Quadratic and cubic invariants of unipotent affine automorphisms

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Abstract

Let K be an arbitrary field of characteristic zero, $P_n := K[x_1, \dots, x_n]$ be a polynomial algebra, and $P_{n,x_1} := K[x_1^{-1}, x_1, \dots, x_n]$, for $n \geq 2$. Let $\sigma' \in \mathrm{Aut}_K(P_n)$ be given by

$$x_1 \mapsto x_1 - 1, \quad x_1 \mapsto x_2 + x_1, \quad \dots, \quad x_n \mapsto x_n + x_{n-1}.$$

It is proved that the algebra of invariants, $F'_n := P_n^{\sigma'}$, is a polynomial algebra in $n-1$ variables which is generated by $[\frac{n}{2}]$ quadratic and $[\frac{n-1}{2}]$ cubic (free) generators that are given explicitly.

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It is well-known that the algebra of invariants, $F_n := P_n^{\sigma}$, is finitely generated (Theorem of Weitzenböck, [5], 1932), has transcendence degree $n-1$, and that one can give an explicit transcendence basis in which the elements have degrees $1, 2, 3, \dots, n-1$. However, it is an old open problem to find explicit generators for F_n. We find an explicit vector space basis for the quadratic invariants, and prove that the algebra of invariants P_{n,x_1}^{σ} is a polynomial algebra over $K[x_1, x_1^{-1}]$ in $n-2$ variables which is generated by $[\frac{n-1}{2}]$ quadratic and $[\frac{n-2}{2}]$ cubic (free) generators that are given explicitly.

The coefficients of these quadratic and cubic invariants throw light on the 'unpredictable combinatorics' of invariants of affine automorphisms and of SL_2-invariants.

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1 Introduction

Throughout the paper, K denotes an *arbitrary* field of characteristic zero. Let $P_n = P^{[n]} = K[x] := K[x_1, \dots, x_n]$ be a polynomial ring in n variables over K. First, we consider the

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K-algebra automorphism σ of P_n given by

$$\sigma: x_1 \mapsto x_1 - 1, \quad x_1 \mapsto x_2 + x_1, \quad \ldots, \quad x_n \mapsto x_n + x_{n-1}.$$

This automorphism can be written in matrix form as $\sigma(x) = J_n(1)x - e_1$, where $J_n(1) = E + \sum_{i=1}^{n-1} E_{i+1,i}$ is the $n \times n$ lower triangular Jordan matrix (E is the identity matrix and E_{ij} are the matrix units), $x = (x_1, \ldots, x_n)^t$, and $e_1 = (1, 0, \ldots, 0)^t$. It is well-known that the algebra of invariants, P_n^σ, is a polynomial algebra in $n-1$ variables and that the generators can be chosen to have degrees $2, 3, \ldots, n$. (Briefly, σ can be presented as $e^\delta := \sum_{i\geq 0} \frac{\delta^i}{i!}$ where $\delta \in \mathrm{Der}_K(P_n)$ is a *locally nilpotent* derivation for which there exists an element $x \in P_n$ such that $\delta(x) = 1$, then $P_n^\sigma = P_n^\delta := \ker(\delta)$ and the result is old and well-known for δ.) A theorem of Weitzenböck [5] states that the algebra of invariants $P_n^{\mathbb{G}_a}$ is finitely generated for every linear action of the additive (algebraic) group $\mathbb{G}_a$ of the field K (see also [4], [1], and also [3]). The same result is true for the algebra of invariants P_n^δ where δ is a linear derivation of P_n; that is, $\delta(x) = Ax$ where A is an $n \times n$ matrix over K. It is an old open problem to find *explicit* generators for the algebras $P_n^{\mathbb{G}_a}$ and P_n^δ. Several cases for small n are considered in [2].

We summarise the main results of the paper below; full proofs are given later.

The proof of the first theorem is 'direct'; that is, it does not use a reduction to the case of δ.

Theorem 1.1 *Let $\sigma(x) = J_n(1)x - e_1 \in \mathrm{Aut}_K(P_n)$, for $n \geq 2$. The algebra of invariants P_n^σ is a polynomial algebra $K[y_2, \ldots, y_n]$ in $n-1$ variables given by*

$$y_{i+1} = \sum_{j=1}^{i} \phi_{-i+j} x_{j+1} + i\sigma^{-1}(\phi_{-i-1}), \quad \text{for } i = 1, \ldots, n-1,$$

where $\phi_0 := 1$ and $\phi_{-i} := \frac{x_1(x_1-1)\cdots(x_1-i+1)}{i!}$, for $i \geq 1$. (Note that $\deg(y_{i+1}) = i+1$.) ∎

The polynomial algebra $P_n = K[x] = \cup_{i\geq 0} K[x]_{\leq i}$ is a filtered algebra by using the *total* degree of variables; so that $K[x]_{\leq i} := \sum_{\deg(p)\leq i} Kp$. The *integer part* of $r \in \mathbb{R}$ is denoted by $[r] := \max\{m \in \mathbb{Z} \mid m \leq r\}$. The next theorem gives an explicit basis for the quadratic invariants of the automorphism σ.

Theorem 1.2 *Let $\sigma(x) = J_n(1)x - e_1$, for $n \geq 2$, and suppose that K is a field of characteristic zero. Then the elements $u_0 = 1$, and*

$$u_k = x_k^2 + \sum_{i=1}^{k-1} \sum_{j=k}^{2k-i} \lambda_{i,j}^k x_i x_j + \sum_{i=k}^{2k} \mu_i^k x_i,$$

where

$$\lambda_{i,j}^k = (-1)^{k-i} \left\{ \binom{k-i}{j-k} + \binom{k-i-1}{j-k-1} \right\}$$

and

$$\mu_i^k = (-1)^{k-1}\left\{\binom{k}{i-k} + \binom{k-1}{i-k-1}\right\},$$

for $k = 1, \ldots, m := [n/2]$*, form a basis of the vector space* $K[x]^\sigma \cap K[x]_{\leq 2}$*. In particular,* $\dim_K(K[x]^\sigma \cap K[x]_{\leq 2}) = m + 1$ *and* $K[x]^\sigma \cap K[x]_{\leq 1} = K$*. Each of the coefficients* $\lambda_{i,j}^k$ *and* μ_i^k *is nonzero.* ▪

Remark. In particular, $u_1 = x_1^2 + x_1 + 2x_2$ and $u_2 = x_2^2 - x_1(x_2 + 2x_3) - x_2 - 3x_3 - 2x_4$. Note that $y_3 = x_3 + x_1x_2 + \frac{x_1^3 - x_1}{3}$. Consider the *cubic* σ-invariant polynomial

$$v_1 := 3y_3 = x_1^3 + 3x_1x_2 - x_1 + 3x_3 \in K[x_1, x_2, x_3]. \tag{1}$$

Theorem 1.3 *Let* $\sigma(x) = J_n(1)x - e_1$*, for* $n \geq 5$*, and suppose that* K *is a field of characteristic zero. Then, for* $k = 2, \ldots, \mu := [(n-1)/2]$*, the following polynomials,* v_k*, belong to* $K[x]^\sigma \cap K[x]_{\leq 3}$*:*

$$v_k = x_1u_k + x_kx_{k+1} + \sum_{i=1}^{k-1}\sum_{j=k+1}^{2k-i+1} \alpha_{i,j}^k x_ix_j + \sum_{i=k+1}^{2k+1} \beta_i^k x_i, \tag{2}$$

where

$$\begin{aligned}\alpha_{i,j}^k &= (-1)^{k-i}\left\{2\binom{k-i-1}{j-k-1} + 3\binom{k-i-1}{j-k-2}\right\} + (k-i-1)\lambda_{i,j-1}^k \\ &= (-1)^{k-i}\left\{2\binom{k-i-1}{j-k-1} + 3\binom{k-i-1}{j-k-2} + (k-i-1)\left[\binom{k-i}{j-k-1} + \binom{k-i-1}{j-k-2}\right]\right\},\end{aligned}$$

and

$$\beta_j^k = \alpha_{1,j-1}^k + \alpha_{1,j}^k + \mu_{j-1}^k, \quad \text{for} \quad j = k+1, \ldots, 2k+1,$$

where $u_k, \lambda_{i,j}^k$ *and* μ_i^k *are as defined in Theorem 1.2. Note that each of the coefficients* $\alpha_{i,j}^k$ *and* β_i^k *is nonzero.* ▪

Remark. In particular,

$$\alpha_{i,2k+1-i}^k = (-1)^{k-i}(1 + 2(k-i)) \quad \text{and} \quad \alpha_{i,k+1}^k = (-1)^{k-i}(k-i+1), \tag{3}$$

and

$$v_2 = x_1u_2 + x_2x_3 - 2x_1x_3 - 3x_1x_4 - 3x_3 - 8x_4 - 5x_5. \tag{4}$$

The quadratic and cubic invariants obtained in the previous two theorems provide a generating set for the algebra of invariants, as the next theorem shows.

Theorem 1.4 *Let $\sigma(x) = J_n(1)x - e_1$, for $n \geq 2$. Set $m := [\frac{n}{2}]$ and $\mu := [\frac{n-1}{2}]$. Then*

1. *$P_n^\sigma = K[u_1, \dots, u_m, v_1, \dots, v_\mu]$ is polynomial ring in $n-1$ ($= m + \mu$) variables.*
2. *$P_n = P_n^\sigma[x_1]$.*

Proof. For each $k \geq 1$, we have

$$u_k = (-1)^{k-1} 2x_{2k} + \dots \qquad \text{and} \qquad v_k = (-1)^{k-1}(1+2k)x_{2k+1} + \dots, \tag{5}$$

where the three dots denote terms from P_{2k-1} and P_{2k} respectively. These imply that $P_n = K[u_1, \dots, u_m, v_1, \dots, v_\mu][x_1]$. It then follows that $P_n^\sigma = K[u_1, \dots, u_m, v_1, \dots, v_\mu]$ and $P_n = P_n^\sigma[x_1]$, since $K[u_1, \dots, u_m, v_1, \dots, v_\mu] \subseteq P_n^\sigma$, $\sigma(x_1) = x_1 - 1$ and $\mathrm{char}(K) = 0$. ∎

Now, consider the K-automorphism $\sigma(x) = J_{n+1}(1)x$ of the polynomial algebra $P_{n+1} := K[x_1, \dots, x_{n+1}]$:

$$\sigma : x_1 \mapsto x_1, \quad x_2 \mapsto x_2 + x_1, \quad \dots, \quad x_{n+1} \mapsto x_{n+1} + x_n.$$

