Generalized Weyl Algebras are Tensor Krull Minimal *

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Abstract

Formulae for calculating the Krull dimension of noetherian rings obtained by the authors and their collaborators are used to calculate Krull dimension for certain classes of algebras. A K-algebra T is said to be Tensor Krull Minimal (TKM) with respect to a class of K-algebras Ω if $\mathcal{K}(T \otimes B) = \mathcal{K}(T) + \mathcal{K}(B)$, for each $B \in \Omega$. We show that Generalized Weyl Algebras over affine commutative K-algebras, where K is an uncountable algebraically closed field, are TKM with respect to the class of countably generated left noetherian K-algebras. This simplifies the task of calculating many Krull dimensions. In addition, we develop an improved formula for the Krull dimension of a skew Laurent extension $D[x, x^{-1}; \sigma]$, where D is a polynomial algebra over an algebraically closed field, and σ is an affine automorphism. Finally, we calculate the Krull dimension of the noetherian Down-Up algebras introduced recently by Benkart.

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Introduction

Computing the Krull dimension of a noncommutative noetherian ring is often a very difficult problem; for example, the Krull dimension of the enveloping algebra of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ was believed to be 3 for several years until S P Smith, [12], showed that the correct value is 2. The value of the Krull dimension of the enveloping algebra of an arbitrary finite dimensional Lie algebra is still unknown. This is a big problem since Krull dimension is one of the most useful invariants for noetherian rings. Thus it is of interest to have ways of calculating the Krull dimension of classes of rings and also of calculating the Krull dimension of specific examples.

In a recent paper, the present authors have developed formulae for calculating the Krull dimension of Generalized Weyl Algebras with noncommutative noetherian coefficient rings, [3]. This work generalizes earlier results of the first author and Van Oystaeyen, [4], and the second author and Goodearl, [6]. In this paper we employ the results of all three of these papers to calculate Krull dimension for several classes of examples.

After recalling the known results, we apply them in three different ways. The first is in Section 2, where we show that Krull dimension behaves well under the process of tensoring an algebra with Generalized Weyl Algebras over affine commutative algebras over uncountable, algebraically closed fields. If A and B are noetherian K-algebras then it is obvious that $\mathcal{K}(A \otimes_K B) \geq \mathcal{K}(A) + \mathcal{K}(B)$, where $\mathcal{K}(-)$ denotes Krull dimension. However, equality can fail, and any result that shows that equality is achieved is important in that it considerably simplifies calculation of Krull dimension of algebras built up by repeated tensor products. A K-algebra A is said to be tensor Krull minimal (TKM) with respect to a class of K-algebras Ω if $\mathcal{K}(A \otimes_K B) = \mathcal{K}(A) + \mathcal{K}(B)$, for all $B \in \Omega$. We show that if $T = \otimes_{1=1}^{n} T_i$ is a tensor product of generalized Weyl algebras of the form $T_i = D_i(\sigma_i, a_i)$, where each D_i is an affine commutative algebra over the uncountable, algebraically closed field K then T is tensor Krull minimal with respect to the class of countably generated left noetherian K-algebras. In particular,

$$\mathcal{K}(\bigotimes_{i=1}^{n} T_i) = \sum_{i=1}^{n} \mathcal{K}(T_i).$$

As an example, the universal enveloping algebra $\mathrm{U}(sl(2))$ is isomorphic to a Generalized Weyl Algebra $K[H, C](\sigma, a)$ with $\sigma(H) = H - 1$, $\sigma(C) = C$ and $a = C - H(H + 1)$, see [4]. The results in Section 2 show that $\mathrm{U}(sl(2))$ is TKM with respect to the class of countably generated left noetherian algebras. In particular, for the enveloping algebra of the direct product of n copies of $sl(2)$ we have

$$\begin{aligned}\mathcal{K}(\mathrm{U}(sl(2)\times\cdots\times sl(2))) &= \mathcal{K}(\mathrm{U}(sl(2))\otimes\cdots\otimes\mathrm{U}(sl(2)))\\ &= n\,\mathcal{K}(\mathrm{U}(sl(2))) = \begin{cases}2n, & \text{if } \mathrm{char}(K)=0;\\ 3n, & \text{if } \mathrm{char}(K)\neq 0.\end{cases}\end{aligned}$$

In Section 3, we consider the problem of calculating the Krull dimension of Generalized Weyl Algebras arising from a base ring which is the algebra of polynomials in several variables and where the automorphism involved in the construction is an affine transformation of the underlying geometric space. In full generality, the problem of calculating Krull dimension for such algebras leads to extremely difficult problems in classical invariant theory (specifically: describe all invariants and semi-invariants for an arbitrary affine automorphism). However, we are able to give criteria for deciding when an affine transformation has a finite orbit, and when we use this in conjunction with the Krull dimension formula obtained by Goodearl-Lenagan, [6], and Hodges, [7], we obtain the following simple formula for the Krull dimension of a skew Laurent extension of a polynomial algebra formed by using an affine automorphism: if $T = D[X, X^{-1}; \sigma]$ is a skew Laurent extension of the polynomial ring, $D = K[X_1, \ldots, X_n]$, over an algebraically closed field K, and $\sigma(x) = \mathcal{A}x + b$ is an affine automorphism of D then

$$\mathcal{K}(T) = \begin{cases} n+1, & \text{if either } b\in \mathrm{Im}(I-\mathcal{A}) \text{ or } \mathrm{char}(K)\neq 0;\\ n, & \text{otherwise.}\end{cases}$$

In the final section, we apply the results of Section 3 to Down-Up algebras, a class of algebras that has recently arisen in the study of partially ordered sets. The Down-Up algebra $A = A(\alpha, \beta, \gamma)$, where $\beta \neq 0$, can be presented as a Generalized Weyl Algebra with the base ring being polynomials in two variables. Because of this fact the Krull dimension is known to be 2 or 3, see [4]. In addition, the automorphism involved is an affine automorphism; so the considerations of Section 3 apply here, and we are able to specify the Krull dimension of an arbitrary noetherian Down-Up algebra in terms of the parameters involved in the original presentation of the algebra. We show that the Krull dimension of the Down-Up algebra $A = A(\alpha, \beta, \gamma)$, where $\beta \neq 0$, is equal to 2 if and only if $\mathrm{char}(K) = 0$, $\gamma \neq 0$, and $\alpha + \beta = 1$; otherwise, the Krull dimension of A is 3.

1 Earlier results

We begin by recalling the definition of Generalized Weyl Algebras, and quoting the known results on Krull dimension of these algebras. Throughout the paper, we will work with left modules; so, unless otherwise stated, all modules are left modules, and Krull dimension means left Krull dimension.

