

In a joint paper with Roger Bielawski ([2]) we used Nahm's equations

$$\frac{dA_1}{dt} = [A_2, A_3]\ \text{(and cyclic permutations)},$$

where $A_i\colon (0,\infty) \to \mathrm{Lie}(G)$ are functions of t subject to suitable boundary conditions $t \to 0$, $t \to \infty$, to prove the existence of a map to G/T. Problems:

1. For $G = U(n)$, is this the same as a map given by polynomials?
2. Is there an explicit algebraic analogue for a map to $G_{\mathbb{C}}/T_{\mathbb{C}}$?.
3. Is there any generalisation of the hyperbolic conjecture from $GL(n;\mathbb{C})$ to other Lie groups?

1.9 Mysterious links with physics

- Origin in Berry-Robbins on spin statistics.
- Link to Dirac equation?
- Generalisation to Minkowski space.
- Nahm's equations and gauge theory.
- Link to Hawking-Gibbons metric?
- Twistor interpretation?

Key fact of physics. The base of the light cone is $\mathbb{CP}^1$. It is Penrose's philosophy that this must be the origin of complex numbers in quantum theory, and it must lie behind any unification of General Relativity and Quantum Mechanics.

What is the physical meaning of our conjectures?

List of conjectures

- Conjecture 1.3.1: The Euclidean conjecture (weak and strong).
- Conjecture 1.5.1: The hyperbolic conjecture (weak and strong).
- The monotonicity conjecture for the normalised determinant.
- Conjecture 1.6.1: The Minkowski space conjecture.
- Conjecture 1.8.1: The Lie group conjecture.

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LECTURE SERIES 2

VECTOR BUNDLES OVER ALGEBRAIC CURVES AND COUNTING RATIONAL POINTS

February 9, 16, 23 and March 2, 2009

2.1 Introduction

There are two themes, both initiated by A. Weil:

1. Extension of classical ideas in algebraic geometry, number theory, physics from Abelian (scalars, $U(1)$) to non-Abelian (matrices, $U(n)$) settings.

2. Connection between homology and counting rational points over finite fields.

2.2 Review of classical theory

(Abel, Jacobi, Riemann, ...) Consider complex projective space

$$\mathbb{CP}^{n-1} \equiv \mathbb{P}^{n-1} := (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^\times$$

with homogeneous coordinates $[z_1 : \ldots : z_n]$. Rational functions on $\mathbb{P}^{n-1}$ are fractions $f(z_1, \ldots, z_n)/g(z_1, \ldots, z_n)$, where f and g are homogeneous polynomials of the same degree.

Note that a meromorphic function on $\mathbb{P}^{n-1}$ is determined up to scale by its zeros and poles (Liouville). On projective space, global complex analysis is just algebraic geometry (Serre).

There is the standard line bundle L over $\mathbb{P}^{n-1}$, i.e. $L \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Holomorphic sections of its k^{th} power $L^k = L \otimes L \otimes \cdots \otimes L$ are just homogeneous polynomials of degree k.

Algebraic curves. Let $n = 3$, so we consider the projective plane $\mathbb{P}^2$. A curve of degree k is given as the locus of points z such that $f(z_1, z_2, z_3) = 0$, where f is a homogeneous polynomial of degree k. Non-singular curves are just compact Riemann surfaces, so topologically they are entirely determined by its genus g. A Riemann surface X of genus g has first Betti number $b_1 = \dim H_1(X; \mathbb{Q}) = 2g$.

If X is a curve of degree k with double points, then removing these double points leaves a Riemann surface. We have a formula

$$g = \frac{1}{2}(k-1)(k-2) - \delta \ ,$$

where δ is the number of double points. If $\delta = 0$, then for $k = 1, 2$ we find that X is a rational curve, i.e. $g = 0$; and for $k = 3$ we get an elliptic curve with genus $g = 1$.

Another interpretation is that g is the dimension of the space of holomorphic differentials (which look locally like $\phi(z)\, dz$, where ϕ is holomorphic). When $g = 0$, the curve is the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, and the differential dz has a pole at infinity, so it is not holomorphic. When $g = 1$, the curve is the torus $\mathbb{C}/\mathbb{Z}^2$, so the differential dz on $\mathbb{C}$ descends to a holomorphic differential on X.

Period matrices. Let $\omega_1, \ldots, \omega_g \in H^1(X; \mathbb{C})$ be a basis of holomorphic differentials and $\alpha_1, \ldots, \alpha_{2g} \in H_1(X; \mathbb{Z})$ a basis for the 1-cycles of a genus-g curve X. The $(g \times 2g)$-matrix with entries $\int_{\alpha_j} \omega_i$ is called the *period matrix* of X.

Divisors. We call a subvariety of codimension 1 a *divisor*. Since curves are 1-dimensional, divisors on curves are just points. The free Abelian group of all divisors of a variety X is denoted by $\mathrm{Div}(X)$, and so if X is a curve, elements of $\mathrm{Div}(X)$ are just formal sums $D = \sum_{i=1}^{N} n_i P_i$, where $P_i \in X$ are points. The *degree* of such a divisor D on a curve is defined as $\deg D := \sum_{i=1}^{N} n_i$.

Jacobians. The *Jacobian* of X, written $J(X)$, is a complex torus of complex dimension g, given as

$$J(X) = \mathbb{C}^g/\text{lattice} = \text{hol. differentials}/\text{differentials with integer periods} \ .$$

The significance of the Jacobian lies in the following observation. Let ϕ be a rational (meromorphic) function on X, and define the divisors

$$\begin{aligned} D_0(\phi) &:= \text{set of zeros of } \phi\text{, with multiplicities,} \\ D_\infty(\phi) &:= \text{set of poles of } \phi\text{, with multiplicities,} \\ D(\phi) &:= D_0(\phi) - D_\infty(\phi) \quad \text{(the \textit{divisor of} } \phi\text{).} \end{aligned}$$

Then $\deg D_0(\phi) = \deg D_\infty(\phi)$. This motivates the question for the converse: Given two divisors D_1 and D_2 of the same degree, when does there exist a function ϕ on X with $D_0(\phi) = D_1$ and $D_\infty(\phi) = D_2$?

This is always true for $g = 0$, but not otherwise. The "gap" between divisors of degree zero and divisors of meromorphic functions is measured precisely by the *divisor class group* $\mathrm{Cl}(X)$. The degree-0 part of it is

$$\mathrm{Cl}^0(X) := \frac{\text{divisors of degree 0}}{\text{divisors of functions}} \; .$$

(Divisors of the form $D = D(\phi)$ are also called *principal divisors*.) For $g = 0$, the group $\mathrm{Div}^0(X)$ is trivial, but for $g = 1$, the divisor class group is precisely the Jacobian (or its dual) – this is the content of the Abel-Jacobi Theorem. Moreover, the group $\mathrm{Cl}^0(X)$ is the group of isomorphism classes of holomorphic line bundles of degree 0, which are just given by elements of

$$\mathrm{Hom}\big(\pi_1(X), U(1)\big)$$

(up to duality and complex structure). Differential geometry shows that a holomorphic line bundle of degree zero (i.e. first Chern class zero) has a unique flat unitary connection.

