Quantum unique factorisation domains

S Launois, T H Lenagan and L Rigal *

Abstract

We prove a general theorem showing that iterated skew polynomial extensions of the type which fit the conditions needed by Cauchon's deleting derivations theory and by the Goodearl-Letzter stratification theory are unique factorisation rings in the sense of Chatters and Jordan. This general result applies to many quantum algebras; in particular, generic quantum matrices and quantized enveloping algebras of the nilpotent part of a semisimple Lie algebra are unique factorisation domains in the sense of Chatters. The result also extends to generic quantum grassmannians (by using noncommutative dehomogenisation) and to the quantum groups $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$.

2000 Mathematics subject classification: 16W35, 16P40, 16S38, 17B37, 20G42.

Key words: Unique factorisation domain, quantum algebra, quantum matrices, quantum grassmannian, quantum enveloping algebras.

Introduction

In [4], Chatters introduced the notion of a noncommutative unique factorisation domain in the following way. An element p of a noetherian domain R is said to be *prime* if (i) $pR = Rp$, (ii) pR is a height one prime ideal of R, and (iii) R/pR is an integral domain. A noetherian domain R is then said to be a *unique factorisation domain*, noetherian UFD for short, if R has at least one height one prime ideal, and every height one prime ideal is generated by a prime element. As well as the usual commutative noetherian UFDs, examples include universal enveloping algebras of finite dimensional solvable Lie algebras over $\mathbb{C}$. However,

*This work was supported by Leverhulme Research Interchange Grant F/00158/X

one of the deficiences of this definition is that the class of noetherian UFDs is not closed under polynomial extensions, as [4, Example 2.11] shows. The problem is that the condition of height one prime factors being domains does not pass up to polynomial extensions.

In order to remedy this deficiency, in a later paper, the notion of a *noetherian unique factorisation ring*, noetherian UFR for short, was introduced by Chatters and Jordan, [5]. For a large class of rings (namely, the noetherian prime rings satisfying the descending chain condition on prime ideals), being a noetherian unique factorisation ring amounts having height one primes principal (that is generated by a normal element). This condition is closed under polynomial extensions, and, indeed, they then are able to prove theorems about skew polynomial extensions of the type $R[x;\sigma]$ and $R[x;\delta]$. However, they do not prove any results about general skew polynomial extensions of type $R[x;\sigma,\delta]$.

In many quantum algebras, in the generic case where the deformation parameter q is not a root of unity, it is known that all prime ideals are completely prime, and then the distinction between a noetherian domain being a noetherian UFD and a noetherian UFR disappears and so the results of [5] on noetherian UFRs also apply to noetherian UFDs in this setting.

The purpose of this paper is to obtain a theorem on unique factorisation for certain extensions of the type $R[x;\sigma,\delta]$ that arise naturally in the study of quantum algebras. Once this theorem is proved, an iterated version is obtained which is sufficient to show that many quantum algebras are noetherian UFDs. In particular, we show that the algebra of generic quantum matrices, $\mathcal{O}_q(M_{m,n})$ is a noetherian UFD, as is the quantized enveloping algebra $U_q^+(\mathfrak{g})$.

Roughly speaking, an iterated skew polynomial extension will be a noetherian UFD provided that the Cauchon theory of deleting derivations can be applied, and that there is a torus action for which the Goodearl-Letzer stratification theory applies. Exact requirements will be given as they become necessary.

In the case of quantum matrices, we can go further, since we can identify the height one prime ideals that are $\mathcal{H}$-primes for the natural torus that acts.

In the two final sections, we show that, in the generic case, quantum grassmannians as well as the quantum groups $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are noetherian UFDs. To deal with the case of the generic quantum grassmannians, we use the idea of noncommutative dehomogenisation, developed in [13].

For general results concerning noetherian rings and localisation, we refer the reader to [12] or [18].

Throughout the paper, k denotes a field.

1 Non commutative unique factorisation rings

This section investigates the behaviour of the notion of a *noetherian unique factorisation ring*, as defined in [5] by Chatters and Jordan, under localisation by normal elements.

To start with, we recall the definition of noetherian unique factorisation ring; further details concerning this notion can be found in [5].

An ideal I in a ring A is called *principal* if there exists a normal element x in A such that $I = \langle x \rangle$ $(= xA = Ax)$.

Definition 1.1 *A ring A is called a noetherian unique factorisation ring (noetherian UFR for short) if: (i) A is a prime noetherian ring, and*
(ii) any nonzero prime ideal in A contains a nonzero principal prime ideal.

Definition 1.2 *A noetherian UFR A is said to be a unique factorisation domain (noetherian UFD for short) if A is a domain and each height one prime ideal P of A is completely prime; that is, A/P is a domain for each height one prime ideal P of A.*

Remark 1.3 If A is a prime noetherian ring that satisfies the descending chain condition for prime ideals, then A is a noetherian UFR if and only if height one primes are principal (see [5]). Hence, the notions of noetherian UFR and noetherian UFD are good generalisations of the usual notion of unique factorisation domain for commutative rings (see in particular Corollaries 10.3 and 10.6 in [7]).

Note that the algebras we are dealing with are all noetherian and have finite Gelfand-Kirillov dimension; so, they satisfy the descending chain condition for prime ideals, see for example, [14, Corollary 3.16].

We start by proving a noncommutative analogue of Nagata's Lemma (in the commutative case, see [7] 19.20 p. 487). The following result is taken from [6], where it appears without proof. We include a proof here, for the convenience of the reader, since it is crucial to a part of our argument.

If A is a prime noetherian ring and x a nonzero normal element of A, we denote by A_x the right localisation of A with respect to the powers of x.

Lemma 1.4 *Let A be a prime noetherian ring and x a nonzero, nonunit, normal element of A such that $\langle x \rangle$ is a completely prime ideal of A.*
(i) If P is a prime ideal of A not containing x and such that the prime ideal PA_x of A_x is

principal, then P is principal.
(ii) If A_x is a noetherian unique factorisation ring, then so is A.
(iii) If A_x is a noetherian unique factorisation domain, then so is A.

Proof. (i) The result is trivial if $P = 0$, so we assume that $P \neq 0$. Since x is a nonzero normal element of the prime ring A one may localise A with respect to the multiplicative set of powers of x and there is canonical embedding $A \hookrightarrow A_x$. Moreover, $Q := PA_x$ is a prime ideal of A_x whose contraction to A is P, since P is a prime ideal of A not containing x . Let us suppose that Q is a principal ideal. Then, clearly, there exists $q \in A$, normal in A_x, such that $Q = qA_x$. Moreover, one may assume the right ideal qA maximal for this property, since A is right noetherian. Suppose that $q \in Ax$. Then there exists p in A such that $q = px$ (in particular $qA \subseteq pA$). But then, $Q = pA_x$ and p is normal in A_x. The maximality of qA leads to $qA = pA$ from which follows the existence of $r \in A$ such that $p = qr$ and hence $q = qrx$. Since q is a non-zero normal element in the prime ring A_x, the above equality gives $1 = rx$ (with $r \in A$), a contradiction, since x is not a unit. Thus, $q \notin Ax$. Now, let $p \in P \subseteq Q$; so that there exist $r \in A$ and $t \in \mathbb{N}$ with $p = qrx^{-t}$, and we may choose t minimal for this property. If $t > 0$ then $r \notin Ax$, by the minimality of t. The above equation then leads to $px^t = qr$; and so either q or r must be in Ax which is a contradiction. Thus, $t = 0$ and so $p \in qA$. Hence, $P \subseteq qA$. Also, $qA \subseteq qA_x \cap A = Q \cap A = P$; so that $P = qA$. A similar argument gives $P = Aq$. Hence $P = Aq = qA$ which proves the first claim.
(ii) Let us now assume that A_x is a noetherian UFR. If Q_0 is a non-zero prime ideal of A not containing x, then Q_0A_x is a non-zero prime ideal of A_x. Since A_x is a noetherian UFR, Q_0A_x contains a nonzero principal prime ideal P which is the extension to A_x of its contraction P_0 in A. By part (i), the ideal P_0 is principal, since P is principal. Thus, P_0 is a nonzero principal prime ideal contained in Q_0. Moreover, if Q_0 is a prime ideal of A containing x, then it contains the nonzero principal prime ideal $\langle x \rangle$. We have proved that each nonzero prime ideal of A contains a nonzero principal prime ideal, which means that A is a noetherian UFR.
(iii) Suppose that A_x is a noetherian UFD. Then part (ii) shows that A is a noetherian UFR. Let P be a prime ideal of height one in A. If $x \in P$ then $P = \langle x \rangle$ and so P is completely prime, by assumption. Otherwise, standard localisation theory shows that PA_x is a prime ideal of height one in A_x and that $P = PA_x \cap A$. Thus, A/P embeds in A_x/PA_x, which is a domain; and so A/P is a domain, as required. □

Proposition 1.6 below will be of central use later. It gives a way to pull back the unique factorisation property from a certain type of localisation to the initial ring. The following lemma is needed in the proof of the proposition.

