LEAVITT PATH ALGEBRAS SATISFYING A POLYNOMIAL IDENTITY

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Abstract. Leavitt path algebras L of an arbitrary graph E over a field K satisfying a polynomial identity are completely characterized both in graph-theoretic and algebraic terms. When E is a finite graph, L satisfying a polynomial identity is shown to be equivalent to the Gelfand-Kirillov dimension of L being at most one, though this is no longer true for infinite graphs. It is shown that, for an arbitrary graph E, the Leavitt path algebra L has Gelfand-Kirillov dimension zero if and only if E has no cycles. Likewise, L has Gelfand-Kirillov dimension one if and only if E contains at least one cycle, but no cycle in E has an exit.

1. Introduction

Leavitt path algebras were introduced in [1, 10] as algebraic analogues of graph C*-algebras and as natural generalizations of Leavitt algebras of type $(1, n)$ constructed by Leavitt [14]. The various ring-theoretical properties of these algebras have been actively investigated in a series of papers, see, for example, [1, 3, 6, 7, 10].

It is straightforward to show that a Leavitt path algebra L of a connected graph E over a field K (see Section 2 for the relevant definitions) is commutative if and only if the graph E is either a single vertex or consists of a single vertex v and an edge e which is a loop at v; namely, the edge e begins and ends at v. In this case, L is isomorphic to K or to the Laurent polynomial ring $K[x, x^{-1}]$. Observing that commutative algebras satisfy the polynomial identity $xy - yx = 0$, it is natural to ask under which conditions a Leavitt path algebra satisfies a polynomial identity. In this paper, we obtain a complete characterization of Leavitt path algebras satisfying a polynomial identity in both algebraic and graph-theoretic terms (Theorem 3.1). In graph-theoretic terms, we show that the Leavitt path algebra L of an arbitrary graph E over the field K satisfies a polynomial identity if and only if no cycle in E has an exit and there is a fixed positive integer d such that for every vertex $v \in E$, the number of distinct paths ending at v having no repeated vertices is at most d. In this case, we show that L is a subdirect product of matrix rings of order $\leq d$ over K and $K[x, x^{-1}]$.

If, in addition, E is row-finite (that is, each edge of E emits only finitely many edges), we get a stronger conclusion for L: the Leavitt path algebra L is isomorphic to a (possibly infinite) direct sum of matrix rings either over K or $K[x, x^{-1}]$, where the order of each matrix ring in this decomposition is less than a fixed positive integer d and is specified in terms of graph-theoretic data (Theorem 3.2).

The first author acknowledges support of NSERC grant 31-611456. The second author acknowledges support of EPSRC grant EP/K035827/1.

Specialising further, when E is a finite graph we obtain several equivalent characterizing properties for L to be a PI algebra (Theorem 4.1) including the property that the Gelfand-Kirillov dimension (GK dimension, for short) of L is at most 1.

In general, if E is an infinite graph with the property that its associated Leavitt path algebra L is PI then L must have GK dimension at most 1. We give examples, however, that show that there exist Leavitt path algebras having GK dimension ≤ 1 which are not PI algebras. We then consider the larger class of Leavitt path algebras having low GK dimension. For instance, a Leavitt path algebra L of a graph E has GK dimension 0 if and only if L is von Neumann regular, equivalently, E has no cycles. Likewise, L will have GK dimension 1 if and only if E contains at least one cycle and no cycle in E has an exit. In this case, L is a directed union of finite direct sums of matrix rings of finite order over K and $K[x, x^{-1}]$.

2. Background and definitions

Here, we give some of the background needed for the paper along with an overview of some of the earlier work on this subject. Unless otherwise stated, all the graphs that we consider are arbitrary in the sense that no restriction is placed either on the number of vertices or on the number of edges emitted by any single vertex. Generally, we follow the notation and terminology for Leavitt path algebras that appears in [1, 10]. We give below a short outline of some of the basic concepts and results that we need.

Definition 2.1. A (directed) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called the *vertices* of E and the elements of E^1 are called the *edges* of E. For each edge $e \in E^1$, there are vertices $s(e), r(e) \in E^0$, not necessarily distinct, such that e begins at $s(e)$ and ends at $r(e)$. The element $r(e)$ is called the *range* of e and the element $s(e)$ is called the *source* of e.