This is the beginning of the link with physics. Maxwell's equations deal with the curvature of a line bundle on space-time.

2.3 Analogy with number theory

Number theory	*Algebraic geometry*
Ring of integers $\mathbb{Z}$	complex (affine) line, the ring $\mathbb{C}[z]$
primes	points
factorisation of integers	factorisation of polynomials
"infinite prime"	point at infinity in $\mathbb{P}^1$
algebraic number field	algebraic curve (covering of a line)
lack of unique factorisation	not all divisors come from functions
ideal class group	divisor class group
Galois group	$\pi_1(X)$

The classical analogy is that between the ring of integers in number theory and the polynomial rings in geometry. A "half-way house" is an *algebraic curve over a finite field.* A finite field is a field $\mathbb{F}_q$ with q elements, where $q = p^n$ for some prime p.

We have "function field analogues" of geometric statements, e.g. a Riemann hypothesis (which is proved for finite fields; also for algebraic varieties of any dimension).

The key fact for algebraic geometry over $\mathbb{F}_q$ is the existence of the *Frobenius map* $x \mapsto x^q$. (Recall that in characteristic p, $(x + y)^p = x^p + y^p$.) There is no such analogue in characteristic zero (but physics suggest rescaling the metric[1]).

2.4 Relation between homology and counting rational points

Definition 2.4.1 (Poincaré series)**.** For any topological space X whose singular homology groups $H_k\big(X; \mathbb{Q}\big)$ are finite-dimensional vector spaces, we define the *Poincaré series* of X to be the formal power series

$$P_X(t) := \sum_{k=0}^{\infty} \dim H_k(X; \mathbb{Q})\ t^k\ .$$

Proposition 2.4.2. *If X is a manifold or homotopy-equivalent to a manifold, then P_X is in fact a polynomial.*

[1]A quick explanation of this remark is in order: In differential geometry, a differential form scales with the power of its degree, so rescaling picks out the degree of the form. In characteristic p, the eigenvalues of the Frobenius map pick out the dimension of the cohomology.

Example. Consider the space $\mathbb{P}^{n-1}$. Over $\mathbb{C}$, this has Poincaré series

$$P(t) = 1 + t^2 + t^4 + \cdots + t^{2n-2} \ . \tag{2.1}$$

Over $\mathbb{F}_q$, the number of points in $\mathbb{P}\mathbb{F}_q^{n-1}$ is

$$\frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1} \ .$$

This agrees with (2.1) if we put $q = t^2$. Note that since we can replace q by $\tilde{q} = q^n$, $n = 1, 2, \ldots$, we can think of q as a variable like t. This extends to all algebraic varieties.

Exercise 2.4.3. Check that a similar relation between the Poincaré series over $\mathbb{C}$ and the number of points over a finite field $\mathbb{F}_q$ holds for the full flag variety $U(n)/T^n$. (Hint: Use successive fibrations by projective spaces.)

Generalisation from $U(1)$ to $U(n)$. This corresponds to generalising from line bundles to vector bundles. In number theory, this corresponds to non-Abelian class field theory. There are representations from the Galois group to $U(n)$, Langlands programme... In physics, this is related to non-Abelian gauge theories and Yang-Mills theory.

Returning to algebraic geometry, we will focus on an algebraic curve X (either over $\mathbb{C}$ or over $\mathbb{F}_q$). The Jacobian is replaced by a "moduli space" of vector bundles over X. There are a few difficulties:

- There is no group structure (the tensor product does not preserve rank for ranks > 1).
- Bundles of rank n can decompose into bundles of lower rank.

There is a moduli space $M_s(X, n, k)$ of holomorphic rank-n bundles of degree k which are *stable*. Here k is the degree of the determinant line bundle, which is the first Chern class; in symbols: $\deg E := \deg \Lambda^n E \equiv c_1(\Lambda^n E)$. The space $M_s(X, n, k)$ is a compact algebraic variety if $\gcd(n, k) = 1$, e.g. if $n = 2$, $k = 1$.

For $k = 0$, the space $M_s(X, n, 0)$ is the space of irreducible representations $\pi_1(X) \to U(n)$. To see this, note that such a representation is a choice of $2g$ unitary matrices $A_1, \ldots, A_g, B_1, \ldots, B_g \in U(n)$ such that $\prod_{i=1}^g [A_i, B_i] = 1$, modulo conjugation by $U(n)$.

For a general k, we replace this condition by $\prod_{i=1}^g [A_i, B_i] = \zeta \, \mathrm{id}$, where ζ is a central element of $U(n)$ and $\zeta^k = 1$. (For example, for $n = 2$, $k = 1$, we have $\prod_i [A_i, B_i] = -\mathrm{id}$.)

The general problem is to study M_s.

1. What does M_s look like topologically?

2. What are its Betti numbers $b_i = \dim H_i\big(M_s;\mathbb{Q}\big)$?

For a connected, oriented manifold X, we have the Poincaré polynomial $P_X(t) = \sum_{i=0}^{\dim X} = b_i t^i$. Note that for any compact manifold X, we have $\deg P_X = \dim X$, and P_X is palindromic (by Poincaré duality). Furthermore, $P_{X\times Y}(t) = P_X(t)P_Y(t)$.

Example. If X is a Riemann surface of genus g, then $P_X(t) = 1 + 2\,g\,t + t^2$, and $P_{J(X)}(t) = (1+t)^{2g}$.

What is $P_{M_s(X,n,k)}(t)$? What is it when $\gcd(n,k) = 1$? Let us consider the special case $n = 2$, $k = 1$ on a curve X of genus $g(x) = 2$. Then

$$P_{M_s(X,2,1)}(t) = \big(1 + t^2 + 4t^3 + t^4 + t^6\big)\big(1+t\big)^4 = \big(1 + t^2 + 4t^3 + t^4 + t^6\big)P_{J(X)}(t)\ .$$

Note. For $n = 2$ and $g \geq 2$, we have $\dim_{\mathbb{C}} M_s(X,2,k) = (3g-3) + g$, and $g = \dim J(X)$.

Let $\det\colon M_s(X,n,0) \to J(X)$ be the determinant map $E \mapsto \det E \equiv \Lambda^n E$, and denote by M^0 the fibre of det over some point. We have a general result.

Theorem 2.4.4 (Formula for general $g \geq 2$)**.**

$$P_{M^0}(t) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)} \tag{2.2}$$

Exercise 2.4.5. This should be a palindromic polynomial of degree $6g-6$, all of whose coefficients are non-negative. Prove this.