Let R be a ring, and let σ be an automorphism of the ring R. Let a be a central element of R. Then the Generalised Weyl Algebra, $T = R(\sigma, a)$, of degree one, is defined to be the ring generated over R by two indeterminates X, Y subject to the relations

$$X\alpha = \sigma(\alpha)X, \quad Y\alpha = \sigma^{-1}(\alpha)Y, \qquad YX = a, \quad XY = \sigma(a),$$

for all $\alpha \in R$. For the definition of Generalised Weyl Algebras of arbitrary degree the reader is referred to [1, 2]. The terminology, Generalised Weyl Algebra, is appropriate, since the Weyl algebras can be presented as Generalised Weyl Algebras.

Generalised Weyl Algebras are $\mathbb{Z}$-graded algebras: $T = \oplus(v_i \otimes R)$, where $v_i = X^i$, for $i > 0$, while $v_0 = X^0 = 1$, and $v_i = Y^{-i}$ for $i < 0$.

A reference for most of the basic notions concerning Krull dimension that we need is [11, Chapter 6]. We denote the Krull dimension of a module M by $\mathcal{K}(M)$. If R is a left noetherian ring, and $T = R(\sigma, a)$ is a Generalised Weyl Algebra then it is known that

$$\mathcal{K}(R) \leq \mathcal{K}(T) \leq \mathcal{K}(R) + 1,$$

see [4, Proposition 2.2], and the hard problem is to decide which of the two possibilities occur.

Since we can consider T as an induced module $T \otimes_R R$, one might suppose that we can control the Krull dimension of T by studying the induced modules of the form $T \otimes_R M$. This is the approach that is successfully taken in [6] to deal with skew Laurent extensions. However, for generalized Weyl algebras, the situation is complicated by the possible existence of the stars and holes, introduced later in this section, which means that the structure of induced modules for Generalized Weyl Algebras is considerably more complicated than in the case of skew Laurent extensions.

If the central element $a \in R$ is not nilpotent, then the (multiplicative) submonoid S of $R \backslash \{0\}$ generated by all $\sigma^i(a)$, $i \in \mathbb{Z}$, satisfies the (left and right) Ore condition in T. In other words, one can form the (left and right) localization $S^{-1}T = T_S$ of the ring T at S. Moreover, σ extends to R_S, and $T_S \simeq R_S[X, X^{-1}; \sigma]$ is a skew Laurent polynomial ring. An R-module M contains the S-**torsion** (or the a-**torsion**, for short) submodule $\mathrm{tor}(M) := \{m \in M \mid sm = 0 \text{ for some } s \in S\}$. An R-module M is called a-**torsion**, if $M = \mathrm{tor}(M)$; and a-**torsionfree**, if $\mathrm{tor}(M) = 0$. If a is a nilpotent element, then, by definition, any R-module is a-torsion.

Let M be an R-module and $\tau \in \mathrm{Aut}(R)$. The **twisted** module ${}^{\tau}M$ as an abelian group coincides with M and the action of R on M is given as follows: $rm := \tau(r)m$. We often write ${}_{\tau^i}M \equiv {}^{\tau^{-i}}M$, in order to avoid sign changes. Note that $v_i \otimes M \simeq {}_{\sigma^i}M$, as R-modules.

Let M be an R-module. The induced T-module

$$T(M) := T \otimes {}_R M = \oplus_{i\in\mathbb{Z}}\, v_i \otimes M$$

is the direct sum of R-submodules ${}_R(v_i \otimes M) \simeq {}_{\sigma^i}M$. The T-module $T(M)$ is $\mathbb{Z}$-graded:

$$T(M) = \oplus_{i\in\mathbb{Z}}\, T(M)_i,$$

where $T(M)_i = v_i \otimes M$.

For an R-module M define the sets

$$St(M) = \{i \in \mathbb{Z} \mid a\, {}_{\sigma^i}M = \sigma^{-i}(a)M = 0\}, \quad Ho(M) = \{i+1 \mid i \in St(M)\}.$$

The elements of $St(M)$ and $Ho(M)$ are the **stars** and **holes** respectively. Set $St_-(M) = \{i \in St(M) \mid i < 0\}$ and $Ho_+(M) = \{j \in Ho(M) \mid j > 0\}$. Denote by $s_-(M)$ the largest element of $St_-(M)$, and by $h_+(M)$ the smallest element of $Ho_+(M)$ (if they exist). The importance of stars and holes is as follows: if i is a positive hole, then $Y \cdot (X^i \otimes M) = 0$; while if i is a negative star then $X \cdot (Y^{-i} \otimes M) = 0$.

An R-module M is called a-**monic** if for every $i \in \mathbb{Z}$ the map $\sigma^i(a)_M : M \to M$, given by $m \to \sigma^i(a)m$ is either injective or zero. It is obvious that a submodule of an a-monic module is a-monic, and easy to check that any critical R-module is a-monic. Let M be an a-monic R-module. Then

$$\ker X_{T(M)} = \oplus_{i\in St_-(M)}\, T(M)_i, \tag{1}$$

$$\ker Y_{T(M)} = \oplus_{j\in Ho_+(M)}\, T(M)_j. \tag{2}$$

If M is a-monic then either M is a-torsionfree, or $St(M) \neq \emptyset$. In the latter case, there is either a negative star, or a positive hole (or both), and so, either $s_-(M)$ or $h_+(M)$ exists (they may both exist).

The R-submodule of $T(M)$

$$\mathcal{L}(M)_- := \sum\nolimits_{i\le s_-(M)} T(M)_i$$

is, in fact, a T-submodule, since $X \cdot (Y^{-s_-(M)} \otimes M) = 0$. We set

$$\mathcal{V}_+(M) = T(M)/\mathcal{L}(M)_- = \oplus_{i>s_-(M)}\, T(M)_i.$$

In a similar manner,

$$\mathcal{L}(M)_+ := \sum\nolimits_{j \geq h_+(M)} T(M)_j,$$

is also a T submodule, and we set

$$\mathcal{V}_-(M) = T(M)/\mathcal{L}(M)_+ = \oplus_{i<h_+(M)} T(M)_i.$$

In order to study the Krull dimension it is then necessary to be able to work with three kinds of induced modules: $T \otimes M$, $\mathcal{V}_+(M)$ and $\mathcal{V}_-(M)$. The point at issue is whether forming one of these modules increases the Krull dimension, or allows it to remain the same.