The algebra of invariants $F := P_{n+1}^\sigma = \oplus_{i\geq 0} F_i$ is a positively graded subalgebra of the polynomial algebra $P_{n+1} = \oplus_{i\geq 0} P_{n+1,i}$ (the natural grading) where $F_i := F \cap P_{n+1,i}$. Let $P_{n+1,x_1} = K[x_1^{-1}, x_1, x_2, \dots, x_{n+1}]$ be the localization of P_{n+1} at the powers of the element x_1. Then $P_{n+1,x_1} = K[x_1^\pm, z_1, \dots, z_n] = Q[x_1^\pm]$ where $Q := K[z_1, \dots, z_n]$ is a polynomial algebra in the n variables $z_i := -\frac{x_{i+1}}{x_1}$, $i = 1, \dots, n$. We denote by the same letter σ the unique extension of the automorphism σ to P_{n+1,x_1}. Then $\sigma(Q) = Q$ and $\sigma(z) = J_n(1)z - e_1$; that is,

$$\sigma(z_1) = z_1 - 1, \quad \sigma(z_2) = z_2 + z_1, \quad \dots, \quad \sigma(z_n) = z_n + z_{n-1}.$$

Define polynomials p_k and q_k in P_{n+1} as follows:

$$p_k := x_1^2 u_k(z) = x_{k+1}^2 + \sum_{i=1}^{k-1} \sum_{j=k}^{2k-i} \lambda_{i,j}^k x_{i+1} x_{j+1} - x_1 \sum_{i=k}^{2k} \mu_i^k x_{i+1}, \quad \text{for } k \geq 1, \tag{6}$$

while

$$q_1 := x_1^3 - x_2^3 + 3x_1x_2x_3 + x_1^2 x_2 - 3x_1^2 x_4, \tag{7}$$

and

$$q_k := x_1^3 v_k(z) = -x_2 p_k + x_1 x_{k+1} x_{k+2} + x_1 \sum_{i=1}^{k-1} \sum_{j=k+1}^{2k-i+1} \alpha_{i,j}^k x_{i+1} x_{j+1} - x_1^2 \sum_{i=k+1}^{2k+1} \beta_i^k x_{i+1}, \tag{8}$$

for $k \geq 2$.

Theorem 1.5 *Let $\sigma(x) = J_{n+1}(1)x$, for $n \geq 2$. Set $m := [\frac{n}{2}]$ and $\mu := [\frac{n-1}{2}]$. Then the set of elements of P^σ_{n+1}:*

$$x_1, p_1, \ldots, p_m, q_1, \ldots, q_\mu$$

is a transcendence basis for the algebra P^σ_{n+1}, with $\deg(p_i) = 2$ and $\deg(q_j) = 3$. Further,

$$P^\sigma_{n+1,x_1} = K[x_1, x_1^{-1}][p_1, \ldots, p_m, q_1, \ldots, q_\mu].$$

Proof. This follows directly from Theorem 1.4. ∎

Corollary 1.6 *Let $\sigma(x) = J_{n+1}(1)x$, for $n \geq 2$, and set $m := [\frac{n}{2}]$. Then $x_1^2, p_1, \ldots, p_m$ is a K-basis of the vector space of quadratic invariants.*

Proof. This follows from Theorem 1.2 and Corollary 5.1(3). ∎

2 σ-exponentials

Let K be a field of characteristic zero, and let σ denote the affine automorphism of $K[x]$ such that $\sigma(x) = x - 1$. Our aim is to choose a basis for $K[x]$ as a K-vector space that facilitates calculations involving σ. The idea is to exploit the fact that $1 - \sigma$ is a σ-derivation, and to choose the basis with this in mind. Accordingly, we define

$$\phi_0 := 1, \quad \phi_i := \phi_i(x) = \frac{x(x+1)\cdots(x+i-1)}{i!} = \frac{x\sigma^{-1}(x)\cdots\sigma^{-i+1}(x)}{i!}, \quad i \geq 1 \tag{9}$$

and

$$\phi_0 := 1, \quad \phi_{-i} := \phi_{-i}(x) = \frac{x(x-1)\cdots(x-i+1)}{i!} = \frac{x\sigma(x)\cdots\sigma^{i-1}(x)}{i!}, \quad i \geq 1. \tag{10}$$

Each of the two sets $\{\phi_i\}$ and $\{\phi_{-i}\}$ forms a K-basis of $K[x]$.

Note that $(1-\sigma)\phi_i = \phi_{i-1}$ and $(1-\sigma)\phi_{-i} = \sigma(\phi_{-i+1})$, for $i \geq 1$, while $(1-\sigma)\phi_0 = 0$. Note also that $\phi_{-i}(-x) = (-1)^i\phi_i(x)$, and that $\sigma^{i-1}(\phi_i) = \phi_{-i}$, for $i \geq 1$.

The choice of bases and the action of $1 - \sigma$ suggests that we should construct exponential functions, twisted by σ. In order to do this, we extend the automorphism σ to an automorphism of the the power series ring $K[x][[\Theta]]$ by defining $\sigma(\Theta) = \Theta$.

Now, define

$$E = E(x) := \sum_{i=0}^{\infty} \phi_{-i}\Theta^i = 1 + \sum_{i=1}^{\infty} x\sigma(x)\cdots\sigma^{i-1}(x)\frac{\Theta^i}{i!} \tag{11}$$

and

$$E_- = E(-x) := \sum_{i=0}^{\infty} \phi_{-i}(-x)\Theta^i = \tag{12}$$

$$\sum_{i=0}^{\infty} (-1)^i \phi_i \Theta^i = 1 + \sum_{i=1}^{\infty} (-1)^i x\sigma^{-1}(x) \cdots \sigma^{-i+1}(x) \frac{\Theta^i}{i!}. \tag{13}$$

The following identities are easily established by direct computation:

$$(1-\sigma)E = \Theta\sigma(E), \qquad \text{and} \qquad (1-\sigma)E_- = -\Theta E_-.$$

Lemma 2.1 *$E(x)^{-1} = E(-x)$ in $K[x][[\Theta]]$.*

Proof. Set $E = E(x)$ and $E_- = E(-x)$. By applying the σ-derivation $(1-\sigma)$ to the product E_-E, and, by using the identities $(1-\sigma)E = \Theta\sigma(E)$ and $(1-\sigma)E_- = -\Theta E_-$, we obtain

$$\begin{aligned} (1-\sigma)(E_-E) &= (1-\sigma)E_- \cdot E + \sigma(E_-)(1-\sigma)E & (14)\\ &= -\Theta(1-\sigma)(E_-E). & (15) \end{aligned}$$

It follows that $(1-\sigma)(E_-E) \in \cap_{n=1}^{\infty} \Theta^n K[x][[\Theta]] = 0$ and so $\sigma(E_-E) = E_-E$. Hence, $E_-E \in k[x][[\Theta]]^\sigma = K[[\Theta]]$, and we may write $E_-E = 1 + \sum_{i=1}^{\infty} \lambda_i \Theta^i$, with each $\lambda_i \in K$.

By setting $x = 0$ in the previous equality, we get

$$1 = 1 \cdot 1 = E_-(0)E(0) = 1 + \sum \lambda_i \Theta^i,$$

and it follows that each $\lambda_i = 0$; so that $E_-E = 1$ in $K[x][[\Theta]]$. ▪

Consider the K-automorphisms σ_i of the polynomial ring $K[x_1, x_2]$ in two variables, defined by $\sigma_i(x_j) := x_j - \delta_{ij}$, for $i, j = 1, 2$, where δ_{ij} is the Kronecker delta symbol. The automorphisms σ_i extend uniquely to automorphisms of the algebra $K[x_1, x_2][[\Theta]]$, by setting $\sigma_i(\Theta) = \Theta$.

Lemma 2.2 *$E(x_1)E(x_2) = E(x_1 + x_2)$ in $K[x][[\Theta]]$.*

Proof. It suffices to show that the product $P := E_-(x_1)E_-(x_2)E(x_1+x_2)$ is equal to one. Note that the identity $(1-\sigma_i)(E(x_1+x_2)) = \Theta\sigma_i(E(x_1+x_2))$ holds, since $\sigma_i(x_1+x_2) = (x_1+x_2)-1$. Hence, by using the same argument as in the proof of (14), one easily obtains $(1-\sigma_i)P = (-\Theta)^n(1-\sigma_i)P$, for all $n \geq 1$, and $i = 1, 2$. Hence,

$$P = 1 + \sum_{i=1}^{\infty} \lambda_i \Theta^i \in \cap_{i=1}^{2} \ker(1-\sigma_i) = K[[\Theta]],$$

so that each $\lambda_i \in K$. Now, set $x_1 = x_2 = 0$ in the previous equality, to obtain

$$1 = E_-(0)E_-(0)E(0) = 1 + \sum_{i=1}^{\infty} \lambda_i \Theta^i.$$

Hence, all $\lambda_i = 0$; and so $P = 1$, as required. ∎

The following useful identity now follows immediately.

Lemma 2.3 *1. For all $k \geq 1$,*

$$(-1)^k \sum_{\substack{i+j=k \\ i\geq 0,\ j\geq 0}} (-1)^j \phi_i \phi_{-j} = \sum_{\substack{i+j=k \\ i\geq 0,\ j\geq 0}} (-1)^i \phi_i \phi_{-j} = 0.$$

2. For all $n, k \geq 1$,

$$(-1)^k \sum_{\substack{i+j=k \\ n\geq i\geq 0,\ n\geq j\geq 0}} (-1)^j \phi_i \phi_{-j} = \sum_{\substack{i+j=k \\ n\geq i\geq 0,\ n\geq j\geq 0}} (-1)^i \phi_i \phi_{-j} = 0.$$

Proof. 1. This follows immediately from the equality

$$1 = E_- E = (\sum_{i\geq 0} (-1)^i \phi_i \Theta^i)(\sum_{j\geq 0} \phi_{-j} \Theta^j) = \sum_{k\geq 0} \left(\sum_{i+j=k} (-1)^i \phi_i \phi_{-j} \right) \Theta^k.$$

2. Follows immediately from the equality above by working modulo θ^n. ∎

In order to study the Jordan blocks occurring in the canonical form of an affine automorphism, we need to consider specializing the above results to the case that Θ is the nilpotent $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Note that $\Theta^{n-1} = 0$, but $\Theta^{n-2} \neq 0$.

