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Mini-Workshop: Non-Negativity is a Quantum Phenomenon

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Abstract. In recent publications, the same combinatorial description has arisen for three separate objects of interest: non-negative cells in the real grassmannian (Postnikov, Williams); torus orbits of symplectic leaves in the classical grassmannian (Brown, Goodearl and Yakimov); and, torus invariant prime ideals in the quantum grassmannian (Launois, Lenagan and Rigal). The aim of this meeting was to explore the reasons for this coincidence in matrices and the grassmannian in particular, and to explore similar ideas in more general settings

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Introduction by the Organisers

There were 15 participants in the mini-workshop, with representatives of the three main areas included.

Before describing the conduct of the meeting, the next few paragraphs outline the background that led to the proposal to have such a mini-workshop and recent results in the three main areas involved in the programme.

Many quantum algebras have a natural action by a torus and a key ingredient in the study of the structure of these algebras is an understanding of the torus invariant objects. For example, the Stratification Theory of Goodearl and Letzter shows that, in the generic case, a complete understanding of the prime spectrum of quantum matrices would start by classifying the (finitely many) torus invariant prime ideals. In a paper in Journal of Algebra in 2003, Cauchon succeeded in counting the number of torus invariant prime ideals in quantum matrices. His method involved a bijection between certain diagrams, now known as Cauchon diagrams, and the torus invariant primes. Considerable progress in the understanding of quantum matrices has been made since that time by using Cauchon diagrams.

The semiclassical limit of quantum matrices is the classical coordinate ring of the variety of matrices endowed with a Poisson bracket that encodes the nature of the quantum deformation which leads to quantum matrices. As a result, the variety of matrices is endowed with a Poisson structure. A natural torus action leads to a stratification of the variety via torus orbits of symplectic leaves. In a paper in Advances in Mathematics in 2006, Brown, Goodearl and Yakimov showed that there were finitely many such torus orbits of symplectic leaves. Each torus orbit is defined by certain rank conditions on submatrices. The classification is given in terms of certain pairs of permutations from the relevant symmetric group with restrictions arising from the Bruhat order.

The non-negative part of the real grassmannian consists of those points in the grassmannian which can be represented by Plücker coordinates that are all non-negative. One can specify a cell decomposition of the non-negative grassmannian by specifying exactly which coordinates are to be zero/non-zero. In a paper posted on the arxiv in 2006, Postnikov classified the non-empty cells by means of a bijection with certain diagrams, known as Le-diagrams. By using this bijection, Williams, in a paper in Advances in Mathematics in 2005, succeeded in obtaining a generating function for the number of cells according to their dimensions, and as a consequence, was able to count the number of non-empty non-negative cells. Many interesting combinatorial structures were used in Postnikov's paper, for example, planar networks, matroids, decorated permutations and Bruhat intervals. Specialisation of the Williams' generating function leads to interesting combinatorial information concerning Eulerian numbers, Narayana numbers and binomial coefficients.

The interesting observation from the point of view of this mini-workshop is that in each of the above three sets of results the combinatorial objects that arise turn out to be the same, although the methods employed to obtain the results are very different! The definitions of Cauchon diagrams and Le-diagrams are the same, and the restricted permutations arising in the Brown-Goodearl-Yakimov study can be seen to lead to Cauchon/Le diagrams via the notion of pipe dreams. Postnikov's work is largely combinatorial, Brown-Goodearl-Yakimov employ algebraic geometry, while Cauchon's work is mainly non-commutive algebra.

Once one is aware of these connections, one can employ notions arising in any of the three areas to suggest means of progress in the others. An example of this is seen in a paper of the two organisers and Rigal which appeared recently in Selecta Mathematica. Here, inspired by the results of Postnikov and Williams on total non-negativity, the authors extended the work of Cauchon in quantum matrices to the quantum grassmannian, established that there are only finitely many torus invariant prime ideals in the quantum grassmannian and counted them by using

Williams' result. Useful structural information about the quantum grassmannian was then obtained via these results.

The intention of this mini-workshop was to bring together experts in each of the three areas to try to understand the underlying reasons for the similarities arising in these investigations. There were indications that this would involve notions of cluster algebras and canonical bases, and so experts from these areas were involved. Finally, combinatorial questions were seen to be important and so there were also combinatorial experts involved.

More specifically, in the longer term, the hope is that one might develop a single setting in which results can be proved and then routinely applied to the separate cases. Alternatively, it may be possible to generate machines that will transfer results from one case to the others. In the case of the link between quantum algebras and their semiclassical limits this is also part of a more general programme that would study many different quantum algebras.

On the first day of the meeting, there were four overview talks. The first, a general introduction to the area was given by one of the organisers, Tom Lenagan. This was followed by three lectures; one devoted to introducing each of the three main areas to be explored during the meeting. Ken Goodearl gave a survey talk about Poisson algebras and symplectic leaves. Lauren Williams introduced the topic of total non-negativity in matrices, the grassmannian and more general classes of varieties. Gérard Cauchon talked about quantum algebras and the deleting derivations algorithm.

On the second day of the meeting, there were three talks, each one introducing a method that might have a bearing on the connection between the three main areas. Robert Marsh introduced the canonical basis, Bernard Leclerc talked about cluster algebras and Francesco Brenti discussed Bruhat intervals.