We now give some graph-theoretic terminology that will be useful. A vertex v is called a *sink* if it emits no edges (that is, $s^{-1}(v)$ is empty) and is called an *infinite emitter* if it emits infinitely many edges (that is, $\#s^{-1}(v) = \infty$). A vertex that is neither a sink nor an infinite emitter is called a *regular vertex*; that is, the regular vertices are the vertices which emit a nonzero finite number of edges. A *path* μ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n-1$. For such a path, we set $s(\mu) := s(e_1)$ and $r(\mu) := r(e_n)$. We consider a vertex to be a path of length 0. The set of all vertices on the path μ is denoted by μ^0.

A path $\mu = e_1 \ldots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be based at the vertex $v = s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once; that is, $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice; that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle $\mu = e_1 e_2 \cdots e_n$ is said to have an *exit* at the vertex $v = s(e_1)$, if there is an edge $f \neq e_1$ such that $s(f) = v = s(e_1)$.

We put a binary relation $\geq$ on the set of vertices as follows. If there is a path from vertex u to a vertex v, we write $u \geq v$. A subset D of vertices is said to be *downward directed* if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and

$v \geq w$. A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set H is *saturated* if $r(s^{-1}(v)) \subseteq H$ implies $v \in H$, for any regular vertex $v \in E^0$.

Intuitively, if there is an edge e with source v and range w then we can think of w as being an immediate descendant of v. The relation $u \geq u'$ just means that u' is a descendant of u and the graph being downward directed means that every pair of vertices share a common descendant. A subset being hereditary then means that if an element is in the set then so are all its descendants and being saturated is the same as saying that if all of the immediate descendants of a regular vertex v are in the set then v is necessarily in the set.

Definition 2.2. Given an arbitrary graph E and a field K, the *Leavitt path algebra*, $L_K(E)$, is defined as follows. For each $e \in E^1$, we create a corresponding *ghost edge*, which we denote e^*. We then define $r(e^*) := s(e)$, and $s(e^*) := r(e)$. With these data, we then define the Leavitt path algebra on E to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$;
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$;
(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$;
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$, we have $v = \sum_{e\in E^1, s(e)=v} ee^*$.

We note that CK stands for Cunz-Krieger, who studied these relations in the context of graph C^*-algebras.

Given a path $\mu = e_1e_2\cdots e_n$ in the graph E, we refer to $\mu^* = e_n^*\cdots e_2^*e_1^*$ as the corresponding *ghost path*.

A Leavitt path algebra carries a natural $\mathbb{Z}$-graded structure, where the vertices have degree zero, the edges have degree one and the ghost edges have degree -1, see [1, Lemma 1.7]. A fact that will prove useful to us is that any nonzero graded ideal must contain a vertex, see [5, Corollary 2.4].

A subgraph F of a graph E is called *complete* in case $s_F^{-1}(v) = s_E^{-1}(v)$, for each regular vertex v in F^0. (In other words, if v emits a nonzero finite number of edges in E, then all these edges must belong to F). If F is a complete subgraph of E then $L_K(F)$ is a subalgebra of $L_K(E)$. This is proved, for example, in [12, Corollary 3.3]. We include a short outline proof here.

Lemma 2.3. *Let F be a complete subgraph of a graph E. Then the inclusion of the graph F in the graph E induces a natural inclusion of $L_K(F)$ as a subalgebra of $L_K(E)$.*

Proof. As the relations for $L_K(F)$ are a subset of the relations for $L_K(E)$, there is a natural induced ring homormorphism from $L_K(F)$ to $L_K(E)$. The kernel of this map is a graded ideal of $L_K(F)$, as the homomorphism respects the $\mathbb{Z}$-grading. If the kernel were nonzero then it would contain a vertex v by [5, Corollary 2.4]. However, in this case the vertex v would be equal to zero in $L_K(E)$, a contradiction. □

For any vertex v, define $T(v) = \{w \in E^0 : v \geq w\}$. We say there is a *bifurcation* at a vertex v, if v emits more than one edge. In a graph E, a vertex v is called a *line point* if there is no bifurcation or a cycle based at any vertex in $T(v)$. Thus, if v is a line point, there will be a single finite or infinite line segment μ starting at v (μ could just be v) and any other path α with $s(\alpha) = v$ will just be an initial sub-segment of μ. It was shown in [11] that v is a line point in E if and only if $L_K(E)v$ (and likewise $vL_K(E)$) is a simple left (right) ideal and that the ideal generated by all the line points in E is the socle, $\mathrm{Soc}(L_K(E))$, of $L_K(E)$.