Consider the matrix

$$\Lambda = \sum_{i=0}^{n-2} (-1)^i \phi_i \Theta^i \tag{16}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\phi_1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \phi_2 & -\phi_1 & 1 & \ddots & & & \\ -\phi_3 & \phi_2 & -\phi_1 & & \ddots & & \\ \vdots & \vdots & & \cdots & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ (-1)^{n-2}\phi_{n-2} & (-1)^{n-3}\phi_{n-3} & \cdots\cdots & & & -\phi_1 & 1 \end{pmatrix} \in \mathrm{SL}_{n-1}(K[x]). \tag{17}$$

The above analysis reveals that

$$\Lambda^{-1} = \sum_{i=0}^{n-2} \phi_{-i}\Theta^i.$$

Set $\Phi := (-\phi_2, \phi_3, \ldots, (-1)^i\phi_{i+1}, \ldots, (-1)^{n-1}\phi_n)^t \in K[x]^{n-1}$. ▪

Lemma 2.4

$$\Lambda^{-1}\Phi = \begin{pmatrix} -\sigma^{-1}(\phi_{-2}) \\ \vdots \\ -i\sigma^{-1}(\phi_{-i-1}) \\ \vdots \\ -(n-1)\sigma^{-1}(\phi_{-n}) \end{pmatrix},$$

where the ith entry is displayed.

Proof. Set

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \vdots \\ \eta_{n-1} \end{pmatrix} := \Lambda^{-1}\Phi = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \phi_{-1} & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \phi_{-2} & \phi_{-1} & 1 & \ddots & & & \\ \phi_{-3} & \phi_{-2} & \phi_{-1} & & \ddots & & \\ \vdots & \vdots & & \cdots & & & \\ \phi_{-i+1} & & & & & & \\ \vdots & \vdots & & & & & \\ \phi_{-n+2} & \phi_{-n+3} & \cdots\cdots & & & \phi_{-1} & 1 \end{pmatrix} \begin{pmatrix} -\phi_2 \\ \phi_3 \\ \vdots \\ (-1)^i\phi_{i+1} \\ \vdots \\ (-1)^{n-1}\phi_n \end{pmatrix}.$$

Then

$$\begin{aligned} \eta_i &= \sum_{j=1}^{i} \phi_{-i+j}(-1)^j\phi_{j+1} = -\sum_{l=2}^{i+1}(-1)^l\phi_l\phi_{-i-1+l} \\ &= -\sum_{l=0}^{i+1}(-1)^l\phi_l\phi_{-i-1+l} + \phi_{-i-1} - \phi_1\phi_{-i} \\ &= 0 + \phi_{-i}\left(\frac{x-i}{i+1} - x\right) = -i\frac{(x+1)\phi_{-i}}{i+1} = -i\sigma^{-1}(\phi_{-i-1}), \end{aligned}$$

as claimed. Note that the 0 in the above calculation arises by applying Lemma 2.3. ∎

3 The invariant polynomials u_i and v_j

Let K be a field of characteristic zero, and let σ be the affine automorphism of $K[x] := K[x_1, \ldots, x_n]$, for $n \geq 2$, defined by the rule

$$\sigma(x_1) = x_1 - 1, \quad \sigma(x_2) = x_2 + x_1, \quad \ldots, \quad \sigma(x_n) = x_n + x_{n-1}.$$

In matrix form,

$$\sigma(x) = J_n(1)x - e_1,$$

where $J_n(1) = E + \sum_{i=1}^{n-1} E_{i+1,i}$ is the $n \times n$ lower triangular Jordan matrix (E is the identity matrix and E_{ij} are the matrix units). Observe that $\sigma(K[x_1, \ldots, x_m]) = K[x_1, \ldots, x_m]$, for each $i \geq 1$. In particular, $\sigma(K[x_1]) = K[x_1]$, and $\sigma(x_1) = x_1 - 1$.

Recall, from the previous section, that the set of polynomials $\{\phi_i := \phi_i(x_1)\}$ defined by

$$\phi_0 := 1, \quad \phi_i := \phi_i(x_1) = \frac{x_1(x_1+1)\cdots(x_1+i-1)}{i!}, \ \ i \geq 1 \tag{18}$$

is a K-basis of $K[x_1]$, and that $(1-\sigma)\phi_i = \phi_{i-1}$, for all $i \geq 1$, while $(1-\sigma)\phi_0 = 0$.

The matrix $\Theta := J_{n-1}(1) - I$ is a nilpotent matrix with $\Theta^{n-1} = 0$, but $\Theta^{n-2} \neq 0$. As in the previous section, we consider the matrix

$$\Lambda = \sum_{i=0}^{n-2} (-1)^i \phi_i \Theta^i$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\phi_1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \phi_2 & -\phi_1 & 1 & \ddots & & & \\ -\phi_3 & \phi_2 & -\phi_1 & & \ddots & & \\ \vdots & \vdots & & \cdots & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ (-1)^{n-2}\phi_{n-2} & (-1)^{n-3}\phi_{n-3} & \cdots\cdots & & & -\phi_1 & 1 \end{pmatrix} \in \mathrm{SL}_{n-1}(K[x_1]).$$

Set

$$x' = (x_2, \ldots, x_{i+1}, \ldots, x_n)^t \quad \text{and} \quad \Phi = (-\phi_2, \phi_3, \ldots, (-1)^i \phi_{i+1}, \ldots, (-1)^{n-1}\phi_n)^t.$$

Define $y = (y_2, \ldots, y_n)^t \in K[x]^{n-1}$ by the linear equation $x' = \Lambda y + \Phi$; so that $y = \Lambda^{-1}(x' - \Phi)$. In more detail, we have

$$x_{i+1} = \sum_{j=1}^{i} (-1)^{i-j} \phi_{i-j} y_{j+1} + (-1)^i \phi_{i+1} \tag{19}$$

and

$$y_{i+1} = \sum_{j=1}^{i} \phi_{-i+j} x_{j+1} + i\sigma^{-1}(\phi_{-i-1}), \tag{20}$$

for $i = 1, \ldots, n-1$, by Lemma 2.4. We extend the action of σ to the $(n-1) \times (n-1)$ matrix ring $M_{n-1}(K[x])$ and to the column space $K[x]^{n-1}$ in the obvious way (that is, elementwise).

The following proposition contains the claim of Theorem 1.1.

Proposition 3.1 *Let $\sigma(x) = J_n(1)x - e_1$, for $n \geq 2$, and suppose that* $\mathrm{char}(K) = 0$. *Then, the fixed ring $K[x]^\sigma$ is equal to the polynomial ring $K[y_2, \ldots, y_n]$ in the $n-1$ variables defined by (20). Further, $K[x] = K[x_1] \otimes K[x]^\sigma = K[x_1, y_2, \ldots, y_n]$.*

Proof. Note that the subalgebra $K[y]$ of $K[x]$, generated by $y_2, \ldots, y_n$, is isomorphic to a polynomial ring in $n-1$ variables, by (20).

Observe that

$$(1-\sigma)x' = -\Theta x' - (\phi_1, 0, \ldots, 0)^t, \qquad (1-\sigma)\Phi = -\Theta\Phi - (\phi_1, 0, \ldots, 0)^t$$

and

$$(1-\sigma)\Lambda^{-1} = \Theta\sigma(\Lambda^{-1}).$$

By applying the σ-derivation $(1-\sigma)$ to the equation $y = \Lambda^{-1}(x' - \Phi)$, we obtain

$$\begin{aligned}(1-\sigma)y &= (1-\sigma)\Lambda^{-1} \cdot (x'-\Phi) + \sigma(\Lambda^{-1}) \cdot (1-\sigma)(x'-\Phi) \\ &= \Theta\sigma(\Lambda^{-1})(x'-\Phi) - \sigma(\Lambda^{-1})\Theta(x'-\Phi) = 0,\end{aligned}$$

since $\Theta\sigma(\Lambda^{-1}) = \sigma(\Lambda^{-1})\Theta$.

Thus, $(1-\sigma)y = 0$, and so $\sigma(y_i) = y_i$, for each $i = 2, \ldots, n$. Hence, $k[y] := k[y_2, \ldots, y_n] \subseteq k[x]^\sigma$, and $k[x] = k[y] \otimes k[x_1]$.

Let $f = \sum_{i=0}^{s} f_i\phi_i \in K[x]^\sigma$, where each $f_i \in K[y]$. Then,

$$0 = (1-\sigma)f = f_1\phi_0 + \cdots + f_s\phi_{s-1},$$

and it follows that $f_i = 0$, for all $i \geq 1$. Thus, $f = f_0 \in K[y]$, and $K[x]^\sigma \subseteq K[y]$, as required. ∎

Let K be a commutative ring and let $Z = I \times \mathbb{Z}$ be a subset of $\mathbb{Z}^2$, where $I = [a, a+1, \ldots, b]$, for some $a < b$. Suppose that $\lambda, \mu : Z \to K$ are functions such that $(i,j) \to \lambda_{i,j}$ and $(i,j) \to \mu_{i,j}$ and such that the relation

$$\lambda_{i,j} = \delta(\lambda_{i+1,j-1} + \lambda_{i+1,j}) + \mu_{i,j-1},$$

holds for all $(i,j) \in Z$ and for some $\delta \in K$, then we write $\mu \overset{\delta}{\rightsquigarrow} \lambda$

Lemma 3.2 *Let K be a commutative ring and suppose that two functions $\lambda, \mu : Z \to K$ given by $(i,j) \mapsto \lambda_{i,j}$ and $(i,j) \mapsto \mu_{i,j}$ satisfy the relation $\lambda_{i,j} = \delta(\lambda_{i+1,j-1} + \lambda_{i+1,j}) + \mu_{i,j-1}$, for all $(i,j) \in Z$ and for some $\delta \in K$. Then,*

1.

$$\lambda_{i,j} = \delta^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{c'=0}^{c-1} \delta^{c'} \sum_{d'=0}^{c'} \binom{c'}{d'} \mu_{i+c',j-1-d'}, \tag{21}$$

for each integer $c \geq 0$, such that $i + c \in [a, b]$.

In particular, when $\mu = 0$, we have

$$\lambda_{i,j} = \delta^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d}, \tag{22}$$

for each integer $c \geq 0$, such that $i + c \in [a, b]$.
2. If, in addition, the function μ satisfies the relation $\mu_{i,j} = \gamma(\mu_{i+1,j-1} + \mu_{i+1,j})$, for all $(i, j) \in Z$ and for some unit $\gamma \in K$, then

$$\lambda_{i,j} = \delta^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \left(1 + \frac{\delta}{\gamma} + \left(\frac{\delta}{\gamma}\right)^2 + \cdots + \left(\frac{\delta}{\gamma}\right)^{c-1}\right) \mu_{i,j-1}. \tag{23}$$

Proof. 1. We use induction on c. The base cases of the induction $c = 0, 1$ are obvious. Suppose that $c \geq 2$ and that the result holds for $c - 1$. Denote by $\Omega = \Omega_c$ the second sum in (21). Then, by induction,

$$\begin{aligned}
\lambda_{i,j} &= \delta^{c-1} \sum_{d=0}^{c-1} \binom{c-1}{d} \lambda_{i+c-1,j-d} + \sum_{c'=0}^{c-2} \delta^{c'} \sum_{d'=0}^{c'} \binom{c'}{d'} \mu_{i+c',j-1-d'} \\
&= \delta^{c} \sum_{d=0}^{c-1} \binom{c-1}{d} (\lambda_{i+c,j-d-1} + \lambda_{i+c,j-d}) + \sum_{c'=0}^{c-1} \delta^{c'} \sum_{d'=0}^{c'} \binom{c'}{d'} \mu_{i+c',j-1-d'} \\
&= \delta^{c} \sum_{d=0}^{c} \left\{ \binom{c-1}{d} + \binom{c-1}{d-1} \right\} \lambda_{i+c,j-d} + \Omega \\
&= \delta^{c} \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \Omega.
\end{aligned}$$

as required.