Definition. A critical R-module M is called T-**clean** if M is a-torsionfree and $T(M)$ is a critical T-module. In a similar manner, a critical R-module M is called $(T,+)$-**clean** if $s_-(M)$ exists and $\mathcal{V}_+(M)$ is a critical T-module, and M is $(T,-)$-**clean** if $h_+(M)$ exists and $\mathcal{V}_-(M)$ is a critical T-module.

Lemma 2.5 of [3] establishes that there are enough $(T,\bullet)$-clean modules, where $\bullet \in \{\emptyset, \pm\}$, in the sense that if M is a noetherian critical R-module then either M or some twist ${}^{\sigma^i}M$ contains a $(T,\bullet)$-clean submodule.

It is also necessary to have a notion of **height** for the simple modules of R. Again the situation is complicated by the three possible types of behaviour.

Definition. Let A be a simple R-module and let M be an arbitrary R-module.

1. If A is a-torsionfree, then $h(A : M)$ is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of T-clean R-modules such that A_i is isomorphic to a minor subfactor of A_{i+1}, for $i = 0, \dots, n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

2. If A is a-torsion, then $h_+(A : M)$, (respectively, $h_-(A : M)$), is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of R-modules such that each of the R-modules $A = A_0, A_1, \dots, A_i$, for some $i \geq 0$, is $(T,+)$-clean, (respectively, $(T,-)$-clean), while each of the R-modules $A_{i+1}, \dots, A_n$ is T-clean; and A_j is isomorphic to a minor subfactor of A_{j+1}, for $j = 0, \dots n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

Remark. Any simple a-torsionfree R-module is T-clean. However, a simple a-torsion R-module may fail to be either $(T,+)$-clean or $(T,-)$-clean. So, $h_\pm(A, M)$ is only defined for the simple a-torsion modules which are $(T,\pm)$-clean.

The sequence of R-modules $A = A_0, A_1, \ldots, A_n$ is called a **$(T, \bullet)$-clean sequence associated with** A in M, where $\bullet$ stands for $\emptyset, +$ or $-$. The corresponding $h_\bullet(A : M)$ is called the **$\bullet$-height** of A in M. If M has Krull dimension, then $h_\bullet(A : M) \leq \mathcal{K}(M)$, since $\mathcal{K}(A_n) > \mathcal{K}(A_{n-1}) > \ldots > \mathcal{K}(A_0) = 0$, so $\mathcal{K}(M) \geq n$.

The following two theorems are from the earlier paper by the authors on Krull dimension of generalized Weyl algebras, see [3, Corollaries 4.3, 4.4].

Theorem 1.1 *Let R be a left noetherian ring with finite Krull dimension and let $T = R(\sigma, a)$ be a generalized Weyl algbera over R. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists a simple $(T, \bullet)$-clean R-module A such that $\mathcal{K}(\mathcal{V}_\bullet(A)) = 1$ and $h_\bullet(A : R) = \mathcal{K}(R)$, in which case

$$\mathcal{K}(T) = \mathcal{K}(R) + 1.$$

Here, $\mathcal{V}_\bullet(A)$ is equal to $T \otimes_R A$ when A is a-torsionfree, and is equal to $\mathcal{V}_\pm(A)$ when A is a-torsion, and $h_\bullet = h$, h_+ or h_-, as appropriate. ∎

Theorem 1.2 *Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists either
(i) a simple a-torsionfree R-module A such that $h(A : R) = \mathcal{K}(R)$ and $\mathcal{O}\{A\}$ is finite; or
(ii) a simple $(T, +)$-clean R-module A such that $h_+(A : R) = \mathcal{K}(R)$ and the set $St_+(A)$ is infinite; or
(iii) a simple $(T, -)$-clean R-module A such that $h_-(A : R) = \mathcal{K}(R)$ and the set $St_-(A)$ is infinite. ∎

In the case that the element a is invertible, the algebra T is a skew Laurent extension of R, there is only one kind of cleanliness, and the above results specialize to the following theorem of Goodearl and the second author, [6, Corollary 3.3].

Theorem 1.3 *Let R be a left noetherian ring with finite Krull dimension and let $T = R[X, X^{-1}; \sigma]$ be a skew Laurent extension of R. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists a simple R-module M such that $\mathcal{K}(T \otimes M) = 1$, and $h(M : R) = \mathcal{K}(R)$, in which case

$$\mathcal{K}(T) = \mathcal{K}(R) + 1. \ \blacksquare$$

If R is a commutative ring then we recover the following theorem of the first author and Van Oystaeyen, [4, Theorem 1.2].

Theorem 1.4 *Let R be a commutative noetherian ring with finite Krull dimension, and let $T = R(\sigma, a)$ be a generalized Weyl algebra over R. Then $\mathcal{K}(T) = \mathcal{K}(R)$ unless there exists a maximal ideal $\mathbf{p}$ of R such that* $\mathrm{height}(\mathbf{p}) = \mathcal{K}(R)$ *and either $\mathbf{p}$ is invariant under some nonzero power of σ, or there are infinitely many $i \in \mathbb{Z}$ with $\sigma^i(a) \in \mathbf{p}$.*

2 Tensor Krull Minimal Algebras

Throughout this section, all rings that are considered are to be taken as algebras over a fixed field K. Whenever we use the tensor product symbol $\otimes$, we are denoting $\otimes_K$.

It is well-known that

$$\mathcal{K}(A \otimes B) \geq \mathcal{K}(A) + \mathcal{K}(B),$$

for all left noetherian K-algebras A, B such that $\mathcal{K}(A \otimes B)$ exists. However, equality may fail to hold: for example, if $K(\mathbf{X}) := K(X_1, \dots, X_n)$ is a field of rational functions in n indeterminates, then $\mathcal{K}(K(\mathbf{X}) \otimes K(\mathbf{X})) = n > 0 = \mathcal{K}(K(\mathbf{X})) + \mathcal{K}(K(\mathbf{X}))$. Thus, it is of interest to establish when this minimal possible Krull dimension of $A \otimes B$ is attained.

Definition. A K-algebra A is said to be **tensor Krull minimal**, written **TKM**, with respect to a class of algebras Ω, if

$$\mathcal{K}(A \otimes B) = \mathcal{K}(A) + \mathcal{K}(B),$$

for all algebras $B \in \Omega$.

In this section, we investigate the TKM property for generalized Weyl algebras. The corresponding problem of the tensor homological minimality of generalized Weyl algebras was addressed in [2].

Example. The polynomial algebra in n variables over K is TKM with respect to the class of all left noetherian K-algebras.

Lemma 2.1 *Let A be a commutative affine algebra over an uncountable, algebraically closed field K. Then*

$$\mathcal{K}(A \otimes B) = \mathcal{K}(A) + \mathcal{K}(B), \tag{3}$$

for any left noetherian K-algebra B.