During the rest of the meeting, each of the other participants gave a talk and there were extensive discussions by various subsets of the participants. Several new collaborations arose from these discussions, including, gratifyingly, some across the boundaries of the three different groups represented.

The final lecture, by the other organiser, Stéphane Launois, announced recent work of the two organisers and Ken Goodearl, giving an application of the restoration algorithm (the inverse process to the deleting derivations algorithm), first developed for use in quantum matrices, to the explicit determination of the totally non-negative cells in totally non-negative matrices and the defining minors of torus orbits of symplectic leaves in the classical variety of matrices endowed with the standard Poisson bracket arising from the semiclassical limit of quantum matrices.

Mini-Workshop: Non-Negativity is a Quantum Phenomenon

Table of Contents

Abstracts

Introduction: Non-negativity is a quantum phenomenon

Tom Lenagan

The opening lecture, by one of the organisers, was a survey lecture introducing the three main areas of interest: quantum algebras, (torus orbits of) symplectic leaves in Poisson algebras arising as semiclassical limits, and total non-negativity. The ideas were illustrated by simple examples, mainly at the level of 2×2 matrices, and no significant proofs were given.

It was pointed out that the initial observation of the similarity between results in these three areas occurred when it was noticed that the diagrams of the non-empty cells in the 2×4 totally non-negative grassmannian (see, for example, Lauren Williams' thesis, [6]) and the $\mathcal{H}$-invariant primes in the 2×4 generic quantum grassmannian were the same (see, for example, [4]). In addition, the combinatorial objects that parameterised the non-empty cells in the case of the totally non-negative grassmannian (Le-diagrams), see [5], and $\mathcal{H}$-invariant primes in the quantum setting (Cauchon diagrams), see [2], were the same!

Torus orbits of symplectic leaves in semiclassical matrix varieties and/or the grassmannian are parameterised by pairs of Weyl group elements satisfying certain restrictions (see [1, 3]) and the connection between Le and Cauchon diagrams and such pairs of permutations was illustrated by means of pipe dreams.

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Poisson algebras and symplectic leaves

K. R. Goodearl

An expository account of basic ideas concerning Poisson algebras, symplectic leaves, and semiclassical limits was presented, in order to describe the landscape in which relations between quantized coordinate rings and Poisson algebras are found. If A is a generic quantized coordinate ring of an affine algebraic variety V, then A is a member of a flat family of algebras with a commutative specialization R, its *semiclassical limit*. As an algebra, R is just the classical coordinate ring (ring of regular functions) of V. By virtue of the semiclassical limit process, R

is also equipped with a Poisson bracket, which records, roughly speaking, a first-order impression of the additive commutators in the noncommutative algebra A. The Poisson algebra structure on R means that V is a Poisson variety, and hence, when working over the complex field, V is partitioned into symplectic leaves. The principles embodied in the Orbit Method descended from Lie theory suggest that there should be some good correspondence between the symplectic leaves of V and the primitive ideals of A, that is, a bijection which also matches other information such as dimensions. (More precisely, the dimension of a symplectic leaf matched with a primitive ideal P should equal the Gelfand-Kirillov dimension of A/P.) Such bijections were first developed for the case when V is a semisimple complex algebraic group by Hodges-Levasseur [5, 6] and Hodges-Levasseur-Toro [7].

Analysis of examples in which the symplectic leaves of the variety V are not algebraic (i.e., not locally closed in the Zariski topology) indicated that symplectic leaves are not always the right objects to be matched with primitive ideals. Instead, the symplectic cores introduced by Brown and Gordon [1], which are viewed as the best algebraic approximation to symplectic leaves, should be used. These correspond precisely to the *Poisson-primitive* ideals of R, which are defined as the largest Poisson ideals contained in the maximal ideals. (The replacement of symplectic leaves by symplectic cores also allows arbitrary base fields to be used.) One thus arrives at the

Conjecture. *If an algebra A is a suitably generic member of a flat family of algebras with semiclassical limit R, the primitive ideal space of A should be homeomorphic to the space of Poisson-primitive ideals of R, and the prime spectrum of A should be homeomorphic to the Poisson prime spectrum of R. These homeomorphisms should preserve appropriate dimensions, and should be equivariant with respect to appropriate group actions.*

The conjecture has only been verified for quantized coordinate rings of algebraic groups in the case of SL_2. Combined work of Letzter and the speaker [4] and Oh-Park-Shin [9] has verified the conjecture for multiparameter quantized coordinate rings of affine spaces. Moreover, Oh has established the conjecture for a class of algebras that includes quantized coordinate rings of euclidean and symplectic spaces [8].

A detailed discussion of most of the material that was presented in the talk can be found in the expository paper [2].

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Introduction to total non-negativity

Lauren Williams

1. Introduction

One says that a matrix is totally positive (respectively, totally non-negative) if all of its minors are positive (respectively, non-negative) real numbers. In the 1930's there was a systematic study of these matrices by Gantmacher, Krein, Schoenberg, Whitney, among others. The field of total positivity was greatly expanded in the 1990's, when Lusztig [6, 7, 8] found a surprising connection between total positivity and canonical bases in quantum groups. This led to his introduction of the totally positive and totally nonnegative parts $G_{>0}, G_{\geq 0}$ in every real reductive group. Similarly, Lusztig introduced the totally positive and totally nonnegative parts of any generalized partial flag variety G/P. He also conjectured, and Rietsch proved [12], that there is a natural cell decomposition of $(G/P)_{\geq 0}$; cells are indexed by pairs of Weyl group elements.

