Poincaré and the Early History of 3-manifolds

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Background in Automorphic Functions

Poincaré in 1889

Poincaré's work on automorphic functions in the early 1880s involved several ingredients that would prove useful for his later work in topology.

In particular:

- groups defined by generators and relations,
- the associated tessellations of the (hyperbolic) plane and space, and
- quotient spaces of the plane or space by discrete groups.

Linear Groups

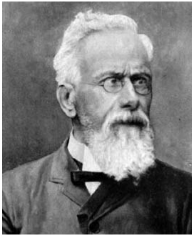
The groups encountered by Poincaré in hyperbolic geometry consisted of *linear fractional transformations*

$$z \mapsto \frac{az+b}{cz+d}.$$

So he and Klein were led to a *geometric classification* of such transformations. (Elliptic, parabolic, hyperbolic.)

This classification was crucial to Poincaré's first breakthrough in the theory of 3-manifolds.

Topology before Poincaré



Enrico Betti

Before 1892, the main tools for distinguishing manifolds were the *Betti numbers*, which count the number of "cuts" (by curves, surfaces, etc) needed to reduce a manifold to a simply connected region. This is now part of what is called *homology theory*.

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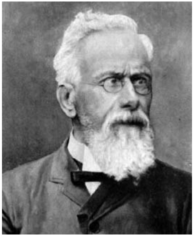

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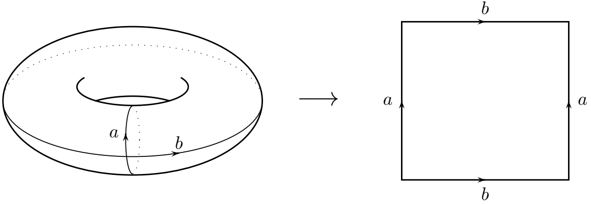
For example, the Betti number of the torus is 2, because two cuts (along the curves a, b shown in the picture) reduce it to a square.



The fundamental group, π_1 (1892)

In 1892, Poincaré introduced the fundamental group, which refines the idea from homology theory—that a manifold can be cut down to a simply connected region—by describing how the manifold may be reassembled from a "fundamental region" by identifying parts of its boundary.

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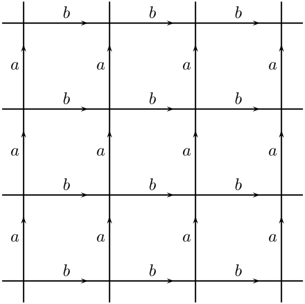
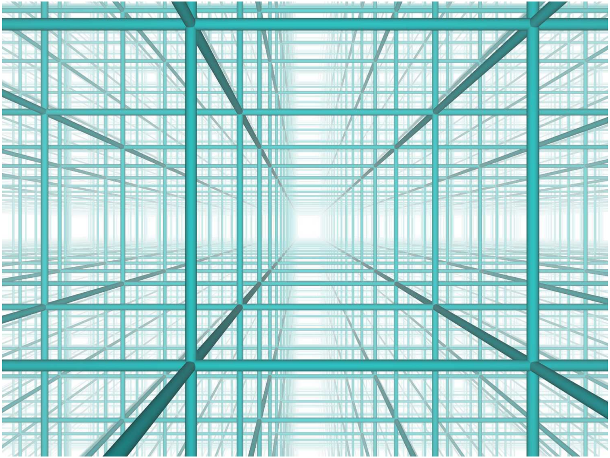
For example, a torus may be constructed from a square by identifying the left side with the right by a horizontal translation, and the bottom side with the top by a vertical translation.



These two translations generate a group (of transformations of the plane) called the *fundamental group of the torus*. Moreover, the translates of the original square make a *tessellation of the plane* that serves as a picture of π_1(torus).

The 3D analogue of the torus is obtained from a cube by pasting left side to right, front to back, and top to bottom.

From inside, the view is like a periodic $\mathbb{R}^3$.