We shall be using the following concepts and results introduced by Tomforde [18]. A *breaking vertex* of a hereditary saturated subset H is an infinite emitter $w \in E^0 \backslash H$ with the property that $0 < \#(s^{-1}(w) \cap r^{-1}(E^0 \backslash H)) < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, we define

$$v^H := v \;-\; \sum_{s(e)=v, r(e) \notin H} ee^*. \tag{1}$$

Given a hereditary saturated subset H and a subset $S \subseteq B_H$, we say that (H, S) is an *admissible pair*. To an admissible pair (H, S), we can associate the ideal generated by $H \cup \{v^H : v \in S\}$. We let $I(H, S)$ denote this ideal. Tomforde [18] showed that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I(H, S)$ for some admissible pair (H, S). Moreover,

$$L_K(E)/I(H, S) \cong L_K(E \backslash (H, S)).$$

Here $E \backslash (H, S)$ is the quotient graph of E, whose vertex set is given by

$$(E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$$

and whose edges are given by

$$\{e \in E^1 \colon r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\},$$

and we extend r, s to $(E \backslash (H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

3. Leavitt path algebras satisfying polynomial identities

In this section we characterize the Leavitt path algebras satisfying a polynomial identity and give explicit isomorphisms in the case that we are working with a finite graph.

Let K be a field and let E be an arbitrary graph. We show that $L_K(E)$ satisfies a polynomial identity (is PI) if and only if no cycle in E has an exit, every path from a vertex in E ultimately ends at a sink or at a vertex on a cycle and there is a positive integer d with the property that whenever μ is a path that does not visit any vertex more than once, then μ necessarily has length at most d. We show that these conditions are in fact equivalent to $L_K(E)$ being a subdirect product of matrix rings of order $\leq d$ over K and $K[x, x^{-1}]$. When E is a row-finite graph, $L_K(E)$ is PI if and only if it decomposes as a direct sum of (possibly infinitely many) matrix rings over K and $K[x, x^{-1}]$ and each matrix ring in this decomposition is of dimension at most d.

We are now ready to describe the Leavitt path algebras satisfying a polynomial identity.

Theorem 3.1. *Let K be a field and let E be an arbitrary graph. Then the following are equivalent:*

(i) *the Leavitt path algebra $L_K(E)$ is a PI algebra;*

(ii) *no cycle in E has an exit, every path from a vertex in E eventually ends at a sink or at a vertex on a cycle, and there is a fixed integer d such that the number of paths that end at any given sink or on a cycle (but not including the cycle) is less than or equal to d.*

(iii) *there exists a fixed positive integer d such that $L_K(E)$ is a subdirect product of matrix rings over K and $K[x, x^{-1}]$ having order at most d.*

Proof. (i)$\Rightarrow$(ii). Suppose that $L := L_K(E)$ satisfies a polynomial identity of degree N and suppose towards a contradiction that there is a cycle c having an exit f at a vertex v. Then vLv is a subring of L with identity v, which satisfies the same polynomial identity. We then have $c^*c = v$ in vLv. Since vLv satisfies a polynomial identity we then must have $cc^* = v$ [15, Chapt II, Proposition 4.3]. But we now observe that $c^*f = 0$ and consequently $cc^*f = 0$. But $cc^*f = vf = f$, and so we get $f = 0$, a contradiction. Hence no cycle in E has an exit.

If there is no integer d satisfying the desired property in (ii), then for each positive integer m there is some vertex v, depending on m, such that there are at least m^2 paths ending at v (with no repeated vertices). Now there are two possibilities: either there is some path of length $> m$ ending at v or for some $i \leq m$ there are m distinct paths of length i ending at v. We pick $m > N$.