Let us write in short $M_g(n,k)$ for $M_s(X,n,k)$, the moduli space of stable vector bundles of rank n and degree k on a smooth curve X of genus g. What can we say for $n \geq 2$? For $\gcd(n,k) = 1$, $M_g(n,k)$ is a complex manifold of dimension $(3g-3)+g$. Topologically, $M_g(n,k)$ is given by $A_1,\dots,A_g,B_1,\dots,B_g \in U(n)$ such that $\prod_{i=1}^{g}[A_i,B_i] = \sigma$, where $\sigma = e^{2\pi i/n}$, modulo conjugation by $U(n)$.

Specific question. What is the homology of $M_g(n,k)$? What is its Poincaré polynomial? Recall:

$$P_M(t) := \sum_{i=0}^{N} \dim H^i\big(M_g(n,k);\ \mathbb{Q}\big)\ t^i$$

Here $N = 8g-6$. For $n=1$, $P_{M_g(1,k)}(t) = (1+t)^{2g}$, independent of k.

For $n=2$, $k=1$, the moduli space decomposes as $M_g(2,1) = M_g^0(2,1) \times J(X)$, and the Poincaré polynomial of $M_g^0(2,1)$ is given by Equation (2.2).

2.5 The approach via Morse theory

2.5.1 Basic Morse theory

Let Y be an n-dimensional manifold and $f\colon Y \to \mathbb{R}$ a function; the points $x \in Y$ where $df(x) = 0$ are called the *critical points of* Y. The Hessian, which we write briefly as "d^2f", is a quadratic form, and we call f a *Morse function* if d^2f is non-degenerate at all critical points of f. By the Morse Lemma, there exist near every critical point p local coordinates $\{x_i\}$ in which f takes the form

$$f(p+x) = f(p) - x_1^2 - x_2^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2\ .$$

The integer r is called the *Morse index* of the critical point. If $r=0$, f has a minimum; if $r=n$, f has a maximum, and if $0<r<n$, f has a saddle point.

If f is a Morse function on Y, the *Morse polynomial* is

$$M_{Y,f}(t) = \sum_Q t^{\gamma(Q)}\ ,$$

the sum over all non-degenerate critical points Q, and $\gamma(Q)$ is the Morse index of Q. It can be shown that

$$M_{Y,f}(t) \geq P_Y(t)\ ,$$

with equality in "good cases".

Examples.

- Let $Y = S^1$ and $f\colon Y \to \mathbb{R}$ the height function. Then $M_{Y,f}(t) = P_Y(t) = 1+t$; this is a "good case".
- Let $Y = S^1$, but "pinched", and f again the height function. Then $M_{Y,f}(t) =$

$2+2t$, a "bad case".

- Let $Y = S^1 \times S^1$ be the torus and f the height function. Then $M_{Y,f}(t) = 1+2t+t^2 = (1+t)^2 = P_Y(t)$, another "good case".

- Let $Y = \mathbb{CP}^{n-1}$ and

$$f(z) = \frac{\sum_{i=1}^n \lambda_i \, |z_i|^2}{\sum_{i=1}^n |z_i|^2} \quad \text{with} \quad \lambda_1 < \cdots < \lambda_n \ .$$

  Then the critical points of f are Q_j where $z_j = 1$ and $z_i = 0$ for $i \neq j$, with indices $\gamma(Q_j) = 2j-2$. Hence $M_{Y,f}(t) = 1+t^2+\cdots+t^{2n-2} = P_Y(t)$, and we have another "good case".

We generalise the notion of non-degeneracy to allow critical submanifolds. $Q \subseteq Y$ is a critical submanifold if $df = 0$ along Q and $d^2 f$ is non-degenerate in normal directions. The Morse index of Q, written again as $\gamma(Q)$, is the number of linearly independent negative normal directions. Such a function will be called a *Morse-Bott function*.

Definition 2.5.1. If $f\colon Y \to \mathbb{R}$ is a Morse-Bott function, the *Morse polynomial* of f is

$$M_{Y,f}(t) = \sum_Q t^{\gamma(Q)} P_Q(t) \ ,$$

where the sum is taken over all non-degenerate critical submanifolds $Q \subset Y$.

Again we have the Morse inequality $M_{Y,f}(t) \geq P_Y(t)$, with equality in good cases.

Examples.

- Let $Y = \mathbb{CP}^{n-1}$ and

$$f(z) = \frac{\sum_{i=1}^n \lambda_i \, |z_i|^2}{\sum_{i=1}^n |z_i|^2} \quad \text{with} \quad \lambda_1 \leq \cdots \leq \lambda_n \ , \ \lambda_1 \neq \lambda_n.$$

  If for example $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} < \lambda_n$, then $Q_{\min} = \mathbb{CP}^{n-2}$ and $Q_{\max} = \{\text{pt.}\} = [0 : \ldots : 0 : 1]$, and so

$$M_{Y,f}(t) = P_{\mathbb{CP}^{n-2}}(t) + t^{2n-2} = P_Y(t) \ .$$

More generally, if $\lambda_1 = \cdots = \lambda_r < \lambda_{r+1} < \cdots \lambda_n$, then

$$M_{Y,f}(t) = P_{\mathbb{CP}^{r-2}}(t) + t^{2r} + \cdots + t^{2n-2} = P_Y(t) \ .$$

- Now take $n = \infty$ in the last example. Then $P_{\mathbb{CP}^\infty}(t) = 1 + t^2 + \cdots = \frac{1}{1-t^2}$. But we still have

$$P_{\mathbb{CP}^\infty}(t) = P_{\mathbb{CP}^{r-1}}(t) + t^{2r} + t^{2r+2} + \cdots \ ,$$

  so we conclude that $P_{\mathbb{CP}^{r-1}}(t) = \frac{1}{1-t^2} - \sum_{k=r}^{\infty} t^{2r}$.

The last example is the prototype of the method to compute $P_{Q_{\min}}(t)$ of some critical manifold $Q_{\min}$ in terms of the (possibly infinite-dimensional) total space and higher critical points. We will use this method again later to compute the Poincaré series of the moduli space of $U(2)$-bundles over a curve of genus g.

2.5.2 Equivariant cohomology, or The effect of symmetry

Let G be a compact Lie group (for instance $U(1)$ or $U(n)$) and suppose G acts on a manifold Y. If the action is free, then Y/G is a manifold and has nice cohomology and Poincaré series. If the action is not free, Y/G has singularities. What to do?

Definition 2.5.2 (Equivariant cohomology)**.** We define $H^*_G(Y) := H^*(Y_G)$ to be the *G-equivariant cohomology of Y*, where Y_G is given by the *Borel construction*

$$Y_G := (EG \times Y)\big/G \ ,$$

where EG is a contractible space with a free G-action, and the action of G on $EG \times Y$ is $g.(e, y) = (g.e, g.y)$. (In fact, EG is the total space of the classifying fibration $G \hookrightarrow EG \twoheadrightarrow BG$.)