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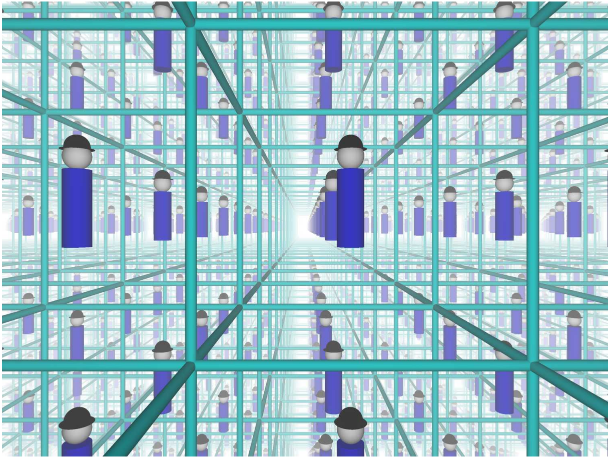

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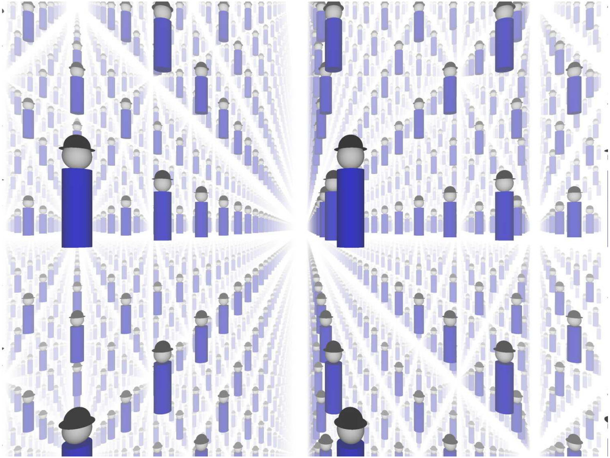

(With apologies to René Magritte)

Poincaré's first family of 3-manifolds

His 1892 example is a family of manifolds indexed by quadruples of integers $(\alpha, \beta, \gamma, \delta)$ with $\alpha\delta - \beta\gamma = 1$. Each is formed from the unit cube in $\mathbb{R}^3$ by

- Pasting the left face to the right by the map $(x, y, z) \mapsto (x + 1, y, z)$
- Pasting the front face to the back by the map $(x, y, z) \mapsto (x, y + 1, z)$
- Pasting the bottom face to the top by the map $(x, y, z) \mapsto (\alpha x + \beta y, \gamma x + \delta y, z + 1)$

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Notice that each horizontal section of this manifold is a torus; namely, a square with opposite sides identified.

The identification of the top face with the bottom is an essentially arbitrary map of the torus onto itself.

The fundamental group trumps the Betti numbers

Poincaré is able to compute the Betti numbers of his manifolds, and finds that there are only three possible sets of values. (Corresponding to the, at most, three generators, arising from the three perpendicular edges of the cube.)

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However, his manifolds have *infinitely many non-isomorphic fundamental groups*, corresponding to the conjugacy classes of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

To prove this, Poincaré draws on his knowledge of linear fractional transformations. The details are provided in Poincaré (1895).

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Thus there are 3-manifolds with the same Betti numbers but different π_1.

Simpler examples with same Betti numbers but different π_1

Poincaré (1895) also noticed examples of orientable 3-manifolds with different *finite* π_1, but necessarily with the same Betti numbers (zero).

For example

1. The 3-sphere has $\pi_1 =$ trivial group.
2. The real projective space has $\pi_1 =$ cyclic group of order 2. (Poincaré arrived at $\mathbb{RP}^3$ by identifying antipodal points of an octahedron. He didn't call it projective space.)

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Thus, in contrast with the case of orientable 2-manifolds, Betti numbers are not adequate to distinguish 3-manifolds, so that stronger invariants (such as π_1) are required.

Moreover, π_1 is "strictly greater" than Betti numbers for 3-manifolds, since the Betti numbers may be extracted from π_1 by "abelianizing" its defining relations.

