Totally nonnegative matrices, quantum matrices and back, via Poisson geometry

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Abstract In this survey article, we describe recent work that connects three separate objects of interest: totally nonnegative matrices; quantum matrices; and matrix Poisson varieties.
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Introduction

In recent publications, the same combinatorial description has arisen for three separate objects of interest: $\mathcal{H}$-prime ideals in quantum matrices, $\mathcal{H}$-orbits of symplectic leaves in matrix Poisson varieties and totally nonnegative cells in the space of totally nonnegative matrices.

Many quantum algebras have a natural action by a torus and a key ingredient in the study of the structure of these algebras is an understanding of the torus-invariant objects. For example, the Stratification Theory of Goodearl and Letzter shows that, in the generic case, a complete understanding of the prime spectrum of quantum matrices would start by classifying the (finitely many) torus-invariant prime ideals. In [8] Cauchon succeeded in counting the number of torus-invariant prime ideals in quantum matrices. His method involved a bijection between certain diagrams, now known as Cauchon diagrams, and the

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torus-invariant primes. Considerable progress in the understanding of quantum matrices has been made since that time by using Cauchon diagrams.

The semiclassical limit of quantum matrices is the classical coordinate ring of the variety of matrices endowed with a Poisson bracket that encodes the nature of the quantum deformation which leads to quantum matrices. As a result, the variety of matrices is endowed with a Poisson structure. A natural torus action leads to a stratification of the variety via torus-orbits of symplectic leaves. In [5], Brown, Goodearl and Yakimov showed that there are finitely many such torus-orbits of symplectic leaves. Each torus orbit is defined by certain rank conditions on submatrices. The classification is given in terms of certain permutations from the relevant symmetric group with restrictions arising from the Bruhat order.

The totally nonnegative part of the space of real matrices consists of those matrices whose minors are all nonnegative. One can specify a cell decomposition of the set of totally nonnegative matrices by specifying exactly which minors are to be zero/non-zero. In [27], Postnikov classified the nonempty cells by means of a bijection with certain diagrams, known as Le-diagrams. The work of Postnikov has recently found applications in Integrable Systems [19] and Theoretical Physics [2].

The interesting observation from the point of view of this work is that in each of the above three sets of results the combinatorial objects that arise turn out to be the same! The definitions of Cauchon diagrams and Le-diagrams are the same, and the restricted permutations arising in the Brown-Goodearl-Yakimov study can be seen to lead to Cauchon/Le diagrams via the notion of pipe dreams.

Once one is aware of these connections, this suggests that there should be a connection between torus-invariant prime ideals, torus-orbits of symplectic leaves and totally nonnegative cells. This connection has been investigated in recent papers by Goodearl and the present authors, [14, 15]. In particular, we have shown that the Restoration Algorithm, developed by the first author for use in quantum matrices, can also be used in the other two settings to answer questions concerning the torus-orbits of symplectic leaves and totally nonnegative cells. The detailed proofs of the results that were obtained in [14, 15] are very technical, and our aim in this survey is to describe the results informally and to compute some examples to illuminate our results. We also present applications of these connections to testing for total nonnegativity through lacunary minors, see Section 6.

1 Totally nonnegative matrices

A real matrix is *totally positive* (TP for short) if each of its minors is positive and is *totally nonnegative* (TNN for short) if each of its minors is nonnegative.

The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$, or $[I \mid J](M)$ if we need to specify the matrix.

An excellent survey of totally positive and totally nonnegative matrices can be found in [11]. In this survey, the authors draw attention to appearance of TP and TNN matrices in many areas of mathematics, including: oscillations in mechanical systems, stochastic processes and approximation theory, Pólya frequency sequences, representation theory, planar networks, ... A good source of examples, especially illustrating the important link with planar networks (discussed below) is [28].

1.1 Checking total positivity and total nonnegativity

In order to specify a $k \times k$ minor of an $n \times n$ matrix, we must choose k rows and k columns. Hence the number of $k \times k$ minors of an $n \times n$ matrix is $\binom{n}{k}^2$; and so the total number of minors is

$$\sum_{k=1}^{n} \binom{n}{k}^2 \quad = \quad \binom{2n}{n} - 1 \quad \approx \quad \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation $n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$. This suggests that we do not want to calculate all of the minors to check for total nonnegativity.

Luckily, for total positivity, we can get away with much less. The simplest example is the 2×2 case.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has *five* minors: $a, b, c, d, \Delta = ad - bc$. Moreover, if $a, b, c, \Delta = ad - bc > 0$ then $d = \frac{\Delta + bc}{a} > 0$, so it is sufficient to check *four* minors.

This observation actually extends to the general situation, and the optimal result is due to Gasca and Peña, [13, Theorem 4.1]: for an $n \times n$ matrix, it is only necessary to check n^2 specified minors.

Definition 1. A minor is said to be an *initial minor* if it is formed of consecutive rows and columns, one of which being the first row or the first column.

For example, a 2×2 matrix has 4 initial minors: a, b, c and Δ. More generally, an initial minor is specified by its bottom right entry; so an $n \times n$ matrix has n^2 initial minors.

Theorem 1. *(Gasca and Peña) The $n \times n$ matrix A is totally positive if and only if each of its initial minors is positive.*

In contrast, there is no such family to check whether a matrix is TNN, see, for example, [10, Example 3.3.1]. However Gasca and Peña do give an efficient algorithm to check TNN, see the comment after [13, Theorem 5.4].