Proof. First, we use standard techniques to reduce to the case that A is a domain. The nil radical N of A is nilpotent, since A is noetherian; suppose that $N^s = 0$, for some $s \geq 1$. Each factor in the chain of $A \otimes B$-modules

$$A \otimes B \geq N \otimes B \geq \ldots \geq N^i \otimes B \geq \ldots \geq N^s \otimes B = 0$$

is a $(A/N) \otimes B$-module, hence,

$$\mathcal{K}(A \otimes B) = \mathcal{K}((A/N) \otimes B).$$

Thus, we may assume that A is semiprime. Let $\mathbf{p}_1, \ldots \mathbf{p}_n$ be the set of minimal prime ideals of A. Note that $\cap \mathbf{p}_i = 0$, since A is semiprime; so there is a ring monomorphism

$$A = A/(\cap \mathbf{p}_i) \longrightarrow A/\mathbf{p}_1 \times \cdots \times A/\mathbf{p}_n, \tag{4}$$

given by $a \mapsto (a + \mathbf{p}_1, \cdots, a + \mathbf{p}_n)$. It follows that

$$\mathcal{K}(A) = \max\{\mathcal{K}(A/\mathbf{p}_i) \mid i = 1, \ldots, n\}. \tag{5}$$

We obtain a ring monomorphism $A \otimes B \to \prod_{i=1}^n (A/\mathbf{p}_i) \otimes B$, by tensoring (4) over K by B. Hence,

$$\mathcal{K}(A \otimes B) = \max\{\mathcal{K}((A/\mathbf{p}_i) \otimes B) \mid i = 1, \ldots, n\}. \tag{6}$$

We conclude, from (5) and (6), that it is sufficient to prove (3) in the case that A is prime. Thus, without loss of generality, A is a domain. Moreover, it is a finitely generated module over a polynomial subalgebra $P = K[x_1, \ldots, x_m]$, where $m = \mathcal{K}(A) = \mathrm{GKdim}(A)$, by Noether normalisation. Thus,

$$\mathcal{K}(A \otimes B) = \mathcal{K}(P \otimes B) = m + \mathcal{K}(B) = \mathcal{K}(A) + \mathcal{K}(B). \quad \blacksquare$$

Theorem 2.2 *Let $T = \otimes_{1=1}^n T_i$ be a tensor product of generalized Weyl algebras of the form $T_i = D_i(\sigma_i, a_i)$, where each D_i is an affine commutative algebra over an uncountable, algebraically closed field K. Then T is Tensor Krull Minimal with respect to the class of countably generated left noetherian algebras; that is,*

$$\mathcal{K}(T \otimes B) = \mathcal{K}(T) + \mathcal{K}(B) = \sum_{i=1}^n \mathcal{K}(T_i) + \mathcal{K}(B) \tag{7}$$

for any countable dimensional, left noetherian K-algebra B.

In particular,

$$\mathcal{K}\Big(\bigotimes_{i=1}^n T_i\Big) = \sum_{i=1}^n \mathcal{K}(T_i). \tag{8}$$

Proof. First, we prove that (7) holds in the case that $n = 1$. Let $T = D(\sigma, a)$ be a generalized Weyl algebra, where D is a commutative affine algebra over the field K. The tensor product $T \otimes B$ can be considered as the generalized Weyl algebra $(D \otimes B)(\sigma, a)$, where the automorphism σ extends trivially on B; that is, $\sigma(1 \otimes b) = b$, for $b \in B$. The algebra $D \otimes B$ is left noetherian, since D is affine commutative and B is left noetherian. Hence, the algebra $T \otimes B$ is also left noetherian. Therefore, we can apply Theorem 1.1. First, observe that every simple D-module is a one dimensional vector space over K, since K is an algebraically closed uncountable field and D is an affine commutative K-algebra. The algebra $D \otimes B$ is countably generated, and D belongs to the centre of $D \otimes B$. Hence, every simple $D \otimes B$-module is isomorphic to a tensor product $K_{\mathbf{m}} \otimes M$, where M is a simple B-module, $K_{\mathbf{m}} := D/\mathbf{m} \simeq K$ is a simple D-module, and $\mathbf{m}$ is a maximal ideal of D. Clearly, $K_{\mathbf{m}} \otimes M = K \otimes_K M = M$, so that every simple $D \otimes B$-module $K_{\mathbf{m}} \otimes M$ is, in fact, a simple B-module which is annihilated by the ideal $\mathbf{m}$ of D. We set $M = M_{\mathbf{m}} := K_{\mathbf{m}} \otimes M$.

Observe that

$$\mathcal{K}(D \otimes B) = \mathcal{K}(D) + \mathcal{K}(B), \tag{9}$$

by Lemma 2.1, and that

$$h_D(K_{\mathbf{m}} : D) = \text{height}(\mathbf{m}) = \mathcal{K}(D), \tag{10}$$

for every maximal ideal $\mathbf{m}$ of D. Now, in the case that $n = 1$, the formula (7) follows immediately from (9), (10) and Theorem 1.2.

For $n > 1$, we use induction on n. We have

$$\mathcal{K}(T \otimes B) = \mathcal{K}(T_1 \otimes (\tilde{T} \otimes B)) = \mathcal{K}(T_1) + \mathcal{K}(\tilde{T} \otimes B)) = \sum \mathcal{K}(T_i) + \mathcal{K}(B),$$

where $\tilde{T} = T_2 \otimes \cdots \otimes T_n$. (Note, we are using the fact that a Generalized Weyl Algebra $T = D(\sigma, a)$ is left noetherian if and only if D is a left noetherian ring.) Hence, (7) holds, and then (8) also holds, by setting $B = K$ in the argument. ∎

3 Affine automorphims

Let $A = D(\sigma, a)$ be a generalized Weyl algebra with $D = K[x_1, \ldots, x_n]$, the polynomial ring in n variables over an algebraically closed field K. Suppose that $\sigma \in \text{Aut}_{Fil}(D)$ is a $K-$automorphism of D which preserves the natural filtration of the polynomial ring. Then σ can be written as

$$\sigma(x) = \mathcal{A}x + b,$$

for some $\mathcal{A} \in GL_n(K)$, $b \in K^n$, where $x = (x_1, \dots, x_n)^T$. The map $\sigma \to (\mathcal{A}, b)$ establishes a one-to-one correspondence between $\mathrm{Aut}_{Fil}(D)$ and $(GL_n(K), K^n)$. In order to study the generalized Weyl algebra A, it is useful to consider a canonical form for such pairs $(\mathcal{A}, b) \in (GL_n(K), K^n)$. This has been done in [2, Section 4], and the following discussion is extracted from there.