2. By (22), $\mu_{i,j-1} = \gamma^{c'} \sum_{d=0}^{c'} \binom{c'}{d'} u_{i+c',j-1-d'}$, for each integer $c' \geq 0$. The element γ is a unit, so

$$\Omega_c = \left(1 + \frac{\delta}{\gamma} + \left(\frac{\delta}{\gamma}\right)^2 + \cdots + \left(\frac{\delta}{\gamma}\right)^{c-1}\right) \mu_{i,j-1},$$

and the equation (23) follows. ▪

Remark. We are setting $\binom{a}{b} = 0$, for each pair $a, b \in \mathbb{Z}$ that does not satisfy $0 \leq b \leq a$.

Corollary 3.3 *Let K be a commutative ring. Suppose that the functions $\lambda, \lambda^1, \ldots, \lambda^n : Z \to K$ satisfy*

$$0 \overset{\delta_n}{\rightsquigarrow} \lambda^n \overset{\delta_{n-1}}{\rightsquigarrow} \lambda^{n-1} \overset{\delta_{n-2}}{\rightsquigarrow} \cdots \overset{\delta_1}{\rightsquigarrow} \lambda^1 \overset{\delta_0}{\rightsquigarrow} \lambda^0 \equiv \lambda,$$

where $\delta_1, \ldots, \delta_n$ *are units in* K.

Then

1.

$$\lambda_{i,j} = \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{k=1}^{n} (-1)^{k-1} \lambda_{i,j-k}^{k} \left(\sum_{c_1=0}^{c-1} \sum_{c_2=0}^{c_1-1} \cdots \sum_{c_k=0}^{c_{k-1}-1} \prod_{l=1}^{k} \left(\frac{\delta_{l-1}}{\delta_l} \right)^{c_l} \right),$$

for each integer $c \geq 0$ *such that* $i + c \in [a, b]$.

2. *In particular, when* $\delta_0 = \delta_1 = \cdots = \delta_n$, *we have*

$$\begin{aligned} \lambda_{i,j} &= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{k=1}^{\min(n,c)} (-1)^{k-1} \phi_k(c-k+1) \lambda_{i,j-k}^{k} \qquad (24) \\ &= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{k=1}^{n} (-1)^{k-1} \binom{c}{k} \lambda_{i,j-k}^{k}. \qquad (25) \end{aligned}$$

Moreover,

$$\begin{aligned} \lambda_{i,j} &= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \left(\sum_{k=0}^{\min(n,c)} \phi_k(c) \lambda_{i+c,j-k-d}^{k} \right) \qquad (26) \\ &= \delta_0^c \sum_{k=0}^{\min(n,c)} \binom{c+k-1}{k} \left\{ \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-k-d}^{k} \right\}. \qquad (27) \end{aligned}$$

Proof. 1. We use induction on n. The base case $n = 1$ was proved in (23). Suppose that $n \geq 2$, and that the result holds for the case $n - 1$. By induction, we have

$$\lambda_{i,j-1}^{1} = \delta_1^{c_1} \sum_{d'=0}^{c_1} \binom{c_1}{d'} \lambda_{i+c_1,j-1-d'}^{1} + \sum_{k=1}^{n-1} (-1)^{k-1} \lambda_{i,j-1-k}^{k+1} \left(\sum_{c_2=0}^{c_1-1} \cdots \sum_{c_{k+1}=0}^{c_k-1} \prod_{l=1}^{k} \left(\frac{\delta_l}{\delta_{l+1}} \right)^{c_{l+1}} \right),$$

for each integer $c_1 \geq 0$ such that $i + c_1 \in [a, b]$. Combining the above equality with (21) in the case λ^1; that is, with $\lambda^1 \overset{\delta_0}{\rightsquigarrow} \lambda$,

$$\lambda_{i,j} = \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{c_1=0}^{c-1} \delta_0^{c_1} \sum_{d'=0}^{c_1} \binom{c_1}{d'} \lambda_{i+c_1,j-1-d'}^{1},$$

we obtain

$$\lambda_{i,j} = \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d}$$
$$+ \sum_{c_1=0}^{c-1} \left(\frac{\delta_0}{\delta_1}\right)^{c_1} \left\{ \lambda^1_{i,j-1} - \sum_{k=1}^{n-1} (-1)^{k-1} \lambda^{k+1}_{i,j-1-k} \left(\sum_{c_2=0}^{c_1-1} \cdots \sum_{c_{k+1}=0}^{c_k-1} \prod_{l=1}^{k} \left(\frac{\delta_l}{\delta_{l+1}}\right)^{c_{l+1}} \right) \right\}$$
$$= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} + \sum_{k=1}^{n} (-1)^{k-1} \lambda^k_{i,j-k} \left(\sum_{c_1=0}^{c-1} \sum_{c_2=0}^{c_1-1} \cdots \sum_{c_k=0}^{c_{k-1}-1} \prod_{l=1}^{k} \left(\frac{\delta_{l-1}}{\delta_l}\right)^{c_l} \right).$$

2. If $\delta_0 = \cdots = \delta_n$, we will prove by induction on k that

$$I_k := \sum_{c_1=0}^{c-1} \sum_{c_2=0}^{c_1-1} \cdots \sum_{c_k=0}^{c_{k-1}-1} \prod_{l=1}^{k} \left(\frac{\delta_{l-1}}{\delta_l}\right)^{c_l} = \begin{cases} \phi_k(c-k+1), & \text{if } c \geq k \\ 0, & \text{if } c < k. \end{cases} \tag{28}$$

Obviously, $I_k = 0$, whenever $c < k$, so we assume that $c \geq k$. Now,

$$I_k = \sum_{c_1=0}^{c-1} \phi_{k-1}(c_1 - k + 2) = \phi_{k-1}(1) + \phi_{k-1}(2) + \cdots + \phi_{k-1}(c-k+1) = \phi_k(c-k+1),$$

since $0, -1, -2, \dots, -(k-2)$ are roots of the polynomial ϕ_{k-1} and $c \geq k$. Since $\sigma^{k-1}(\phi_k) = \phi_{-k}$, for all $k \geq 1$, we see that

$$\phi_k(c-k+1) = \left(\sigma^{k-1}(\phi_k)\right)(c) = \phi_{-k}(c) = \frac{c(c-1)\dots(c-k+1)}{k!},$$

for $c \geq k$. Hence, $I_k = \binom{c}{k}$, since we are setting $\binom{a}{b} = 0$, for each pair $a, b \in \mathbb{Z}$ that does not satisfy $0 \leq b \leq a$. Thus, the formula (24) follows.

Since

$$\phi_k(c) = \frac{c(c+1)\dots(c+k-1)}{k!} = \binom{c+k-1}{k},$$

for $k \geq 0$, the second equality in (26) follows from the first; so, it remains to prove the first equality in (26). We use induction on n. The case $n = 0$ is evident, see (22), so we suppose that $n \geq 1$. By (24),

$$\lambda_{i,j} = \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} - \sum_{k=1}^{\min(n,c)} (-1)^k \phi_{-k}(c) \lambda^k_{i,j-k}.$$

By induction on n, setting $m := \min(n, c)$, we have

$$\begin{aligned}
\lambda_{i,j} &= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \lambda_{i+c,j-d} - \sum_{k=1}^{\min(n,c)} (-1)^k \phi_{-k}(c) \left\{ \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \left(\sum_{l=0}^{\min(n-k,c)} \phi_l(c) \lambda_{i+c,j-(k+l)-d}^{k+l} \right) \right\} \\
&= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \left\{ \lambda_{i+c,j-d} - \sum_{k=1}^{\min(n,c)} \sum_{l=0}^{\min(n-k,c)} (-1)^k \phi_{-k}(c) \phi_l(c) \lambda_{i+c,j-(k+l)-d}^{k+l} \right\} \\
&= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \left\{ \lambda_{i+c,j-d} - \sum_{s=1}^{m} \left(\sum_{\substack{k+l=s \\ m \geq k \geq 1, m \geq l \geq 0}} (-1)^k \phi_{-k}(c) \phi_l(c) \right) \lambda_{i+c,j-s-d}^{s} \right\} \\
&= \delta_0^c \sum_{d=0}^{c} \binom{c}{d} \left\{ \lambda_{i+c,j-d} + \sum_{s=1}^{\min(n,c)} \phi_s(c) \lambda_{i+c,j-s-d}^{s} \right\},
\end{aligned}$$

as required, since

$$\sum_{\substack{k+l=s \\ m \geq k \geq 1, m \geq l \geq 0}} (-1)^k \phi_{-k}(c) \phi_l(c) = -\phi_s(c) + \sum_{\substack{k+l=s \\ m \geq k \geq 0, m \geq l \geq 0}} (-1)^k \phi_{-k}(c) \phi_l(c) = -\phi_s(c) + 0 = -\phi_s(c),$$

by Lemma 2.3. ▪

We are now in a position to prove Theorem 1.2 in which we find a basis for the quadratic invariants.

Proof of Theorem 1.2. Any element of the set $K[x]_{\leq 2}$ which has constant term equal to zero can be written as a sum

$$u = \sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{i=1}^{n-1} x_i \left(\sum_{j=i+1}^{n} \lambda_{ij} x_j \right) + \sum_{j=1}^{n} \mu_j x_j.$$

The element u is uniquely determined by the upper triangular $n \times n$ matrix $\Lambda = (\lambda_{ij}) \in M_n(K)$ and the vector $(\mu_1, \dots, \mu_n) \in K^n$. For the sake of convenience, we will set $\lambda_i = \lambda_{ii}$.