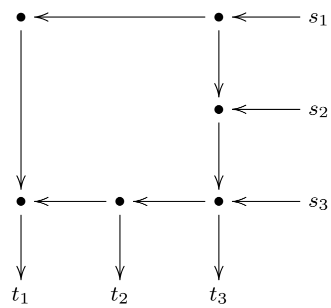
1.2 Planar networks

We refer the reader to [28] for the definition of a planar network. Consider a directed planar graph with no directed cycles, m sources, s_i, and n sinks, t_j. Set $M = (m_{ij})$ where m_{ij} is the number of paths from source s_i to sink t_j. The matrix M is called the *path matrix* of this planar network. See Figure 1 for an example. Planar networks give an easy way to construct TNN matrices.

Theorem 2. *(Lindström's Lemma, [25]) The path matrix of any planar network is totally nonnegative. In fact, the minor $[I \mid J](M)$ is equal to the number of families of non-intersecting paths from sources indexed by I and sinks indexed by J.*

For example, the matrix M from Figure 1 is totally nonnegative, by Lindström's Lemma. If we allow weights on paths then even more is true.

Theorem 3. *(Brenti, [3]) Every totally nonnegative matrix is the weighted path matrix of some planar network.*



$$M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Fig. 1 An example of a planar network and its associated path matrix

1.3 Cell decomposition

Our main concern in this section is to consider the possible patterns of zeros that can occur as the values of the minors of a totally nonnegative matrix. The following example shows that one cannot choose a subset of minors arbitrarily and hope to find a totally nonnegative matrix for which the chosen subset is precisely the subset of minors with value zero.

Example 1. There is no 2×2 totally nonnegative matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d = 0$, but the other four minors nonzero. For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is TNN and $d = 0$. Then $a, b, c \geq 0$ and also $ad - bc \geq 0$. Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$. □

Remark 1. The argument of Example 1 can be used to prove that if $M = (x_{ij})$ is a totally nonnegative matrix and $x_{st} = 0$ for some s, t, then, either $x_{it} = 0$ for all $i < s$, or $x_{sj} = 0$ for all $j < t$. For, suppose not, and that there are entries $x_{it} \neq 0, x_{st} = 0$ for some $i < s$, and consider x_{sj} for any $j < t$. If $x_{sj} > 0$ then the minor coming from rows i, s and columns j, t is equal to $-x_{it}x_{sj} < 0$, a contradiction. This observation motivates the definition of Le-diagram which is given below.

Let $\mathcal{M}_{m,p}^{\text{tnn}}$ be the set of totally nonnegative $m \times p$ real matrices. Let Z be a subset of minors. The *cell* S_Z° is the set of matrices in $\mathcal{M}_{m,p}^{\text{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero). Some cells may be empty. The space $\mathcal{M}_{m,p}^{\text{tnn}}$ is partitioned by the nonempty cells.

Example 2. It can easily be checked that of the 32 zero patterns for minors in $\mathcal{M}_2^{\text{tnn}}$, only 14 produce nonempty cells.

The question is then to describe the patterns of minors that represent nonempty cells in the space of totally nonnegative matrices. In [27], Postnikov defines *Le-diagrams* to solve this problem. An $m \times p$ array with entries either 0 or 1 is said to be a *Le-diagram* it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0. (Compare with Remark 1.)

Figure 2 shows an example and a non-example of a Le-diagram on a 3×3 array

Theorem 4. *(Postnikov) There is a bijection between Le-diagrams on an $m \times p$ array and nonempty cells S_Z° in $\mathcal{M}_{m,p}^{\text{tnn}}$.*

1	0	1
0	0	1
1	1	1

1	0	1
0	0	0
1	0	0

Fig. 2 An example and a non-example of a Le-diagram on a 3×3 array

In fact, Postnikov proves this theorem for the totally nonnegative grassmannian, and we are interpreting the result on the big cell, which is the space of totally nonnegative matrices.

In view of Example 2, there should be 14 2×2 Le-diagrams. At this stage, the interested reader should draw the 16 possible fillings of a 2×2 array with either 0 or 1 and identify the two non-Le-diagrams.

In [27], Postnikov describes an algorithm that starts with a Le-diagram and produces a planar network from which one generates a totally nonnegative matrix which defines a nonempty cell in the space of totally nonnegative matrices. The procedure to produce the planar network is as follows. In each 1 box of the Le-diagram, place a black dot. From each black dot draw a hook which goes to the right end of the diagram and the bottom of the diagram. Label the right ends of the horizontal part of the hooks as the sources of a planar network, numbered from top to bottom, and label the bottom ends of the vertical part of the hooks as the sinks, numbered from left to right. Then consider the resulting graph to be directed by allowing movement from right to left along horizontal lines and top to bottom along vertical lines. By Lindström's Lemma (see Theorem 2) the path matrix of this planar network is a totally nonnegative matrix, and so the pattern of its zero minors produces a nonempty cell in the space of totally nonnegative matrices. The above procedure that associates to any Le-diagram a nonempty cell provides a bijection between the set of $m\times p$ Le-diagrams and nonempty cells in the space of totally nonnegative $m \times p$ matrices (see Theorem 4).

Example 3. One can easily check that Postnikov's procedure applied to the 3×3 Le-diagram on the left of Figure 2 leads to the planar network and the path matrix from Figure 1.

The minors that vanish on this path matrix are:

$$[1,2|2,3], \quad [1,3|2,3], \quad [2,3|2,3], \quad [2,3|1,3], \quad [2,3|1,2], \quad [1,2,3|1,2,3].$$

The cell associated to this family of minors is nonempty and this is the nonempty cell associated to the Le-diagram above. □

In fact, by allowing suitable weights on the edges of the above planar network, one can obtain all of the matrices in this cell as weighted path matrices of the planar network.