If we take another generating set for the algebra D, given by $x' = Sx + c$, where $S \in GL_n(K)$, and $c \in K^n$, then σ acts on x' via

$$\sigma(x') = S\mathcal{A}S^{-1}x' + Sb + (I - S\mathcal{A}S^{-1})c. \tag{11}$$

Pairs $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are called *equivalent*, denoted by $(\mathcal{A}, b) \sim (\mathcal{A}', b')$, if

$$\mathcal{A}' = S\mathcal{A}S^{-1} \text{ and } b' = Sb + (I - S\mathcal{A}S^{-1})c, \tag{12}$$

for some $S \in GL_n(K)$, $c \in K^n$. The aim is to develop a canonical form to which $(\mathcal{A}, b)$ can be reduced by the transformation (12), in line with the Jordan Canonical Form of $\mathcal{A}$.

Since K is algebraically closed, we can choose $S \in GL_n(K)$ such that $\mathcal{A}$ is reduced by the transformation $\mathcal{A} \to S\mathcal{A}S^{-1}$ to the block diagonal matrix

$$\mathcal{A} = diag(J(\lambda_1), \dots, J(\lambda_s)), \tag{13}$$

where $\lambda_1, \dots, \lambda_s$ are distinct eigenvalues of A and every matrix $J(\lambda_i)$ consists of Jordan matrices

$$J_m(\lambda_i) = \lambda_i E + \sum\nolimits_{i=1}^{m-1} e_{k,k+1} \in GL_m(K),$$

where the e_{ij} are matrix units. We can write $b = (b_1, \dots, b_s)^T$, for some vectors b_i corresponding to the decomposition in (13).

If $\lambda_i \neq 1$, then the matrix $I - J(\lambda_i)$ is non-singular, therefore by (11), the vector b_i can be chosen to be 0.

It remains to consider the case when, say, $\lambda_1 = 1$. By using (13), for the sake of simplicity, we put $\mathcal{A} = J(1)$; that is, 1 is the unique eigenvalue of $\mathcal{A}$. Then

$$J(1) = diag(J_1^{(m_1)}, J_2^{(m_2)}, \dots, J_k^{(m_k)}, \dots), \tag{14}$$

where

$$J_k^{(m_k)} = diag(J_k(1), \dots, J_k(1)) \tag{15}$$

has m_k occurences of the $k \times k$ block $J_k(1)$. Corresponding to (14), the vector b is written as $b = (b_1, \dots, b_k, \dots)$, with each $b_k = (b_{k1}, \dots, b_{km_k})$, corresponding to the block decomposition (15).

Let $\mathcal{A}$ be an $m \times m$ Jordan matrix $\mathcal{A} = J_m(1)$ and let $e_1 = (1, 0, \dots, 0) \dots, e_m = (0, \dots, 0, 1)$ be the standard basis of K^m. Then $e_1, \dots, e_{m-1}$ is a basis of Im $(I - \mathcal{A})$; and so the vector b can be reduced to $b = (0, \dots, 0, \lambda)$, for some $\lambda \in K$, by the transformation $b \to b + (I - \mathcal{A})c$. If $\lambda \neq 0$, then by replacing $\{x_i\}$ by $\{-\lambda^{-1}x_i\}$, for $i = 1, \dots, n$, we reduce $(J_m(1), (0, \dots 0, \lambda))$ to $(J_m(1), (0, \dots, 0, -1))$.

Suppose that $\mathcal{A} = J_m^{(k)} = diag(J_m(1), \dots, J_m(1))$, with k occurences of $J_m(1)$. By using the above arguments b can be reduced to $b = (b_1, \dots, b_k)$ where each b_i is either of the form $(0, \dots, 0, -1)$ or 0. If $b \neq 0$, then, by re-arrangement, if necessary, we may suppose that $b_k = (0, \dots, 0, -1)$. Let S be the $k \times k$ matrix with elements from the matrix ring $M_m(K)$:

$$S = diag(E, \dots, E) + \sum_{i=1}^{k-1} S_{ik} e_{ik},$$

where $E = diag(1, \dots, 1)$, and $S_{ik} = -E$, if $b_i \neq 0$, while $S_{ik} = 0$, if $b_i = 0$. Choosing $c = 0$ in (11), we see that $(\mathcal{A}, b) \sim (\mathcal{A}, b = (0, \dots, 0, b_k))$ where $b_k = (0, \dots, 0, -1)$. The vector $b = (0, \dots, 0, (0, \dots, 0, -1))$ is called exceptional. Let $\mathcal{A} = J(1) = diag(J_1^{(m_1)}, \dots, J_k^{(m_k)}, \dots)$. The vector b is written as $b = (b_1, \dots, b_k, \dots)$, with respect to (14), and we may suppose that each b_k is either 0 or exceptional. The maximal k such that b_k is exceptional is also said to be exceptional. Let k be exceptional: then it is easy to show that all b_i, for $i < k$ can be made equal to 0. This analysis establishes the following Lemma.

Lemma 3.1 *With the notation as above, a pair $(\mathcal{A}, b)$ is equivalent to a pair*

$$(diag(J(\lambda_1), \dots, J(\lambda_s)), b = (b_1, \dots, b_s)),$$

where $b = 0$, if all $\lambda_i \neq 1$. Otherwise, with $\lambda_1 = 1$, say, then, with respect to the decomposition of $J(1)$ in (14), the vector b_1 can be written as $b_1 = (c, 0, \dots, 0)$, with $c_k = (0, \dots, 0, (0, \dots, 0, -1))$ or $c_k = 0$ for some k. ∎

We say that a pair $(\mathcal{A}, b)$ is *exceptional* if and only if it is equivalent to a pair $(\mathcal{A}', b')$, in the canonical form given by Lemma 3.1, with $b' \neq 0$.