As the Jacobson radical of $L_K(E)$ is zero, see [2, Proposition 6.3], there is a primitive ideal Q such that $v \notin Q$. Since the prime ring $L_K(E)/Q$ satisfies the same polynomial identity as $L_K(E)$, we appeal to Amitsur's theorem [15, Chapt. II, Theorem 3.1] to conclude that $L_K(E)/Q$ embeds in a matrix ring $M_s(F)$ for some $s \geq 1$ and some field F, and where s is bounded above by $N/2$.

Suppose first that we have m distinct paths $\alpha_1, \dots, \alpha_m$, all having the same length $i \leq m$ and ending at v. Now, the elements $e_j := \alpha_j \alpha_j^*$, for $j = 1, \dots, m$, are m mutually orthogonal idempotents in L. We claim that their images are nonzero and distinct in L/Q. If not, suppose $e_j \in Q$ or $e_j - e_i \in Q$, for some $i \neq j$. Then $e_j = e_j(e_j - e_i) \in Q$, in either case. Hence,

$$v = v^2 = (\alpha_j^* \alpha_j)^2 = \alpha_j^*(\alpha_j \alpha_j^*)\alpha_j = \alpha_j^* e_j \alpha_j \in Q,$$

a contradiction that establishes our claim. However, $s \leq N/2$ and $M_s(F)$ cannot have more than s orthogonal idempotents. As $m > N > s$, we get a contradiction in this case.

On the other hand, if we have a path α of length $> m$ ending at v, let $\beta_1, \dots, \beta_m$ be the terminal paths of α such that β_i has length i for each $i = 1, \dots, m$, and $r(\beta_i) = r(\alpha) = v$. If $f_i := \beta_i \beta_i^*$, then the images of the f_i in $L_K(E)/Q$ form a set of m nonzero orthogonal idempotents in $L_K(E)/Q$. As before, we get a contradiction.

Thus we conclude that there is a fixed positive integer d such that, given any vertex v, the number of paths ending at v and having no repeated vertices is at most d. Since, in addition, cycles to not have exits, this means that the only infinite paths in E are paths that are eventually of the form $ggg\cdots$ for some cycle g. In other words, every path from any vertex in E eventually ends at a sink or at a vertex on a cycle. This proves that (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii). Assume (ii). Given any vertex v in E, set

$$H(v) := \{u \in E^0 : u \not\geq v\}$$

and set

$$M(v) := E^0 \backslash H(v) = \{u \in E^0 : u \geq v\}.$$

If v is a sink or a vertex on a cycle, then $H(v)$ is a hereditary saturated set and since $M(v)$ is downward directed, the ideal $P_v = I(H(v), B_{H(v)})$ is a prime ideal not containing v, by [16, Theorem 3.12]. Let $\mathcal{S}$ denote the set of all such (graded) prime ideals P_v, where v ranges over all vertices that are either a sink or that lie on a cycle in E. Observe that if v is a sink E then v is still a sink in the quotient graph $E\backslash(H(v), B_{H(v)})$ and, moreover, $(E\backslash(H(v), B_{H(v)})^0 = M(v)$. Hence, in $E\backslash(H(v), B_{H(v)})$ there are no cycles, every vertex is connected to v by a path and the number of paths ending at v is at most d. This implies that $E\backslash(H(v), B_{H(v)})$ is a finite acyclic graph with a unique sink v. Hence $L_K(E)/P_v \cong L_K(E\backslash(H(v), B_{H(v)}) \cong M_{n_v}(K)$ with $n_v \leq d$ (see [4, Lemma 3.4]). Similarly, if v is a vertex on a cycle, then, by similar argument, $E\backslash(H(v), B_{H(v)})$ is a finite graph in which every vertex connects to v and v lies on a unique cycle and so, for the corresponding P_v, $L_K(E)/P_v \cong L_K(E\backslash(H(v), B_{H(v)}) \cong M_{m_v}(K[x, x^{-1}])$ with $m_v \leq d$ (see [4, Proposition 3.6]). Now

$$\bigcap_{P_v \in \mathcal{S}} P_v = (0).$$

This is immediate due to the fact that every vertex u in E belongs to $M(v)$ for some v where v is a sink or lies on a cycle, and so for this v we have $u \notin P_v$. It follows that the intersection of the P_v cannot contain any vertices. Now, any nonzero graded ideal must contain a vertex, by [5, Proposition 2.4]; so the intersection of the (graded ideals) P_v is 0. Thus $L_K(E)$ is a subdirect product of $\{L/P_v : P_v \in \mathcal{S}\}$, proving (iii).