Lemma 1.5 *Let R be a prime noetherian ring and suppose that d, s are normal elements of R such that dR is prime and $s \notin dR$. Then, there exist units $u, v \in R$ such that $ds = sdu$ and $sd = vds$.*

Proof. If either d or s is zero, then the result is trivial; so we assume that $d, s \neq 0$. Since s is normal in a prime ring, s is regular and we can associate to it an automorphism $\sigma : R \longrightarrow R$ such that $xs = s\sigma(x)$, for all $x \in R$. Set $P := dR = Rd$. Then $s\sigma(P) = Ps \subseteq P$; and so $\sigma(P) \subseteq P$, since s is normal and not in P. Hence, $P \subseteq \sigma^{-1}(P)$, and it follows that there is an ascending chain $P \subseteq \sigma^{-1}(P) \subseteq \sigma^{-2}(P) \subseteq \ldots$ of ideals of R. The noetherian hypothesis then ensures that there exists $n \in \mathbb{N}$ such that $\sigma^{-n}(P) = \sigma^{-(n+1)}(P)$, and so $\sigma(P) = P$; that is, $\sigma(d)R = dR$. From this it follows that $dsR = sdR$, which gives the existence of $u, u' \in R$ such that $ds = sdu$ and $sd = dsu'$. But then, $ds = sdu = dsu'u$; and so $u'u = 1$ which shows u is a unit in R. We also have $Rds = Rsd$, since d and s are normal, and it follows in a similar manner that there exists a unit v in R such that $sd = vds$. $\square$

Proposition 1.6 *Let R be a prime noetherian ring and suppose that $d_1, \ldots, d_t$ are nonzero normal elements of R such that the ideals $d_1R, \ldots, d_tR$ are completely prime and pairwise distinct. Denote by T the right quotient ring of R with respect to the right denominator set generated by $d_1, \ldots, d_t$. If T is a noetherian UFR then so is R. Also, if T is a noetherian UFD then so is R.*

Proof. We proceed by induction on t, the result being true for $t = 1$ by Lemma 1.4 (ii). Assume that the result is true up to order $t \in \mathbb{N}^*$. We will work in the right quotient ring of fractions of R in which all the localisations of R are naturally embedded. Denote by $\mathcal{S}_{t+1}$ the multiplicative subset of R generated by $d_1, \ldots, d_{t+1}$ and by $\mathcal{S}_t$ the multiplicative subset of R generated by $d_1, \ldots, d_t$. Hence $T = R\mathcal{S}_{t+1}^{-1}$. We first show, using the above lemma, that d_{t+1} is a nonzero normal element of $R\mathcal{S}_t^{-1}$. Let $(a, s) \in R \times \mathcal{S}_t$; hence s is normal in R and, due to the hypothesis that the ideals d_iR are completely prime and pairwise distinct, $s \notin d_{t+1}R$ (by the principal ideal theorem). So, by the lemma above, there exist elements $u, v \in R$ such that $d_{t+1}s = sd_{t+1}u$ and $sd_{t+1} = vd_{t+1}s$. In addition, since d_{t+1} is normal in R, there exist $b, c \in R$ such that $ad_{t+1} = d_{t+1}b$ and $d_{t+1}a = cd_{t+1}$. Hence, we have $as^{-1}d_{t+1} = ad_{t+1}us^{-1} = d_{t+1}bus^{-1}$ and $d_{t+1}as^{-1} = cd_{t+1}s^{-1} = cs^{-1}vd_{t+1}$. It follows that d_{t+1} is indeed a nonzero normal element of $R\mathcal{S}_t^{-1}$.

Let $\mathcal{S}$ be the multiplicative subset of $R\mathcal{S}_t^{-1}$ generated by d_{t+1}. Notice that, $R\mathcal{S}_{t+1}^{-1} = (R\mathcal{S}_t^{-1})\mathcal{S}^{-1}$, as is easily verified. Of course, $R\mathcal{S}_t^{-1}$ is prime noetherian and the ideal $d_{t+1}R\mathcal{S}_t^{-1}$ is completely prime since $d_{t+1}R$ is completely prime and does not intersect $\mathcal{S}_t$.

Now assume that $T = (R\mathcal{S}_t^{-1})\mathcal{S}^{-1}$ is a noetherian UFR. By the comments above, Lemma 1.4 (ii) may be applied and we get that $R\mathcal{S}_t^{-1}$ is a noetherian UFR. Now, the induction hypothesis gives that R is a noetherian UFR, as required.

Finally, suppose that T is a noetherian UFD. Then T is certainly a noetherian UFR; and so R is a noetherian UFR, by the first part of this result. That R is a noetherian UFD then follows by standard localisation theory (cf. the proof of Lemma 1.4 (iii)). $\square$

2 Height one $\mathcal{H}$-primes in Cauchon extensions

Most of the algebras that we are considering in this paper have groups acting on them in natural ways. The study of the prime spectra of such algebras is often facilitated by first studying ideals invariant under the natural group action. We begin this section by recalling some standard terminology concerning ideals invariant under group actions. A convenient reference is [1, II.1.8, II.1.9]. Let $\mathcal{H}$ be a group acting by automorphisms on a ring R. An ideal I of R is an $\mathcal{H}$*-ideal* provided that $h(I) = I$ for all $h \in \mathcal{H}$. A proper $\mathcal{H}$-ideal is an $\mathcal{H}$*-prime ideal* provided that whenever $IJ \subseteq P$ for $\mathcal{H}$-ideals I, J of R then either $I \subseteq P$ or $J \subseteq P$. The set of $\mathcal{H}$-prime ideals of R is denoted by $\mathcal{H}-\mathrm{Spec}(R)$. It is obvious that a prime ideal P that is an $\mathcal{H}$-ideal is an $\mathcal{H}$-prime ideal. The converse is not true in general; however, it will usually be true for the algebras that interest us in this paper (see comments after Definition 3.1).

Hypothesis 2.1 *Let A be a domain that is a noetherian k-algebra and suppose that σ is a k-automorphism of A. Suppose that there is a group $\mathcal{H}$ acting as automorphisms on the skew Laurent extension $A[X^{\pm 1}; \sigma]$ in such a way that X is an $\mathcal{H}$-eigenvector and A is stable under $\mathcal{H}$. Further, suppose that the action of σ on A coincides with the action of an element $h_0 \in \mathcal{H}$. Finally, suppose that there is a non root of unity λ_0 in k^* such that $h_0.X = \lambda_0 X$.*

Given the conditions of this hypothesis, we are going to show that there is a bijection between the $\mathcal{H}$-ideals of A and the $\mathcal{H}$-ideals of $A[X^{\pm 1}; \sigma]$, and, consequently, there is a bijection between $\mathcal{H}-\mathrm{Spec}(A)$ and those $\mathcal{H}$-primes of $A[X; \sigma]$ that do not contain X.

Lemma 2.2 *Assume Hypothesis 2.1 and let I be an $\mathcal{H}$-ideal of $A[X^{\pm 1};\sigma]$. Suppose that $x = a_1 X^{k_1} + \cdots + a_n X^{k_n} \in I$, with $a_i \in A$ and k_i all distinct. Then, each $a_i \in I \cap A$. Consequently, $I = \oplus_{i\in\mathbb{Z}} (I \cap A)X^i$.*

Proof. The proof is by induction on n. If $n = 1$ the result is trivial, since X is invertible. Suppose now that $n > 1$. Since I is an $\mathcal{H}$-ideal, the element $Xx - \lambda_0^{-k_n} h_0(x)X$ belongs to I. However,

$$\begin{aligned} Xx - \lambda_0^{-k_n} h_0(x)X &= \sum_{i=1}^{n} h_0(a_i)X^{k_i+1} - \sum_{i=1}^{n} \lambda_0^{k_i-k_n} h_0(a_i)X^{k_i+1} \\ &= \sum_{i=1}^{n-1} (1 - \lambda_0^{(k_i-k_n)})h_0(a_i)X^{k_i+1}; \end{aligned}$$

so that $\sum_{i=1}^{n-1}(1 - \lambda_0^{(k_i-k_n)})h_0(a_i)X^{k_i+1} \in I$. By the induction hypothesis, we see that $(1-\lambda_0^{(k_i-k_n)})h_0(a_i) \in I$ for each $1 \le i \le n-1$. The elements $(1-\lambda_0^{(k_i-k_n)})$ are nonzero, since λ_0 is not a root of unity and the k_i are distinct. Thus, each $h_0(a_i)$ is in the $\mathcal{H}$-ideal $I \cap A$, and so each $a_i \in I \cap A$ for $1 \le i \le n-1$. Finally, $a_n X^{k_n} = x - a_1 X^{k_1} - \cdots - a_{n-1}X^{k_{n-1}} \in I$; and so $a_n \in I \cap A$ also. $\square$

The next result follows easily from this lemma.