2 Quantum matrices

We denote by $R := O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ the standard quantisation of the ring of regular functions on $m \times p$ matrices with entries in $\mathbb{C}$; the algebra R is the $\mathbb{C}$-algebra generated by the $m \times p$ indeterminates $X_{i,\alpha}$, for $1 \le i \le m$ and $1 \le \alpha \le p$, subject to the following relations:

$$\begin{array}{ll} X_{i,\alpha}X_{i,\beta} = qX_{i,\beta}X_{i,\alpha} & (\alpha < \beta) \\ X_{i,\alpha}X_{j,\alpha} = qX_{j,\alpha}X_{i,\alpha} & (i < j) \\ X_{j,\beta}X_{i,\alpha} = X_{i,\alpha}X_{j,\beta} & (i < j,\ \alpha > \beta) \\ X_{i,\alpha}X_{j,\beta} - X_{j,\beta}X_{i,\alpha} = (q-q^{-1})X_{i,\beta}X_{j,\alpha} & (i < j,\ \alpha < \beta). \end{array}$$

It is well known that R can be presented as an iterated Ore extension over $\mathbb{C}$, with the generators $X_{i,\alpha}$ adjoined in lexicographic order. Thus, the ring R is a noetherian domain; its skew-field of fractions is denoted by F.

It is easy to check that the torus $\mathcal{H} := \left(\mathbb{C}^{\times}\right)^{m+p}$ acts on R by $\mathbb{C}$-algebra automorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).X_{i,\alpha} = a_i b_\alpha X_{i,\alpha} \quad \text{for all} \quad (i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!].$$

We refer to this action as the *standard action* of $\left(\mathbb{C}^{\times}\right)^{m+p}$ on $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$.

The algebra R possesses a set of distinguished elements called *quantum minors* that we now define. If $I \subseteq [\![1, m]\!]$ and $\Lambda \subseteq [\![1, p]\!]$ with $|I| = |\Lambda| = k \geq 1$, then we denote by $[I|\Lambda]_q$ the corresponding *quantum minor* of R. This is the element of R defined by:

$$[I|\Lambda]_q = [i_1, \ldots, i_k|\alpha_1, \ldots, \alpha_k]_q := \sum_{\sigma \in S_k} (-q)^{l(\sigma)} X_{i_1,\alpha_{\sigma(1)}} \cdots X_{i_k,\alpha_{\sigma(k)}},$$

where $I = \{i_1, \ldots, i_k\}$, $\Lambda = \{\alpha_1, \ldots, \alpha_k\}$ and $l(\sigma)$ denotes the length of the k-permutation σ. Note that quantum minors are $\mathcal{H}$-eigenvectors.

In this survey, we will assume that q is not a root of unity. As a consequence, the stratification theory of Goodearl and Letzter applies, so that the prime spectrum of R is controlled by those prime ideals of R that are $\mathcal{H}$-invariant, the so-called $\mathcal{H}$*-primes*. We denote by $\mathcal{H} - \mathrm{Spec}(R)$ the set of $\mathcal{H}$-primes of R. The set $\mathcal{H} - \mathrm{Spec}(R)$ is finite and all $\mathcal{H}$-primes are completely prime, see [4, Theorem II.5.14].

The aim is to parameterise/study the $\mathcal{H}$-prime ideals in quantum matrices.

In [8], Cauchon showed that his theory of deleting derivations can be applied to the iterated Ore extension R. As a consequence, he was able to parametrise the set $\mathcal{H}$-Spec(R) in terms of combinatorial objects called *Cauchon diagrams*.

Definition 2. [8] An $m \times p$ *Cauchon diagram* C is simply an $m \times p$ grid consisting of mp squares in which certain squares are coloured black. We require that the collection of black squares have the following property: if a square is black, then either every square strictly to its left is black or every square strictly above it is black.

We denote by $\mathcal{C}_{m,p}$ the set of $m \times p$ Cauchon diagrams.

Note that we will often identify an $m \times p$ Cauchon diagram with the set of coordinates of its black boxes. Indeed, if $C \in \mathcal{C}_{m,p}$ and $(i, \alpha) \in [\![1, m]\!] \times [\![1, p]\!]$, we will say that $(i, \alpha) \in C$ if the box in row i and column α of C is black. Recall [8, Corollaire 3.2.1] that

Fig. 3 An example and a non-example of a 3×3 Cauchon diagram

Cauchon has constructed (using the deleting derivations algorithm) a bijection between $\mathcal{H}$-Spec$(O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right))$ and the collection $\mathcal{C}_{m,p}$. As a consequence, Cauchon [8] was able to give a formula for the size of $\mathcal{H}-$Spec$(O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right))$. This formula was later re-written by Goodearl and McCammond (see [22]) in terms of Stirling numbers of second kind and poly-Bernoulli numbers as defined by Kaneko (see [18]).

Notice that the definitions of Le-diagrams and Cauchon diagrams are the same! Thus, the nonempty cells in totally nonnegative matrices and the $\mathcal{H}$-prime ideals in quantum matrices are parameterised by the same combinatorial objects. Much more is true, as we will now see in the 2×2 case.

The algebra of 2×2 quantum matrices may be presented as

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$
$$bd = qdb \qquad cd = qdc \qquad ad - da = (q - q^{-1})bc.$$

This algebra has 5 quantum minors: a, b, c, d and the *quantum determinant* $D_q := [12|12]_q = ad - qbc$.