(iii)$\Rightarrow$(i). This is immediate from the Amitsur-Levitzki theorem [15]. □

In the case that the graph E is row-finite, Theorem 3.1(iii) can be sharpened further, leading to a structure theorem for PI Leavitt path algebras over row-finite graphs.

Theorem 3.2. *Let E be a row-finite graph. Then the following are equivalent for the Leavitt path algebra $L := L_K(E)$:*

(i) *$L_K(E)$ is a PI algebra;*

(ii) *no cycle in E has an exit, every path from a vertex in E eventually ends at a sink or at a vertex on a cycle, and there is a fixed integer d such that the number of paths that end at any given sink or on a cycle (but not including the cycle) is less than or equal to d.*

(iii) *there is a fixed integer d and an isomorphism*

$$L_K(E) \cong \bigoplus_{v \in \Lambda} M_{n_v}(K) \oplus \bigoplus_{C \in \Lambda'} M_{m_C}(K[x, x^{-1}]),$$

where Λ is the collection of sinks in E and Λ' is the collection of cycles in E, and for each $v \in \Lambda$ and $C \in \Lambda'$ we have that n_v and m_C are at most d;

(iv) *there exists a fixed positive integer d such that, for each minimal prime ideal P of $L_K(E)$, L/P is isomorphic to a matrix ring over K or $K[x, x^{-1}]$ of size at most d.*

Proof. The implication (i)$\Rightarrow$(ii) follows from Theorem 3.1.

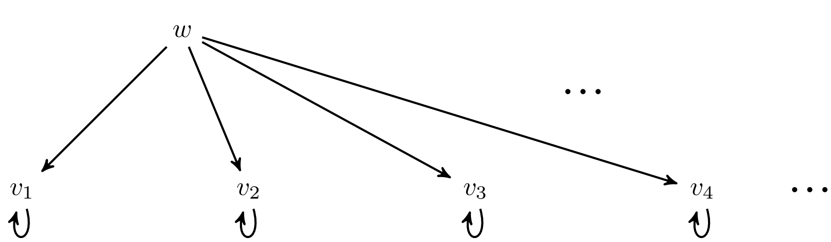
(ii)⇒(iii). Assume (ii). One can obtain (iii) from (ii) by following the proof of Theorem 3.9 of [3]. Here we outline a proof. If $(w_i : i \in I\}$ is the set of all the sinks in E, then the socle of L is $S = \bigoplus\limits_{i \in I} S_i$ where each ideal $S_i = \bigoplus\limits_{r=1}^{n_i} L w_i \alpha_{i_r}^*$ with n_i being the number of paths α_{i_r} in E that end at the sink w_i, see [11]. Now $S_i \cong M_{n_i}(K)$, as S_i is a direct sum of n_i isomorphic simple modules (whose endomorphism ring is the field K). Likewise, if $\{c_j : j \in J\}$ is the set of all distinct cycles (with no exits) in E and if T the ideal generated by all the vertices in these cycles, then $T = \bigoplus\limits_{j \in J} T_j$. Here T_j is the ideal generated by the vertices on a single no-exit cycle c_j based at a vertex v_j and so $T_j = M_{l_j}(K[x, x^{-1}])$ where l_j represents the number of paths that end at v_j but do not include that cycle c_j [4, Propositions 3.6, 3.7]. Consider the ideal $S + T$. Now $H = (S + T) \cap E^0$ is a hereditary saturated set. Since every path from a vertex u in E eventually ends in a sink or at a vertex on a cycle, the row-finiteness of E implies (by a simple induction on the length of a path of maximum length from u to a vertex in H) that u belongs to the hereditary saturated set H. Thus $E^0 \subset H$ and we conclude that the ideal $S + T = L$. Also $S \cap T = 0$ since if a vertex belongs to the hereditary saturated set $S \cap T$, this will give rise to a cycle c which will have an exit, a contradiction. Thus $L = S \oplus T = \bigoplus\limits_{i \in \Lambda} M_{n_i}(K) \oplus \bigoplus\limits_{j \in \Lambda'} M_{l_j}(K[x, x^{-1}])$ with the desired properties for $n_i, l_j, \Lambda, \Lambda'$.