Corollary 3.2 *A pair $(\mathcal{A}, b)$ is equivalent to $(\mathcal{A}', \mathbf{0})$ (so that $(\mathcal{A}, b)$ is not exceptional) if and only if $b \in \mathrm{Im}(I - \mathcal{A})$.*

Proof. The vector b in $(\mathcal{A}, b)$ is specified only up to an element of $\mathrm{Im}(I - \mathcal{A})$, by (14). Hence, $(\mathcal{A}, b)$ is equivalent to $(\mathcal{A}', \mathbf{0})$ if and only if $b \in \mathrm{Im}(I - \mathcal{A})$. ∎

Any maximal ideal $\mathbf{m}$ of D is given by

$$\mathbf{m} := \mathbf{m}_\alpha = (\mathbf{x} - \alpha) = (x_1 - \alpha_1, \dots, x_n - \alpha_n),$$

for some uniquely defined $\alpha = (\alpha_1, \dots, \alpha_n) \in K^n$, since K is algebraically closed. We use this to identify $\mathrm{MaxSpec}(D)$ with K^n, via the assignment $\mathbf{m}_\alpha \mapsto \alpha$. We will use the same letter σ for the affine bijection

$$\sigma : K^n \to K^n, \qquad \sigma(\alpha) = \mathcal{A}\alpha + b \tag{16}$$

Now,

$$\begin{aligned} \sigma(\mathbf{m}_\alpha) &= (\sigma(\mathbf{x}) - \alpha) = (\mathcal{A}\mathbf{x} + b - \alpha) = (\mathcal{A}(\mathbf{x} - \mathcal{A}^{-1}(b - \alpha))) \\ &= (\mathbf{x} - \mathcal{A}^{-1}(\alpha - b)) = (\mathbf{x} - \sigma^{-1}(\alpha)) = \mathbf{m}_{\sigma^{-1}(\alpha)} \end{aligned}$$

so, under the above identification, the orbit, $\mathcal{O}(\mathbf{m}_\alpha)$, of the maximal ideal $\mathbf{m}_\alpha$ in D can be identified with the orbit $\mathcal{O}(\alpha) := \{\sigma^i(\alpha) \mid i \in \mathbb{Z}\}$, of the corresponding vector α under the action of the group $G = \langle \sigma \rangle$, defined in (16) above.

The following formulae are easy to establish by induction.

$$\sigma^i(\alpha) = \mathcal{A}^i\alpha + (I + \mathcal{A} + \dots + \mathcal{A}^{i-1})b, \tag{17}$$

$$\sigma^{-i}(\alpha) = \mathcal{A}^{-i}\alpha - (I + \mathcal{A}^{-1} + \dots + \mathcal{A}^{-i+1})b. \tag{18}$$

Lemma 3.3 *The following three statements are equivalent.*

1. *There is a finite orbit in* $\mathrm{MaxSpec}(D)$ *under the action of the group* G.
2. $\mathrm{Im}(I - \mathcal{A}) \cap \{b + \ker(I + \mathcal{A} + \dots + \mathcal{A}^{i-1})\} \neq \emptyset$, *for some* $i \geq 1$.
3. *Either* $b \in \mathrm{Im}(I - \mathcal{A})$ *or* $\mathrm{char}(K) = p > 0$ *(or both).*

Proof. Denote the intersection in statement 2. by I_i.

(1⇔2). The first statement is true if and only if $\sigma^i(\alpha) = \alpha$, for some $\alpha \in K^n$. However,

$$\alpha - \sigma^i(\alpha) = (I - \mathcal{A}^i)\alpha - (I + \mathcal{A} + \dots + \mathcal{A}^{i-1})b = (I + \mathcal{A} + \dots + \mathcal{A}^{i-1})((I - \mathcal{A})\alpha - b);$$

so the equivalence follows.

(3⇒2). Note that if $b \in \mathrm{Im}(I - \mathcal{A})$ then $b \in I_i$, for each i. Suppose that $\mathrm{char}(K) = p > 0$ and that $b \notin \mathrm{Im}(I - \mathcal{A})$; so that the pair $(\mathcal{A}, b)$ is exceptional, by Corollary 3.2. In view of Lemma 3.1, we may restrict ourselves to the case that $\mathcal{A} = J_m(1)$ and $b = (0, \dots, 0, -1)$. Set $\Theta := \mathcal{A} - I$, and note that $\Theta^m = 0$. Hence,

$$\mathcal{A}^{p^m} = (I + \Theta)^{p^m} = I + \Theta^{p^m} = I.$$

Let S be the minimal positive integer satisfying $\mathcal{A}^S = I$. Then

$$I + \mathcal{A} + \ldots + \mathcal{A}^{pS-1} = \sum_{i=0}^{p-1} \mathcal{A}^{iS}(I + \mathcal{A} + \ldots + \mathcal{A}^{S-1}) = p(I + \mathcal{A} + \ldots + \mathcal{A}^{S-1}) = 0,$$

so $I_{pS-1} = \operatorname{Im}(I - \mathcal{A}) \neq \emptyset$.

(2$\Rightarrow$3). Suppose that statement 3. does not hold, so that $b \notin \operatorname{Im}(I-\mathcal{A})$ and $\operatorname{char}(K) = 0$. It suffices to show that each of the sets I_i is empty. As in the previous case, we may assume that $\mathcal{A} = J_m(1)$ and that $b = (0, \ldots, 0, -1)$. For each $i \geq 1$, let B_i be the matrix such that $I + \mathcal{A} + \ldots + \mathcal{A}^{i-1} = iI + B_i$. Note that B_i is a strictly upper triangular matrix, so that $\ker(iI + B_i) \subseteq U := \sum_{i=1}^{m-1} Ke_i$, since $\operatorname{char}(K) = 0$. Observe that $\operatorname{Im}(I - \mathcal{A}) = U$, and that $b \notin U$, hence $I_i = \emptyset$, for each $i \geq 1$. ∎

The above results enable us to give a good description of the Krull dimension of skew Laurent extensions which involve affine automorphisms.

Theorem 3.4 *Let $T = D[X, X^{-1}; \sigma]$ be a skew Laurent extension of the polynomial ring, $D = K[X_1, \ldots, X_n]$, over an algebraically closed field K, and suppose that $\sigma(x) = \mathcal{A}x + b$. Then*

$$\mathcal{K}(T) = \begin{cases} n+1, & \textit{if either } b \in \operatorname{Im}(I - \mathcal{A}) \textit{ or } \operatorname{char}(K) \neq 0; \\ n, & \textit{otherwise.} \end{cases}$$

Proof. This follows immediately from the above lemma and [6, Theorem 4.3]. ∎

Many interesting Generalized Weyl Algebras can be constructed starting from the base ring of polynomials in two variables. Next we summarize the above results in this case, for use in the next section.

Let $D = K[x_1, x_2]$ be the polynomial ring in two variables over an algebraically closed field K. The considerations earlier in the section show that, up to an affine change of variables, there are four possible equivalence types of automorphism σ to consider:

$$\begin{array}{rl} \text{(F)} & \sigma(x_1) = \lambda x_1, \quad \sigma(x_2) = \mu x_2; \\ \text{(G)} & \sigma(x_1) = \lambda x_1 + x_2, \quad \sigma(x_2) = \lambda x_2; \\ \text{(H)} & \sigma(x_1) = x_1 - 1, \quad \sigma(x_2) = \lambda x_2; \\ \text{(I)} & \sigma(x_1) = x_1 - 1, \quad \sigma(x_2) = x_2 + x_1; \end{array}$$

where λ and μ are nonzero elements of K.