In the late 1990's, Fomin and Zelevinsky [2] further developed Lusztig's theory of total positivity in G and attempted to get a concrete understanding of Lusztig's dual canonical basis. Their efforts led to the introduction of cluster algebras in 2002 [3], which has since then spawned a tremendous amount of activity.

Postnikov [10] studied in detail the totally non-negative part of the Grassmannian: he gave an explicit cell decomposition (which coincides with Rietsch's) and introduced various new classes of combinatorial objects which are in one-to-one correspondence with the cells. One of these classes was the so-called Ⅎ-diagrams (now also known as *Cauchon diagrams*), which were independently and simultaneously introduced by Cauchon [1] in the context of H-primes in quantum matrices. See also the paper of Launois, Lenagan, and Rigal [5], for the generalization of Cauchon's work to prime ideals in the quantum Grassmannian.

2. Enumeration of totally positive Grassmann cells

The *totally nonnegative part* of the Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$ is defined to be the quotient $Gr^+_{k,n} = \mathrm{GL}^+_k \setminus \mathrm{Mat}^+(k,n)$, where $\mathrm{Mat}^+(k,n)$ is the space of real $k \times n$-matrices of rank k with nonnegative maximal minors and GL^+_k is the group of real matrices with positive determinant. By partitioning the space into pieces based on which maximal minors are strictly positive and which are zero, one obtains a cell decomposition of $Gr^+_{k,n}$ [10]. We refer to the cells in this decomposition as *totally positive cells.*

Fix k and n. Then a ⅃-*diagram* $(\lambda, D)_{k,n}$ is a partition λ contained in a $k\times(n-k)$ rectangle, together with a filling $D : Y_\lambda \to \{0,1\}$ which has the ⅃-*property*: there is no 0 which has a 1 above it and a 1 to its left. (Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.) In Figure 1 we give an example of a ⅃-diagram. [1]

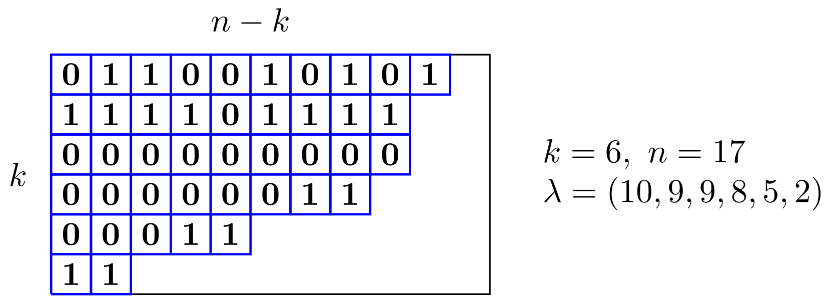

FIGURE 1. A ⅃-diagram $(\lambda, D)_{k,n}$

Postnikov proved that there is a one-to-one correspondence between ⅃-diagrams (λ, D) contained in a $k \times (n-k)$ rectangle, and totally positive cells in $Gr^+_{k,n}$, such that the dimension of a totally positive cell is equal to the number of 1's in the corresponding ⅃-diagram. Define the polynomial $A_{k,n}(q) = \sum a_r q^r$ by letting a_r be the number of cells of dimension r in the cell decomposition of $Gr^+_{k,n}$.

By counting ⅃-diagrams, we were able to find an explicit expression for $A_{k,n}(q)$. See Table 1 for the values of $A_{k,n}(q)$ for small k and n.

Theorem 2.1. [16] *The value of* $A_{k,n}$ *is equal to:*

$$q^{-k^2}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}\left(q^{ki}[k-i]^i[k-i+1]^{n-i} - q^{(k+1)i}[k-i-1]^i[k-i]^{n-i}\right)$$

$$=\sum_{i=0}^{k-1}\binom{n}{i}q^{-(k-i)^2}\left([i-k]^i[k-i+1]^{n-i} - [i-k+1]^i[k-i]^{n-i}\right).$$

3. On the topology of $(G/P)_{\geq 0}$

It has been believed for some time that the topology of $(G/P)_{\geq 0}$ should be very simple. A notion of total positivity for toric varieties also exists [4, Section 4.1], so it is natural to compare the properties of the spaces $(G/P)_{\geq 0}$ to their toric counterparts. And the non-negative part of a toric variety is homeomorphic to a closed ball – more specifically, the moment map provides a homeomorphism to its moment polytope [4, Section 4.2].

[1]The symbol ⅃ is meant to remind the reader of the shape of the forbidden pattern, and should be pronounced as [le], because of its relationship to the letter L.