Nevertheless, Poincaré is not done with homology theory

Betti numbers do not *quite* suffice to distinguish 2-manifolds—one also needs to know whether the manifold is orientable or not.

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In his 1898 thesis, Heegaard discovered that there is a similar extra ingredient in 3-dimensional homology theory, and Poincaré called it *torsion*.

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Heegaard's example was the 3-dimensional projective space, $\mathbb{RP}^3$, obtained by identifying antipodal points on the ball.

Like $\mathbb{S}^3$, $\mathbb{RP}^3$ has Betti number 0, but $\pi_1(\mathbb{RP}^3) =$ cyclic order 2, so $\mathbb{RP}^3$ has torsion number 2.

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Poincaré (1900) reworked his homology theory so as to extract "torsion numbers" as well as Betti numbers.

$\pi_1 \geq$ homology for 3-manifolds

Tietze (1908) observed that the 1-dimensional Betti and torsion numbers can be extracted from the abelianization of π_1.
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For 3-manifolds,

$$\text{2-dimensional Betti number} = \text{1-dimensional Betti number}$$

by Poincaré duality (which comes from dual polyhedra).
So all homological invariants can be extracted from π_1.

Thus $\pi_1 \geq$ homology for 3-manifolds.

When the homology concept is expanded to include torsion, the real projective space $\mathbb{RP}^3$ becomes distinguished from the 3-sphere $\mathbb{S}^3$, because $\mathbb{RP}^3$ has torsion and $\mathbb{S}^3$ does not.

In his 1900 paper, Poincaré conjectured that any 3-manifold with trivial homology is homeomorphic to the 3-sphere, $\mathbb{S}^3$. This was the first version of the Poincaré conjecture.

The Poincaré conjecture

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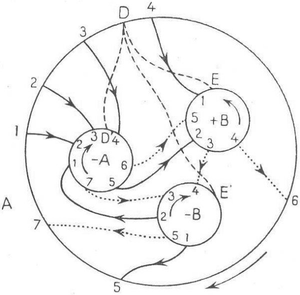
In 1904 he found a counterexample to this conjecture, the so-called Poincaré homology sphere, which has trivial H_1 (hence trivial homology) but nontrivial (yet finite) π_1.

He then stated his amended version of the Poincaré conjecture: any 3-manifold with trivial π_1 is homeomorphic to $\mathbb{S}^3$.

The Poincaré homology sphere



By a strange, asymmetric, unmotivated construction, Poincaré arrives at the π_1 with defining relations

$$a^4ba^{-1}b = b^{-2}a^{-1}ba^{-1} = 1$$

On the one hand, this group is nontrivial because setting $(a^{-1}b)^2 = 1$ gives the *icosahedral group*

$$a^5 = b^3 = (a^{-1}b)^2 = 1.$$

Indeed, π_1(Poincaré homology sphere) has 120 elements.

On the other hand, abelianizing gives $a = b = 1$, so $H_1 = \{1\}$.

If we lived in a Poincaré homology sphere ...

Space would seem to be periodic, with everything seen 120 times.

In a 3-sphere divided into dodecahedral cells.

Something like this:

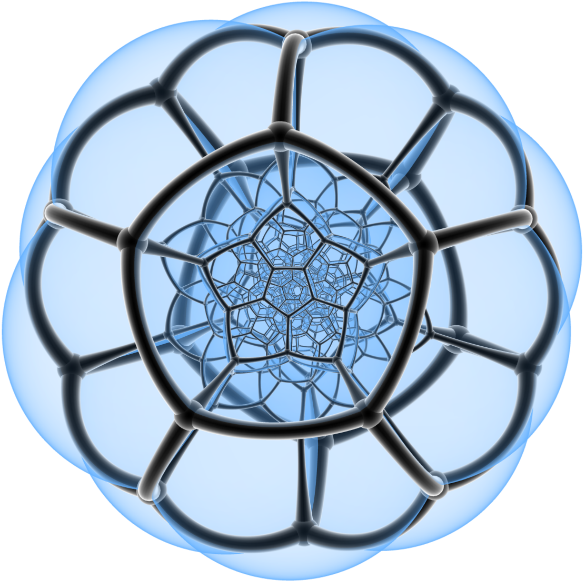
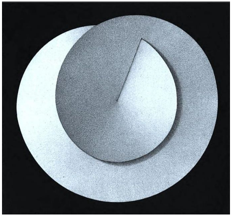


Image thanks to Jenn3D.