Example. Let $G = U(1)$ and $EG = \mathbb{C}^\infty \setminus \{0\} = \varinjlim\big(\mathbb{C}^N \setminus \{0\}\big)$. Then $BG := EG\big/G = \mathbb{CP}^\infty$. We compute:

$$H^*_G(\text{pt.}) = H^*\big(\mathbb{CP}^\infty\big) \qquad \text{and} \qquad P(t) = 1 + t^2 + \cdots = \frac{1}{1-t^2} \ .$$

Note that the projection

$$Y_G = (EG \times Y)\big/G \to EG\big/G =: BG \simeq \{\text{pt.}\}_G$$

gives a homomorphism

$$H_G^*\big(\{\text{pt.}\}\big) = H^*\big(BG\big) \longrightarrow H_G^*\big(Y\big) \ ,$$

which turns $H_G^*\big(Y\big)$ into a graded module over the graded cohomology ring $H_G^*(\text{pt.})$. We saw from the example that for $G = U(1)$, the equivariant cohomology $H_G^*(\text{pt.}) = H^*\big(\mathbb{CP}^\infty\big)$ is a polynomial ring in one variable u of degree 2, and we may take u to be the Chern class of the tautological line bundle on $\mathbb{CP}^\infty$. More generally, for $G = U(n)$ the equivariant cohomology $H_G^*(\text{pt.}) = H^*\big(BU(n)\big)$ is a polynomial ring in n variables $u_1, \ldots, u_n$ of degrees $2, 4, \ldots, 2n$, and again the u_i may be interpreted as the Chern classes of the tautological n-plane bundle over $BU(n) = \text{Gr}_n\big(\mathbb{C}^\infty\big)$.

Definition 2.5.3. Let Y be a manifold with an action of a compact Lie group G as above. The *equivariant Poincaré series* of Y is

$$P_Y^G(t) = \sum_{k=0}^{\infty} \dim H_G^k\big(Y\big)\, t^k \ .$$

Remark 2.5.4. If the action of G on Y is free, then $Y_G \cong (EG \times Y)\big/G \simeq Y\big/G$, and so $P_Y^G(t) = P_{Y/G}(t)$ is a polynomial. In general, however, $P_Y^G(t)$ is only a power series which is the expansion of a rational function. If Y is contractible, then $Y_G \simeq BG$, and so $H_G^*\big(Y\big) = H^*\big(BG\big)$.

Equivariant Morse theory. Suppose G acts on Y and $f\colon Y \to \mathbb{R}$ is a G-invariant Morse-Bott function, i.e. f is a Morse-Bott function and $f(g.y) = f(y)$ for all $g \in G$. If G acts freely on Y, then f induces a function $f_G\colon Y\big/G \to \mathbb{R}$, and we can apply Morse theory to f_G. Otherwise, consider f on Y, but remember the G-action and use H_G, that is, consider f as a Morse function on Y_G.

Example. Let $Y = S^2$ and $G = U(1)$, acting by a simple rotation with two fixed points, and let f be the height function. Then

$$M_{Y,f}^G(t) = \underbrace{\frac{1}{1-t^2}}_{\text{min.}} + \underbrace{\frac{t^2}{1-t^2}}_{\text{max}} = \frac{1+t^2}{1-t^2} \ .$$

This is a "good case", since we also have $P_Y^G(t) = (1+t^2)\big/(1-t^2)$.

Some criteria for a good Morse(-Bott) function. The following conditions allow us to conclude that a Morse polynomial (or power series) is "good", i.e. equal to the Poincaré series.

- If all Morse indices and all Betti numbers are even. (E.g. for $\mathbb{CP}^{n-1}$.)
- In the equivariant case: If each critical submanifold is point-wise fixed by a some $U(1) \subset G$ which has no fixed vectors in the negative normal bundle.

We will use these criteria in gauge-theoretical computations in the following section.

2.5.3 Application to infinite dimensions (gauge theory)

Let X be a surface of genus $g \geq 2$ and A a G-connection for a vector bundle of rank n over X, where $G = U(n)$. For the trivial bundle $X \times \mathbb{C}^n$,

$$A = \sum_{i=1}^{2} A_i(x)\, dx_i \ ,$$

where (x_1, x_2) are local coordinates on X and $A_i \in \mathfrak{u}(n)$, the Lie algebra of skew-Hermitian $(n \times n)$-matrices. The *curvature* of the connection is (locally, or globally in the case of the trivial bundle)

$$F_A = dA + A \wedge A \in \Omega^2\big(X; \mathfrak{u}(n)\big) \ .$$

The Lie algebra $\mathfrak{u}(n)$ admits an invariant inner product, so we can define a norm $\|-\|$ on it. The *Yang-Mills functional* of the connection A is

$$\phi(A) := \int_X \|F_A\|^2 d\,\mathrm{Vol} \ .$$

The key idea is to apply Morse theory to ϕ.

1. The function ϕ is a function on the infinite-dimensional space $\mathcal{A}$ of all connections. This is an affine-linear space, hence contractible.
2. The function ϕ is invariant under the infinite-dimensional symmetry group of all bundle automorphisms $\mathcal{G} = \mathrm{Map}(X, G)$, the so-called group of *gauge transformations.*
3. Inside $\mathcal{G}$ we have the subgroup $\mathcal{G}_0 \subset \mathcal{G}$ of *based maps* $X \to G$, which is the kernel of $\mathrm{ev}\colon \mathcal{G} \to G$, the evaluation at a base point $x_0 \in X$ given by

ev$(f) = f(x_0)$. That is, $\mathcal{G}_0$ consists of all those gauge transformations which are the identity at x_0.

The restricted group $\mathcal{G}_0$ acts freely on $\mathcal{A}$, and so we can reduce to a G-action on $\mathcal{A}/\mathcal{G}_0$. Moreover, $\mathcal{G}$-equivariant cohomology on $\mathcal{A}$ becomes G-equivariant cohomology on $\mathcal{A}/\mathcal{G}_0$.

4. We will apply $\mathcal{G}$-equivariant Morse theory to the Yang-Mills functional ϕ on the space $\mathcal{A}$.

The critical connections for ϕ are the those for which the curvature F_A is covariantly constant. The absolute minimum appears when $F_A = 0$, i.e. when A is *flat* (or more generally *central harmonic*). For higher critical points, A decomposes.

Example. Let us consider the simplest case, $n = 2$. That is, we consider rank-2 bundles, or $U(2)$-bundles, on a Riemann surface X. The determinant line bundle $\det E$ of a rank-2 bundle E has degree $k = c_1(E) = c_1(\Lambda^2 E)$, and E is topologically non-trivial whenever $k \neq 0$. Let us assume $k = 1$; so we are in a different component of the moduli space than for $k = 0$.