Example 4. Let P be an $\mathcal{H}$-prime ideal that contains d. Then

$$(q - q^{-1})bc = ad - da \in P$$

and, as $0 \neq (q - q^{-1}) \in \mathbb{C}$ and P is completely prime, we deduce that either $b \in P$ or $c \in P$. Thus, there is no $\mathcal{H}$-prime ideal in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ such that d is the only quantum minor that is in P. □

You should notice the analogy with the corresponding result in the space of 2×2 totally nonnegative matrices: the cell corresponding to d being the only vanishing minor is empty (see Example 1).

The algebra $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ has 14 $\mathcal{H}$-prime ideals, as there are 14 Cauchon/Le-diagrams. It is relatively easy to identify these $\mathcal{H}$-primes: they are $\langle 0\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle D_q\rangle$, $\langle a,b\rangle$, $\langle a,c\rangle$, $\langle b,c\rangle$, $\langle b,d\rangle$, $\langle c,d\rangle$, $\langle a,b,c\rangle$, $\langle a,b,d\rangle$, $\langle a,c,d\rangle$, $\langle b,c,d\rangle$, $\langle a,b,c,d\rangle$,

It is easy to check that 13 of the ideals are prime. For example, let P be the ideal generated by b and d. Then $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))/P \cong \mathbb{C}[a,c]$ and $\mathbb{C}[a,c]$ is an iterated Ore extension of $\mathbb{C}$ and so a domain. The only problem is to show that the determinant generates a prime ideal. This was originally proved by Jordan, and, independently, by Levasseur and Stafford. A general result that includes this as a special case is [16, Theorem 2.5].

Recently, Casteels, [7], has shown that all $\mathcal{H}$-prime ideals are generated by the quantum minors that they contain, following on from a similar result by the first author with the restriction that the parameter q be transcendental over $\mathbb{Q}$, [21] (see also [29]).

Comparing Example 2 with the above list reveals that the sets of all quantum minors that define $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ are exactly the quantum versions of the sets of vanishing minors for nonempty cells in the space of 2×2 totally nonnegative matrices. This coincidence also occurs in the general case and an explanation of this coincidence is obtained in [14, 15]. However, in order to explain the coincidence, we need to introduce a third setting, that of Poisson matrices, and this is done in the next section.

3 Poisson matrix varieties

In this section, we study the standard Poisson structure of the coordinate ring $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ coming from the commutators of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Recall that a *Poisson algebra* (over $\mathbb{C}$) is a commutative $\mathbb{C}$-algebra A equipped with a Lie bracket $\{-,-\}$ which is a derivation (for the associative multiplication) in each variable. The derivations $\{a,-\}$ on A are called *Hamiltonian derivations*. When A is the algebra of complex-valued C^∞ functions on a smooth affine variety V, one can use Hamiltonian derivations in order to define Hamiltonian paths in V. A smooth path $\gamma : [0,1] \to V$ is a *Hamiltonian path in* V if there exists $H \in C^\infty(V)$ such that for all $f \in \mathcal{C}^\infty(V)$:

$$\frac{d}{dt}(f \circ \gamma)(t) = \{H, f\} \circ \gamma(t), \tag{1}$$

for all $0 < t < 1$. In other words, Hamiltonian paths are the integral curves (or flows) of the Hamiltonian vector fields induced by the Poisson bracket. It is easy to check that the relation "connected by a piecewise Hamiltonian path" is an equivalence relation. The equivalence classes of this relation are called the *symplectic leaves* of V; they form a partition of V.

3.1 The Poisson algebra $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$

Denote by $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ the coordinate ring of the variety $\mathcal{M}_{m,p}(\mathbb{C})$; note that $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ is a (commutative) polynomial algebra in mp indeterminates $Y_{i,\alpha}$ with $1 \leq i \leq m$ and $1 \leq \alpha \leq p$.

The variety $\mathcal{M}_{m,p}(\mathbb{C})$ is a Poisson variety: there is a unique Poisson bracket on the coordinate ring $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ determined by the following data. For all $(i,\alpha) < (k,\gamma)$, we set:

$$\{Y_{i,\alpha}, Y_{k,\gamma}\} = \begin{cases} Y_{i,\alpha}Y_{k,\gamma} & \text{if } i = k \text{ and } \alpha < \gamma \\ Y_{i,\alpha}Y_{k,\gamma} & \text{if } i < k \text{ and } \alpha = \gamma \\ 0 & \text{if } i < k \text{ and } \alpha > \gamma \\ 2Y_{i,\gamma}Y_{k,\alpha} & \text{if } i < k \text{ and } \alpha < \gamma. \end{cases}$$

This is the standard Poisson structure on the affine variety $\mathcal{M}_{m,p}(\mathbb{C})$ (cf. [5, §1.5]); the Poisson algebra structure on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ is the semiclassical limit of the noncommutative algebras $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Indeed one can easily check that

$$\{Y_{i,\alpha}, Y_{k,\gamma}\} = \frac{[X_{i,\alpha}, X_{k,\gamma}]}{q-1} \;|_{q=1} .$$

In particular, the Poisson bracket on $\mathcal{O}(\mathcal{M}_2(\mathbb{C})) = \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by:

$$\begin{array}{lll} \{a,b\} = ab & \{a,c\} = ac & \{b,c\} = 0 \\ \{b,d\} = bd & \{c,d\} = cd & \{a,d\} = 2bc. \end{array}$$

Note that the Poisson bracket on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ extends uniquely to a Poisson bracket on $\mathcal{C}^\infty(\mathcal{M}_{m,p}(\mathbb{C}))$, so that $\mathcal{M}_{m,p}(\mathbb{C})$ can be viewed as a Poisson manifold. Hence, $\mathcal{M}_{m,p}(\mathbb{C})$ can be decomposed as the disjoint union of its symplectic leaves.