Observe that $u \in K[x]^\sigma$ if and only if $\partial(u) = 0$, where $\partial = 1 - \sigma$. In order to calculate $\partial(u)$, we perform elementary computations using the fact that ∂ is a σ-derivation, and the following facts: $\partial(x_1) = 1$ and $\partial(x_1^2) = 2x_1 - 1$; while $\partial(x_i) = -x_{i-1}$ and $\partial(x_i^2) = -2x_{i-1}x_i - x_{i-1}^2$, for

$i = 2, \dots, n$. We obtain

$$\begin{aligned}
\partial(u) = &-\sum_{j=1}^{n-1}(\lambda_{j,j+1} + \lambda_{j+1})x_j^2 \\
&-\sum_{i=2}^{n-2} x_i \left\{ (\lambda_{i,i+2} + \lambda_{i+1,i+2} + 2\lambda_{i+1})x_{i+1} + \sum_{j=i+2}^{n-1} (\lambda_{i+1,j} + \lambda_{i,j+1} + \lambda_{i+1,j+1})x_j + \lambda_{i+1,n}x_n \right\} \\
&- x_1 \left\{ (\lambda_{2,3} + \lambda_{1,3} + 2\lambda_2)x_2 + \sum_{j=3}^{n-1}(\lambda_{2,j} + \lambda_{1,j+1} + \lambda_{2,j+1})x_j + \lambda_{2,n}x_n \right\} \\
&- 2\lambda_n x_{n-1}x_n + (2\lambda_1 + \lambda_{1,2} - \mu_2)x_1 \\
&+ \sum_{j=3}^{n}(\lambda_{1,j-1} + \lambda_{1,j} - \mu_j)x_{j-1} + \lambda_{1,n}x_n + (\mu_1 - \lambda_1).
\end{aligned} \tag{29}$$

Thus, $\partial(u) = 0$ if and only if each of the coefficients in the expression above are zero. This gives the system of linear equations below (see (30), (31), (32), (33), (34) below).

We can immediately see from the coefficients that the entries in the last column of the matrix Λ must all be zero for a solution to $\partial(u) = 0$. Also, the linear terms are specified by the last few coefficients, viz:

$$\mu_1 = \lambda_1, \quad \mu_2 = 2\lambda_1 + \lambda_{1,2}, \quad \mu_j = \lambda_{1,j-1} + \lambda_{1,j}, \quad j = 3, \dots, n. \tag{30}$$

The remaining equations can be separated into four classes:

$$\lambda_{2,3} + \lambda_{1,3} + 2\lambda_2 = 0, \tag{31}$$

$$\lambda_{j,j+1} + \lambda_{j+1} = 0, \quad j = 1, \dots, n-1, \tag{32}$$

$$\lambda_{i+1,j} + \lambda_{i,j+1} + \lambda_{i+1,j+1} = 0, \quad i = 1, \dots, n-2, \quad j = i+2, \dots, n-1, \tag{33}$$

$$\lambda_{i,i+2} + \lambda_{i+1,i+2} + 2\lambda_{i+1} = 0, \quad i = 2, \dots, n-2. \tag{34}$$

Obviously, the elements u_i are linearly independent; so, it suffices to prove that an element $u \in K[x]^\sigma \cap K[x]_{\leq 2}$, with zero constant term, is a linear combination of the elements u_i. In order to do this, we will use induction on $n \geq 2$. In the case that $n = 2$, we see that the element $u_1 = x_1^2 + x_1 + 2x_2$ is the unique solution (up to non-zero scalar multiple) of the system $\partial(u) = 0$. The same is true for $n = 3$, since $\lambda_2 = \lambda_{12} = 0$, by using (31) and (32).

Thus, we may assume that $n \geq 4$, and that the result is true for all n' strictly less than n.

The last column of the matrix (λ_{ij}) is zero. By using (33) with $i = 1, \ldots, n-2$ and $j = n-1$, we see that $\lambda_{i,n-1} = 0$, for $i = 2, \ldots, n-2$. Since $\lambda_{n-1} = -\lambda_{n-2,n-1} = 0$, by (32) with $j = n-2$, it follows that $\lambda_{i,n-1} = 0$, for all $i > 1$. By using similar arguments, it follows that all of the elements of the matrix (λ_{ij}) lying below and on the anti-diagonal are zero; that is,

$$\lambda_{i,j} = 0, \quad i + j \geq n + 1. \tag{35}$$

By passing from u to a suitable linear combination of the form $u + \sum_{i=1}^{m'} \alpha_i u_i$, where $m' = [(n-1)/2]$ and $\alpha_i \in K$, we may assume that

$$\lambda_1 = \cdots = \lambda_{m'} = 0. \tag{36}$$

In more detail, we will solve the system assuming that the conditions above hold, as a result we will have the polynomials u_i, and this justifies our assumption. By (32), we have

$$\lambda_{j,j+1} = 0, \quad j = 1, \ldots, m' - 1,$$

and

$$\lambda_{j,j+2} = 0, \quad j = 1 \ldots, m' - 2,$$

by (31) and (34). Now, by (33), we obtain

$$\lambda_{i,j} = 0, \quad i, j = 1, \ldots, m'. \tag{37}$$

In the case that n is even, it is enough to prove that the element u_m is the unique solution (up to a nonzero scalar multiple) satisfying (36), and in the case that n is odd, that the only solution satisfying (36) is 0.

Suppose first that n is even, with $n = 2m$, and $m \geq 2$ (the cases where $n = 2, 3$ have been considered earlier). Suppose that $n = 4$. By (32), we see that $\lambda_{1,2} = -\lambda_2$, and by (31), we obtain $\lambda_{1,3} = -2\lambda_2$; and so, $u = \lambda_2 u_2$, by (30).

Suppose now that $n = 6$. By (32), we see that $\lambda_{2,3} = -\lambda_3$, and by (31), we obtain $\lambda_{1,3} = \lambda_3$. Now, $\lambda_{2,4} = -2\lambda_3$, by (34). By (33), $\lambda_{1,4} = 3\lambda_3$ and $\lambda_{1,5} = 2\lambda_3$, hence $u = \lambda_3 u_3$, by (30). Finally (for the even case), suppose that $n = 2m$, with $m \geq 4$. In this case, $m' = m - 1$, so all of the diagonal elements of the matrix (λ_{ij}), are zero, except for $\lambda_m = \lambda_{mm}$. By (32), we obtain $\lambda_{m-1,m} = -\lambda_m$, and, by using (34), we get $\lambda_{m-2,m} = \lambda_m$; then, by (32),

$$\lambda_{i,m} = (-1)^{m-i}\lambda_m, \quad i = 1, \ldots, m. \tag{38}$$

By (34), $\lambda_{m-1,m+1} = -2\lambda_m$, and then, by (33),

$$\lambda_{m-i,m+i} = (-1)^i 2\lambda_m, \quad i = 1, \ldots, m-1. \tag{39}$$

Now, all of the entries of the first $m-1$ rows satisfy (33), and $\lambda_{m-1,m} = -\lambda_m, \lambda_{m-1,m+1} = -2\lambda_m$; also, all of the other entries of the $(m-1)$-st row are zero. If we apply Lemma 3.2 to the entries of the first $m-1$ rows, and put $\delta = -1$, we obtain

$$\begin{aligned}\lambda_{ij} &= (-1)^{m-1-i} \sum_{k=0}^{m-1-i} \binom{m-1-i}{k} \lambda_{m-1,j-k} \\ &= (-1)^{m-1-i} \lambda_m \left\{ -\binom{m-1-i}{j-m} - 2\binom{m-1-i}{j-m-1} \right\} \\ &= (-1)^{m-i} \lambda_m \left\{ \binom{m-i}{j-m} + \binom{m-1-i}{j-m-1} \right\},\end{aligned} \tag{40}$$

for $i = 1, \ldots, m-1,\ j = m, \ldots, 2m-i$.

Thus,

$$\begin{aligned}\mu_j &= (-1)^{m-1} \lambda_m \left\{ \binom{m-1}{j-m-1} + \binom{m-2}{j-m-2} + \binom{m-1}{j-m} + \binom{m-2}{j-m-1} \right\} \\ &= (-1)^{m-1} \lambda_m \left\{ \binom{m}{j-m} + \binom{m-1}{j-m-1} \right\},\end{aligned} \tag{41}$$

by (30) and (40). This finishes the even case.

Now, suppose that n is odd, with $n = 2m+1$, for some $m \geq 2$. In this case, $m' = m$, and so $\lambda_{m+1} = 0$, by (35); and $\lambda_{m,m+1} = 0$, by (32). If $m = 2$, then $\lambda_{1,3} = 0$, by (31), and $\lambda_{1,4} = 0$, by (33); so that $\lambda_{ij} = 0$, for all i, j. If $m \geq 3$, then $\lambda_{m-1,m+1} = 0$, by (34). It then follows that all λ_{ij} are zero, by using (35), (36) and (33). ▪

Theorem 1.3, which presents the cubic invariants, can now be proved.

Proof of Theorem 1.3. The element u_k of Theorem 1.2 can be written as the sum, $u_k = u'_k + u''_k$, of the quadratic terms u'_k and the linear terms u''_k. The element $v_k = x_1 u_k + v'_k + v''_k$, where

$$v'_k = a x_k x_{k+1} + \sum_{i=1}^{k-1} \sum_{j=k+1}^{2k-i+1} \alpha^k_{i,j} x_i x_j \quad \text{and} \quad v''_k = \sum_{i=k+1}^{2k+1} \beta^k_i x_i.$$

Clearly,

$$\partial(v_k) = u_k + \partial(v'_k) + \partial(v''_k) = (u'_k + \partial(v'_k)) + (u''_k + \partial(v''_k)).$$

Thus, by using (29), we see that $\partial(v_k) = 0$ if and only if the coefficient a and the $\alpha^k_{i,j}, \beta^k_i$ satisfy the following system of linear equations:

$$\begin{aligned} -a + 1 &= 0 \\ -a - \alpha^k_{k-1,k+1} - 1 &= 0 \\ -a - \alpha^k_{k-1,k+2} - 2 &= 0 \\ -(\alpha^k_{i+1,j} + \alpha^k_{i,j+1} + \alpha^k_{i+1,j+1}) + \lambda^k_{i,j} &= 0, \end{aligned}$$

for $i = 1, \dots, k-1$ and $j = k, \dots, 2k-i$; and

$$\beta^k_j = \alpha^k_{1,j-1} + \alpha^k_{1,j} + \mu^k_{j-1},$$

for $j = k+1, \dots, 2k+1$ (note that we set $\alpha^k_{1,2k+1} = 0$).

Equivalently,

$$a = 1, \quad \alpha^k_{k-1,k+1} = -2, \quad \alpha^k_{k-1,k+2} = -3$$

and

$$\alpha^k_{i,j} = \delta(\alpha^k_{i+1,j-1} + \alpha^k_{i+1,j}) + \lambda^k_{i,j-1},$$

for $i = 1, \dots, k-1$ and $j = k+1, \dots, 2k-i+1$, where $\delta = -1$.