At the absolute minimum, $\mathcal{G}$ acts freely. The moduli space $M_g(2,1)$ is a manifold and contributes $P_{M_g(2,1)}(t)$. At higher critical points, the bundle is a direct sum of line bundles, $E \cong L_1 \oplus L_2$, and $\deg L_1 + \deg L_2 = 1$. Assume without loss of generality that $\deg L_2 > \deg L_1$. Now $\mathcal{G}$ acts with isotropy subgroup $U(1)$ and contributes

$$P^{U(1)}_{J(X)\times J(X)} = \frac{(1+t)^{4g}}{1-t^2} \ .$$

What is the contribution of the total space $\mathcal{A}$? We know that $H^*_{\mathcal{G}}\big(\mathcal{A}\big) = H^*\big(B\mathcal{G}\big)$, but how do we calculate this? Following Atiyah and Bott [4, §2] we have:

1. $B\mathcal{G} = \mathrm{Map}\big(X, B\mathcal{G}\big)$.

2. For $G = U(1)$, we have $BG = \mathbb{CP}^\infty$. So

$$\mathrm{Map}\big(X, \mathbb{CP}^\infty\big) = \mathbb{Z} \times \prod_{2g} S^1 \times \mathbb{CP}^\infty \ ,$$

and

$$P_{B\mathcal{G}}(t) = (1+t)^{2g}/(1-t^2) \ .$$

3. For $G = U(n)$, we have $U(n) \sim U(1) \times S^3 \times \cdots \times S^{2n-1}$, so

$$P_{B\mathcal{G}}(t) = \frac{\prod_{i=1}^{n} \left(1 + t^{2i-1}\right)^{2g}}{\left(\prod_{i=1}^{n-1} (1 - t^{2i})\right) (1 - t^{2n})} . \tag{2.3}$$

All of these are "good cases". We finish with a computation to prove Theorem 2.4.4.

$$\frac{t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)} = \frac{1}{1-t^2} \sum_{i=1}^{\infty} t^{2g+4i}(1+t)^{2g} \tag{2.4}$$

On the right-hand side we recognise the factors $(1-t^2)^{-1} = P_{\mathbb{CP}^\infty}(t)$ and $(1+t)^{2g} = P_{J(X)}(t)$. We obtain one big equation

$$\left\{\text{minimum}\right\} + \left\{\text{higher critical points}\right\} = \left\{\text{total space}\right\} ,$$

where

$$\begin{aligned} \text{minimum} &= P_{M_g^0(2,1)} \text{ , the series of the space of interest,} \\ \text{higher points} &= \text{the expression (2.4), and} \\ \text{total space} &= (1+t^3)^{2g}/(1-t^2)(1-t^4) \text{ from Equation (2.3) with } n = 2, \end{aligned}$$

for the Yang-Mills functional ϕ on the space of all connections on $U(2)$-bundles with fixed degree 1. The contribution from the higher critical points is given by the $L_1 \oplus L_2$ (with fixed total degree), which is the origin of the Jacobian factor $P_{J(X)}(t)$.

Remark 2.5.5. For $n \geq 3$, even if we only want to deal with the co-prime case $\gcd(n, k) = 1$, the inductive step will need a general case (e.g. $n = 3$, $k = 1$ can decompose into $E_2 \oplus E_1$ with $\operatorname{rk} E_i = i$ and $\deg E_2 = 0$, $\deg E_1 = 1$). But Morse theory still works to give induction if we use equivariant cohomology and equivariant Poincaré series. (The Poincaré series $P_M(t)$ will not be a polynomial).

2.6 Counting rational points

2.6.1 Finite fields

Fields with finitely many elements are either the integers modulo some prime p, written $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, or some algebraic extension thereof, written $\mathbb{F}_q$ with $q = p^n$

for some $n \geq 1$. Note that every field is a vector space over its prime subfield $\mathbb{F}_p$, and the characteristic is in each case the prime p. We can consider an algebraic variety V defined over any field, in particular over $\mathbb{F}_q$ – for example by considering as the defining equations of V polynomials with integer coefficients and reducing modulo p.

Example (Projective spaces)**.** Let $V := \mathbb{P}\big(\mathbb{F}_q^n\big) = \big(\mathbb{F}_q^n \setminus \{0\}\big)/\mathbb{F}_q^\times$. The number of points in V is

$$N_q(V) = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1} \ .$$

Observe:

1. Over the field $\mathbb{F}_{q^m}$, the number of points is

$$N_{q^m}(V) = 1 + q^m + q^{2m} + \cdots + q^{m(n-1)} \ ,$$

   so varying m determines a polynomial in q via $m \mapsto N_{q^m}(V) \in \mathbb{Z}[q]$.

2. Setting $q = t^2$ gives the Poincaré polynomial of $\mathbb{P}(\mathbb{C}^n) = \mathbb{CP}^{n-1}$. This indicates a relation between counting rational points over finite fields and Betti numbers of complex varieties.

3. Replacing q by q^{-1} gives

$$N_q(V) = \frac{q^n(1 - q^{-n})}{q(1 - q^{-1})} = q^{n-1}\big(1 + q^{-1} + \cdots q^{-(n-1)}\big) \ ,$$

   and

$$\frac{N_q(V)}{q^{n-1}} = 1 + q^{-1} + q^{-2} + \cdots + q^{-(n-1)} \text{ (Poincaré Duality).}$$

4. Let $n \to \infty$. We get $1/\big(1 - q^{-1}\big)$, and putting $q = t^{-2}$ we get $1/\big(1 - t^2\big) = P_{\mathbb{CP}^\infty}(t)$.

Zeta functions. The ζ-function of an algebraic variety V over $\mathbb{F}_q$ is

$$Z_V(t) = \exp\left(\sum_{m=1}^{\infty} N_{q^m}(V)\frac{t^m}{m}\right) \ ,$$

where $N_{q^m}(V)$ is the number of points of V over the finite field $\mathbb{F}_{q^m}$. We define further

$$\zeta_V(s) := Z_V(q^{-s}) \ ,$$

which is the analogue of the Riemann ζ-function. Note that $|q^{-s}| = q^{-\Re(s)}$. In the special case where V is a single point, $Z_V(t) = \frac{1}{1-t}$.

2.6.2 The Weil conjectures

(The Weil conjectures were proved by A. Grothendieck and P. Deligne.)

Theorem 2.6.1. *Let V be a non-singular projective algebraic variety over a finite field $\mathbb{F}_q$. Then*

1. *$Z_V(t)$ is a rational function of t.*
2. *If $n = \dim V$, then*

$$Z_V(t) = \frac{p_1(t)\, p_3(t)\, \cdots\, p_{2n-1}(t)}{p_0(t)\, p_2(t)\, \cdots\, p_{2n}(t)} \ ,$$

   where each root ω of p_i has $|\omega| = q^{-i/2}$.