Theorem 2.3 *Assume Hypothesis 2.1. Then there is an inclusion preserving bijection from the set of $\mathcal{H}$-ideals of A to the set of $\mathcal{H}$-ideals of $A[X^{\pm 1};\sigma]$ given by $I \mapsto \oplus_{i\in\mathbb{Z}} IX^i$; its inverse is defined by $J \mapsto J \cap A$. Furthermore, these bijections induce order preserving bijections between $\mathcal{H}-\mathrm{Spec}(A)$ and $\mathcal{H}-\mathrm{Spec}A[X^{\pm 1};\sigma]$.*

Let $\mathcal{H}$ be a group acting by automorphisms on a noetherian ring R and suppose that X is a normal $\mathcal{H}$-eigenvector. Then there is a bijective correspondence between the $\mathcal{H}$-prime ideals of R that do not contain X and the $\mathcal{H}$-prime ideals of $R[X^{-1}]$, cf [1, Exercise II.1.J]. Using this fact, the next corollary follows easily.

Corollary 2.4 *Assume Hypothesis 2.1. Then contraction $P \mapsto P \cap A$ and extension $P \mapsto \oplus_{i\ge 0} PX^i$ provide inverse order preserving bijections between the $\mathcal{H}$-prime ideals of $A[X;\sigma]$ that do not contain X and $\mathcal{H}-\mathrm{Spec}(A)$.*

Definition 2.5 Let A be a domain that is a noetherian k-algebra and let $R = A[X;\sigma,\delta]$ be a skew polynomial extension of A. We say that $R = A[X;\sigma,\delta]$ is a *Cauchon Extension* provided that

- σ is a k-algebra automorphism of A and δ is a k-linear locally nilpotent σ-derivation of A. Moreover we assume that there exists $q \in k^*$ which is not a root of unity such that $\sigma \circ \delta = q\delta \circ \sigma$.

- There exists an abelian group $\mathcal{H}$ which acts on R by k-algebra automorphisms such that X is an $\mathcal{H}$-eigenvector and A is $\mathcal{H}$-stable.

- σ coincides with the action on A of an element $h_0 \in \mathcal{H}$.

- Since X is an $\mathcal{H}$-eigenvector and since $h_0 \in \mathcal{H}$, there exists $\lambda_0 \in k^*$ such that $h_0.X = \lambda_0 X$. We assume that λ_0 is not a root of unity.

- Every $\mathcal{H}$-prime ideal of A is completely prime.

Note that the conditions of [1, II.5.3] are satisfied by any Cauchon extension; and so, for example, every $\mathcal{H}$-prime of R is also completely prime, by [1, Proposition II.5.11].

In a Cauchon extension $R = A[X; \sigma, \delta]$ the set $S = \{X^n \mid n \in \mathbb{N}\}$ is a right and left Ore set in R, [2, Lemme 2.1]; and so we can form the Ore localization $\widehat{R} := RS^{-1} = S^{-1}R$.

For each $a \in A$, set

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{-n} \in \widehat{R}$$

Note that $\theta(a)$ is a well-defined element of $\widehat{R}$, since δ is locally nilpotent, q is not a root of unity, and $0 \neq 1 - q \in k$.

The following facts are established in [2, Section 2]. The map $\theta : A \longrightarrow \widehat{R}$ is a k-algebra monomorphism. Let $A[Y; \sigma]$ be a skew polynomial extension. Then θ extends to a monomorphism $\theta : A[Y; \sigma] \longrightarrow \widehat{R}$ with $\theta(Y) = X$. Set $B = \theta(A)$ and $T = \theta(A[Y; \sigma]) \subseteq \widehat{R}$. Then $T = B[X; \alpha]$, where α is the automorphism of B defined by $\alpha(\theta(a)) = \theta(\sigma(a))$.

The element X is a normal element in T, and so the set S is an Ore set in T and Cauchon shows that $TS^{-1} = S^{-1}T = \widehat{R}$.

Since X is an $\mathcal{H}$-eigenvector, it follows from [1, Exercise II.1.J] that $\mathcal{H}$ also acts by automorphisms on $\widehat{R}$. Moreover, the following result shows that the group $\mathcal{H}$ also acts by automorphisms on T and B by restriction.

Note, for later use, that, since each element of $B = \theta(A)$ is of the form $\theta(a) = \sum_{i=0}^{n} a_i X^{-i}$ for some $a_i \in A$, and each element of R is of the form $\sum_{i=0}^{n} c_i X^i$ for some $c_i \in A$, it follows that $B \cap R \subseteq A$.

The next result shows that the action of $\mathcal{H}$ can be transferred to B via θ. This result is essentially a generalisation of [2, Proposition 2.1].

Lemma 2.6 *Let $R = A[X;\sigma,\delta]$ be a Cauchon extension and let $h \in \mathcal{H}$. Then $h.\theta(a) = \theta(h.a)$ for each $a \in A$.*

Proof. We start by showing inductively that $h.\delta^n(a) = \lambda_h^n \delta^n(h.a)$ for all $n \in \mathbb{N}$, $a \in A$ and $h \in \mathcal{H}$, where λ_h denotes the $\mathcal{H}$-eigenvalue associated to the $\mathcal{H}$-eigenvector X.

If $n = 0$, there is nothing to prove. Now we assume that $n \geq 1$. Then, since $\delta^n(a) = X\delta^{n-1}(a) - \sigma \circ \delta^{n-1}(a)X = X\delta^{n-1}(a) - q^{n-1}\delta^{n-1}(\sigma(a))X$, we deduce from the induction hypothesis that

$$h.\delta^n(a) = \lambda_h X \lambda_h^{n-1}\delta^{n-1}(h.a) - q^{n-1}\lambda_h^{n-1}\delta^{n-1}(h.\sigma(a))\lambda_h X.$$

Since $h.\sigma(a) = hh_0.a = h_0h.a = \sigma(h.a)$, this leads to

$$\begin{aligned} h.\delta^n(a) &= \lambda_h^n \left[X\delta^{n-1}(h.a) - q^{n-1}\delta^{n-1}(\sigma(h.a))X\right] \\ &= \lambda_h^n \left[X\delta^{n-1}(h.a) - \sigma \circ \delta^{n-1}(h.a)X\right] = \lambda_h^n\delta^n(h.a). \end{aligned}$$

This achieves the induction.

Now, let $a \in A$. Then, using the notations of [2], we have

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q}\delta^n \circ \sigma^{-n}(a)X^{-n}.$$

Hence we get

$$h.\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} h.\left[\delta^n \circ \sigma^{-n}(a)\right] h.X^{-n}.$$

Then the previous study shows that

$$h.\theta(a) = \sum_{n=0}^{+\infty} \lambda_h^n \frac{(1-q)^{-n}}{[n]!_q}\delta^n(h.\sigma^{-n}(a))\lambda_h^{-n}X^{-n}.$$

Now, since σ coincide with the action of $h_0 \in \mathcal{H}$ on A, we have $h.\sigma^{-n}(a) = hh_0^{-n}.a = h_0^{-n}h.a = \sigma^{-n}(h.a)$, so that

$$h.\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q}\delta^n \circ \sigma^{-n}(h.a)X^{-n},$$

that is, $h.\theta(a) = \theta(h.a)$ as desired.

□

Note that for $b \in B$ with $b = \theta(a)$, we have $\alpha(b) = \alpha(\theta(a)) = \theta(\sigma(a)) = \theta(h_0.a) = h_0.\theta(a) = h_0.b$; so that the action of α on B coincides with the action of h_0.

The above lemma shows that the action of $\mathcal{H}$ on $\widehat{R}$ by automorphisms induces an action of $\mathcal{H}$ on B by automorphisms. Further, since $T = B[X;\alpha]$ and since X is an $\mathcal{H}$-eigenvector, this observation also proves that the action of $\mathcal{H}$ on $\widehat{R}$ by automorphisms induces an action of $\mathcal{H}$ on T by automorphisms. Moreover, since every $\mathcal{H}$-prime ideal of A is completely prime, we deduce that every $\mathcal{H}$-prime ideal of $B = \theta(A)$ is completely prime. Then, it follows from [1, Proposition II.5.11] that every $\mathcal{H}$-prime ideal of $T = B[X;\alpha]$ is also completely prime.

Let $b \in B$ be an $\mathcal{H}$-eigenvector, say $h.b = \lambda_h b$ for $\lambda_h \in k$, and suppose that $b = \theta(a)$. Then $\theta(h.a - \lambda_h a) = h.\theta(a) - \lambda_h \theta(a) = h.b - \lambda_h b = 0$; so $h.a = \lambda_h a$ and a is an $\mathcal{H}$-eigenvector with the same action of $\mathcal{H}$ on a as on b.

Definition 2.7 Suppose that A is a noetherian domain that is a k-algebra and suppose that $\mathcal{H}$ is a group acting on A via k-automorphisms. Then A is an *$\mathcal{H}$-UFD* if each nonzero $\mathcal{H}$-prime Q of A contains a nonzero normal $\mathcal{H}$-eigenvector x such that the $\mathcal{H}$-ideal $xA = Ax$ is completely prime.

Remark 2.8 In particular, in an $\mathcal{H}$-UFD, all $\mathcal{H}$-primes of height one as $\mathcal{H}$-primes have height one as ordinary prime ideals, by the principal ideal theorem. Thus, an ideal is an $\mathcal{H}$-prime of height one as an $\mathcal{H}$-prime if and only if it is a prime $\mathcal{H}$-ideal of height one as an ordinary prime ideal. Also, in an $\mathcal{H}$-UFD, the $\mathcal{H}$-primes of height one are principal, generated by a normal element, and completely prime.