We finish this section by stating an analogue of Examples 1 and 4 in the Poisson setting.

Proposition 1. *Let $\mathcal{L}$ be a symplectic leaf such that $d(M) = 0$ for all $M \in \mathcal{L}$. Then, either $b(M) = 0$ for all $M \in \mathcal{L}$ or $c(M) = 0$ for all $M \in \mathcal{L}$.*

3.2 $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$

Notice that the torus $\mathcal{H} := \left(\mathbb{C}^\times\right)^{m+p}$ acts rationally on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ by Poisson automorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \text{for all} \quad (i,\alpha) \in [\![1,m]\!] \times [\![1,p]\!].$$

At the geometric level, this action of the algebraic torus $\mathcal{H}$ comes from the left action of $\mathcal{H}$ on $\mathcal{M}_{m,p}(\mathbb{C})$ by Poisson isomorphisms via:

$$(a_1,\dots,a_m,b_1,\dots,b_p).M := \mathrm{diag}(a_1,\dots,a_m)^{-1}\cdot M\cdot \mathrm{diag}(b_1,\dots,b_p)^{-1}.$$

This action of $\mathcal{H}$ on $\mathcal{M}_{m,p}(\mathbb{C})$ induces an action of $\mathcal{H}$ on the set $\mathrm{Sympl}(\mathcal{M}_{m,p}(\mathbb{C}))$ of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ (cf. [5, §0.1]). As in [5], we view the $\mathcal{H}$-orbit of a symplectic leaf $\mathcal{L}$ as the set-theoretic union $\bigcup_{h\in\mathcal{H}} h.\mathcal{L} \subseteq \mathcal{M}_{m,p}(\mathbb{C})$, rather than as the family $\{h.\mathcal{L} \mid h\in\mathcal{H}\}$. We denote the set of such orbits by $\mathcal{H}\text{-}\mathrm{Sympl}(\mathcal{M}_{m,p}(\mathbb{C}))$.

As the symplectic leaves of $\mathcal{M}_{m,p}(\mathbb{C})$ form a partition of $\mathcal{M}_{m,p}(\mathbb{C})$, so too do the $\mathcal{H}$-orbits of symplectic leaves.

Example 5. The symplectic leaf $\mathcal{L}$ containing $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ is the set $\mathcal{E}$ of those 2×2 complex matrices $M=\begin{pmatrix}x&y\\z&t\end{pmatrix}$ with $y-z=0$, $xt-yz=0$ and $y\neq 0$. In other words,

$$\mathcal{E} := \{M\in\mathcal{M}_2(\mathbb{C}) \mid \Delta(M)=0,\ (b-c)(M)=0 \text{ and } b(M)\neq 0\},$$

where a,b,c,d denote the canonical generators of the coordinate ring of $\mathcal{M}_2(\mathbb{C})$ and $\Delta := ad-bc$ is the determinant function. It easily follows from this that the $\mathcal{H}$-orbit of symplectic leaves in $\mathcal{M}_2(\mathbb{C})$ that contains the point $\begin{pmatrix}1&1\\1&1\end{pmatrix}$ is the set of those 2×2 matrices M with $\Delta(M)=0$ and $b(M)c(M)\neq 0$. Moreover the closure of this $\mathcal{H}$-orbit coincides with the set of those 2×2 matrices M with $\Delta(M)=0$. □

The $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ have been explicitly described by Brown, Goodearl and Yakimov, [5, Theorems 3.9, 3.13, 4.2].

Theorem 5. *Set $\mathcal{S} := \{w\in S_{m+p} \mid -p\leq w(i)-i\leq m$ for all $i=1,2,\dots,m+p\}$.*

1. *The $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ are smooth irreducible locally closed subvarieties.*
2. *There is an explicit $1:1$ correspondence between $\mathcal{S}$ and $\mathcal{H}\text{-}\mathrm{Sympl}(\mathcal{M}_{m,p}(\mathbb{C}))$.*
3. *Each $\mathcal{H}$-orbit is defined by some rank conditions.*

Before going any further let us look at the 2×2 case: by the above theorem, there is a $1:1$ correspondence between $\mathcal{H}\text{-}\mathrm{Sympl}(\mathcal{M}_2(\mathbb{C}))$ and

$$\mathcal{S} = \{w\in S_4 \mid -2\leq w(i)-i\leq 2 \text{ for all } i=1,2,3,4\},$$

in other words, with the set of those permutations w in S_4 such that $w(1)\neq 4$ and $w(4)\neq 1$. One may be disappointed not to retrieve 2×2 Cauchon diagrams, but a direct inspection shows that there are exactly 14 such restricted permutations in the 2×2 case! This is not at all a coincidence as we will see in the following section.

The rank conditions that define the $\mathcal{H}$-orbits of symplectic leaves and their closures are explicit in [5]. The reader is refered to [5] for more details.

For $w\in\mathcal{S}$, we denote by $\mathcal{P}_w$ the $\mathcal{H}$-orbit of symplectic leaves associated to the restricted permutation w. To finish, let us mention that the set $\mathcal{M}(w)$ of all minors that vanish on the closure of $\mathcal{P}_w$ has been described in [14, Definition 2.6].