(iii)⇒(iv). Assume (iii). Now any minimal prime ideal P of L is the complement of a single matrix summand in L, namely, $L = P \oplus Q$ where $Q \cong M_{n_i}(K)$ or $M_{l_j}(K[x, x^{-1}])$ where $n_i, l_j \leq d$. This proves (iv).

(iv)⇒(i). Assume (iv). Since the Jacobson radical of L is 0, the intersection of all minimal prime ideals P of L is 0. Thus L embeds in the direct product of L/P for various minimal prime ideals P of L. By hypothesis, each L/P is a matrix ring of size $\leq d$ over K or $K[x, x^{-1}]$ and so satisfies the standard polynomial identity S_{2d} by the Amitsur-Levitzki theorem [15]. Then L itself satisfies the polynomial identity S_{2d}, thus proving (i). □

We give an example of a graph E that is not row-finite whose Leavitt path algebra $L_K(E)$ satisfies a polynomial identity but is not a direct sum of matrix rings over K and $K[x, x^{-1}]$.

Example 3.3. Consider the graph E whose vertex set consists of a vertex w along with a countably infinite set of vertices $v_1, v_2, \ldots$, and with a directed edge from w to each v_i and with a loop based at each v_i. The graph E is given below.



Using the notation of Theorem 3.1, for each i, $M(v_i) = \{w, v_i\}$,

$$H(v_i) = \{v_j : j \neq i\}$$

and P_{v_i} is the (prime) ideal generated by

$$H(v_i) \cup \left\{ w^{H(v_i)} \right\}$$

and P_w is the (maximal) ideal generated by $\{v_1, v_2, \ldots\}$. For each i, $L/P_{v_i} \cong M_2(K[x, x^{-1}])$ and $L/P_w \cong K$. Hence, L is the subdirect product of K and infinitely many copies of $M_2(K[x, x^{-1}])$. Thus L is a PI algebra, but L cannot decompose as a direct sum of matrix rings over K and $K[x, x^{-1}]$, since if this were the case, w would lie in a direct sum of finitely many matrix rings of finite order; since the ideal generated by w is all of $L_K(E)$, we would then necessarily have that $L_K(E)$ embeds in a finite direct sum of matrix rings of finite order over K and $K[x, x^{-1}]$, contradicting the fact that $L_K(E)$ has infinitely many orthogonal idempotents.

In the next section, we explore the connection between polynomial identities and GK dimension for Leavitt path algebras.

4. Leavitt path algebras with GK dimension ≤ 1

We first show that if E is a finite graph, then $L_K(E)$ is a PI algebra if and only if the Gelfand-Kirillov dimension (for short, GK dimension) of $L_K(E)$ is ≤ 1. (Note that, in view of Theorem 3.2, this can be deduced from work in [8, 9].) Examples are constructed showing that this statement is no longer true if E is an infinite graph.

Let K be a field, let A be a finitely generated K-algebra, and let V be a finite-dimensional subspace of A that generates A as a K-algebra. Then the *Gelfand-Kirillov dimension* of A (GK dimension, for short) is defined by

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \log_n(V + V^2 + \cdots + V^n),$$

where V^i denotes the subspace of A spanned by all products of i elements from V. We note that $\mathrm{GKdim}(A)$ is independent of the choice of generating subspace V.

If A is not finitely generated as a K-algebra, then we define the GK dimension of A by

$$\mathrm{GKdim}(A) = \sup_B \mathrm{GKdim}(B),$$

where B runs over all finitely generated K-subalgebras of A. For basic properties and results about GK dimension, we refer the reader to [13]. For Leavitt path algebras of finite graphs, we show that having Gelfand-Kirillov dimension at most one is equivalent to satisfying a polynomial identity.