Proposition 2.9 *Let $R = A[X;\sigma,\delta]$ be a Cauchon extension. Suppose that A is an $\mathcal{H}$-UFD. Then R is an $\mathcal{H}$-UFD.*

Proof. Since B is isomorphic to A via θ and θ preserves the $\mathcal{H}$-action, we know that every non-zero $\mathcal{H}$-prime of B contains a non-zero normal $\mathcal{H}$-eigenvector b such that $bB = Bb$ is a completely prime ideal; that is, B is an $\mathcal{H}$-UFD. We start by showing that such an element b of B can be used to produce, in a natural way, an element of R with similar properties.

Note, that every $\mathcal{H}$-prime ideal of A and B is completely prime, since this is one of the properties of A being part of a Cauchon extension and $B \cong A$ via a map compatible with the $\mathcal{H}$-actions.

Let $b \in B$. Then $b \in B \subseteq T \subseteq \widehat{R} = RS^{-1}$; and so there exists $n \geq 0$ with $bX^n \in R$.

Now, suppose that $0 \neq b \in B$ is a normal $\mathcal{H}$-eigenvector such that $bB = Bb$ is a completely prime ideal. Choose $s \geq 0$ minimal such that $x := bX^s \in R$. We will show that x is a normal $\mathcal{H}$-eigenvector in R such that $xR = Rx$ is a completely prime ideal.

First, note that x is an $\mathcal{H}$-eigenvector, since each of b and X is an $\mathcal{H}$-eigenvector. Next,

$$Xb = \alpha(b)X = h_0(b)X = \eta bX$$

for some $0 \neq \eta \in k$, since b is an $\mathcal{H}$-eigenvector.

Hence, b is normal in $T = B[X;\alpha]$. Also, $bT = Tb$ is a completely prime $\mathcal{H}$-ideal of T.

It follows that $b\widehat{R} = \widehat{R}b$ is a completely prime $\mathcal{H}$-ideal of $\widehat{R}$. However, $x\widehat{R} = bX^s\widehat{R} = b\widehat{R}$; and so $x\widehat{R}$ is a completely prime $\mathcal{H}$-ideal of $\widehat{R}$. Thus, $I := x\widehat{R} \cap R = b\widehat{R} \cap R$ is a completely prime $\mathcal{H}$-ideal of R. We will show that $I = Rx$.

It is obvious that $Rx \subseteq I$. For the reverse inclusion, let $y \in I$. Then $y \in b\widehat{R}$ and so there exists $u \geq 0$ such that $yX^u \in bT = Tb$. Thus, there exists $c \in T$ such that $yX^u = cb$. Next, since $c \in T \subseteq RS^{-1}$, there exists $v \geq 0$ such that $cX^v \in R$. Set $r := \eta^{-v}cX^v \in R$. Then, by using the fact that $Xb = \eta bX$, we get $yX^{u+v+s} = cbX^{v+s} = \eta^{-v}cX^v bX^s = rx$; and so there exists $t \geq 0$ such that $yX^t = rx$ with $r \in R$. Choose such a t minimal.

Assume that $t \geq 1$. Express r, y and x as elements in the Ore extension $R = A[X;\sigma,\delta]$, say,

$$r = \sum_{i=0}^{d} r_i X^i, \quad y = \sum_{i=0}^{d} y_i X^i \quad \text{and} \quad x = \sum_{i=0}^{d} x_i X^i,$$

where $d \geq 0$ and $r_i, y_i, x_i \in A$ for all $0 \leq i \leq d$. If $s = 0$, then $x = b \in B \cap R \subseteq A$, and so $x_0 = b \neq 0$. If $s \geq 1$ then $x_0 = 0$ would give

$$bX^{s-1} = (bX^s)X^{-1} = xX^{-1} = \sum_{i=1}^{d} x_i X^{i-1} \in R,$$

contradicting the minimality of s. Thus, $x_0 \neq 0$ whatever the value of $s \geq 0$.

Recall that $Xb = \eta bX$, so that

$$rx = \sum_{i=0}^{d} r_i X^i bX^s = \sum_{i=0}^{d} \eta^i r_i bX^{i+s} = \sum_{i=0}^{d} \eta^i r_i xX^i;$$

that is,

$$rx = \sum_{i,j=0}^{d} \eta^i r_i x_j X^{i+j}.$$

Also, $rx = yX^t = \sum_{i=0}^{d} y_i X^{i+t}$; and so we obtain the following equality

$$\sum_{i,j=0}^{d} \eta^i r_i x_j X^{i+j} \quad = \quad \sum_{i=0}^{d} y_i X^{i+t} \tag{1}$$

in $R = A[X;\sigma,\delta]$.

Since $t \geq 1$, the term of degree 0 in the left hand side of (1) must be zero; that is, $r_0x_0 = 0$. Since $x_0 \neq 0$, this gives $r_0 = 0$. Hence $r = \sum_{i=1}^{d} r_iX^i = wX$ with $w = \sum_{i=1}^{d} r_iX^{i-1} \in R$. Consequently, the equality $yX^t = rx$ can be rewritten as

$$yX^t = wXx = wXbX^s = \eta wbX^{s+1} = \eta wxX.$$

It follows that $yX^{t-1} = \eta wx$, with $\eta w \in R$, contradicting the minimality of t.

Hence $t = 0$ and $y = rx$ with $r \in R$; so that $y \in Rx$, as required.

To sum up, we have established that $I = Rx$.

It remains to show that $xR = I$. First, note that, since $Xb = \eta bX$, we have $x = \eta^{-s}X^sb \in R$ and it is clear that $\min\{i \in \mathbb{N} \mid X^ib \in R\} = \min\{i \in \mathbb{N} \mid bX^i \in R\} = s$. Now by writing elements of R as polynomials with coefficients on the right, a very similar calculation (which we omit) to that done above shows that $xR = I$. Thus, $x = bX^s$ is a nonzero $\mathcal{H}$-eigenvector of R such that $I = xR = Rx$ is a completely prime ideal. This finishes the first part of the proof.

Now, let J be any nonzero $\mathcal{H}$-prime ideal of R, and note that J is completely prime.

First, assume that $X \notin J$. Then $JS^{-1} \cap T$ is a nonzero $\mathcal{H}$-ideal of T which is prime and it follows that $JS^{-1} \cap B$ is a nonzero $\mathcal{H}$-prime ideal of B, by Corollary 2.4. Thus, there exists $0 \neq b \in JS^{-1} \cap B$ such that b is a normal $\mathcal{H}$-eigenvector and $bB = Bb$ is a completely prime ideal of B. As in the earlier part of the proof, set $x := bX^s$, where s is minimal such that $bX^s \in R$. Note that $x \in JS^{-1} \cap R = J$, and that x is a nonzero normal $\mathcal{H}$-eigenvector of R such that $xR = Rx$ is a completely prime ideal of R.

Next, assume that $X \in J$. If $\delta = 0$ then X is a nonzero normal $\mathcal{H}$-eigenvector such that $XR = RX$ is completely prime (since A is a domain), as required. Thus, we may assume that $\delta \neq 0$.

Choose $c \in A$ such that $\delta(c) \neq 0$, and note that $0 \neq \delta(c) = Xc - \sigma(c)X \in J$; and so $J \cap A \neq 0$. It is clear that the map $b \mapsto \theta^{-1}(b) + J$ defines a homomorphism from B to R/J, and this homomorphism extends to a homorphism g from T to R/J such that $g(X) = 0$. This map, given by $g(\sum b_iX^i) = \theta^{-1}(b_0) + J$, commutes with the action of $\mathcal{H}$. Set $J' = \ker(g)$; so that J' is a completely prime $\mathcal{H}$-ideal of T. With $c \in A$ as above, note that $g(\theta(\delta(c))) = \delta(c) + J = 0_{R/J}$. Thus, $J' \cap B$ is a nonzero $\mathcal{H}$-prime ideal of B. Thus, there is a nonzero normal $\mathcal{H}$-eigenvector $b \in J' \cap B$ such that $bB = Bb$ is a completely prime $\mathcal{H}$-ideal of B. Set $x := bX^s$, where s is minimal such that $bX^s \in R$. Then, as in the earlier part of the proof, we know that x is a nonzero normal $\mathcal{H}$-eigenvector of R such that $xR = Rx$ is a completely prime ideal of R. In order to finish this case, we will show that

$x \in J$. Now, $b = \theta(a)$ for some $0 \neq a \in A$. We use the explicit formula for $\theta(a)$ to finish the calculation:

$$b = \theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{-n}.$$

(The sum on the right hand side exists since δ is locally nilpotent). Since δ is locally nilpotent, there exists $d \in \mathbb{N}$ such that $\delta^d(a) \neq 0$ and $\delta^{d+1}(a) = 0$. Then, since $q\delta \circ \sigma = \sigma \circ \delta$, we have

$$b = \theta(a) = \sum_{n=0}^{d} \frac{(1-q)^{-n}}{[n]!_q} q^{n^2} \sigma^{-n} \circ \delta^n(a) X^{-n},$$

and so the smallest integer i such that $bX^i \in R$ is equal to d. In other words, $s = d$ and $x = bX^d = \sum_{n=0}^{d} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{d-n}$, that is:

$$x = \frac{(1-q)^{-d}}{[d]!_q} \delta^d \circ \sigma^{-d}(a) + \left(\sum_{n=0}^{d-1} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{d-1-n} \right) X.$$

Since $X \in J$, in order to prove that $x \in J$, it is so sufficient to prove that $\delta^d \circ \sigma^{-d}(a)$ belongs to J.