Heegard's 1898 thesis not only raised the issue of torsion; it introduced some new constructions of 3-manifolds, via branched coverings and Heegaard diagrams.



Orientable 2-manifolds (Riemann surfaces) can be obtained by covering $\mathbb{S}^2$ by copies of itself ("sheets") that meet at *branch points*. This example, from Neumann (1865), shows a branch point of a 2-sheeted covering.

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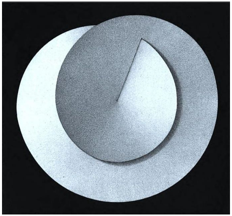

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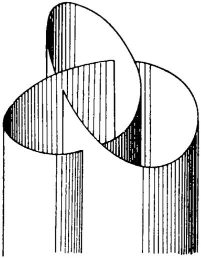

Just as $\mathbb{S}^2$ has coverings with branch *points*, $\mathbb{S}^3$ has coverings with branch *curves* (and "sheets" which are copies of $\mathbb{S}^3$). If the branch curve is a circle, then the covering is again $\mathbb{S}^3$.

But if the branch curve is *knotted* the covering can be a 3-manifold $\neq \mathbb{S}^3$.

Heegaard discovered that if the branch curve is a *trefoil knot*, then the double cover of $\mathbb{S}^3$ over this curve is *not* $\mathbb{S}^3$. (Thus he inadvertently proved that the trefoil knot really is knotted.)

The first theorem of knot theory

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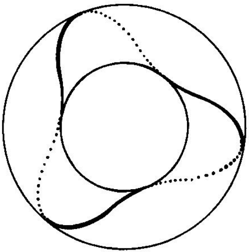

The double cover is also obtained by pasting two solid tori together. This Heegaard diagram shows where the meridian curve on one solid torus must be pasted on the other.

The double cover has torsion number 3. That is, it contains a curve which becomes a boundary when taken 3 times.

Max Dehn (1878 – 1952) and Heegaard

Dehn entered topology while still under the influence of Hilbert's *Grundlagen*. In 1899 he gave a proof of the polygonal Jordan curve theorem from Hilbert's axioms of incidence and order.

With Heegaard he set up combinatorial foundations for topology in their *Encyclopädie* article "Analysis situs" of 1907.

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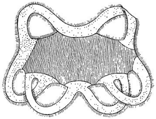

Fig. 8.

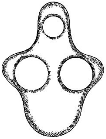

Fig. 9.

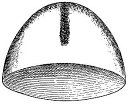

Fig. 10.

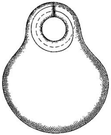

Fig. 11.

This led to more rigorous proofs of known results—for example, the classification of 2-manifolds—but not to any striking new results.

For example, they discuss knots, but only descriptively.

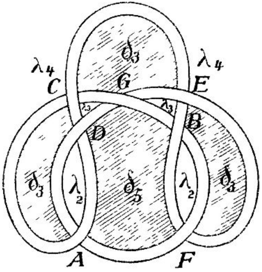

Fig. 17.

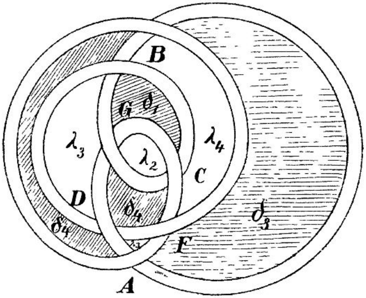

Fig. 18.