Example 6. When $m=p=3$ and $w=(2\ 3\ 5\ 4)$, then we obtain

$$\mathcal{M}(w) = \{[1,2|2,3],[1,3|2,3],[2,3|2,3],[2,3|1,3],[2,3|1,2],[1,2,3|1,2,3]\}.$$

Observe that this family of minors defines a nonempty cell in $\mathcal{M}_3^{\mathrm{tnn}}(\mathbb{R})$ by Example 3. □

In [14], the following result was obtained thanks to previous results of [5] and [12].

Theorem 6. *Let $w \in \mathcal{S}$. The closure of the $\mathcal{H}$-orbit $\mathcal{P}_w$ is given by:*

$$\overline{\mathcal{P}_w} = \{x \in \mathcal{M}_{m,p}(\mathbb{C}) \mid [I|J](x) = 0 \text{ for all } [I|J] \in \mathcal{M}(w)\}.$$

Moreover, the minor $[I|J]$ vanishes on $\overline{\mathcal{P}_w}$ if and only if $[I|J] \in \mathcal{M}(w)$.

4 From Cauchon diagrams to restricted permutations, via pipe dreams

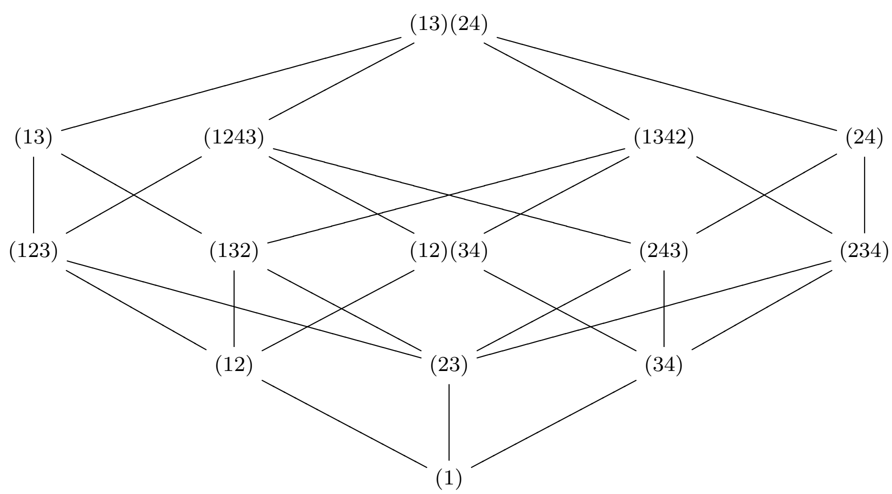
In the previous section, we have seen that the $\mathcal{H}$-orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ are parameterised by the restricted permutations in S_{m+p} given by

$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

In the 2×2 case, this subposet of the Bruhat poset of S_4 is

$$\mathcal{S} = \{w \in S_4 \mid -2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and is shown below.

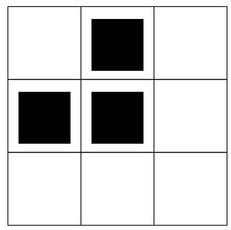


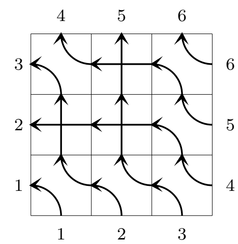
Inspection of this poset reveals that it is isomorphic to the poset of the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$ in Section 2, partially ordered by inclusion; and so to a similar poset of the Cauchon diagrams corresponding to the $\mathcal{H}$-prime ideals.

More generally, it is known that the numbers of $\mathcal{H}$-primes in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ (and so the number of $m \times p$ Cauchon diagrams) is equal to $|\mathcal{S}|$ (see [23]). This is no coincidence, and the connection between the two posets can be illuminated by using *Pipe Dreams*.

The procedure to produce a restricted permutation from a Cauchon diagram goes as follows. Given a Cauchon diagram, replace each black box by a cross, and each white box by an elbow joint, that is:

For example, the Cauchon diagram on the left below produces the pipe dream on the right





We obtain a permutation σ from the pipe dream in the following way. To calculate $\sigma(i)$, locate the i either on the right hand side or the bottom of the pipe dream and trace through the pipe dream to find the number $\sigma(i)$ that is at the end of the pipe starting at i. In the example displayed, we find that $\sigma = 135246$ (in one-line notation).

It is easy to check that this produces a restricted permutation of the required type by using the observation that as you move along a pipe from source to image, you can only move upwards and leftwards; so, for example, in any 3×3 example $\sigma(2)$ is at most 5 (the number directly above 2).

This procedure provides an explicit bijection between the set of $m \times p$ Cauchon diagrams and the poset $\mathcal{S}$ (see [27, 9]).

5 The Unifying Theory

In the previous sections we have seen that the nonempty cells in $\mathcal{M}^{\text{tnn}}_{m,p}$, the torus-invariant prime ideals in $O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ and the closure of the $\mathcal{H}$-orbits of symplectic leaves are all parametrised by $m \times p$ Cauchon diagrams. This suggests that there might be a connection between these objects. Going a step further, all these objects are characterised by certain families of (quantum) minors.

Totally nonnegative cells are defined by the vanishing of families of minors. Some of the TNN cells are empty. So it is natural to introduce the following definition. A family of minors is *admissible* if the corresponding TNN cell is nonempty. Three obvious questions, which we discuss in the next section, are:

Question 1: what are the admissible families of minors?