Observe that, since $b \in J'$, we have $0 = g(b) = a + J$ and thus $a \in J$. Hence, if $d = 0$, then $x = b = a$, and so $x \in J$ as desired. Assume now that $d \geq 1$. Then $\delta^d \circ \sigma^{-d}(a) = \delta\left(\delta^{d-1} \circ \sigma^{-d}(a)\right)$. Set $e := \delta^{d-1} \circ \sigma^{-d}(a) \in A$. Then $\delta^d \circ \sigma^{-d}(a) = \delta(e) = Xe - \sigma(e)X \in J$, since $X \in J$. This was what we needed to conclude that $x \in J$, as required.

$\square$

3 CGL extensions

In this section, we develop a suitable context in which to apply the results of the previous section to establish that certain iterated skew polynomial extensions are $\mathcal{H}$-UFDs. The next problem is to use this information, the Goodearl-Letzter stratification theory and the noncommutative version of Nagata's lemma that we have established, Proposition 1.6, to deduce that these extensions are, in fact, noetherian UFDs

The next definition contains all of the conditions that are necessary for this programme to succeed. The definition is unwieldy, but is justified by the fact that many of the quantum algebras that we wish to study satisfy all of these conditions.

Definition 3.1 An iterated skew polynomial extension

$$A = k[x_1][x_2; \sigma_2, \delta_2] \ldots [x_n; \sigma_n, \delta_n]$$

is said to be a *CGL extension* (after Cauchon, Goodearl and Letzter) provided that the following list of conditions is satisfied:

- With $A_j := k[x_1][x_2; \sigma_2, \delta_2] \ldots [x_j; \sigma_j, \delta_j]$ for each $1 \leq j \leq n$, each σ_j is a k-automorphism of A_{j-1}, each δ_j is a locally nilpotent k-linear σ_j-derivation of A_{j-1}, and there exist nonroots of unity $q_j \in k^*$ with $\sigma_j \delta_j = q_j \delta_j \sigma_j$;
- For each $i < j$ there exists a λ_{ji} such that $\sigma_j(x_i) = \lambda_{ji} x_i$;
- There is a torus $\mathcal{H} = (k^*)^r$ acting rationally on A by k-algebra automorphisms;
- The x_i for $1 \leq i \leq n$ are $\mathcal{H}$-eigenvectors;
- There exist elements $h_1, \ldots, h_n \in \mathcal{H}$ such that $h_j(x_i) = \sigma_j(x_i)$ for $j > i$ and such that the h_j-eigenvalue of x_j is not a root of unity.

If, in addition, the subgroup of k^* generated by the λ_{ji} is torsionfree then we will say that A is a *torsionfree CGL extension.*

For a discussion of rational actions of tori, see [1, Chapter II.2].

Note that any CGL extension will be a noetherian domain with finite GK dimension, cf. [1, Lemma II.9.7]; and so will satisfy the descending chain condition on prime ideals, as mentioned earlier.

Notice that, if A is a CGL extension, then the action of $\mathcal{H}$ on $k[x_1]$ is such that $h_1.x_1 = \lambda x_1$, where $\lambda \in k^*$ is not a root of unity. From this, it follows easily that the only nonzero $\mathcal{H}$-prime of $k[x_1]$ is $\langle x_1 \rangle$, which is (completely) prime. Using [1, II.5.11], we deduce that, if A is a CGL extension then each of the extensions $A_j = A_{j-1}[x_j; \sigma_j, \delta_j]$ is a Cauchon extension; so the results of the previous section are available. Also, any CGL extension satisfies the conditions of [1, II.5.1] and so there are only finitely many $\mathcal{H}$-primes in A and they are all completely prime, by [1, Theorem II.5.12]. Further, if A is a torsionfree CGL extension, then all prime ideals of A are completely prime, by [1, Theorem II.6.9]. In particular, if such an A is a noetherian UFR then it is a noetherian UFD.

Proposition 3.2 *Let A be a CGL extension. Then A is an $\mathcal{H}$-UFD; that is, each nonzero $\mathcal{H}$-prime Q of A contains a nonzero normal $\mathcal{H}$-eigenvector a such that the $\mathcal{H}$-ideal $P := aA = Aa$ is completely prime.*

Proof. As already mentioned, the only nonzero $\mathcal{H}$-prime of $k[x_1]$ is $\langle x_1 \rangle$ and it follows immediately that $k[x_1]$ is an $\mathcal{H}$-UFD. Now, each of the extensions $A_j = A_{j-1}[x_j; \sigma_j, \delta_j]$ is a Cauchon extension; so apply Proposition 2.9 repeatedly. $\square$

The main aim in this section is to show that any CGL extension is in fact a noetherian UFR. It then follows that any torsionfree CGL extension is a noetherian UFD. Since a CGL extension A is an $\mathcal{H}$-UFD, the prime ideals of height one that are $\mathcal{H}$-ideals are principal, generated by elements that are normal and $\mathcal{H}$-eigenvectors. Also, as noted above, there are only finitely many $\mathcal{H}$-primes, by [1, Theorem II.5.12], and they are all completely prime. Thus, in order to show that such an extension is a noetherian UFD, we have to deal with the primes of height one that are not $\mathcal{H}$-primes. In the language of Goodearl and Lezter, these primes are in the *stratum* of the zero ideal; that is, if P is a prime ideal of height one that is not an $\mathcal{H}$-prime, then the largest $\mathcal{H}$-ideal contained in P is the zero ideal. The Goodearl-Letzter stratification theory enables us to deal with these primes. The idea is simple. The stratification theory shows that, once we invert all the regular $\mathcal{H}$-eigenvectors, the prime ideals in the stratum of the zero ideal become centrally generated. In fact, the height one primes in the zero stratum become principal, generated by a central element in this localisation; this shows this localisation is a noetherian UFR. However, Proposition 1.6 is valid only when we are inverting a multiplicative set generated by finitely many normal elements. To deal with this point, it turns out, and this is what we show first, that it is enough to invert the multiplicative set generated by the finitely many generators of the $\mathcal{H}$-primes of height one in order to get a picture similar to that of the stratification theory.

Lemma 3.3 *Let I be an $\mathcal{H}$-ideal in a CGL extension A. Then the prime ideals minimal over I are all $\mathcal{H}$-prime ideals.*

Proof. Since A is noetherian, there are finitely many primes minimal over I. Let Q be a prime minimal over I. The $\mathcal{H}$-orbit of Q consists of primes minimal over I and hence is finite. Now, [1, II.2.9] shows that Q is an $\mathcal{H}$-ideal. $\square$

Corollary 3.4 *Suppose that A is a CGL extension and that $P_i = a_iA$ for $1 \le i \le m$ are the prime ideals of height one that are $\mathcal{H}$-primes, where the a_i are normal $\mathcal{H}$-eigenvectors. Then, each nonzero $\mathcal{H}$-ideal of A contains a product of the a_i (repetitions allowed).*

Proof. Let I be a nonzero $\mathcal{H}$-ideal of A. Since A is noetherian, there are only a finite number of prime ideals that are minimal over I; denote these primes by $Q_1, \dots, Q_s$. By the previous lemma, these are all $\mathcal{H}$-primes. Since A is noetherian, the ideal I contains a product of the Q_i. However, each Q_i contains some P_j, by Proposition 3.2; and so I contains a product of the P_i, hence a product of the a_i. $\square$

Set T to be the localisation of A with respect to the multiplicatively closed set generated by the normal $\mathcal{H}$-eigenvectors a_i. Then the rational action of $\mathcal{H}$ on A extends to an action of $\mathcal{H}$ on the localisation T by k-algebra automorphisms, since we are localising with respect to $\mathcal{H}$-eigenvectors, and this action of $\mathcal{H}$ on T is also rational, by using [1, II.2.7]. We have the following proposition.

Proposition 3.5 *The ring T is $\mathcal{H}$-simple; that is, the only $\mathcal{H}$-ideals of T are 0 and T.*

Proof. Let J be an $\mathcal{H}$-ideal of T and let $I = J \cap A$. Clearly, I is an $\mathcal{H}$-ideal of A. In addition, $J = IT$, by [18, 2.1.16]. If $I = 0$, then $J = 0$. Otherwise, $J = T$, by the previous corollary. $\square$

We are now in position to show that the CGL extension A is a noetherian UFR.