In characteristic zero, Case (I) can be simplified somewhat by making a non-affine change of variables. Suppose that σ is of the form occuring in Case (I). Note that the polynomial $\theta := x_1^2 + x_1 + 2x_2$ is invariant under σ; that is, $\sigma(\theta) = \theta$.

Lemma 3.5 *Suppose that σ is of the form occuring in Case (I).*
1. If $\mathrm{char}(K) \neq 2$*, then* $K[x_1, x_2] = K[x_1, \theta]$ *and* $\sigma(x_1) = x_1 - 1$*,* $\sigma(\theta) = \theta$*.*
2. If $\mathrm{char}(K) = 2$*, then* σ^4 *is the identity automorphism.*

Proof. Straightforward. ▪

4 Down-up algebras

In this section, we calculate the Krull dimension of the **Down-Up** algebras introduced by Benkart, [5], and recently studied in papers by Kulkarni, [10], and Kirkman, Musson and Passman, [9], Jordan, [8], and others. Let K be a field, and choose elements $\alpha, \beta, \gamma \in K$, with $\beta \neq 0$. Set $A := A(\alpha, \beta, \gamma)$ to be the K-algebra generated by indeterminates d and u and subject to the relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d = (\alpha du + \beta ud + \gamma)d \\ du^2 &= \alpha udu + \beta u^2d + \gamma u = u(\alpha du + \beta ud + \gamma). \end{aligned}$$

Kirkman, Musson and Passman show that $A(\alpha, \beta, \gamma)$ is isomorphic to a generalized Weyl algebra $R(\sigma, a)$, where R is the polynomial ring $K[x_1, x_2]$ and $a = x_1$. The correspondence is given by

$$x_1 \leftrightarrow du, \quad x_2 \leftrightarrow ud, \quad X \leftrightarrow d, \quad Y \leftrightarrow u,$$

and the automorphism σ is given by

$$\sigma(x_1) = x_2 \qquad \sigma(x_2) = \alpha x_2 + \beta x_1 + \gamma.$$

As a consequence, $A(\alpha, \beta, \gamma)$, with $\beta \neq 0$ is a noetherian domain, by [1]. Hence, the Krull dimension of $A(\alpha, \beta, \gamma)$ is either two or three, since $\mathcal{K}(K[x_1, x_2]) = 2$. If $\beta = 0$, then Kirkman, Musson and Passman show that $A(\alpha, \beta, \gamma)$ is neither noetherian nor a domain.

Note that if we set

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

then

$$\sigma(\mathbf{x}) = \mathcal{A}\mathbf{x} + b.$$

In order to calculate the Krull dimension of noetherian Down-Up algebras, we need to identify which of the four possible equivalence classes of automorphisms (F), (G), (H) and (I), introduced at the end of the previous section, arises for the various choices of the parameters α, β, γ. Observe that if 1 is a root of the characteristic polynomial $\det(\mathcal{A} - tI) = t^2 - \alpha t - \beta$ of the matrix $\mathcal{A}$, then $\det(\mathcal{A} - tI) = (t-1)(t-\alpha+1)$, and so $\alpha - 1$ is the other root of $\det(\mathcal{A} - tI)$.

Lemma 4.1 *Let* $b = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \in K^2$.
1. $b \in \mathrm{Im}(I - \mathcal{A})$ *if and only if* $\gamma = 0$ *or* $\gamma \neq 0$ *and* $\alpha + \beta \neq 1$.
2. Suppose that $b \neq 0$. *Then* $b \notin \mathrm{Im}(I - \mathcal{A})$ *if and only if* $\alpha + \beta = 1$ *if and only if* 1 *is a root of the characteristic polynomial* $\det(\mathcal{A} - tI)$.
3. Suppose that $b \neq 0$ *while* $\beta = -1$ *and* $\alpha = 2$. *Then the pair* $(\mathcal{A}, b)$ *belongs to Case (I).*
4. Suppose that $b \neq 0$ *while* $\alpha + \beta = 1$ *but* $\beta \neq -1, 0$. *Then the pair* $(\mathcal{A}, b)$ *belongs to Case (H).*

Proof. 1. Note that $\begin{pmatrix} 0 \\ \gamma \end{pmatrix} \in \mathrm{Im}(I - \mathcal{A})$ if and only if there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in K^2$ such that $(I - \mathcal{A})\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$ if and only if $x_0 = y_0$ and $(1 - \alpha - \beta)x_0 = \gamma$, and the result then follows easily.
2. This is evident, since $\det(\mathcal{A} - tI) = t^2 - \alpha t - \beta$.
3. Suppose that $b \neq 0$ while $\beta = -1$ and $\alpha = 2$. Set $H = \gamma^{-1}x_1 - \gamma^{-1}x_2$ and $C = -\gamma^{-1}x_1$. Then

$$\sigma(H) = \gamma^{-1}x_2 - \gamma^{-1}(-x_1 + 2x_2 + \gamma) = \gamma^{-1}x_1 - \gamma^{-1}x_2 - 1 = H - 1$$

and

$$\sigma(C) = -\gamma^{-1}x_2 = \gamma^{-1}x_1 - \gamma^{-1}x_2 - \gamma^{-1}x_1 = H + C;$$

so that Case (I) applies.
4. Suppose that $b \neq 0$ while $\alpha + \beta = 1$ but $\beta \neq -1$. Set $H = -\gamma^{-1}\beta x_1 - \gamma^{-1}x_2$ and $C = x_1 - x_2 + \gamma/(1+\beta)$. Then

$$\sigma(H) = -\gamma^{-1}\beta x_2 - \gamma^{-1}(x_2 - \beta x_2 + \beta x_1 + \gamma) = -\gamma^{-1}\beta x_1 - \gamma^{-1}x_2 - 1 = H - 1$$

and

$$\sigma(C) = x_2 - (x_2 - \beta x_2 + \beta x_1 + \gamma) + \gamma/(1+\beta) = -\beta(x_1 - x_2 + \gamma/(1+\beta)) = -\beta C.$$

Thus, Case (H) applies, with $\lambda = -\beta$. ∎

Theorem 4.2 *Let* K *be an arbitrary field (not necessarily algebraically closed) and let* $\beta \neq 0$. *Then the noetherian Down-Up algebra* $A = A(\alpha, \beta, \gamma)$ *has Krull dimension equal to* 2 *if and only if* $\mathrm{char}(K) = 0$, $\gamma \neq 0$, *and* $\alpha + \beta = 1$. *Otherwise, the Krull dimension of* A *is* 3.