Theorem 4.1. *Let E be a finite graph and $L = L_K(E)$. Then the following are equivalent:*

(i) *$L_K(E)$ is a PI algebra;*
(ii) *no cycle in E has an exit;*
(iii) *L is a direct sum of finitely many matrix rings of finite order over K and $K[x, x^{-1}]$;*
(iv) *L has GK dimension ≤ 1;*
(v) *L is a finite module over its centre;*

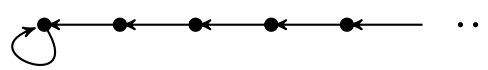
(vi) *if* $a, b \in L$ *satisfy* $ab = 1$ *then* $ba = 1$

Proof. The equivalence of (i)–(iii) follows from Theorem 3.2 specialized to the case of a finite graph. Since $\mathrm{GKdim}(K) = 0$ and $\mathrm{GKdim}(K[x, x^{-1}]) = 1$, we immediately have (iii) $\Longrightarrow$ (iv). Similarly, (iii) implies (v). The fact that (iv) implies (i) follows immediately from the Small-Stafford-Warfield theorem [17] and (v) implies (i) is immediate from basic facts about PI algebras, so that (i)–(v) are equivalent.

Another general fact about PI algebras, [15, Chapter II, Proposition 4.3], shows that (i)⇒(vi). Finally, we show that (vi)⇒(ii). Suppose that (vi) holds but that there is a cycle c based at v with an edge e that starts at v and does not lie on c. Note that $c^*c = v$. Set u to be the sum of all vertices in the graph other than v, and note that $u + v = 1$. Using the fact that the distinct vertices give orthogonal idempotents in L, we see that $(u + c^*)(u + c) = u + v = 1$, and so $(u + c)(u + c^*) = 1$, by (vi). Thus, $u + v = 1 = (u + c)(u + c^*) = u + cc^*$ and so $v = cc^*$. Hence, $e = ve = c(c^*e) = 0$, a contradiction. □

In general the conclusion to Theorem 4.1 need not hold for arbitrary infinite graphs. Specifically, there exist infinite graphs E for which the GK dimension of $L_K(E)$ is either zero or one, but where $L_K(E)$ is not a PI algebra.

Example 4.2. Let E be the infinite row-finite graph given below.



As E is row finite, but fails to satisfy Condition (ii) of Theorem 3.2, since there is no suitable integer d, we see that L is not a PI algebra.

For each $n \geq 2$, let E_n be the subgraph of E that contains the first n vertices from the left, and all edges to the left of the nth vertex. Then E_n is a finite graph with one cycle and this cycle has no exit; hence, $\mathrm{GKdim}(E_n) = 1$, by [8, Theorem 5]. As each E_n is a complete subgraph of E, each $L_K(E_n)$ can be naturally viewed as a subalgebra of $L_K(E)$. Moreover, $L_K(E)$ is the directed union of the $L_K(E_n)$ and so $\mathrm{GKdim}(E) = 1$.

Example 4.3. Let F be the graph obtained from E in the above example by removing the vertex with the loop and the two edges that end at this vertex. Then F contains no cycles and arguments similar to those used in the preceding example shows that the corresponding Leavitt path algebra $L_K(F)$ has GK dimension 0, but is not a PI algebra.

In view of the preceding examples, one would like to investigate the nature of a Leavitt path algebra having GK dimension 0 or 1.

We first consider the Leavitt path algebras whose GK dimension is 0.

Theorem 4.4. *Let E be an arbitrary graph. Then the following are equivalent for the Leavitt path algebra $L := L_K(E)$:*

(i) *L has GK dimension zero;*
(ii) *the graph E has no cycles;*
(iii) *L is von Neumann regular and is a directed union of subalgebras each of which is a direct sum of finitely many matrix rings of finite order over K.*

Proof. (i)⇒(ii). Suppose that E has a cycle c based at a vertex v. Let $V := Kv \oplus Kc$. As the powers of c are linearly independent over K, we see that $\dim(V^n) \geq n$. This forces $\mathrm{GKdim}(L) \geq 1$, and so (i)⇒(ii) follows.