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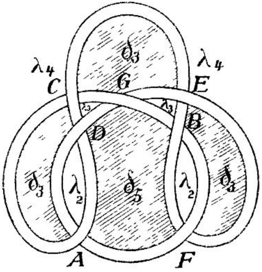

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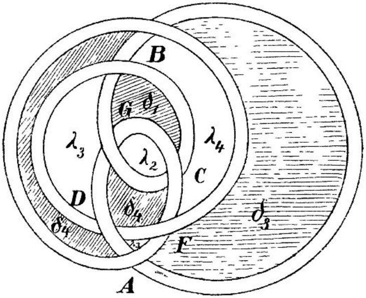

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Apparently, Heegaard did not realize the importance of knots, despite having used them to construct 3-manifolds.

In particular, the fundamental group is barely mentioned. And there is a faulty attempt to prove that the Poincaré homology sphere $\neq \mathbb{S}^3$ *without* using its fundamental group.

Dehn's first homology sphere

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The proof that the resulting manifold $\neq \mathbb{S}^3$ relies on the claim (proved only in 1924 by Alexander) that any torus in $\mathbb{S}^3$ bounds a solid torus on at least one side.

Ein sehr übersichtliches Beispiel für solche merkwürdigen Mannigfaltigkeiten kann man folgendermaßen konstruieren: Eine im gewöhnlichen Raume liegende in demselben *verknotete* Ringfläche T_2 zerlegt denselben in einen gewöhnlichen Ringraum T_3 und einen nicht mit T_3 homöomorphen Teil K_3. Mögen die auf T_2 liegenden, T_2 nicht zerstückelnden Kurven C resp. Γ in T_3 resp. K_3 begrenzen. Mit K_3 verschmelze man einen mit ihm homöomorphen Körper K_3', der von T_2' (mit den Kurven C' und Γ') begrenzt sein möge, und zwar derart, daß T_2 mit T_2', C' mit Γ, C mit Γ' zusammenfällt. In der entstehenden geschlossenen M_3 begrenzt jede Kurve einmal genommen. Aber sie ist nicht mit dem gewöhnlichen Raum homöomorph, denn die M_3 wird durch eine Ringfläche T_2 in zwei Teile K_3 und K_3' zerlegt, von denen keiner mit einem gewöhnlichen Ringraum homöomorph ist.

M. Dehn.

Dehn and the Poincaré conjecture

Another way to fill the gap in this proof is by the so-called *Dehn's lemma*, which Dehn introduced (though also with a faulty proof) in 1910.

In 1908, Dehn believed that he had proved the Poincaré conjecture, and submitted the proposed proof to Hilbert. However, Tietze found a mistake in it, and the paper was withdrawn.

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It seems that, at this stage, Tietze had a better grasp of topology than Dehn.

Where did Tietze gain his expertise?

Wilhelm Wirtinger (1865-1945)

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Wirtinger discovered, apparently in the 1890s, that *knots* occur in the singularities of algebraic curves.

For example, the intersection of the curve $y^2 = x^3$ with a small 3-sphere centered on its singularity at $x = y = 0$ is a *trefoil knot*.

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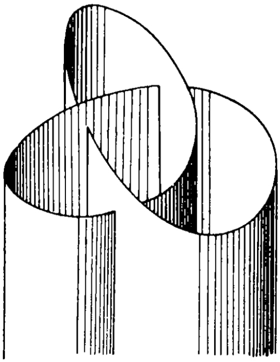
The results were not published by Wirtinger, but they gradually became known through his students and colleagues, among them Tietze, Reidemeister, and Schreier. Wirtinger's results were still coming to light as late as Brauner (1928).

The Wirtinger presentation



Die Erzeugenden der Fundamentalgruppe und die definierenden Relationen findet man nun wohl am einfachsten mit Hilfe einer Methode, die Herr WIRTINGER in seinen Vorlesungen entwickelt hat und deren Kenntnis ich Herrn SCHREIER verdanke.