3. *The roots of p_i are interchanged with the roots of p_{2n-i} under the substitution $t \to 1/q^n\, t$.*
4. *If V is the reduction of an algebraic variety over a subfield of $\mathbb{C}$, then the Betti numbers b_i of the variety $V(\mathbb{C})$ are $b_i = \deg p_i$.*

Remark 2.6.2. Part (2) of Theorem 2.6.1 is the Riemann hypothesis for function fields. Part (3) is the functional equation for $\zeta(s)$.

Steps in the proof.

1. Define cohomology groups $H^i\big(V\big)$ which are the analogues to $H^i\big(V(\mathbb{C})\big)$. (Done by Grothendieck.)
2. Use the Frobenius map $\phi\colon V \to V$, $x \mapsto x^q$. This maps preserves both multiplication and addition. The fixed points of ϕ^m are the points of $V(\mathbb{F}_{q^m})$, and there are $N_{q^m}(V)$ of them.
3. Apply the Lefschetz fixed point theorem: The number of fixed points of a map $f\colon X \to X$ is

$$\sum_{i=0}^{\dim X} (-1)^i \operatorname{tr}\big(f^*\colon H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{Z})\big) \ .$$

Take $X = V$, $f = \phi$ and H^i to be Grothendieck cohomology:

$$N_{q^m}(V) = \sum_i \operatorname{tr}\big((\phi^m)^*\colon H^i(V) \to H^i(V)\big) = \sum_i (-1)^i \sum_j \omega_{ij}^m \ ,$$

where the ω_{ij} are the eigenvalues of ϕ_* acting on $H_i(V)$.

4. Now compute:

$$\begin{aligned} Z_V(t) &= \exp\left(\sum_{m=1}^{\infty} N_{q^m}(V)\frac{t^m}{m}\right) \\ &= \exp\Big(\sum_i (-1)^i \sum_j -\log(1-\omega_{ij}\,t)\Big) = \prod_{i \text{ odd}} p_i(t) \Big/ \prod_{i \text{ even}} p_i(t) \ , \end{aligned}$$

where $p_i(t) = \prod_j (1 - \omega_{ij}\,t)$. This proves the theorem subject to

5. Poincaré duality, and

6. the Riemann hypothesis: $|\omega_{ij}| = q^{i/2}$ for all i, j (done by Deligne).

Example. Let $V = X_g$ be an algebraic curve of genus g. Then

$$Z_V(t) = \frac{\prod_{j=1}^{2g}\big(1-\omega_j t\big)}{\big(1-t\big)\big(1-qt\big)}$$

and

$$\zeta_V(s) = \frac{\prod_{j=1}^{2g}\big(1-\omega_j q^{-s}\big)}{\big(1-q^{-s}\big)\big(1-q^{-s+1}\big)} \ .$$

Example. Let $V = M_g(n,k)$ with $\gcd(n,k) = 1$ be the moduli of stable vector bundles over X_g of rank n and degree k. If we can compute $N_{q^m}(V)$ for all m, then Theorem 2.6.1 gives the Betti numbers of $V(\mathbb{C})$, i.e. the Poincaré polynomial of $M_g(n,k)$ over $\mathbb{C}$.

How do we compute the number of points of $M_g(n,k)$ over $\mathbb{F}_q$? We use two key ideas:

1. All bundles are trivial if we allow poles (of all orders), i.e. if we work with the field of rational functions on X_g.

2. The vector space A of power series over $\mathbb{F}_q$ of the form

$$\sum_{j=0}^{\infty} a_j t^j \in \mathbb{F}_q[\![t]\!] \tag{2.5}$$

is infinite-dimensional but *compact*, since it is a product of finite (hence compact) sets.

The space A has a natural measure μ, which is normalised such that $\mu(A) = 1$. Let $A_r \leq A$ be the linear subspace of power series of the form (2.5) which satisfy $a_0 = a_1 = \cdots = a_{r-1} = 0$. Then the quotient space A/A_r has q^r points, so $\mu(A_r) = q^{-r}$.

We define the infinite projective space over $\mathbb{F}_q$ to be

$$\mathbb{P}(\mathbb{F}_q^\infty) \equiv \mathbb{F}_q\mathbb{P}^\infty := (A \setminus \{0\})/\mathbb{F}_q^\times \ .$$

Since $\{0\}$ has measure zero,

$$\mu\big(\mathbb{F}_q P^\infty\big) = \mu(A)/\big|\mathbb{F}_q^\times\big| = \frac{1}{q-1} \ .$$

(Compare this with the Poincaré series $P_{\mathbb{CP}^\infty}(t) = \frac{1}{1-t}$.)

The way in which we just dealt with infinite dimensions and computed measures is our inspiration for counting points in moduli spaces over a finite field $\mathbb{F}_q$: Allowing poles and using measures we can compute the number of points as ratios of measures.

Example. The group of isomorphism classes of line bundles over X_g is isomorphic to the divisor class group $\mathrm{Cl}(X_g)$ of X_g, which is

$$\mathrm{Cl}(X_g) := \mathrm{Div}(X_g)/\big(D \sim D + (f)\big) \ .$$

A divisor D is a formal finite sum $D = \sum_j k_j Q_j$, where the $Q_j \in X_g$ are points and $k_j \in \mathbb{Z}$. Now pick a local coordinate u near a point Q and let f be a local power series

$$f(u) = \sum_{k=-N}^{\infty} a_k\, u^k \ , \text{ with } a_{-N} \neq 0 \ .$$

Multiplication by elements of a compact group $\mathcal{K}_Q$ reduces this to $f(u) = u^{-N}$. (The group is the group of holomorphic power series around Q with non-vanishing constant term, i.e. the invertible elements.) So the group $\mathrm{Div}(X_g)$ of all divisors on $X_g(\mathbb{F}_q)$ is

$$\mathrm{Div}(X_g) = \prod_{x \in X_g} \mathcal{K}_x \backslash \mathcal{A}_x = \mathcal{K} \backslash \mathcal{A} \ ,$$

and the group of divisor classes of degree 0, written $\mathrm{Cl}^0(X_g)$, is

$$\mathrm{Cl}^0(X_g) = \mathcal{K} \backslash \mathcal{A} / K^\times \ ,$$

where $K = K(X_g)$ is the function field of X_g. The measure $\mu\big(\mathcal{A}/K^\times\big)$ is finite, and counting points gives the answer q^{2g}.

Bundles of higher rank. To study the moduli space $M_g(n,k)$ for $n > 1$, i.e. the moduli space of bundles of higher rank, we can use the same method, provided we fix the determinant. We have

$$\frac{1}{\mu(\mathcal{K})} = (q-1) \sum_E \frac{1}{|\mathrm{Aut}(E)|} \ ,$$

where μ is the Tamagawa measure (with $c = 1$). We have further

$$\frac{1}{\mu(\mathcal{K})} = q^{(n^2-1)(g-1)} \zeta_{X_g}(2) \cdots \zeta_{X_g}(n) \ .$$

In particular, for $n = 2$ and $k = 1$ the sum over all bundles E splits into a sum over stable bundles and a sum over unstable bundles, where for a stable bundle E we have $\mathrm{Aut}(E) = \{1\}$. Therefore

$$\sum_E \frac{1}{|\mathrm{Aut}(E)|} = \big|M_g^0(2,1)\big| + \sum_{r=1}^{\infty} \frac{1}{|\mathrm{Aut}(E)|} \ ,$$

where the last sum is a geometric series running over all bundles $E = L^r \oplus L^{1-r}$ and extensions. This gives an explicit formula for $\big|M_g^0(2,1)\big|$, and hence by the Weil conjectures for $P_{M_g(2,1)}(t)$.