$A_{1,1}(q)$	1
$A_{1,2}(q)$	$q+2$
$A_{1,3}(q)$	q^2+3q+3
$A_{1,4}(q)$	q^3+4q^2+6q+4
$A_{2,4}(q)$	$q^4+4q^3+10q^2+12q+6$
$A_{2,5}(q)$	$q^6+5q^5+15q^4+30q^3+40q^2+30q+10$
$A_{2,6}(q)$	$q^8+6q^7+21q^6+50q^5+90q^4+120q^3+110q^2+60q+15$
$A_{3,6}(q)$	$q^9+6q^8+21q^7+56q^6+114q^5+180q^4+215q^3+180q^2+90q+20$

TABLE 1. $A_{k,n}(q)$

A series of papers [13, 17, 11, 14, 15] analyzing the cell decomposition has provided a better understanding of the topology of $(G/P)_{\geq 0}$. In [13], Rietsch described the face poset of $(G/P)_{\geq 0}$, that is, the partially ordered set (poset) of closures of cells. Shortly thereafter, [17] showed that this poset is actually the poset of cells of a *regular CW complex* which is homeomorphic to a ball. Here, *regular* means that the closure of each cell is homeomorphic to a closed ball and the boundary of each cell is homemorphic to a sphere. A *CW complex* is a cell complex together with the additional data of *attaching maps*. Note that while the result of [17] is very suggestive (prompting us to conjecture that the cell decomposition of $(G/P)_{\geq 0}$ is a regular CW complex homeomorphic to a ball), it is a purely combinatorial result, and says nothing about the topology of $(G/P)_{\geq 0}$.

In subsequent work, [11] proved for the Grassmannian and [14] for an arbitrary (G/P) that the cell decomposition of $(G/P)_{\geq 0}$ is a CW complex. Using this result together with *discrete Morse theory*, joint work with Rietsch [15] proved the following theorem, answering a question of Lusztig [9].

Theorem 3.1. *The closure of each cell of $(G/P)_{\geq 0}$ is homotopy-equivalent to a closed ball (i.e. it is contractible). Furthermore, the boundary of each cell is homotopy-equivalent to a sphere.*

This proves the conjecture that the cell decomposition is regular up to homotopy equivalence. It also generalizes Lusztig's result that $(G/P)_{\geq 0}$ itself – the closure of the top-dimensional cell – is contractible.

Remark 3.2. *It would be interesting to know if any of these results on the topology of $(G/P)_{\geq 0}$ can be interpreted in terms of quantum algebras or Poisson geometry.*

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Quantum algebras (deleting derivations algorithm)

Gérard Cauchon

Denote by $\mathfrak{g}$ a complex simple Lie algebra, by Φ it's (irreducible) root system, by Π a basis of Φ, by Φ^+ the subset of positive roots, by $\mathbb{Z}\Pi$ the root lattice, and set $n = |\Pi|$, $\mathrm{N} = |\Phi^+|$ $(1 \leq n \leq N)$. $(\ ,\ )$ is the unique scalar product on the real vector space V generated by Φ, such that $(\beta, \beta) = 2$ for all short roots β in Φ. W is the Weyl group of Φ and, for any β in Φ, we denote by s_β the reflection with respect to β.
k denotes a (commutative) field and $q \in k^* := k \setminus \{0\}$ is not a root of unity. The k-algebra $U_q(\mathfrak{g})$ and it's canonical generators E_α, F_α, $K_\alpha^{\pm 1}$ $(\alpha \in \Pi)$ are defined as in [3]. We denote by U^+ the subalgebra generated by the E_α $(\alpha \in \Pi)$. For any $\rho = \sum_{\alpha \in \Pi} m_\alpha \alpha$ $(m_\alpha \in \mathbb{Z})$ in the root lattice $\mathbb{Z}\Pi$, we set $h_\rho = \prod_{\alpha \in \Pi} K_\alpha^{m_\alpha}$.
The multiplicative group $H = \{h_\rho \mid \rho \in \mathbb{Z}\Pi\}$ is called the Torus (of $U_q(\mathfrak{g})$). It acts on the algebra $U_q(\mathfrak{g})$ by

$$h_\rho.u = h_\rho^{-1} u h_\rho \quad (\forall u \in U_q(\mathfrak{g}))$$

Consider any $w \in W$, set $t = l(w)$ and consider a reduced expression

$$w = s_{\alpha_1} \circ \ ... \ \circ s_{\alpha_t} \qquad (\alpha_i \in \Pi \ \ for \ \ 1 \leq i \leq t) \tag{1}$$

It is well known that $\beta_1 = \alpha_1$, $\beta_2 = s_{\alpha_1}(\alpha_2)$, ... , $\beta_t = s_{\alpha_1} \dots s_{\alpha_{t-1}}(\alpha_t)$ are distinct positive roots and that the set $\{\beta_1, \ ..., \ \beta_t\}$ does not depend on the reduced expression (1) of w. For any $\alpha \in \Pi$, define the braid automorphism T_α of the algebra $U_q(\mathfrak{g})$ as in ([3], p. 153), set

$$X_1 = E_{\alpha_1},\ X_2 = T_{\alpha_1}(E_{\alpha_2}),\ \dots\ ,\ X_t = T_{\alpha_1}\ \dots\ T_{\alpha_{t-1}}(E_{\alpha_t})$$

and denote by $R = \mathcal{U}_q^+(w)$ the algebra generated by $X_1, \dots, X_t$.
We know ([3], chapter 8) that $X_1, \dots, X_t$ are all in U^+, so that $R \subset U^+$. Each X_i $(1 \le i \le t)$ is homogeneous of degree β_i (ie. $h_\rho(X_i) = q^{-(\rho,\beta_i)}X_i$ for any $\rho \in \mathbb{Z}\Pi$), so that the torus H acts on R . The algebra $R = \mathcal{U}_q^+(w)$ does not depend on the reduced expression (1) of w. It is a noetherian domain and the ordered monomials $X^{\underline{a}} := X_1^{a_1} \dots X_t^{a_t}$, $\underline{a} = (a_1, \dots, a_1) \in \mathbb{N}^{\,t}$, are a PBW basis of R . If $1 \le i < j \le t$, denote by $[X_j, X_i]_q := X_jX_i - q^{-(\beta_i,\beta_j)}X_iX_j$ the q-bracket of X_j and X_i and recall the following straightening formula due to Levendorskii and Soibelman:

$$(2) \qquad [X_j, X_i]_q = P_{j,i} = \sum_{\underline{a} = (a_{i+1},\dots,a_{j-1})} c_{\underline{a}} X_{\beta_{i+1}}^{a_{i+1}} \dots X_{\beta_{j-1}}^{a_{j-1}},$$

with $\underline{a} \in \mathbb{N}^{j-i-1}$, $c_{\underline{a}} \in k$, and $c_{\underline{a}} \neq 0$ for only finitely many $\underline{a}$.

Denote by F the division ring of fractions of R. The deleting derivations algorithm is a recursive process (see [1]) which constructs, for $m = t+1, t, \dots, 3, 2$, a sequence of new variables $(X_1^{(m)}, \dots, X_t^{(m)})$ in F such that each algebra $R^{(m)} := k < X_1^{(m)}, \dots, X_t^{(m)} >$ satisfies the following properties: It is a noetherian domain and the ordered monomials $(X^{(m)})^{\underline{a}}$, $\underline{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, are a PBW basis of $R^{(m)}$. If $1 \le i < j \le t$, we have the following simplifed straightening formula: $[X_j^{(m)}, X_i^{(m)}]_q = P_{j,i}^{(m)}$ with

$$P_{j,i}^{(m)} = 0 \quad if \ \ m \le j,$$

$$P_{j,i}^{(m)} = \sum_{\underline{a} = (a_{i+1},\dots,a_{j-1})} c_{\underline{a}} (X_{i+1}^{(m)})^{a_{i+1}} \dots (X_{j-1}^{(m)})^{a_{j-1}} \quad if \ \ j < m,$$

where the coefficients $c_{\underline{a}}$ are the same as in (2).
At the first step, we have $(X_1^{(t+1)}, \dots, X_t^{(t+1)}) = (X_1, \dots, X_t)$, so that $R^{(t+1)} = R$. At he last step, all the polynomials $P_{j,i}^{(2)}$ are zero, so that $R^{(2)}$ is a quantum dimension t affine space. It turns out that, for each $m \in [\![2, \dots, t]\!]$, there exists an (explicit) embedding $\phi^{(m)} : Spec(R^{(m+1)}) \to Spec(R^{(m)})$ (see [1]), so that $\phi := \phi^{(2)} \circ \dots \circ \phi^{(t)} : Spec(R^{(t+1)}) = Spec(R) \to Spec(R^{(2)})$ is also an embedding. A subset Δ of $[\![1, \dots, t]\!]$ is called an *admissible diagram* (or a Cauchon diagram) if there exists $\mathcal{P} \in Spec(R)$ such that

$$(3) \qquad \phi(\mathcal{P}) \cap \{X_1^{(2)}, \dots, X_t^{(2)}\} = \{X_i^{(2)} \mid i \in \Delta\}.$$

If we denote by $Spec_\Delta(R)$ the set of all prime ideals $\mathcal{P}$ in $Spec(R)$ which satisfy (3), we can prove [1] that the sets $Spec_\Delta(R)$ (Δ admissible diagram) coincide with the H-strata of $Spec(R)$. Moreover, each stratum $Spec_\Delta(R)$ is a torus and the shape of Δ enables to compute the dimension of this torus via an algorithmic process.

If $\mathfrak{g}$ has type A_n, we can choose $w \in W$ such that $R = \mathcal{U}_q^+(w)$ is a quantum matrices algebra $O_q(M_{l,p}(k))$ and, in this case, we can describe the admissible diagrams [2]: They coincide with the Ⅎ diagrams (also called the Postnikov diagrams). Morover, for each admissible diagram Δ, all prime ideals in $Spec_{\Delta}(R)$ contain the same quantum minors an (if q is transcendental) the shape of Δ enables to construct, via an algorithmic process, the list of those quantum minors.
In the general case, we can prove [4] that the admissible diagrams coincide with the subsets of $[\![1, \ ..., \ t]\!]$ corresponding to subexpressions of

$$w^{-1} = s_{\alpha_t} \circ \ ... \ \circ s_{\alpha_1} \quad (4)$$

which are positive in the sense of R. Marsh and K. Rietsch (ie. which have defect 0 in the sense of V. Deodhar).

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The dual canonical basis of a quantized enveloping algebra

Robert J. Marsh

Let $U = U_q(\mathbf{g})$ be the quantized enveloping algebra over the field $k = \mathbb{Q}(q)$, associated to a simply-laced semisimple Lie algebra by Drinfel'd and Jimbo. The *canonical basis* (or *global crystal basis*) $\mathbb{B}$ of U was introduced independently by Kashiwara [5] and Lusztig [4] and has very good properties. See, for example, [3] for an introduction. This talk was an introduction to the canonical basis (via Lusztig's approach) and its dual.