Theorem 3.6 *Let $A = k[x_1][x_2; \sigma_2, \delta_2] \dots [x_n; \sigma_n, \delta_n]$ be a CGL extension. Then A is a noetherian UFR.*

Proof. By Proposition 1.6, it is enough to prove that the localisation T is a noetherian UFR. Now, as proved in Proposition 3.5, T is an $\mathcal{H}$-simple ring. Thus, using [1, II.3.9], it is a noetherian UFR, as required. $\square$

Theorem 3.7 *Let A be a torsionfree CGL-extension. Then A is a noetherian UFD.*

Proof. Use Theorem 3.6 and the fact that all prime ideals are completely prime in a torsionfree CGL-extension. $\square$

This theorem applies to many quantum algebras. A selection of such algebras of current interest is given in the following corollary. For exact definitions of those of the algebras that are not explicitly defined in this paper, consult [8] or [2, Section 6.2]

Corollary 3.8 *The following algebras are noetherian UFDs:*

- *The algebra of quantum matrices $\mathcal{O}_q(M_{m,n})$, with q not a root of unity, (see also the next section for more information about $\mathcal{O}_q(M_{m,n})$), and, more generally, the multiparameter version $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(k))$, with λ not a root of unity and the group $\langle \lambda, p_{ij} \rangle$ torsionfree.*

- *The quantized enveloping algebra $U_q(\mathfrak{n}^+)$, with q not a root of unity, of the nilpotent subalgebra $\mathfrak{n}^+$ of a complex semisimple Lie algebra $\mathfrak{g}$.*

- *The quantized enveloping algebra $U_q(\mathfrak{b}^+)$, with q not a root of unity, of the Borel subalgebra $\mathfrak{b}^+$ of a complex semisimple Lie algebra $\mathfrak{g}$.*

- *The quantum affine space $\mathcal{O}_{\mathbf{q}}(k^n)$, with $\langle q_{ij} \rangle$ torsionfree.*

- *The quantized Weyl algebra $A_n^{Q,\Gamma}(k)$ with each q_i not a root of unity and $\langle q_i, \gamma_{ij} \rangle$ torsionfree.*

- *The quantum grassmannian $\mathcal{G}_q(m,n)$, with q not a root of unity.*

- *The quantum groups $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$, with q not a root of unity.*

Proof. The algebras $\mathcal{O}_q(M_{m,n}), \mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(k)), \mathcal{O}_{\mathbf{q}}(k^n), A_n^{Q,\Gamma}(k)$ are described in [8] as iterated skew polynomial extensions with appropriate torus actions, and can easily be checked to be torsionfree CGL-extensions. (The only awkward point is to check that the first condition holds, and, in particular, to check that the δ_i involved all act locally nilpotently. The lemma below, which is easy to prove, helps deal with this point.)

The algebra $U_q(\mathfrak{n}^+)$ is described in [2, Section 6.2] and is easily seen to be a CGL-extension.

The algebra $U_q(\mathfrak{b}^+)$ is described in [8] as a localisation of an algebra that is an iterated skew polynomial extension with a torus action. This algebra is easily checked to be a CGL-extension.

The algebra $\mathcal{G}_q(m,n)$ is shown to be a noetherian UFD in section 5 of this paper. The quantum groups $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are shown to be noetherian UFDs in section 6 of this paper.

$\square$

Lemma 3.9 *Let R be a k-algebra, τ a k-algebra automorphism, δ a left τ-derivation, which we assume to be k-linear and set $S = R[x;\tau,\delta]$. In addition, let $X \subseteq R$ be a generating set of the k-algebra R. Then, the following holds.*
(i) Assume that there exists $q \in k$ such that, for all $x \in X$, $\delta\tau(x) = q\tau\delta(x)$, then $\delta\tau = q\tau\delta$.
(ii) Assume that there exists $q \in k$ such that $\delta\tau = q\tau\delta$. If, for all $x \in X$, there exists $d \in \mathbb{N}^*$ such that $\delta^d(x) = 0$, then δ is locally nilpotent.

4 Height one $\mathcal{H}$-primes in $\mathcal{O}_q(M_{m,n})$

In this section, we identify generators for each of the height one primes which are $\mathcal{H}$-ideals of the algebra of quantum matrices, in the generic case.

Throughout, k is a field and q is a nonzero element of k that is not a root of unity. Let m, n be positive integers. Recall that the algebra of $m \times n$ quantum matrices, $\mathcal{O}_q(M_{m,n})$, is the k-algebra generated by mn indeterminates x_{ij}, with $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the relations

$$\begin{array}{ll} x_{ij}x_{il} = qx_{il}x_{ij}, & (j < l); \\ x_{ij}x_{kj} = qx_{kj}x_{ij}, & (i < k); \\ x_{ij}x_{kl} = x_{kl}x_{ij}, & (i < k, j > l); \\ x_{ij}x_{kl} - x_{kl}x_{ij} = (q - q^{-1})x_{il}x_{kj}, & (i < k, j < l). \end{array}$$

In the case that $m = n$, we write $\mathcal{O}_q(M_n)$ for $\mathcal{O}_q(M_{m,n})$.

In view of the restriction that q is not a root of unity, we refer to $\mathcal{O}_q(M_{m,n})$ as the algebra of *generic quantum matrices.*

Let $\mathcal{H}$ be the $(m+n)$-torus $(k^*)^m \times (k^*)^n$. The torus $\mathcal{H}$ acts on $\mathcal{O}_q(M_{m,n})$ by k-algebra automorphisms in the following way:

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \cdot x_{ij} := \alpha_i \beta_j x_{ij}.$$

The algebra $\mathcal{O}_q(M_{m,n})$ can be presented as an iterated skew polynomial extension with the variables added in lexicographical order. With this presentation, and with the group $\mathcal{H}$ above acting, $\mathcal{O}_q(M_{m,n})$ is a torsionfree CGL extension; and so is a noetherian UFD by the results of the previous section. There are only finitely many height one prime ideals which are $\mathcal{H}$-primes, and the purpose of this section is to identify these $\mathcal{H}$-primes.

In the literature, many results are only stated for $\mathcal{O}_q(M_n)$ but are easily translated to $\mathcal{O}_q(M_{m,n})$, by using arguments based on the following easy observations. First, if I is a set of row indices and J is a set of column indices then the subalgebra of $\mathcal{O}_q(M_n)$ or $\mathcal{O}_q(M_{m,n})$ generated by the x_{ij} with $i \in I$ and $j \in J$ is isomorphic to another quantum matrix algebra in a natural way. Secondly, let $A = \mathcal{O}_q(M_n)$, and let $B = \mathcal{O}_q(M_{m,n})$, with $m \leq n$, be the quantum matrix algebra generated by generators in the first m rows of A, then there is an algebra epimorphism $\pi : A \longrightarrow B$ defined by the projection given by $x_{ij} \mapsto x_{ij}$ if $i \leq m$ and $x_{ij} \mapsto 0$ otherwise. By using the first observation, we may think of $\mathcal{O}_q(M_{m,n})$ and $\mathcal{O}_q(M_{n,m})$ being embedded in a common $\mathcal{O}_q(M_n)$. Then, there is an isomorphism between $\mathcal{O}_q(M_{m,n})$ and $\mathcal{O}_q(M_{n,m})$ given by transposition of the generators in $\mathcal{O}_q(M_n)$; that is, $x_{ij} \mapsto x_{ji}$, see [19, Proposition 3.7.1]. For this reason, we will assume that

$m \leq n$. In view of the restriction that q is not a root of unity, we will refer to $\mathcal{O}_q(M_{m,n})$ as a *generic quantum matrix algebra*.

The algebra $\mathcal{O}_q(M_n)$ has a special element, $\det_q$, the *quantum determinant*, defined by

$$\det_q := \sum_\sigma (-q)^{l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where the sum is taken over the permutations of $\{1, \ldots, n\}$ and $l(\sigma)$ is the usual length function on such permutations. The quantum determinant is a central element of $\mathcal{O}_q(M_n)$, see, for example, [19, Theorem 4.6.1]. If I is a t-element subset of $\{1, \ldots, m\}$ and J is a t-element subset of $\{1, \ldots, n\}$ then the quantum determinant of the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by $\{x_{ij}\}$, with $i \in I$ and $j \in J$, is denoted by $[I \mid J]$. The elements $[I \mid J]$ are the *quantum minors* of $\mathcal{O}_q(M_{m,n})$. They are not in general central; however, they do possess good commutation properties: in particular, in what follows, we will identify several quantum minors that are normal elements. Two elements a, b are said to *q-commute* if there is an integer s such that $ab = q^s ba$. An element that q-commutes with each of the generators of a quantum matrix algebra is easily seen to be normal, and this is a standard way to demonstrate normality. In many sources, such commutation relations are established for $\mathcal{O}_q(M_n)$. Usually, it is easy to transfer such results to $\mathcal{O}_q(M_{m,n})$, by including this quantum matrix algebra as a subalgebra of a suitable $\mathcal{O}_q(M_n)$ by including extra rows or columns of generators: obviously, if an element q-commutes with each of the generators in this larger algebra then it q-commutes with the generators of the original algebra. In addition, we will use the transposition isomorphism to derive further q-commutation results, with little comment.