Computing measures. Let α run over all points of $M_g^0(n,k)$, i.e. orbits of $\mathcal{K}$ acting on $\mathcal{A}^*\big/K^\times$. Then

$$\sum_\alpha \mu\big(\mathcal{K}/K_\alpha\big) = \mu\big(\mathcal{A}^*/K^\times\big) = C \ ,$$

or

$$\sum_\alpha \frac{1}{|\mathcal{K}_\alpha|} = \frac{C}{\mu(\mathcal{K})} \ .$$

The only automorphisms of line bundles are scalars, so $|\mathcal{K}_\alpha| = q-1$. Also,

$$\sum_\alpha \frac{1}{|\mathcal{K}_\alpha|} = \frac{|J(X_g)|}{q-1} \ .$$

We need to know the value of C and $\mu(\mathcal{K})$. Both depend on the precise normalisation of μ. If we choose $C=1$, then we get $1\big/\mu(\mathcal{K}) = |J(X_g)|$.

2.7 Comparison of equivariant Morse theory and counting rational points

We obtain the same formula for $P_M(t)$ and agreement term by term in the method of the proof. This also works for all n, k and other groups than $U(n)$. The key points are the following:

- The total space is "trivial": The space of connections is affine-linear, hence contractible, and the Tamagawa measure of $SL(n;\mathbb{C})$ is 1.
- Let I be the isotropy group. We can compute $P_{BI}(t)$ and divide by $\mu(I)$.

With $\mathcal{G} = \mathrm{Map}\big(X_g, U(n)\big) \cong \mathcal{K}$,

$$P_{B\mathcal{G}}(t) = \prod_{k=1}^{n} (1+t^{2k-1})^{2g} \Big/ (1-t^{2n}) \prod_{k=1}^{n-1} (1-t^{2k})^2 \ ,$$

and

$$\frac{1}{\mu(\mathcal{K})} = q^{(n^2-1)(g-1)} \zeta_{X_g}(2) \cdots \zeta_{X_g}(n) \ .$$

These agree using the formula

$$\zeta_{X_g}(s) = \prod_{i=1}^{2g} (1-\omega_i q^{-s}) \big/ (1-q^{-s})(1-q^{1-s}) \ .$$

Questions.

1. Why do these two formulae agree? ("Quantum analogue of the Weil conjectures")
2. Is there an extension of the Weil conjectures to infinite dimensions?

3. Is computing measures on adèlic spaces analogous to Feynman integration in gauge theories?

2.8 Relation to physics

Does physics help us understand the questions we raised in the last section? Is there a relation to the original ζ-function? (This leads to arithmetic algebraic geometry (Arakelov theory) and further speculations.)

The Yang-Mills functional came from physics over 4-dimensional space-time. It can be considered formally over a compact Riemannian manifold X of any dimension d. In particular,

- if $d = 2$, X is a Riemann surface and we have many results about moduli spaces;
- if $d = 4$ we have *Donaldson theory*.

Quantum field theory.

1. *Hamiltonian approach:* Consider space and time separately. We have a Hilbert space $\mathcal{H}$ of states, and a self-adjoint "Hamiltonian" operator H acting on $\mathcal{H}$. The *evolution* is given by the unitary operator e^{itH} on $\mathcal{H}$.

2. *Lagrangian formulation* (relativistically invariant): Let L be a functional on some space of functions f on space-time, e.g. $L(f) = \int |\nabla f|^2$.

3. The *Feynman integral* is $\int \exp\big(\frac{i}{\hbar}L(f)\big)$, integrated over all functions f on $\mathbb{R}^3 \times [0, \tau]$ with $f(0) = u$ and $f(\tau) = v$, determines the value $\langle u, e^{i\tau H} v \rangle$. This relates to the Hamiltonian approach. (Recall that the Lagrangian and Hamiltonian are related via the Legendre transform.)

Topological quantum field theories. For some special Lagrangians, we get $H = 0$, and so time evolution is just the identity. In this case, the Feynman integrals give topological information, and we call these cases *topological quantum field theories*. There are many interesting examples of topological QFTs in dimensions 2, 3 and 4.

In four dimensions, we get Donaldson theory and Seiberg-Witten theory, but these have no parameters.

In three dimensions, we get Chern-Simons theory, which does have an interesting parameter. Let A be a G-connection over X, where $G = U(n)$. Let

$$L = CS(A) = \frac{2\pi}{k} \int_X \mathrm{tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big) \ .$$

The Hilbert space is the space of holomorphic sections of a line bundle L^k over $M_n(X_g)$, where X_g is a Riemann surface. (This three-dimensional theory is related to two-dimensional conformal field theory.) We get topological invariants of 3-manifolds and knots inside them (Jones, Witten).

In two dimensions, there is also a Yang-Mills theory with Lagrangian $L(A) = \|F_A\|^2 = \int_{X_g} |F_A|^2$. (This is the function on the space of connections to which we applied equivariant Morse theory.) This theory is physical and not just topological, but we can solve it exactly. A coupling constant ϵ is introduced and the Feynman integral is formally

$$Z(\epsilon) = \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{A}} \exp\Big(-\frac{1}{2\epsilon} \|F_A\|^2\Big)\, dA \ .$$

This has a non-trivial dependence on ϵ and can be used to compute the multiplicative structure on the cohomology ring $H^*\big(M(X_g, n)\big)$ (Witten).

This quantum field theory looks promising, but does not give the Poincaré series of $M(X_g, n)$. Question: Is there an analogue over a finite field (where the Frobenius map is related to scaling ϵ)? Another possibility is to use a (super-symmetric) variant of Chern-Simons theory for a 3-manifold $S^1 \times X_g$ (or more generally a circle bundle or a Seifert fibration). The Hilbert space is $\Omega^*\big(M(X_g, n)\big)$, the space of all differential forms on $M(X_g, n)$, which comes equipped with a differential d and its adjoint (with respect to the symplectic structure) d^*. However, this seems to involve integration for functions on $S^1 \times X_g$, while we want just functions on X_g (for the analogy with finite fields).

A possible idea is contained in Witten-Beasley for another theory of Chern-Simons type, where integration is reduced to $X_g \subset S^1 \times X_g$ as the fixed-point set of a symmetry.

2.9 Finite-dimensional approximations

We can use approximations to link topology with finite fields and then pass to a limit. Let us consider approximations to BG.