Question 2: which families of quantum minors generate $\mathcal{H}$-prime ideals?

Question 3: which families of minors define closures of $\mathcal{H}$-orbits of symplectic leaves?

5.1 An algorithm to rule them all

In [8], Cauchon developed and used an algorithm, called the *deleting derivations algorithm* in order to study the $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Roughly speaking, in the 2×2 case, this algorithm consists in the following change of variable:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}.$$

Let us now give a precise definition of the deleting derivations algorithm.

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$ with K a skew-field, then we set $g_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} - x_{i,\beta}x_{j,\beta}^{-1}x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)} := g_{j,\beta} \circ \cdots \circ g_{m,p-1} \circ g_{m,p}(M)$ where the indices are taken in lexicographic order.

The matrix $M^{(1,1)}$ is called the matrix obtained from M at the end of the deleting derivations algorithm.

The deleting derivations algorithm has an inverse that is called the *restoration algorithm*. It was originally developed in [21] to study $\mathcal{H}$-primes in quantum matrices. Roughly speaking, in the 2×2 case, the restoration algorithm consists of making the following change of variable:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}.$$

Let us now give a precise definition of the restoration algorithm.

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, then we set $f_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} + x_{i,\beta}x_{j,\beta}^{-1}x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)^+} := f_{j,\beta} \circ \cdots \circ f_{1,2} \circ f_{1,1}(M)$ where the indices are taken in the reverse of the lexicographic order and where $(j,\beta)^+ \in \{1,\dots,m\} \times \{1,\dots,p\} \cup \{(m+1,p)\}$ is the successor of (j,β) in the lexicographic order.

The matrix $M^{(m,p)^+} = M^{(m+1,p)}$ is called the matrix obtained from M at the end of the restoration algorithm.

Example 7. Set $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then, applying the restoration algorithm to M, we get $M^{(3,3)^+} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is the matrix obtained from M at the end of the restoration algorithm. □

5.2 The restoration algorithm and TNN matrices

The matrix $M^{(3,3)^+}$ obtained from M by the restoration algorithm in Example 7 is not TNN, as the minor $[1,2|2,3]$ is negative. The reason for this failure to be TNN is that the starting matrix M has a negative entry. In general, one can express the (quantum) minors of $M^{(j,\beta)^+}$ in terms of the (quantum) minors of $M^{(j,\beta)}$ (see [14, 15]). As a consequence, one is able to prove the following result that gives a necessary and sufficient condition for a real matrix to be TNN.

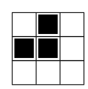
Theorem 7. *[14]*

1. *If the entries of M are nonnegative and its zeros form a Cauchon diagram, then the matrix $M^{(m,p)^+}$ obtained from M at the end of the restoration algorithm is TNN.*
2. *Let M be a matrix with real entries. Let N be the matrix obtained at the end of the deleting derivations algorithm applied to M. Then M is TNN if and only if the matrix N is nonnegative and its zeros form a Cauchon diagram. (That is, the zeros of N correspond to the black boxes of a Cauchon diagram.)*

Example 8. Use the deleting derivations algorithm to test whether the following matrices are TNN: $M_1 = \begin{pmatrix} 11 & 7 & 4 & 1 \\ 7 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 7 & 5 & 4 & 1 \\ 6 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

5.3 Main result

Let C be an $m \times p$ Cauchon diagram and $T = (t_{i,\alpha})$ be a matrix with entries in a skew-field K. Assume that $t_{i,\alpha} = 0$ if and only if (i,α) is a black box of C. Set $T_C := f_{m,p} \circ \cdots \circ f_{1,2} \circ f_{1,1}(T)$, so that T_C is the matrix obtained from T by the restoration algorithm.

Example 9. Let $m = p = 3$ and consider the Cauchon diagram  .

Then $T = \begin{pmatrix} t_{1,1} & 0 & t_{1,3} \\ 0 & 0 & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}$ and $T^{(j,\beta)^+} := f_{j,\beta} \circ \cdots \circ f_{1,1}(T)$, so that

$$T_C = T^{(3,3)^+} = \begin{pmatrix} t_{1,1} + t_{1,3}t_{3,3}^{-1}t_{3,1} & t_{1,3}t_{3,3}^{-1}t_{3,2} & t_{1,3} \\ t_{2,3}t_{3,3}^{-1}t_{3,1} & t_{2,3}t_{3,3}^{-1}t_{3,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}.$$

The above construction can be applied in a variety of situations. In particular, we have the following.

• If $K = \mathbb{R}$ and T is nonnegative, then T_C is TNN.
• If the nonzero entries of T commute and are algebraically independent, and if $K = \mathbb{C}(t_{ij})$, then the minors of T_C that are equal to zero are exactly those that vanish on the closure of a given $\mathcal{H}$-orbit of symplectic leaves. (See [14].)
• If the nonzero entries of T are the generators of a certain quantum affine space over $\mathbb{C}$ and K is the skew-field of fractions of this quantum affine space, then the quantum minors of T_C that are equal to zero are exactly those belonging to the unique $\mathcal{H}$-prime in $O_q(\mathcal{M}_{m,p}(\mathbb{C}))$ associated to the Cauchon diagram C. (See [21] for more details.)
• The families of (quantum) minors we get depend only on C in these three cases, and if we start from the same Cauchon diagram in these three cases, then we get exactly the

same families.

As a consequence, we get the following comparison result (see [14, 15]).