Cauchon's theory of deleting derivations,[2, 3], has been applied to quantum matrices with great success. In fact, in [3], Cauchon works with $\mathcal{O}_q(M_n)$; however, the methods extend to $\mathcal{O}_q(M_{m,n})$ and the details are worked out in [15]. Let w denote an $m \times n$ array of square boxes in which each box is coloured either black or white. A *Cauchon diagram* is such an array with the following property: if a square is coloured black then either every square to the left of this square is also coloured black, or every square above this square is also coloured black. Cauchon [3] and Launois [15] prove that the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_{m,n})$ are in bijection with the $m \times n$ Cauchon diagrams. In addition, if P is an $\mathcal{H}$-prime, then the height of P (as a prime ideal) is equal to the number of black boxes in the corresponding diagram, by [3], Théorème 6.3.3 (which is easily adapted to the rectangular case), and [15], Proposition 1.3.2.2. (Recall that, by [1, II.2.9], any $\mathcal{H}$-prime is prime.)

For $1 \leq i \leq m$, let c_i denote the $i \times i$ quantum minor $[m-i+1, \ldots, m \mid 1, \ldots, i]$ of

$\mathcal{O}_q(M_{m,n})$ and let b_i denote the $i\times i$ quantum minor $[1,\dots,i \mid n-i+1,\dots,n]$ of $\mathcal{O}_q(M_{m,n})$, while for $m<i\le n$, let b_i denote the quantum minor $[1,\dots,m \mid n-i+1,\dots,n+m-i]$. Note that $c_m = b_n$; in particular, note that in $\mathcal{O}_q(M_n)$ we have $b_n = c_n = \det_q$. For orientation, note that the b_i are the minors coming from the top right of the matrix of generators of $\mathcal{O}_q(M_{m,n})$, while the c_i come from the bottom left.

The quantum minors are $\mathcal{H}$-eigenvectors; and so, for example, the ideals generated by each of the elements b_i and c_i, defined above, are $\mathcal{H}$-ideals. We will show below that they are $\mathcal{H}$-prime ideals.

Lemma 4.1 *The elements b_i, with $1\le i\le n$, and c_i, with $1\le i\le m$, are normal elements of $\mathcal{O}_q(M_{m,n})$.*

Proof. Let $1\le i\le m$; it follows easily from [9, Corollary A.2] that c_i q-commute with each generator of $\mathcal{O}_q(M_{m,n})$ and, using the transpose automorphism, that the same is true for b_i. For b_i, with $m<i\le n$, a slightly more complicated argument is required. Fix an i with $m<i\le n$. Consider a generator x_{kl}. If $l\le n+m-i$ then x_{kl} and b_i belong to the quantum matrix algebra $\mathcal{O}_q(M_{m,n+m-i})$ obtained from the generators in the first $n+m-i$ columns of $\mathcal{O}_q(M_{m,n})$. In fact, b_i is b_m in this subalgebra, and so x_{kl} and b_i q-commute. If $l>n+m-i$ then x_{kl} and b_i belong to the quantum matrix algebra $\mathcal{O}_q(M_{m,i})$ obtained from the generators in the last i columns of $\mathcal{O}_q(M_{m,n})$. In this case, b_i is c_m in this subalgebra and so again we see that x_{kl} and b_i q-commute. Thus b_i q-commutes with each of the generators of $\mathcal{O}_q(M_{m,n})$ and so is a normal element in this algebra. $\square$

Proposition 4.2 *There are precisely $m+n-1$ height one primes that are $\mathcal{H}$-primes in the generic quantum matrix algebra $\mathcal{O}_q(M_{m,n})$. They are the ideals generated by $b_1,\dots,b_n$ and $c_1,\dots,c_{m-1}$ (recall that $c_m=b_n$).*

Proof. It is easily seen that the elements $b_1,\dots,b_n,c_1,\dots,c_{m-1}$ generate pairwise distinct ideals.

The height one primes that are $\mathcal{H}$-primes are in bijection with the Cauchon diagrams with precisely one black box. Such Cauchon diagrams arise by filling in one box either in the first row of the array, or the first column. There are $m+n-1$ ways of doing this; and so there are $m+n-1$ height one primes that are $\mathcal{H}$-primes.

That the ideals specified are $\mathcal{H}$-ideals is due to the fact that the b_i and c_i are $\mathcal{H}$-eigenvectors. That the ideals are prime comes about in the following way. If we restrict to the quantum submatrix algebra A, say, specified by the rows and columns of a b_i or c_i, then when we factor out b_i or c_i from A we are factoring out the quantum determinant of A, and so the factor A/b_iA or A/c_iA is a domain, see, for example, [9, Theorem 2.5]. Since the b_i or c_i q-commute with the remaining x_{ij} we can add the remaining x_{ij} in such a way that at any stage if we have reached a subalgebra B then B/b_iB, say, is an iterated skew polynomial algebra over A/b_iA and so is a domain. For example, if we are in the case that $m < i \leq n$, then we can add the x_{ij} to the left of the rows and columns used by b_i by moving from right to left along each row, starting with the bottom row and moving upwards row by row. We then can add the x_{ij} to the right of the rows and columns used by b_i in lexicographic order. Thus each $\mathcal{O}_q(M_{m,n})/b_i\mathcal{O}_q(M_{m,n})$ and $\mathcal{O}_q(M_{m,n})/c_i\mathcal{O}_q(M_{m,n})$ is a domain and so each ideal of $\mathcal{O}_q(M_{m,n})$ generated by a b_i or c_i is a completely prime ideal. Since these ideals are $\mathcal{H}$-ideals, they are also $\mathcal{H}$-primes. Since we have precisely $m+n-1$ elements b_i or c_i this gives all of the height one primes that are $\mathcal{H}$-primes. $\square$

5 Generic quantum grassmannians are UFD

Recall that the *quantum grassmannian* subalgebra, $\mathcal{G}_q(m,n)$, of $\mathcal{O}_q(M_{m,n})$ is the subalgebra generated by the $m\times m$ maximal quantum minors of $\mathcal{O}_q(M_{m,n})$ (recall that we are assuming that $m \leq n$). The algebra $\mathcal{G}_q(m,n)$ is a noetherian domain, see, for example, [13, Theorem 1.1]. Our usual restriction that q is not a root of unity applies in this section; so we refer to $\mathcal{G}_q(m,n)$ at the *generic quantum grassmannian.*

In view of the fact that each of the quantum minors that generates $\mathcal{G}_q(m,n)$ is of the form $[1,\dots,m \mid J]$ we will denote such a minor by $[J]$. The two extreme quantum minors, $[1,\dots,m]$ and $[n-m+1,\dots,n]$ are normal in $\mathcal{G}_q(m,n)$, see, for example, [13, Corollary 1.1, Lemma 1.1].

We will use the fact that generic quantum matrices are noetherian UFD, and the dehomogenisation isomorphism

$$\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi] \longrightarrow \mathcal{G}_q(m,n)[[n-m+1,\dots,n]^{-1}]$$

of [13, Corollary 4.1] to show that $\mathcal{G}_q(m,n)$ is a noetherian UFD. Note that the automorphism ϕ used in the dehomogenisation isomorphism acts on generators via $\phi(x_{ij}) = q^{-1}x_{ij}$, see [13, Corollary 4.1].

To show that $\mathcal{G}_q(m,n)$ is a noetherian UFD, we proceed as follows. First, we show that the problem reduces to proving that the localisation $\mathcal{G}_q(m,n)[[n-m+1,\dots,n]^{-1}]$ is a noetherian UFD. Once this is done, by the dehomogenisation theorem, the problem transfers to showing that $\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi]$ is a noetherian UFD, and this is the second step.

The first step is easy and essentially amounts to proving Lemma 5.1 below.

Lemma 5.1 *The ideal of $\mathcal{G}_q(m,n)$ generated by $[n-m+1,\dots,n]$ is a completely prime ideal.*

Proof. Let $R=\mathcal{G}_q(m,n)$. The isomorphism discussed immediately before [13, Lemma 1.1] shows that the result follows provided that we show that $a:=[1,\dots,m]$ generates a completely prime ideal in $\mathcal{G}_q(m,n)$.

Note that $u:=[n-m+1,\dots,n]$ is left regular modulo aR, see the proof of Theorem 6.1 of [13]. Hence, it is enough to prove that a generates a completely prime ideal in the localisation $R[u^{-1}]$. We use the dehomogenisation isomorphism introduced above. Set $v:=[1,\dots,t|1,\dots,t]$ with $t=m$ if $m\le n-m$ and $t=n-m$ otherwise. By [16, Lemma 3.5.1], it is enough to show that v generates a completely prime ideal of $\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi]$. However, v generates a completely prime ideal of $\mathcal{O}_q(M_{m,n-m})$ that is left invariant by ϕ, since $\phi(v)$ is a scalar multiple of v. Thus v generates a completely prime ideal of $\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi]$, as required. $\square$

To achieve the second step, we observe first that $\mathcal{O}_q(M_{m,n-m})[y;\phi]$ is a torsionfree CGL-extension (in $s:=m(n-m)+1$ steps) as follows. The torus $\mathcal{H}=(k^*)^n=(k^*)^m\times(k^*)^{n-m}$ acts on $\mathcal{O}_q(M_{m,n-m})$ as defined at the beginning of the previous section, and we have already observed that this makes $\mathcal{O}_q(M_{m,n-m})$ a CGL-extension (in $s-1=m(n-m)$ steps). In order to deal with the last step (extension by y) we proceed as follows. We extend this action of $\mathcal{H}$ to $\mathcal{O}_q(M_{m,n-m})[y;\phi]$ by setting $(a_1,\dots,a_n).y=a_1\dots a_n y$. The element h_s needed for the final extension is given by $h_s:=(q^{-1},\dots,q^{-1},1,\dots,1)\in\mathcal{H}$ (with m occurences of q^{-1}), since we require that $h_s(x_{ij})=\phi(x_{ij})=q^{-1}x_{ij}$. Moreover $h_s.y=q^{-m}y$, and q^{-m} is not a root of unity, since q is not. With this information provided, it is easy to check the remaining conditions and conclude that $\mathcal{O}_q(M_{m,n-m})[y;\phi]$ is a torsionfree CGL-extension.