At some stage, apparently after seeing Heegaard's thesis, Wirtinger gave a method for finding generators and relations for π_1 of a knot complement—the so-called *Wirtinger presentation*.

He uses Heegard's semicylinder to study loops around the knot and to find relations between them.

The method (and the diagram) did not come to light until Artin published it in his *Theorie der Zöpfe* in 1925.

The first major paper to combine ideas of Poincaré and Wirtinger was the Vienna *Habilitationschrifft* of Tietze (1908).

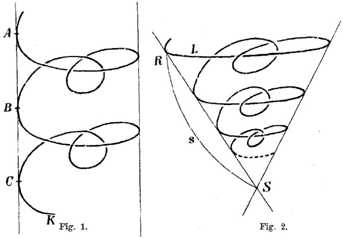


Fig. 1. Fig. 2.

Tietze gives several examples involving knots, including *wild knots* (left).

Also the first published proof that the trefoil knot *is* knotted, by showing that π_1 of the trefoil knot complement is not abelian.

From Wirtinger to Tietze (1880 – 1964)

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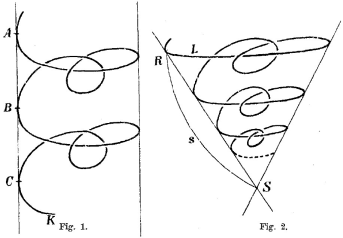

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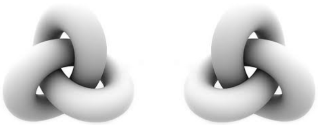

However, Tietze is unable to prove that the two trefoil knots are different, even though he thinks it “completely out of the question” for them to be the same.

Tiezte introduced a family of new 3-manifolds with finite π_1 —the (m, n) lens spaces—members of which had been obtained by Heegaard and Wirtinger as branched coverings.

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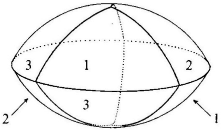

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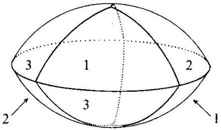

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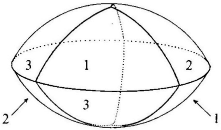

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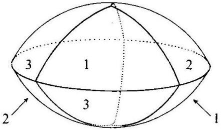

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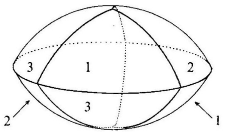
Each lens is a union of solid tori, i.e. it has Heegaard genus 1.

Decomposing a (3,1) lens space into solid tori



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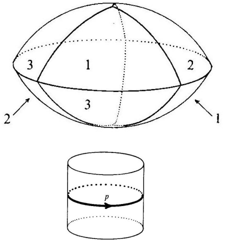

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The “core” of the lens becomes a solid torus.
Mark the curve p on it.

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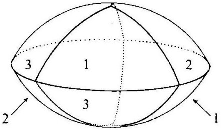

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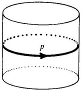

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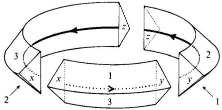

The curve p meets the "rim" pieces along the dark curve shown.

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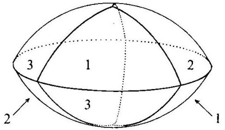

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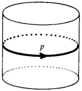

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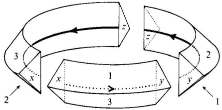

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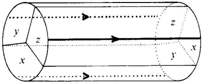

When the "rim" pieces are assembled, they too form a solid torus, and the dark curve p winds around it three times.