Proof. If $b := \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \in \mathrm{Im}(I - \mathcal{A})$, then the pair $(\mathcal{A}, b)$ is equivalent to the pair $(\mathcal{A}', 0)$; that is, there exist elements $H, C \in K[x_1, x_2] = D$ such that $D = K[H, C]$ and $\sigma \begin{pmatrix} H \\ C \end{pmatrix} = \mathcal{A}' \begin{pmatrix} H \\ C \end{pmatrix}$. In this case, the maximal ideal J of D that is generated by H and C is σ-invariant and has height 2. Hence, $\mathcal{K}(A) = 3$, by Theorem 1.4.

Thus, we may assume that $b \notin \mathrm{Im}(I - \mathcal{A})$, or equivalently, that $\gamma \neq 0$ and $\alpha + \beta = 1$, by Lemma 4.1.(2). Then 1 is a root of the characteristic polynomial, $\det(\mathcal{A} - tI)$, of $\mathcal{A}$ and $\lambda := \alpha - 1$ is the other root so that $\det(\mathcal{A} - tI) = (t-1)(t-\lambda)$.

Suppose first that $\mathrm{char}(K) = p > 0$. If $\beta = -1$ then Case (I) applies, by Lemma 4.1.(3), and with $H = \gamma^{-1}x_1 - \gamma^{-1}x_2$, $C = -\gamma^{-1}x_1$ we have $\sigma(H) = H - 1$ and $\sigma(C) = C + H$. By using Lemma 3.5, we see that the maximal ideal generated by C and H is invariant under σ^p, when p is odd, and invariant under σ^4 when $p = 2$. Thus, $\mathcal{K}(A) = 3$, by Theorem 1.4. If $\beta \neq -1$, then again, the maximal ideal generated by H and C, as in the proof of Lemma 4.1.(4), is invariant under σ^p, and so $\mathcal{K}(A) = 3$, by Theorem 1.4.

Thus, we may assume that $\mathrm{char}(K) = 0$, and we aim to show that $\mathcal{K}(A) = 2$. It is enough to show that $\mathcal{K}(A) \leq 2$, since $\mathcal{K}(A)$ is either 2 or 3. Let $\overline{K}$ denote the algebraic closure of K and let $\overline{A}$ denote the Down-Up algebra constructed over $\overline{K}$ using the same parameters as for A. Then there is an injective map from the lattice of left ideals of A into the lattice of left ideals of $\overline{A}$ given by $I \to \overline{K} \otimes I$. Hence, $\mathcal{K}(A) \leq \mathcal{K}(\overline{A})$. Thus, it suffices to prove the result when K is algebraically closed.

Suppose first that $\beta = -1$, so that Case (I) applies, by Lemma 4.1.(3), and with $H = \gamma^{-1}x_1 - \gamma^{-1}x_2$, $C = -\gamma^{-1}x_1$ we have $\sigma(H) = H - 1$ and $\sigma(C) = C + H$. Note that $a = x_1 = -\gamma C$. It is easy to check, by induction, that $\sigma^n(C) = C + nH - n(n-1)/2$, for $n \geq 1$, and it then follows that $\sigma^{-n}(C) = C - nH - n(n+1)/2$, for $n \geq 1$. Suppose that $\mathcal{K}(A) = 3$. Then, by Theorem 1.4, there exists a maximal ideal $\mathbf{m}$ of D and an infinite subset $I \subseteq \mathbb{Z}$ such that $\sigma^i(a) \in \mathbf{m}$, for $i \in I$, or, equivalently, that $\sigma^i(C) \in \mathbf{m}$, for $i \in I$. Suppose that $\mathbf{m}$ is generated by the elements $H - h$ and $C - c$, for $h, c \in K$. Then, for each positive $i \in I$, we have $c + ih - i(i-1)/2 = 0$ and, for each negative $i \in I$, we have $c - ih - i(i+1)/2 = 0$. Thus, the possible i are among the solutions of these two quadratic equations, and so I cannot be infinite, a contradiction. Thus $\mathcal{K}(A) = 2$ in this case.

Now suppose that $\beta \neq -1$, so that Case (H) applies, by Lemma 4.1.(4). Again, supposing that $\mathcal{K}(A) = 3$, there is a maximal ideal $\mathbf{m}$ generated by elements $H - h$ and $C - c$, for $h, c \in K$ and an infinite set I such that

$$\sigma^i(x_1) \in \mathbf{m} = (H - h, C - c) \quad \text{for } i \in I \tag{19}$$

Write $x_1 = x_1(H,C) = \mu H + \nu C + \xi$, for some $\mu, \nu, \xi \in K$. Note that $\mu \neq 0$, since otherwise $x_2 = \sigma(x_1) = \sigma(\nu C + \xi) = \nu\lambda C + \xi = \lambda x_1 + \xi - \xi\lambda$, with $\lambda = -\beta$, a contradiction since x_1 and x_2 are algebraically independent. Thus, (19) is equivalent to

$$\mu(h-i) + \nu\lambda^i c + \xi = 0 \quad \text{for } i \in I. \tag{20}$$

Clearly, $c \neq 0$ and $\nu \neq 0$; so $i - j = \delta(\lambda^i - \lambda^j)$, for $i, j \in I$, where $\delta := \nu c/\mu$. So,

$$\frac{i-j}{k-j} = \frac{\lambda^{i-j}-1}{\lambda^{k-j}-1} \quad \text{for } i,j,k \in I \text{ with } j \neq k. \tag{21}$$

In view of (21), and since $(i-j)/(k-j) \in \mathbb{Q}$ and $\mathrm{char}(K) = 0$, we may suppose that $\lambda \in \mathbb{C}$, and that the nonzero element $\delta = (i-j)/(\lambda^i - \lambda^j) \in \mathbb{C}$. Without loss of generality, since I is an infinite set, we may assume that $I \cap \mathbb{N}$ is infinite. If j and k are fixed, and i is chosen sufficiently large, then the absolute value of the complex number λ will be greater than 1, which implies that $\delta = \lim_{i\to\infty} \frac{i-j}{\lambda^i - \lambda^j} = 0$, a contradiction. Thus, $\mathcal{K}(A) = 2$ in this case also. In summary, $\mathcal{K}(A) = 2$ if $\mathrm{char}(A) = 0$, $\gamma \neq 0$ and $\alpha + \beta = 1$; while otherwise $\mathcal{K}(A) = 3$. ∎

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