Theorem 5.2 *Suppose that $q\in k^*$ is not a root of unity. Then $\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi]$ is a noetherian UFD.*

Proof. That $\mathcal{O}_q(M_{m,n-m})[y;\phi]$ is a noetherian UFD follows from Theorem 3.7, since $\mathcal{O}_q(M_{m,n-m})[y;\phi]$ is a torsionfree CGL-extension. It follows that $\mathcal{O}_q(M_{m,n-m})[y,y^{-1};\phi]$ is a noetherian UFD. $\square$

Theorem 5.3 *The generic quantum grassmannian, $\mathcal{G}_q(m,n)$, is a noetherian UFD.*

Proof. The previous result shows that $\mathcal{G}_q(m,n)[[n-m+1,\dots,n]^{-1}]$ is a noetherian UFD, by using the dehomogenisation isomorphism. By Lemma 5.1, we know that $[n-m+1,\dots,n]$ generates a completely prime ideal. Thus the result follows from Lemma 1.4. $\square$

6 Generic $\mathcal{O}_q(SL_n)$ is a UFD.

In this section we prove that, in the generic case, $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ are noetherian UFDs.

For this purpose, we will use the fact that there exists a k-algebra isomorphism between the Laurent extension $\mathcal{O}_q(SL_n)[z^{\pm1}]$ of $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$. This isomorphism appears in [17] (see also [1, I.2.9]).

Proposition 6.1 *If $q \in k^*$ is not a root of unity, $\mathcal{O}_q(SL_n)$ is a noetherian UFD.*

Proof. Recall that, since q is not a root of unity, every prime ideal of $\mathcal{O}_q(SL_n)$ is completely prime ([1, II.6.10]. Hence it remains to prove that all the height one primes of $\mathcal{O}_q(SL_n)$ are principal.

By Corollary 3.8, $\mathcal{O}_q(M_n)$ is a noetherian UFD, and it follows easily that the same holds for $\mathcal{O}_q(GL_n)$. Hence, by the isomorphism above, $\mathcal{O}_q(SL_n)[z^{\pm1}]$ is a noetherian UFD. It remains to use Lemma 1.4 to conclude that $\mathcal{O}_q(SL_n)[z]$ is a noetherian UFD. Hence, in $\mathcal{O}_q(SL_n)[z]$, height one primes are principal.

Let P be a height one prime of $\mathcal{O}_q(SL_n)$ and put $P[z] := \oplus_{i\in\mathbb{N}} Pz^i$. Clearly, we have an isomorphism of k-algebras $\mathcal{O}_q(SL_n)[z]/P[z] \cong (\mathcal{O}_q(SL_n)/P)[z]$. Hence, $P[z]$ is a (completely) prime ideal of $\mathcal{O}_q(SL_n)[z]$. In addition, recall from [1, II.9.18] that $\mathcal{O}_q(SL_n)$ satisfies Tauvel's height formula, so that $\mathrm{GKdim}((\mathcal{O}_q(SL_n)/P)) = n^2-2$. Now, by using [14, Corollary 3.16], we get that the height of $P[z]$ is bounded above by $\mathrm{GKdim}(\mathcal{O}_q(SL_n)[z]) - \mathrm{GKdim}(\mathcal{O}_q(SL_n)[z]/P[z]) = n^2-(n^2-2+1) = 1$. Thus, $P[z]$ is a height one prime ideal of $\mathcal{O}_q(SL_n)[z]$. Hence there exists a normal element x in $\mathcal{O}_q(SL_n)[z]$ such that $P[z] = \langle x\rangle$.

We now show that x is in fact a normal element of $\mathcal{O}_q(SL_n)$ that generates P. Indeed, let $p \in P\setminus\{0\}$. There exists $a \in \mathcal{O}_q(SL_n)[z]$ such that $p = ax$. But $\mathcal{O}_q(SL_n)$ being a domain, by degree considerations, it follows that $a, x \in \mathcal{O}_q(SL_n)$. So, we have proved that $x \in \mathcal{O}_q(SL_n)$ and $p \in \mathcal{O}_q(SL_n)x$. That is, $P = \mathcal{O}_q(SL_n)x$. A similar argument yields $P = x\mathcal{O}_q(SL_n)$. This finishes the proof. $\square$

Remark 6.2 Let G be a connected, complex, semisimple algebraic group. We denote by $\mathcal{O}_q(G)$ the quantized coordinate ring of G (as defined in [1, I.7.5]). It is shown in [20] that, if G is simply connected, the ring of regular functions on G is a unique factorisation domain. The result above then leads to ask whether the same holds for $\mathcal{O}_q(G)$ for q not a root of unity.

Acknowledgment We thank Gerard Cauchon for many useful comments.

References

[1] K A Brown and K R Goodearl, Lectures on algebraic quantum groups. Advanced Courses in Mathematics-CRM Barcelona. Birkhäuser Verlag, Basel, 2002.

[2] G Cauchon, *Effacement des dérivations et spectres premiers des algèbres quantiques*, J Algebra 260 (2003), 476-518.

[3] G Cauchon, *Spectre premier de* $\mathcal{O}_q(M_n(k))$ *image canonique et séparation normale*, J Algebra 260 (2003), 519–569

[4] A W Chatters, *Non-commutative unique factorisation domains*, Math Proc Camb Phil Soc, 95 (1984), 49-54

[5] A W Chatters and D A Jordan, *Non-commutative unique factorisation rings*, J London Math Soc 33 (1986), 22-32

[6] F Dumas and L Rigal, *Prime spectrum and automorphisms for* 2×2 *jordanian matrices*, Comm Alg 30 (6) (2002), 2805–2828.

[7] D Eisenbud, Commutative Algebra, Graduate Texts in Mathematics 150, Springer-Verlag, New-York, 1995.

[8] K R Goodearl, *Prime spectra of quantized coordinate rings*, in: F Van Oystaeyen, M Saorin (Eds), Interactions between Ring Theory and Representations of Algebras, Murcia, 1998, Dekker, New York, 2000, pp 205-237

[9] K R Goodearl and T H Lenagan, *Quantum determinantal ideals*, Duke Math J 103 (2000), 165-190

[10] K R Goodearl and E S Letzter, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans Amer Math Soc 352 (2000), 1381-1403

[11] K R Goodearl and E S Letzter, *Prime factor algebras of the coordinate ring of quantum matrices*, Proc Amer Math Soc 121 (1994), 1017-1025

[12] K R Goodearl and R B Warfield, An introduction to noncommutative Noetherian rings. Second edition. London Mathematical Society Student Texts, 61. Cambridge University Press, Cambridge, 2004

[13] A Kelly, T H Lenagan and L Rigal, *Ring theoretic properties of quantum grassmannians*, Journal of Algebra and its Applications 3 (2004), 9-30

[14] G R Krause and T H Lenagan, Growth of algebras and Gelfand-Kirillov dimension, Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000

[15] S Launois, *Idéaux premiers $\mathcal{H}$-invariants de l'algèbre des matrices quantiques*, Thèse, Université de Reims, 2003

[16] T H Lenagan and L Rigal, *Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum grassmannians*, posted at math.QA/0403021

[17] T Levasseur and J T Stafford, *The quantum coordinate ring of the special linear group*, J Pure Applied Algebra 86 (1993), 181-186

[18] J C McConnell and J C Robson, Noncommutative Noetherian Rings, Wiley-Interscience, Chichester, 1987.

[19] B Parshall and J Wang, *Quantum linear groups*, Mem Amer Math Soc 89 (1991), no. 439

[20] V L Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles*, Math USSR Izvestija 8 (1974), 301-327

S Launois:
Laboratoire de Mathématiques - UMR6056, Université de Reims
Moulin de la Housse
BP 1039
51687 REIMS Cedex 2
France
E-mail : stephane.launois@univ-reims.fr

T H Lenagan:
School of Mathematics, University of Edinburgh,
James Clerk Maxwell Building, King's Buildings, Mayfield Road,
Edinburgh EH9 3JZ, Scotland
E-mail: tom@maths.ed.ac.uk

L Rigal:
Université Jean Monnet (Saint-Étienne),
Faculté des Sciences et Techniques,
Département de Mathématiques,
23 rue du Docteur Paul Michelon,
42023 Saint-Étienne Cédex 2,
France
E-mail: Laurent.Rigal@univ-st-etienne.fr