Let Φ be the root system of $\mathbf{g}$ with simple system $\Pi = \{\alpha_i : i \in I\}$, where $I = \{1, 2, \ldots, n\}$. Let W be the Weyl group, with Coxeter generators s_i, $i \in I$. Let $Q = \mathbb{Z}\Pi$ be the root lattice containing $Q^+ = \mathbb{N}\Pi$. Let E_i, F_i, $K_i^{\pm 1}$, $i \in I$ be the usual generators of U. Let U_+, (respectively, U_0; $U_{\geq 0}$) be the subalgebra of U generated by the E_i (respectively, the K_i; the K_i and the E_i) for $i \in I$. Let $\bar{\ }$ denote the $\mathbb{Q}$-algebra automorphism of U fixing the E_i and the F_i and inverting the K_i and q. Then U is known to be a Hopf algebra with sub-Hopf algebra $U_{\geq 0}$.

Let B denote the Artin braid group associated to $\mathbf{g}$ with generators T_i, $i \in I$ and action on $U_q(\mathbf{g})$ as in [4] (denoted $\widetilde{T}_i$ there). Let w_0 be the longest element of W (of length ν) and define:

$$\chi = \{\mathbf{i} = (i_1, i_2, \ldots, i_\nu) : s_{i_1} s_{i_2} \cdots s_{i_\nu} = w_0\},$$

(the set of reduced expressions for w_0). Let $\mathbf{i} \in \chi$. For $\mathbf{c} \in \mathbb{N}^\nu$ let

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)} T_{i_1}(E_{i_2}^{(c_2)}) \cdots T_{i_1} T_{i_2} \cdots T_{i_{\nu-1}}(E_{i_\nu}^{(c_\nu)}).$$

Let $B_{\mathbf{i}} = \{E_{\mathbf{i}}^{\mathbf{c}} : \mathbf{c} \in \mathbb{N}^\nu\}$ denote the basis of U_+ of PBW-type corresponding to $\mathbf{i}$.
Theorem (Lusztig [4])
(a) The $\mathbb{Z}[q^{-1}]$-lattice $\mathcal{L}$ spanned by $B_{\mathbf{i}}$ is independent of $\mathbf{i}$.
(b) The image B of any $B_{\mathbf{i}}$ under the natural map $\pi : \mathcal{L} \to \mathcal{L}/q^{-1}L$ is independent of $\mathbf{i}$.
(c) The restriction $\pi' := \pi_{\mathcal{L}\cap\overline{\mathcal{L}}} : \mathcal{L} \cap \overline{\mathcal{L}}$ is an isomorphism of $\mathbb{Z}$-algebras.
Set $\mathbf{B} = (\pi')^{-1}(B)$.
(d) Then $\mathbf{B}$ is a k-basis of U_+, known as the *canonical basis* of U_+. Furthermore, $\overline{b} = b$ for all $b \in \mathbf{B}$.

For example [4], if $\mathbf{g}$ has type A_1, $\mathbf{B} = \{E_1^{(a)} : a \in \mathbb{N}\}$, and if $\mathbf{g}$ has type A_2 then

$$\mathbf{B} = \{E_1^{(a)} E_2^{(b)} E_1^{(c)} : b \geq a + c\} \cup \{E_2^{(a)} E_1^{(b)} E_2^{(c)} : b \geq a + c\},$$

where $E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)}$ are the only elements which coincide.

Following [1], we can define the dual of U_+ as follows. For $\gamma \in Q^+$ let $U_+(\gamma)$ denote the subspace of U_+ consisting of elements of weight γ, and let $U_+^*(\gamma)$ denote the dual vector space. Let $U_+^* = \bigoplus_{\gamma \in Q^+} U_+^*(\gamma)$. Define $U_{\geq 0}(\gamma)$, $U_{\geq 0}^*(\gamma)$ and $U_{\geq 0}^*$ similarly. Then $U_{\geq 0}^*$ is an associative algebra using the adjoint of the comultiplication on $U_{\geq 0}$. The map $\iota : U_+^* \to U_{\geq 0}^*$ is an embedding of vector spaces whose image is a subalgebra of $U_{\geq 0}^*$. This structure is transferred back to U_+^* via ι making it into an associative algebra. This can be considered to be the quantized coordinate ring $O_q(N)$ of a maximal unipotent subgroup N of the simply connected semisimple algebraic group corresponding to $\mathbf{g}$. Using the Rosso-Tanisaki bilinear form (see e.g. [2] for the definition) it is possible to show that $O_q(N)$ is isomorphic to U_+ (first observed by Drinfel'd). The *dual canonical basis* $\mathbf{B}^*$ is the basis of $O_q(N)$ dual to the canonical basis of U_+. Let b^* denote the element of $\mathbf{B}^*$ corresponding to $b \in \mathbf{B}$.

It follows from [1] that, for $b \geq a + c$,

$$(E_1^{(a)} E_2^{(b)} E_1^{(c)})^* = q^{r(a,b)} \Delta_q(12;13)^{b-a-c} \Delta_q(1,3)^c \Delta_q(12;23)^a;$$

$$(E_2^{(a)} E_1^{(b)} E_2^{(c)})^* = q^{r(a,b)} \Delta_q(1;2)^{b-a-c} \Delta_q(12;23)^c \Delta_q(1,3)^a,$$

where $r(a,b) = \begin{pmatrix} a \\ 2 \end{pmatrix} + \begin{pmatrix} b-a \\ 2 \end{pmatrix}$ and $\Delta_q(I;J)$ denotes the quantum minor with row set I and column set J. Note that this also demonstrates the two adapted algebras in type A_2 from [2].