Theorem 8. *Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:*

1. *The totally nonnegative cell associated to $\mathcal{F}$ is nonempty.*
2. *$\mathcal{F}$ is the set of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$.*
3. *$\mathcal{F}_q$ is the set of quantum minors that belong to an $\mathcal{H}$-prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.*

This result has several interesting consequences.

First, it easily follows from Theorem 8 that the TNN cells in $\mathcal{M}^{\text{tnn}}_{m,p}$ are the traces of the closure of $\mathcal{H}$-orbits of symplectic leaves on $\mathcal{M}^{\text{tnn}}_{m,p}$.

Next, the sets of all minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ have been explicitly described in [14] (see also Theorem 6). So, as a consequence of the previous theorem, *the sets of minors that define nonempty totally nonnegative cells are explicitly described:* these are the families $\mathcal{M}(w)$ of [14, Definition 2.6] for $w \in \mathcal{S}$.

On the other hand, the torus-invariant primes in $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ are generated by the quantum minors that they contain, and so we deduce from the above theorem *explicit generating sets of quantum minors for the torus-invariant prime ideals of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$*. Recently and independently, Yakimov [29] also described explicit families of quantum minors that generate $\mathcal{H}$-primes. However his families are smaller than ours and so are not adapted to the TNN world. The problem of deciding whether a given quantum minor belongs to the $\mathcal{H}$-prime associated to a Cauchon diagram C was studied by Casteels [6] who gave a combinatorial criterion inspired by Lindström's Lemma.

6 Lacunary sequences

It follows from Theorem 7 that each totally nonnegative matrix is associated to a Cauchon diagram via the deleting derivation algorithm. In other words, we have a mapping $\pi : M \mapsto C$ from $\mathcal{M}^{\text{tnn}}_{m,p}$ to the set $\mathcal{C}_{m,p}$ of $m \times p$ Cauchon diagrams, where C is the Cauchon diagram associated to the matrix N deduced from M by the deleting derivations algorithm. In [24], we have proved that the nonempty totally nonnegative cells are precisely the fibres of π, so that the nonempty totally nonnegative cells in $\mathcal{M}^{\text{tnn}}_{m,p}$ are precisely the sets

$$S^0_C := \{M \in \mathcal{M}^{\text{tnn}}_{m,p} \mid \pi(M) = C\},$$

where C runs through the set of $m \times p$ Cauchon diagrams.

When C is the all white diagram, S^0_C is the cell of all totally positive matrix. Gasca and Peña's result, Theorem 1, specifies a set of n^2 minors that need to be checked to guarantee that an $n \times n$ matrix is totally positive; that is, in S^0_C.

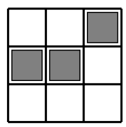
Here we outline a generalisation of this result to arbitrary nonnegative cells: full details are in [24]. The inspiration for lacunary minors comes from Cauchon's work on $\mathcal{H}$-primes in quantum matrices and Theorem 8.

Given an $m \times p$ Cauchon diagram C, we specify for each box (i, j) a *lacunary minor*, $\Delta_{i,j}$, given by [24, Algorithm 1]. Then a matrix A is totally nonnegative and in the cell corresponding to the Cauchon diagram C if and only if each of the lacunary minors of A

corresponding to a black box is zero, while each of the lacunary minors of A corresponding to a white box is greater than zero.

Note that this test only involves mp minors. In the case that C is the Cauchon diagram with all boxes coloured white, the test states that a real matrix M is totally positive if and only if each final minor of M is strictly positive. (A minor $[I|J]$ is a final minor if I and J consist of consecutive entries and either $m \in I$ or $p \in J$.) This is the well-known Gasca and Peña test, but applied to final minors rather than initial minors.

Example 10. A real matrix M is TNN and belongs to the cell associated to the Cauchon diagram C on the left below if and only if the nine lacunary conditions on the right below are satisfied. (The lacunary minors have all been obtained by using [24, Algorithm 1].)

$C =$ 

$$\Delta_{1,1} = [13|12] > 0,\ \Delta_{1,2} = [12|23] > 0,\ \Delta_{1,3} = [1|3] = 0$$
$$\Delta_{2,1} = [23|12] = 0,\ \Delta_{2,2} = [23|23] = 0,\ \Delta_{2,3} = [2|3] > 0$$
$$\Delta_{3,1} = [3|1] > 0, \quad \Delta_{3,2} = [3|2] > 0, \quad \Delta_{3,3} = [3|3] > 0.$$

It is easy to check that the matrix $M = \begin{pmatrix} 16 & 5 & 0 \\ 12 & 6 & 3 \\ 4 & 2 & 1 \end{pmatrix}$ satisfies the above nine conditions. Hence, we deduce from the comments above that M is TNN and belongs to the TNN cell S_C^0 associated to C. □

Lacunary minors have been used in recent work by Adm and Garloff on intervals of totally nonnegative matrices, [1].

Acknowledgements Some of results in this paper were announced during the mini-workshop "Nonnegativity is a quantum phenomenon" that took place at the Mathematisches Forschungsinstitut Oberwolfach, 1–7 March 2009, [26]; we thank the director and staff of the MFO for providing the ideal environment for this stimulating meeting. We thank Konni Rietsch, Laurent Rigal, Lauren Williams, Milen Yakimov and, especially, our co-author Ken Goodearl for discussions and comments concerning the results presented in this survey paper both at the workshop and at other times. We also thank Natalia Iyudu for the organisation of a Belfast conference in August 2009 at which we presented a preliminary version these results and the organisers of the INdAM Intensive Research Period "Perspectives in Lie Theory" at Centro de Giorgi in Pisa (Italy) for the opportunity to publish this survey article.

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