We know, from the proof of Theorem 1.2, that $\lambda^k_{i,j} = \delta(\lambda^k_{i+1,j-1} + \lambda^k_{i+1,j})$, for $i = 1, \dots, k-1$ and $j = k, \dots, 2k-1$. Thus, by using Lemma 3.2.(2), we have

$$\begin{aligned} \alpha^k_{i,j} &= (-1)^{k-i-1}\left\{\binom{k-i-1}{j-k-1}\alpha^k_{k-1,k+1} + \binom{k-i-1}{j-k-2}\alpha^k_{k-1,k+2}\right\} + (k-i-1)\lambda^k_{i,j-1} \\ &= (-1)^{k-i}\left\{2\binom{k-i-1}{j-k-1} + 3\binom{k-i-1}{j-k-2} + (k-i-1)\left[\binom{k-i}{j-k-1} + \binom{k-i-1}{j-k-2}\right]\right\}. \end{aligned}$$

■

4 $\mathcal{F}$-direct sums

Let $Q = \cup_{i\in\mathbb{Z}} Q_i$ be a $\mathbb{Z}$-filtered algebra with a filtration $\mathcal{F} = \{Q_i\}$. We will always assume that the filtration is **separated**; that is, $\cap_{i\in\mathbb{Z}} Q_i = 0$. Any subspace U of Q has an induced filtration $U = \cup_{i\in\mathbb{Z}} U_i$, where $U_i := U \cap Q_i$. In this case, the associated graded space $\mathrm{gr}_{\mathcal{F}} U = \oplus_{i\in\mathbb{Z}} U_i/U_{i-1}$ is a *subspace* of the associated graded algebra $\mathrm{gr}_{\mathcal{F}} Q = \oplus_{i\in\mathbb{Z}} Q_i/Q_{i-1}$ in a natural manner.

Given a separated filtration $\mathcal{F} = \{Q_i\}$ of Q, then, for any nonzero element $u \in Q$, there exists a unique $i \in \mathbb{Z}$ such that $u \in Q_i \backslash Q_{i-1}$. The integer i is called the **$\mathcal{F}$-degree** of u, and is denoted by $\mathrm{fdeg}(u)$.

Definition 4.1 *Let $\{U_j, j \in \mathcal{J}\}$ be a set of subspaces of the $\mathcal{F}$-filtered algebra Q. We say that the sum $\sum_{j\in\mathcal{J}} U_j$ is* **$\mathcal{F}$-direct** *if $\sum_{j\in\mathcal{J}} \mathrm{gr}_{\mathcal{F}} U_j = \bigoplus_{j\in\mathcal{J}} \mathrm{gr}_{\mathcal{F}} U_j$ in $\mathrm{gr}_{\mathcal{F}} Q$.*

The concept of $\mathcal{F}$-directness is extremely useful in finding a K-basis of a ring of invariants and in proving that relations for a ring of invariants are *defining* relations. For a separated filtration $\mathcal{F}$ it follows easily that any $\mathcal{F}$-direct sum $\sum_{j\in\mathcal{J}} U_j$ is the direct sum, $\oplus_{j\in\mathcal{J}} U_j$, of the subspaces U_j.

Lemma 4.2 *Let $Q = \cup_{i\in\mathbb{Z}} Q_i$ be a filtered algebra with separated filtration $\mathcal{F} = \{Q_i\}$ and $\sum_{j\in\mathcal{J}} U_j$ be an $\mathcal{F}$-direct sum of subspaces $\{U_j\}$. Then*

1. *if $u_j \in U_j$, then $\mathrm{fdeg}(\sum u_j) = \max\{\mathrm{fdeg}(u_j)\}$.*
2. *$(\sum_{j\in\mathcal{J}} U_j) \cap Q_i = \sum_{j\in\mathcal{J}} U_j \cap Q_i$.*

Proof. 1. This is evident.

2. Denote by L and R the left and right hand side vector spaces in the equality that we are trying to establish. Clearly, $L \supseteq R$. If $u = \sum u_j \in (\sum U_j) \cap Q_i$, for some $u_j \in U_j$, then each $u_j \in U_j \cap Q_i$, by statement 1, so $R \subseteq L$. ∎

A K-basis $\{U_i, i \in J\}$ of the filtered algebra $Q = \cup_{i\in\mathbb{Z}} Q_i$ is called an $\mathcal{F}$-*basis* if the sum of 1-dimensional subspaces $\sum_{j\in J} K u_j$ is $\mathcal{F}$-*direct*. In this case, $\{\mathrm{gr}\, u_j, j \in J\}$ is a K-basis for the associated graded algebra $\mathrm{gr}\, Q := \oplus_{i\in\mathbb{Z}} Q_i/Q_{i+1}$. If, in addition, the algebra $Q = \cup_{i\geq 0} Q_i$ is *positively* graded then the converse is true: a basis $\{u_j, j \in J\}$ of Q is an $\mathcal{F}$-basis of Q if and only if $\{\mathrm{gr}\, u_j, j \in J\}$ is a basis for $\mathrm{gr}\, Q$; and a basis $\{u_j, j \in J_i\}$ of Q_i is an $\mathcal{F}$-basis of Q if and only if $\{\mathrm{gr}\, u_j, j \in J_i\}$ is a basis for $\oplus_{\nu=0}^{i} Q_\nu/Q_{\nu-1}$.

Similarly, elements $\{u_j, j \in J\}$ of Q are $\mathcal{F}$-*independent* if the sum of 1-dimensional subspaces $\sum_{j\in J} K u_j$ is $\mathcal{F}$-direct. In this case, elements $\{\mathrm{gr}\, u_j, j \in J\}$ are linearly independent elements of $\mathrm{gr}\, Q$. The converse is obviously true; so, the elements $\{u_i, i \in J\}$ of Q are $\mathcal{F}$-independent if and only if the elements $\{\mathrm{gr}\, u_j, j \in J\}$ are linearly independent in $\mathrm{gr}\, Q$.

5 Number of variables ≤ 5

Let K be a field of characteristic zero. The polynomial ring, $P \equiv P^{[n+1]} \equiv K[x] \equiv K[x_1, \ldots, x_{n+1}]$, in $n+1$ variables, is a positively graded K-algebra $P = \oplus_{i\geq 0} P_i$, where $P_i \equiv K[x]_i$ consists of the homogeneous polynomials of degree i, together with zero.

Consider the graded automorphism $\sigma \in \mathrm{Aut}_{\mathrm{gr}}(P)$ defined by $\sigma(x) = \mathcal{J}_{n+1}(1)x$, where $\mathcal{J}_{n+1}(1)$ is the $(n+1) \times (n+1)$ lower triangular Jordan matrix with 1 in each diagonal entry; that is,

$$\sigma(x_1) = x_1, \quad \sigma(x_2) = x_2 + x_1, \quad \dots, \quad \sigma(x_{n+1}) = x_{n+1} + x_n.$$

We use the results in the earlier sections of the paper to give explicit generators and defining relations for P^σ for some small values of n.

The algebra $F = P^\sigma$ of invariants is a positively graded algebra $F = \oplus_{i \geq 0} F_i$, where $F_i = P_i \cap F$. In the case that $n = 0$ we have that σ is the identity map, so that $F = K[x_1]$. Thus, we assume that $n \geq 1$. The element x_1 is σ-invariant. Denote by P_{x_1} the localization of P at the powers of x_1, that is,

$$P_{x_1} = S^{-1}K[x] = K[x_1, x_1^{-1}, x_2, \dots, x_{n-1}, x_n, x_{n+1}],$$

where $S = \{x_1^i \mid i \geq 0\}$. Set $z_i := -\frac{x_{i+1}}{x_1}$, for $i = 1, \dots, n$, so that

$$P_{x_1} = K[z, x_1^{\pm 1}] = K[z_1, \dots, z_n, x_1^{\pm 1}] = Q[x_1^{\pm 1}], \tag{42}$$

where $Q = Q^{[n]} = K[z] = K[z_1, \dots, z_n]$ is the polynomial ring in n variables. The algebra $Q = \oplus_{i \geq 0} Q_i$ is a positively graded K-algebra, using the degree of the polynomials. The filtration $\mathcal{F} = \{Q_{\leq i} := \oplus_{j \leq i} Q_j\}$, for $i \geq 0$, associated with this grading, satisfies $Q_{\leq i} = P_i x_1^{-i}$, for $i \geq 0$, and so $Q = \sum_{i \geq 0} P_i x_1^{-i}$.

Let $p(x_1, \dots, x_{n+1}) \in P$ be a homogeneous polynomial. Then

$$p(x_1, \dots, x_{n+1}) = (-x_1)^{\deg(p)} p(-1, -\frac{x_2}{x_1}, \dots, -\frac{x_{n+1}}{x_1}) = (-x_1)^{\deg(p)} p(-1, z_1, \dots, z_n), \tag{43}$$

where $p(-1, z_1, \dots, z_n) \in Q_{\leq \deg(p)}$.

Denote by the same letter σ the *unique extension* of the automorphism σ to the localized algebra P_{x_1}. Then $\sigma(Q) = Q$, and $\sigma(z) = \mathcal{J}_n(1)z - e_1$; that is,

$$\sigma(z_1) = z_1 - 1, \ \sigma(z_2) = z_2 + z_1, \dots, \sigma(z_n) = z_n + z_{n-1}.$$

Theorem 1.1 (or Proposition 3.1) now becomes available for use.

The case $n = 1$

Clearly,

$$(K[x_1, x_2]_{x_1})^\sigma = K[z_1, x_1^{\pm 1}]^\sigma = K[x_1^{\pm 1}],$$

since $\sigma(z_1) = z_1 - 1$ and the characteristic of K is zero. Hence,

$$K[x_1, x_2]^\sigma = (K[x_1, x_2]_{x_1})^\sigma \cap K[x_1, x_2] = K[x_1]. \tag{44}$$

Thus, we may assume that $n \geq 2$. The first part of the next corollary follows immediately from Proposition 3.1 (statement 3 follows from (43); statements 4 and 5 follow from the definition of $\mathcal{F}$-basis).

Corollary 5.1 *Let $n \geq 2$. Then*
1. $Q^\sigma = K[y_2, \dots, y_n]$ is a polynomial ring in the $n-1$ variables y_i given by

$$y_{i+1} = y_{i+1}(z) = \sum_{j=1}^{i} \phi_{-i+j}(z_1) z_{j+1} + i\sigma^{-1}(\phi_{-i-1}(z_1)),$$

for $i = 1, \dots, n-1$, where the ϕ_l are defined in (10) and $Q = K[z_1] \otimes Q^\sigma$.