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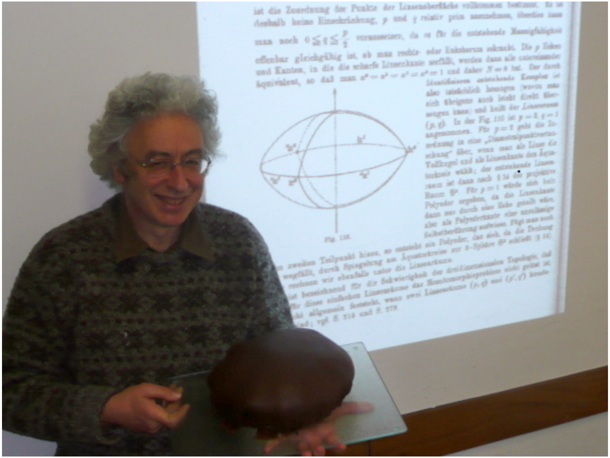

Tietze's critique of Poincaré

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(Found to be correct by Alexander in 1919).
- He introduces the *mapping class group* as another kind of group that is topologically invariant, and computes the mapping class group of the torus.

The isomorphism problem

The key difference between homology and π_1 is a matter of computability. The homology invariants, Betti and torsion numbers, are *integers* that we can compute and compare.

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Tietze (1908) is the first to realize that groups presented by generators and relations can be hard to analyse. He raises the *isomorphism problem*: given finite presentations of groups G_1 and G_2, decide whether $G_1 \cong G_2$.

He solves this problem in one direction, using what we now call *Tietze transformations*: if $G_1 \cong G_2$, then there is a finite sequence of Tietze transformations that converts the presentation of G_1 to the presentation of G_2.

Not solvable?

Nevertheless , Tietze dares to state that the isomorphism problem is in general "not solvable" (decades before a suitable definition of "solvable" became available, via Turing machines).

From page 80 of Tietze's *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*:

Da es anderseits, wie Poincaré[2]) gezeigt hat, Mannigfaltigkeiten gibt, welche gleiche Bettische Zahlen und Torsionszahlen, aber verschiedene Fundamentalgruppen aufweisen, so dient die Angabe der Fundamentalgruppe mehr zur Charakterisierung einer zweiseitigen geschlossenen dreidimensionalen Mannigfaltigkeit als die aller übrigen bis jetzt bekannten topologischen Invarianten zusammengenommen. Eine Einschränkung ist bei dieser Aussage allerdings zu machen. Während sich nämlich die Gleichheit von zwei Zahlenreihen stets feststellen läßt, ist (vgl. § 11) die Frage, ob zwei Gruppen isomorph seien, nicht allgemein lösbar. [redacted]

Dehn 1910, 1912

Dehn's most creative period opens when he learns some group theory, around 1910.

- He introduces his "Gruppenbilder," which allow groups to be viewed as geometric objects, often in one of the familiar geometries (spherical, euclidean or hyperbolic).

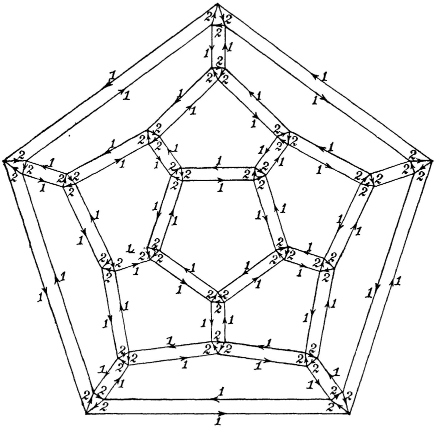

Fig. 7.

Gruppenbild for the rotation group of the dodecahedron

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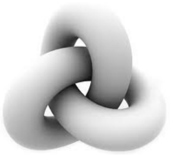
- Solves these problems for π_1 of 2-manifolds, first using hyperbolic geometry, and later by purely combinatorial arguments (Dehn's algorithm).
- Also, he finally becomes the master of homology spheres:



He finds a new construction of 3-manifolds by Dehn surgery, including an infinite class of homology spheres, and proves they are $\neq \mathbb{S}^3$ by finding their fundamental groups.

The two trefoil knots, 1914

Tietze (1908) remarked on the apparently obvious, but unproven, fact that a trefoil knot is different from its mirror image.

Dehn (1914) proved this, using the most sophisticated study of π_1 up to that time.