(ii)⇒(iii). This is proved in [7, Theorem 1].

(iii)⇒(i). This follows from the defintion of the GK dimension of an arbitrary K-lagebra in terms of finitely generated subalgebras, and the fact that matrix rings over K have GK dimension zero. □

Next, we consider the Leavitt path algebras with GK dimension one.

Theorem 4.5. *Let E be an arbitrary graph. Then the following are equivalent:*

(i) *$L_K(E)$ has GK dimension at most one;*
(ii) *no cycle in E has an exit;*
(iii) *$L_K(E)$ is a directed union of subalgebras each of which is a direct sum of finitely many matrix rings of finite order over K and $K[x, x^{-1}]$;*
(iv) *$L_K(E)$ is locally PI.*

Moreover, if $L_K(E)$ has GK dimension ≤ 1, then it has GK dimension zero if and only if E has no cycles; otherwise, it has GK dimension one.

Proof. (i)⇒(ii). Suppose that E has a cycle c based at a vertex v and that there is an edge f with source v that does not lie on c. We will show that the set $S := \{c^i(c^*)^j : i, j \geq 0\}$ is a linearly independent set. The implication (i)⇒(ii) follows immediately, as the number of pairs (i, j) with $i + j \leq n$ is quadratic in n; so that $\mathrm{GKdim}(L_K(E) \geq 2$.

We will use the following, easily checked, facts: $c^*c = v$, $vf = f$, $cv = c$, $c^*f = 0$, and note that, by convention, $c^0 = (c^*)^0 = v$.

Suppose that S is not a linearly independent set and consider a nontrivial relation

$$\sum_{j=0}^{n} a_j (c^*)^j = 0$$

where each a_j is in the K-subalgebra generated by c. Let t be the least integer such that $a_t \neq 0$. Multiply the above equation on the right by $c^t f$ to obtain

$$a_t f + \sum_{j=t+1}^{n} a_j (c^*)^{j-t} f = 0$$

Now, each $(c^*)^{j-t} f = 0$, as $j - t > 0$ and $c^*f = 0$. Hence, $a_t f = 0$. Write $a_t = \sum_{i=1}^{m} k_i c^i$ with $k_m \neq 0$; so that

$$\sum_{i=1}^{m} k_i c^i f = 0.$$

Now, $(c^*)^m c^i f = (c^*)^{m-i} f = 0$, for each $i < m$; so multiplying the above equation on the left by $(c^*)^m$ gives $k_m f = 0$, contradicting the fact that $k_m \neq 0$. Thus, S is a linearly independent set, and (i)⇒(ii) is established.

(ii)⇒(iii). In Proposition 2 of [7], it was shown that a Leavitt path algebra L over an arbitrary graph E is a directed union of subalgebras B, where each $B = im(\theta) \oplus$ (a finite direct sum of copies of K) and where θ is a graded monomorphism $L_K(E_F) \longrightarrow L$. Here E_F is a finite graph constructed from a prescribed finite set

F of edges in E. Moreover, if no cycle in E has an exit, then it is clear from its construction that the finite graph E_F also has the same property. Thus given any finite set F of edges in E, no cycle in the corresponding finite graph E_F has an exit in E_F. We then appeal to Theorem 4.1 to conclude that $im(\theta) \cong L_K(E_F)$ is a direct sum of finitely many matrix rings over K and $K[x, x^{-1}]$, and hence so is the subalgebra B.

(iii)$\Rightarrow$(iv). Assume that $L_K(E)$ is a directed union of subalgebras each of which is a direct sum of finitely many matrix rings of finite order over K and $K[x, x^{-1}]$. Then any finitely generated subalgebra of $L_K(E)$ will be contained in a such a subalgebra and so will be PI.

(iv)$\Rightarrow$(i). This follows from the definition of the GK dimension of an arbitrary K-algebra and the fact that direct sums of matrix rings over a field K have GK dimension 0 and direct sums of matrix rings over $K[x, x^{-1}]$ have GK dimension one.

Finally, Theorem 4.4 shows that if $\mathrm{GKdim}(L_K(E)) \leq 1$ then $\mathrm{GKdim}(L_K(E)) = 0$ if and only if E has no cycles. □

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