2. $y_i \equiv z_i \pmod{z_1}$, for $i = 2, \dots, n$.

3. $F = \oplus_{i \geq 0} F_i$, with $F_i = x_1^i Q^\sigma_{\leq i}$, where $Q^\sigma_{\leq i} = Q^\sigma \cap Q_{\leq i}$, for $i \geq 0$.

4. If $\{b_j, j \in J_i\}$ is an $\mathcal{F}$-basis for the vector space $Q^\sigma_{\leq i}$ then $\{x_1^{i-\deg_z(b_j)}(x_1^{\deg_z(b_j)} b_j), j \in J_i\}$ is a K-basis for F_i. If $\{b_j, j \in J\}$ is an $\mathcal{F}$-basis for the vector space Q^σ then $\{x_1^{i-\deg_z(b_j)}(x_1^{\deg_z(b_j)} b_j), j \in J\}$ is a K-basis for F_i.

5. Given $g_1, \dots, g_m \in Q^\sigma$ such that, for each $i \geq 0$, $\{g^\alpha := g_1^{\alpha_1} \cdots g_m^{\alpha_m} \mid \alpha = (\alpha_1, \dots, \alpha_m) \in J_i \subseteq \mathbb{N}^m\}$ is an $\mathcal{F}$-basis for $Q^\sigma_{\leq i}$ then $\{x_1^{i-\sum_{j=1}^m \alpha_j \deg_z(g_j)} \prod_{j=1}^m (x_1^{\deg_z(g_j)} g_j)^{\alpha_j}, j \in J_i\}$ is a K-basis for F_i; and if $J_1 \subseteq J_2 \subseteq \cdots$ then

$$\{x_1^{i-\sum_{j=1}^m \alpha_j \deg_z(g_j)} \prod_{j=1}^{m} (x_1^{\deg_z(g_j)} g_j)^{\alpha_j} \mid j \in \cup_{i \geq 1} J_i,\ i - \sum_{j=1}^{m} \alpha_j \deg_z(g_j) \geq 0\}$$

is a K-basis for F. ∎

Corollary 5.2 *1. Let $u = u(z_1, \dots, z_n) \in Q^\sigma$. Then*

$$u(z_1, \dots, z_n) = u(0, y_2, \dots, y_n).$$

2. In particular, for the elements u_k and v_k in Theorems 1.2 and 1.3,

$$u_k = y_k^2 + \sum_{i=2}^{k-1} \sum_{j=k}^{2k-i} \lambda_{i,j}^k y_i y_j + \sum_{i=k}^{2k} \mu_i^k y_i,$$

$$v_k = y_k y_{k+1} + \sum_{i=2}^{k-1} \sum_{j=k+1}^{2k-i+1} \alpha_{i,j}^k y_i y_j + \sum_{i=k+1}^{2k+1} \beta_i^k y_i.$$

Proof. Note that $Q = Q^\sigma \oplus Qz_1$, $Q^\sigma = K[y_2, \dots, y_n]$ and $y_i \equiv z_i \pmod{z_1}$, for $i = 2, \dots, n$, by Corollary 5.1. Thus, for any $u(z_1, \dots, z_n) \in Q^\sigma$, we have

$$u(z_1, z_2, \dots, z_n) = u(0, y_2, \dots, y_n) + z_1 v,$$

fore some polynomial $v \in Q$. However,

$$u(z_1, \dots, z_n) - u(0, y_2, \dots, y_n) = v z_1 \in Q^\sigma \cap Q z_1 = 0,$$

since $Q = Q^\sigma \oplus Qz_1$. Hence, $u(z_1, \dots, z_n) = u(0, y_2, \dots, y_n)$.

2. Evident. ▪

The polynomials

$$f_{i+1} := x_1^{\deg(y_{i+1})} y_{i+1} = x_1^{i+1} y_{i+1},$$

for $i = 1, \dots, n-1$, belong to the algebra F of invariants. Note that

$$f_{i+1} = \sum_{j=1}^{i} (-1)^{i-j+1} x_1^j \frac{x_2(x_2 + x_1) \dots (x_2 + (i-j-1)x_1)}{(i-j)!} x_{j+2} + (-1)^{i+1} i \frac{(x_2 - x_1)x_2(x_2 + x_1) \dots (x_2 + (i-1)x_1)}{(i+1)!}.$$

Corollary 5.3 *For each homogeneous polynomial $f(x_1, \dots, x_{n+1}) \in F = K[x_1, \dots, x_{n+1}]^\sigma$, where $\sigma(x) = J_{n+1}(1)x$:*

$$f(x_1, \dots, x_{n+1}) = (-x_1)^{\deg(f)} f(-1, 0, f_2/x_1^2, \dots, f_n/x_1^n).$$

Proof. This follows immediately from the equality (43) and Corollary 5.2.(1):

$$\begin{aligned} f(x_1, \dots, x_{n+1}) &= (-x_1)^{\deg(f)} f(-1, z_1, \dots, z_n) \\ &= (-x_1)^{\deg(f)} f(-1, 0, y_2, \dots, y_n) \\ &= (-x_1)^{\deg(f)} f(-1, 0, f_2/x_1^2, \dots, f_n/x_1^n). \end{aligned}$$ ▪

The case $n = 2$

By Corollary 5.1.(1), we know that $Q^\sigma = K[u_1]$, where $u_1 := 2y_2 = z_1^2 + z_1 + 2z_2$. Clearly, $\{u_1^i, i \geq 0\}$ is an $\mathcal{F}$-basis for Q^σ. By Corollary 5.1(5), the algebra

$$K[x_1, x_2, x_3]^\sigma = K[x_1, p_1], \tag{45}$$

is the polynomial ring in the two variables x_1 and $p_1 := x_1^2 u_1^2 = x_2^2 - x_1(x_2 + 2x_3)$.

The case $n = 3$

Recall that
$$y_2 = z_2 + \frac{(z_1+1)z_1}{2} \quad \text{and} \quad y_3 = z_3 + z_1 z_2 + \frac{z_1^3 - z_1}{3}.$$
Set
$$v_1 := 3y_3 = z_1^3 + 3z_1z_2 - z_1 + 3z_3 = z_1^3 + \dots, \tag{46}$$
Clearly,
$$\begin{aligned} v_1 &= z_1u_1 - z_1^2 + z_1z_2 - z_1 + 3z_2 \\ &= z_1u_1 - u_1 + z_1z_2 + 5z_2. \end{aligned}$$
Consider the element
$$\theta := v_1^2 - u_1^3 + 3v_1u_1 + 2u_1^2. \tag{47}$$
Direct computation gives
$$\begin{aligned} \theta &= -3z_1^2\{z_2^2 - z_1(z_2+2z_3)\} + 9z_1(z_1+2z_2)z_3 - 8z_2^3 + z_1z_2(5z_1+6z_2) \\ &\quad + 9z_3^2 + 3(z_1+6z_2)z_3 + 8z_2^2 + 2z_1z_2 \\ &= -3z_1^2\{z_2^2 - z_1(z_2+2z_3)\} + \cdots . \end{aligned} \tag{48}$$

The leading terms of the elements $u_1 = z_1^2 + \cdots$ and θ are *algebraically independent*, so the subalgebra $U := K[u_1, \theta]$ of $Q^\sigma = K[u_1, v_1]$ is isomorphic to a polynomial ring in two variables, and the elements $\{u_1^i \theta^j \mid i,j \geq 0\}$ are $\mathcal{F}$-independent.

Lemma 5.4 *The sum $Q^\sigma = U + Uv_1$ is an $\mathcal{F}$-direct sum in the filtered algebra Q with the filtration $\mathcal{F} = \{Q_{\leq i}\}$. In particular, $Q^\sigma = U \oplus Uv_1$.*

Proof. It follows from (47) that $Q^\sigma = U + Uv_1$. Observe that the degree of the leading term of any element from U is *even*, and the degree of the leading term of any element of Uv_1 is *odd*. The result then follows. ∎

Corollary 5.5 *For $i \geq 0$, we have*
$$Q^\sigma_{\leq i} = U_{\leq i} \oplus U_{\leq (i-3)}\, v_1,$$
where $U_{\leq i} = U \cap Q_{\leq i} = \bigoplus\{Ku_1^{i_2}\theta^{i_4} \mid i_2, i_4 \geq 0, 2i_2 + 4i_4 \leq i\}$, and $\{u_1^{i_2}v_1^{i_3}\theta^{i_4} \mid 2i_2 + 3i_3 + 4i_4 \leq i;\ i_3 = 0,1;\ i_2, i_4 \geq 0\}$ is an $\mathcal{F}$-basis for $Q^\sigma_{\leq i}$.

Theorem 5.6 *The fixed algebra $F = K[x_1, x_2, x_3, x_4]^\sigma$ is generated by the four elements*

$$x_1, \quad p_1 := x_1^2 u_1 = x_2^2 - x_1(x_2 + 2x_3), \quad q_1 := x_1^3 v_1 = -x_2^3 + 3x_1x_2x_3 + x_1^2 x_2 - 3x_1^2 x_4$$

and

$$s := x_1^4 \theta = -3x_2^2 \left\{x_3^2 - x_2(x_3 + 2x_4)\right\} - 9x_1x_2(x_2 + 2x_3)x_4 + 8x_1x_3^3 \\ - x_1x_2x_3(5x_2 + 6x_3) + 9x_1^2x_4^2 + 3x_1^2(x_2 + 6x_3)x_4 + 8x_1^2x_3^2 + 2x_1^2x_2x_3.$$

of degrees $1, 2, 3, 4$, *respectively, subject to the defining relation*

$$x_1^2 s = q_1^2 + 3x_1p_1q_1 - p_1^3 + 2x_1^2p_1^2. \tag{49}$$

For $i \geq 0$, $F_i = \oplus\{Kx_1^{i_1}p_1^{i_2}q_1^{i_3}s^{i_4} \mid i_1 + 2i_2 + 3i_3 + 4i_4 = i;\ i_3 = 0, 1;\ i_1, i_2, i_4 \geq 0\}$.