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Introduction to cluster algebras

Bernard Leclerc

Cluster algebras were introduced in 2001 by S. Fomin and A. Zelevinsky [1], to provide an algebraic and combinatorial setup for studying Lusztig's theories of total positivity and canonical bases.

I have given a quick introduction to the definition and basic properties of cluster algebras (Laurent phenomenon, classification of cluster algebras of finite type), and I have stated some open problems (positivity of cluster expansions, linear independence of cluster monomials).

Finally I have given some examples, including the following one. Let $\mathcal{F}$ denote the variety of partial flags

$$\mathfrak{f} = L \subset E \subset \mathbb{C}^6,$$

where L is a line, and E is a 3-dimensional subspace. Let R be the multi-homogeneous coordinate ring of $\mathcal{F}$, generated by the 6 + 20 Plücker coordinates $D_1, D_2, \ldots, D_6$ of L, and $D_{123}, D_{124}, \ldots, D_{456}$ of E. Here we think of $\mathfrak{f}$ as being given by a 3×6 matrix and the D_i are the entries on the first row, while the D_{ijk} are the 3×3 minors. These homogeneous coordinates satisfy some Plücker-type relations. In [2] we have shown that R has a cluster algebra structure, with finitely many cluster variables (R has cluster type E_6 in the classification of Fomin and Zelevinsky). Each cluster consists of 6 cluster variables, together with 7 frozen variables which belong to every cluster. There are 42 cluster variables in total. The frozen variables are

$$D_1, \quad D_6, \quad D_{123}, \quad D_{234}, \quad D_{345}, \quad D_{456}, \quad D_2D_{156} - D_1D_{256}.$$

It is interesting to note that the last one is *not* a Plücker coordinate, thus, in contrast with the case of Grassmannians, there is no cluster consisting only of Plücker coordinates.

More generally, if G is a simple algebraic group of simply-laced type, and $P \subset G$ is an arbitrary parabolic subgroup with unipotent radical N_P, we have constructed in [2] some cluster algebra structures in the coordinate rings of N_P and G/P, using an appropriate Frobenius subcategory $\mathcal{C}_P$ of $\operatorname{mod}\Lambda$, where Λ is the preprojective algebra attached to the Dynkin diagram of G.

It is a pleasure to acknowledge that the article [2] was written during a RIP stay at MFO in 2006. I am very grateful to this institution for its hospitality, its support, and for providing ideal working conditions.

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Bruhat intervals

Francesco Brenti

The purpose of this talk has been to give a survey on Bruhat intervals. After reviewing the main definitions I have described the main properties of Bruhat order including the subword property, the fact that Bruhat order is graded and Eulerian, that the order complex of any Bruhat interval is PL-homeomorphic to a sphere, that Bruhat intervals avoid $K_{2,3}$, and that to any Bruhat interval $[u, v]$ there is associated a uniquely defined regular CW-complex whose face poset is isomorphic to $[u, v]$. I have concluded by defining the Kazhdan-Lusztig and the R-polynomial of a Bruhat interval and explaining their geometric significance.

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Symplectic cores, symplectic leaves and the orbit method

Ken A. Brown

1. Stratifications of Poisson varieties.

We work over the complex numbers, denoted by k. Basing our discussion on [3], we reviewed three stratifications of the maximal ideal spectrum $\mathcal{Z}$ of an affine commutative k-algebra R admitting a Poisson bracket $\{-,-\}$, namely the *rank stratification*, the stratification by *symplectic leaves*, and the stratification by *symplectic cores*. The definitions of the first two being more familiar, we recall here only the third of these.

For $\mathfrak{m} \in \mathcal{Z}$, the *Poisson core* $\mathcal{P}(\mathfrak{m})$ is the unique biggest Poisson ideal in $\mathfrak{m}$. It is easy to check that $\mathcal{P}(\mathfrak{m})$ exists and is a prime ideal. The ideals $\mathcal{P}(\mathfrak{m})$, $\mathfrak{m} \in \mathcal{Z}$, are called the *Poisson primitive* ideals of R. Now define an equivalence relation on $\mathcal{Z}$ by: $\mathfrak{m} \sim \mathfrak{n} \Leftrightarrow \mathcal{P}(\mathfrak{m}) = \mathcal{P}(\mathfrak{n})$. Denote the $\sim$-class of $\mathfrak{m}$ by $\mathcal{C}(\mathfrak{m})$, the *symplectic core* of $\mathfrak{m}$. One shows (see [3]) that each $\mathcal{C}(\mathfrak{m})$ consists of smooth points of $\overline{\mathcal{C}(\mathfrak{m})}$, and that the rank is constant across $\mathcal{C}(\mathfrak{m})$, so that the stratification of $\mathcal{Z}$ by cores refines the rank stratification. We ask:

Question 1. *Are symplectic cores locally closed?*

This is closely related to another open question: namely, whether the so-called Dixmier-Moeglin equivalence holds for Poisson primitive ideals.