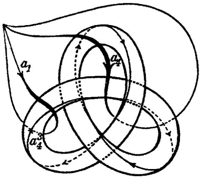
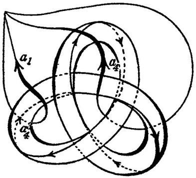


Fig. 9a.



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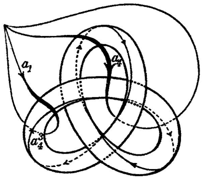

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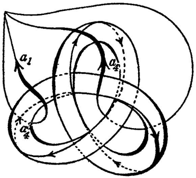
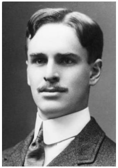


Fig. 9b.

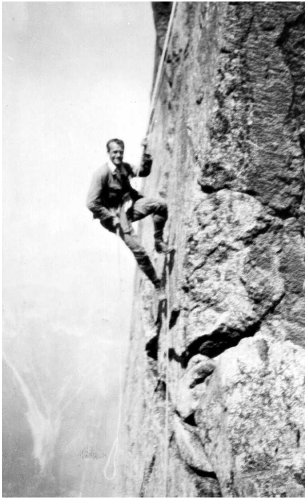
He showed that:

1. An isotopy from one trefoil knot to the other induces a certain kind of *automorphism* of its group.
2. By finding all automorphisms of the trefoil knot group, he is able to show that the required automorphism does not exist.



Alexander was a student of Veblen at Princeton, but his research career began in Paris and Bologna during World War I, and ended at the Institute of Advanced Study.

L to R: Alexander, Morse, Einstein, Aydelotte, Weyl, Veblen



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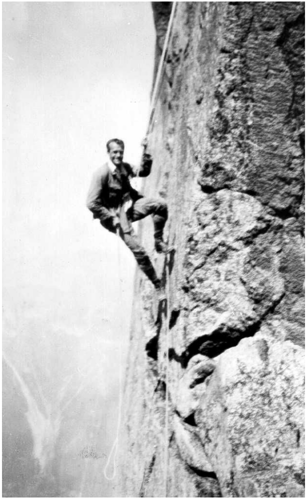

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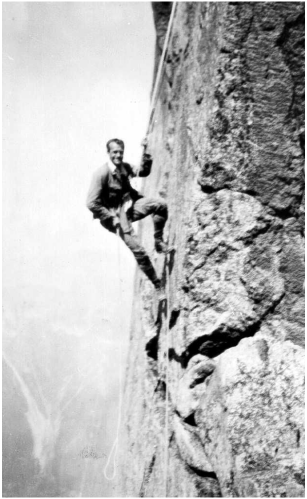

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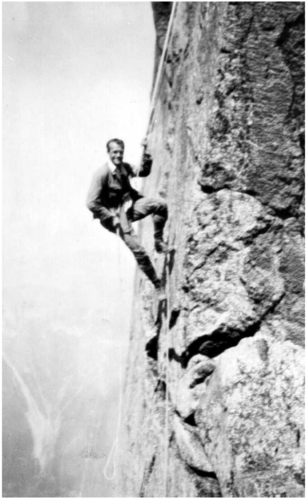
Clearing up problems left by Poincaré, Tietze, and Dehn



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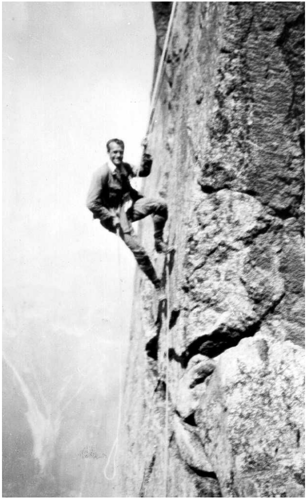

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- 1920 The first computable knot invariants—torsion numbers of cyclic covers of $\mathbb{S}^3$ branched over a knot.
- 1924 Any torus embedded in $\mathbb{S}^3$ bounds a solid torus on at least one side.