Proof. By Corollary 5.1, $F = \oplus_{i\geq 0} F_i$, where $F_i = x_1^i Q^\sigma_{\leq i}$. By Corollary 5.1(5) and Corollary 5.5, (using $\mathcal{F}$-directness) $F_i = \bigoplus Kx_1^{i_1}p_1^{i_2}q_1^{i_3}s^{i_4}$ where all $i_\nu \geq 0$, and $i_3 = 0, 1$ while $i_1 + 2i_2 + 3i_3 + 4i_4 = i$. We conclude that the fixed algebra F is generated by the four elements above, subject to the relation (49). The relation is *the* defining relation for F (as the algebra that satisfies this defining relation has the *same* basis $\{x_1^{i_1}p_1^{i_2}q_1^{i_3}s^{i_4}\}$ as F). ∎

The case $n = 4$

Recall that (Corollary 5.1 and Theorem 1.2)

$$y_4 = z_4 + z_1z_3 + \frac{z_1(z_1-1)}{2!}z_2 + \frac{3(z_1+1)z_1(z_1-1)(z_1-2)}{4!},$$

and

$$u_2 = z_2^2 - z_1(z_2 + 2z_3) - z_2 - 3z_3 - 2z_4 = z_2^2 - z_1(z_2 + 2z_3) + \cdots . \tag{50}$$

By Corollary 5.2,

$$u_2 := y_2^2 - y_2 - 3y_3 - 2y_4 = \frac{1}{4}u_1^2 - \frac{1}{2}u_1 - v_1 - 2y_4. \tag{51}$$

Consider the element

$$\tilde{\theta} := \theta + 3u_1u_2 = v_1^2 - u_1^3 + 3v_1u_1 + 2u_1^2 + 3u_1u_2. \tag{52}$$

Direct computation gives

$$\begin{aligned}\tilde{\theta} &= -6z_1^2z_4 + 6z_1(z_2 - z_1)z_3 - 2z_2^3 + z_1z_2(3z_2 - z_1) \\ &\quad - 6(z_1 + 2z_2)z_4 + 9z_3^2 - 6z_1z_3 + 2z_2^2 - z_1z_2 \\ &= -6z_1^2z_4 + 6z_1(z_2 - z_1)z_3 - 2z_2^3 + z_1z_2(3z_2 - z_1) + \cdots .\end{aligned} \tag{53}$$

The leading terms of the elements $u_1 = z_1^2 + \cdots$, $u_2 = z_2^2 - z_1(z_2 + 2z_3) + \cdots$, and $\tilde{\theta}$ are *algebraically independent*; so the algebra $Q^\sigma = K[y_2, y_3, y_4]$ contains the subalgebra $U := K[u_1, u_2, \tilde{\theta}]$ which is a polynomial ring in three variables.

For $l \geq 0$, we set

$$U_{\leq l} := U \cap Q_{\leq l} = \bigoplus_{i,j,k \geq 0} \{K u_1^i u_2^j \tilde{\theta}^k \mid 2i + 2j + 3k \leq l\}.$$

The set $\{u_1^i u_2^j \tilde{\theta}^k \mid 2i + 2j + 3k \leq l\}$ is an $\mathcal{F}$-basis for $U_{\leq l}$, and $\{u_1^i u_2^j \tilde{\theta}^k \mid i, j, k \geq 0\}$ is an $\mathcal{F}$-basis for U. Now,

$$Q^\sigma = K[u_1, u_2, \tilde{\theta}] + K[u_1, u_2, \tilde{\theta}]y_3 = U + Uv_1. \tag{54}$$

In more detail,

$$\begin{aligned} Q^\sigma &= K[y_2, y_3, y_4] = K[u_1, v_1, y_4] = K[u_1, v_1, u_2] \ \ \text{(by (51))} \\ &= K[u_1, u_2, \tilde{\theta}] + K[u_1, u_2, \tilde{\theta}]v_1 \ \ \text{(by (52))}. \end{aligned}$$

Lemma 5.7 *The sum $Q^\sigma = U + Uv_1$ is an $\mathcal{F}$-direct sum in the filtered algebra Q with the filtration $\mathcal{F} = \{Q_{\leq i}\}$. Hence, $\{u_1^m u_2^j \tilde{\theta}^k v_1^l \mid 2m + 2j + 3k + 3l \leq i,\ l = 0, 1\}$ is an $\mathcal{F}$-basis for $Q^\sigma_{\leq i}$, and $\{u_1^m u_2^j \tilde{\theta}^k v_1^l \mid m, j, k \geq 0,\ l = 0, 1\}$ is an $\mathcal{F}$-basis for Q^σ.*

Proof. This follows directly from the equality

$$Q^\sigma = U + Uv_1 = \bigoplus_{i,j \geq 0} \{K[u_1] + K[u_1]v_1\} u_2^i \tilde{\theta}^j$$

and the fact that the leading terms of the elements $u_2, \tilde{\theta}$ and z_1 are algebraically independent, and from the fact that $v_1 = z_1^3 + \cdots$ and $u_1 = z_1^2 + \cdots$. ∎

Theorem 5.8 *The fixed algebra $F = K[x_1, x_2, x_3, x_4, x_5]^\sigma$ is generated by the five elements*

$$x_1, \quad p_1, \quad q_1, \quad p_2 := x_1^2 u_2 = x_3^2 - x_2(x_3 + 2x_4) + x_1(x_3 + 3x_4 + 2x_5)$$

and

$$t := x_1^3\tilde{\theta} = 6x_2^2x_5 - 6x_2(x_3 - x_2)x_4 + 2x_3^3 - x_2x_3(3x_3 - x_2) \\ + x_1\left\{-6(x_2 + 2x_3)x_5 + 9x_4^2 - 6x_2x_4 + 2x_3^2 - x_2x_3\right\} \quad (55)$$

of degrees $1, 2, 3, 2, 3$*, respectively, subject to the defining relation*

$$x_1^3 t = q_1^2 - p_1^3 + 3x_1p_1q_1 + 2x_1^2p_1^2 + 3x_1^2p_1p_2. \quad (56)$$

For $i \geq 0$*,*

$$F_i = x_1^i Q^\sigma_{\leq i} = \bigoplus\{Kx_1^a p_1^b p_2^c t^d q_1^e \,|\, a + 2b + 2c + 3t + 3e = i,\ e = 0, 1\}$$

where $a, b, c, d \geq 0$*.*

Proof. By Corollary 5.1 and Lemma 5.7, $F = \oplus_{i\geq 0} F_i$, where

$$F_i = x_1^i Q^\sigma_{\leq i} = \bigoplus\{Kx_1^a p_1^b p_2^c t^d q_1^e \,|\, a + 2b + 2c + 3t + 3e = i,\ e = 0, 1\}.$$

By (52), the relation (56) holds. This relation is *the* defining relation for the algebra F since the algebra that satisfies this defining relation has the same basis as F (see (56)). ∎

Finally, we make a comment on the case where $n \geq 5$. Assume that $n \geq 5$. Set $m := \left[\frac{n}{2}\right]$ and $\mu := \left[\frac{n-1}{2}\right]$; then $m = \mu$, when $n = 2m + 1$, and $m = \mu + 1$, when $n = 2m$. The elements $u_k = u_k(z)$ and $v_k = v_k(z)$ of Theorems 1.2, 1.3 can be written as sums $u_k = u'_k + u''_k$ and $v_k = z_1u_k + v'_k + v''_k$, where u'_k, v'_k and u''_k, v''_k are the *quadratic* terms and *linear* terms, respectively. The elements

$$w_k := v_k^2 - u_1u_k^2 = u_k(2z_1v'_k - (z_1 + 2z_2)u_k) + (v'_k)^2 + 2z_1u_kv''_k + 2v'_kv''_k + (v''_k)^2, \quad (57)$$

for $k = 2, \ldots, \mu$, have degree 5 and leading terms

$$l(w_k) = u'_k(2z_1v'_k - (z_1 + 2z_2)u'_k), \quad (58)$$

for $k = 2, \ldots, \mu$. Observe that u'_k is the leading term of the polynomial u_k.

The polynomials

$$u_k = (-1)^{k-1}2z_{2k} + \ldots \qquad \text{and} \qquad v_k = (-1)^{k-1}(1 + 2k)z_{2k+1} + \ldots, \quad (59)$$

where by three dots we denote the terms from $Q^{[2k-1]}$ and $Q^{[2k]}$, respectively. Hence,

$$Q \equiv Q^{[n]} = Q^{[4]}[v_2, \ldots, v_\mu, u_3, \ldots, u_m], \quad (60)$$

and

$$Q^\sigma = (Q^{[4]})^\sigma[v_2, \dots, v_\mu, u_3, \dots, u_m] = (Q^{[4]})^\sigma \otimes V, \tag{61}$$

where $V := K[v_2, \dots, v_\mu, u_3, \dots, u_m]$. Note that

$$\begin{aligned}
(Q^{[5]})^\sigma &= (Q^{[4]})^\sigma[v_2], \\
(Q^{[6]})^\sigma &= (Q^{[4]})^\sigma[v_2, u_3], \\
(Q^{[7]})^\sigma &= (Q^{[4]})^\sigma[v_2, v_3, u_3], \\
(Q^{[8]})^\sigma &= (Q^{[4]})^\sigma[v_2, v_3, u_3, u_4].
\end{aligned}$$

The polynomial ring V contains the polynomial subalgebra $W := K[w_2, \dots, w_\mu, u_3, \dots, u_m]$ and

$$V = \bigoplus_{I \in \mathbb{Z}_2^{\mu-1}} W v^I, \tag{62}$$

is a free W-module of rank $2^{\mu-1}$, where $\mathbb{Z}_2^{\mu-1} = \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ is the product of $\mu - 1$ copies of $\mathbb{Z}_2 = \{0, 1\}$, and $v^I = v_2^{i_2} \dots v_\mu^{i_\mu}$, for $I = (i_3, \dots, i_\mu) \in \mathbb{Z}_2^{\mu-1}$.

The leading terms of the polynomials $w_2, u_3, w_3, u_4, \dots,$ are *algebraically independent*, since

$$l(u_k) = u_k' = (-1)^{k-1} 2 z_1 z_{2k-1} + \dots \qquad \text{and} \qquad l(v_k') = (-1)^{k-1}(2k-1) z_1 z_{2k} + \dots, \tag{63}$$

where three dots denotes terms from $Q^{[2k-2]}$ and $Q^{[2k-1]}$, respectively. Thus, the sum

$$W = \sum K u_3^{i_3} \dots u_m^{i_m} w_2^{j_2} \dots w_\mu^{j_\mu}, \tag{64}$$

for $i_2, \dots, j_\mu \geq 0$, is an $\mathcal{F}$-direct sum.

Lemma 5.9 *The sum (62) is an $\mathcal{F}$-direct sum in the filtered algebra Q with the filtration $\mathcal{F} = \{Q_{\leq i}\}$.*

Proof. This is evident, since $l(v_k) = z_1 l(u_k), l(u_k) = u_k'$ and $l(w_k)$ is as defined in (58). ∎

This means that one cannot produce new invariants from the elements of V; that is, in any new invariants generators of $Q^{[4]}$ must necessarily occur.

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