For $G = U(n)$,

$$BG = \varinjlim_{N\to\infty} \frac{U(N)}{U(n)\times U(N-n)} = \varinjlim_{N\to\infty} \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{Gr}_n(\mathbb{C}^\infty) \ .$$

For maps $f\colon X_g \to BG$, fix a degree $\deg(f) = m$ (and then let $m \to \infty$). For fixed N, m, the space of holomorphic maps $f\colon X_g \to \mathrm{Gr}_n(\mathbb{C}^N)$ of degree m forms a finite-dimensional algebraic variety $V(N,m)$.

The idea of finite-dimensional approximations is the following: Holomorphic maps are determined by their behaviour at "poles", and the Graßmannians Gr_n can be embedded in projective space. We can study whether continuous maps can be approximated by holomorphic maps, apply the Weil conjectures to $V(N,m)$ and take limits.

This is a reasonable programme.

2.10 Relation of ζ-functions for finite fields and Riemann's ζ-function

The original Riemann ζ-function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{p\ \text{prime}} \Big(1 - \frac{1}{p^s}\Big)^{-1} \ ,$$

where the last expression is also known as the *Euler product*, whose factors are so-called *local factors*. (They are called thus with reference to the closed points (p) of the scheme $\mathrm{Spec}(\mathbb{Z})$.) The ζ-function ostensibly contains information about the set of primes.

Now let V be an algebraic variety over a finite field $\mathbb{F}_p$. We want to define a ζ-function for V. If $V = \{*\} = \mathrm{Spec}(\mathbb{F}_p)$ is a single point, let

$$\zeta_V(s) := \big(1 - p^{-s}\big)^{-1} \ .$$

In general, if V is any variety defined over $\mathbb{Z}$, we define

$$\zeta_V(s) := \prod_p \zeta_{V_p}(s) \ ,$$

where V_p is the reduction of V modulo p. We need to look out for special "bad" primes and add a term for the "infinite prime" (arising in valuation theory).

By the Weil conjectures, $\zeta_{V_p}(s)$ is given by a rational function of $t = p^{-s}$ in terms of the Frobenius action on cohomology.

Example. Let V be an elliptic curve (i.e. of genus 1) defined over $\mathbb{Z}$. The Weil formula gives

$$Z(t) = \frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-pt)} \ ,$$

where α, β are eigenvalues of the Frobenius map ϕ on H^1, and further we have $|\alpha| = |\beta| = p^{-1/2}$, $\beta = \alpha^{-1}$ and $\alpha + \beta = a = \operatorname{tr}\big(\phi^*|_{H^1(V)}\big)$. Put

$$L_p(s) = \Big(\text{numerator of } Z(t) \text{ with } t = p^{-s}\Big) = 1 - a_p p^{-s} + p^{1-2s} \ ,$$

and

$$L_V(s) = c. \prod_p L_p(s) \ .$$

Theorem 2.10.1 (Hasse-Weil Conjecture)**.** *With V as above and with suitable choices for the infinite prime and for bad primes, the function $L_p(s)$ extends holomorphically to all $s \in \mathbb{C}$, and $L_V(s) = \pm L_V(2s)$.*

The Hasse-Weil Conjecture has now been proved by Wiles, Taylor and others. Similar conjectures exist for all V and all H^i. (There is one L-function for each i.)

Remark 2.10.2 (The adèlic picture for $\mathbb{Q}$ or number fields)**.** This is a comment on the double coset space $\mathcal{K}\backslash G_A/G_K$ used for an algebraic curve over $\mathbb{F}_p$. For $\mathbb{Q}$ or $\mathbb{Z}$ and for $SL(2)$ we have $SO(2;\mathbb{R})\backslash SL(2;\mathbb{R})$, which is the upper-half plane (or hyperbolic plane). The double coset space is

$$\mathcal{M} :== SO(2;\mathbb{R})\backslash SL(2;\mathbb{R})/SL(2;\mathbb{Z}) \ ,$$

the moduli space of elliptic curves. To compute the area of $\mathcal{M}$. we start with $SL(2;\mathbb{R})/SL(2;\mathbb{Z})$, which is a three-dimensional manifold with an invariant volume form. We decompose it into $SO(2;\mathbb{R})$-orbits and integrate.

2.11 Arithmetic algebraic geometry (Arakelov theory)

Suppose we have an algebraic variety V of dimension d defined over the integers $\mathbb{Z}$. We can either embed $\mathbb{Z}$ into $\mathbb{C}$ and consider $V(\mathbb{C})$ as a complex variety, or we can form the residues $\mathbb{Z} \to \mathbb{Z}/p$ and get a corresponding variety V_p. So in fact we get a family V_p over the primes in $\mathbb{Z}$, and we include V_∞ sitting over the infinite prime. This family, a scheme over $\operatorname{Spec}\mathbb{Z}$, is an algebraic variety of dimension $d+1$. If $d = 0$ we get a number field, if $d = 1$ we get a so-called *arithmetic surface.*

"In the big picture, physics is at infinity, and number theory at the finite points."

We may try to extend theorems from surfaces to their arithmetic analogues.

Non-Abelian theories. For $d = 0$ and $G = SL(n)$, we have the Langlands programme, also known as non-Abelian class-field theory. For $d = 1$ we study the local theory at p. For $p = \infty$, we have the *geometric* Langlands programme, which has been related by Witten to quantum field theories over $V(\mathbb{C})$. What is the ultimate goal? Perhaps quantum field theories over arithmetic varieties? One would start with the case $d = 1$.

2.12 Other questions

Can we extend our results from curves to varieties of higher dimensions? Recall that for a curve X_g and gauge group $G = SU(n)$, we know the Poincaré series

$$P_{\mathrm{Map}(X_g,BG)}(t) = \prod_{k=1}^{n}(1+t^{2k-1})^{2g} \Big/ (1-t^{2n}) \prod_{k=1}^{n-1}(1-t^{2k})^2 \ .$$

Over $\mathbb{F}_q$,

$$\mathrm{vol}(\mathcal{K})^{-1} = q^{(n^2-1)(g-1)} \zeta_{X_g}(2) \cdots \zeta_{X_g}(n) \ ,$$

where

$$\zeta_{X_g}(s) = \prod_{i=1}^{2g}(1-\omega_i q^{-s}) / (1-q^{-s})(1-q^{1-s})$$

and $\mathcal{K}$ is the maximal compact subgroup of G_{A_X}. Both formulae extend from curves to varieties V of all dimensions and still appear to be closely related. We may study, for example, bundles over

- $\mathbb{P}^2$,
- $(\mathbb{P}^2, \mathbb{P}^1)$,
- $\mathbb{P}^1 \times X_g$ (here Morse theory is trickier),
- X_g with gauge group $\Omega(G)$, this is related to the previous point,
- and also Weil theory for some infinite-dimensional cases.

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