Let us call the Poisson bracket *algebraic* if the symplectic leaves $\mathcal{L}(\mathfrak{m})$ are locally closed for all $\mathfrak{m} \in \mathcal{Z}$. For $\mathfrak{m} \in \mathcal{Z}$,

$$\mathcal{L}(\mathfrak{m}) \quad \subseteq \quad \mathcal{C}(\mathfrak{m});$$

in general this inclusion is strict; however equality holds when the leaves are algebraic. It follows that in this case the above question has a positive answer, and the Dixmier-Moegiln equivalence for Poisson primitives holds. It is therefore of interest to find criteria sufficient to ensure that the leaves are algebraic; this is the case when they are finite in number, and also for the coordinate rings of semisimple groups and of $m \times n$ matrices equipped in each case with the *standard bracket*. But no general criterion seems to be known.

2. The orbit method: solvable Lie algebras.

Recall the famous success, due to many authors, of the orbit method for enveloping algebras of finite dimensional solvable k-Lie algebras $\mathfrak{g}$: the map

$$\text{Dix} : \mathfrak{g}^*/G \longrightarrow \text{Primspec}(U(\mathfrak{g})$$

is a homeomorphism. Here, Primspec($U(\mathfrak{g})$) denotes the space of primitive ideals of the enveloping algebra $U(\mathfrak{g})$, and G is the adjoint *algebraic* group for $\mathfrak{g}$. Thus the domain of Dix consists of the symplectic leaves of $\mathfrak{g}^*$ only if $\mathfrak{g}$ is an *algebraic* Lie algebra. As Goodearl observed in [4], this "fudge" to permit the orbit method to work can be repaired if one takes the domain of Dix to be the space of symplectic cores of $\mathfrak{g}^*$, rather than the leaves. That is, there is evidence to suggest that the cores may constitute the best "algebraic" approximation to the leaves in situations where the leaves lie outwith the "algebbraic category".

3. The orbit method: quantised function algebras.

In a series of papers in the 1990s, Hodges, Joseph, Levasseur and Toro established a bijection (preserving structural information) between the leaves of a semisimple group G and the primitive ideals of its (generic) quantised function algebra; see [5] and references there. Similar correspondences for $m \times n$ matrices follow from more recent work in [2]. However, in all these cases the existence of a homeomorphism remains beyond reach at present. We ended the talk by speculating about possible ways to imitate the methods used in the solvable Lie algebras setting in defining a Dixmier map for quantised function algebras.

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Links between prime spectra of quantum algebras and the geometry of their "totally positive" counterpart : some significant examples.

Laurent Rigal

(joint work with Stéphane Launois, Tom Lenagan)

Let $\Bbbk$ be a field, $q \in \Bbbk \setminus \{0\}$, and assume q is not a root of unity. Fix integers $0 < m \leq n$. The quantum deformation of the coordinate ring on the space of $m \times n$ matrices with entries in $\Bbbk$, denoted $\mathcal{O}_q(M_{m,n})$, is generated by X_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ subject to certain relations (see [4] for details) depending on the *deformation parameter* q. To $1 \leq t \leq m$ and two sets $I = \{i_1 < \cdots < i_t\} \subseteq \{1, \ldots, m\}$ and $J = \{j_1 < \cdots < j_t\} \subseteq \{1, \ldots, n\}$ we associate the $t \times t$ *quantum minor* in $\mathcal{O}_q(M_{m,n})$:

$$[I|J] = \sum_{\sigma \in \mathfrak{S}_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)} j_1} \ldots X_{i_{\sigma(t)} j_t}.$$

We let $\Pi_{m,n}$ denote the set of $m \times m$ quantum minors and define the quantum deformation of the coordinate ring on the grassmannian of m-dimensional subspaces of $\Bbbk^n$, denoted $\mathcal{O}_q(G_{m,n})$, to be the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by $\Pi_{m,n}$. These algebras are non-commutative (or quantum) deformations of coordinate rings on the affine space $M_{m,n}$ and on the grassmannian $G_{m,n}$ in the sense that they specialize to their usual coordinate ring when $q = 1$. They are fundamental objects in the theory of *quantum groups*.

In algebraic Lie theory, a classical problem for non-commutative deformations of classical algebras is the study of their primitive ideals (seen as a first approximation to their representations).

For quantum algebras which are deformations of polynomial rings (*i.e.* iterated Ore extensions of quantum type), Goodearl and Letzter (see [1]) have proposed the following general approach. Consider a convenient torus action on such an algebra A. One can partition the prime spectrum, $\mathrm{Spec}(A)$, of A into so called *strata* which are naturally indexed by the H-prime spectrum, $H - \mathrm{Spec}(A)$, of A (*i.e.* the set of H-invariant prime ideals of A). A significant interest of this stratification is that, in many cases, it locates primitive ideals as being those prime ideals which are maximal in their own stratum. In this approach, a main goal is thus the classification of $H - \mathrm{Spec}(A)$. For the forementioned class of algebras, Cauchon has developed a very efficient method to reach this goal : the *deleting derivations* method (see [2], [3]).

However, these two main theories do not apply to non-commutive deformations of quotient varieties, such as $G_{m,n}$ for instance. The original motivation of [4] is to extend these approaches to the quotient variety $G_{m,n}$. More precisely, put

$A = \mathcal{O}_q(G_{m,n})$. This paper addresses the following problem: classify $H-\mathrm{Spec}(A)$, where $H = (\mathbb{k}^*)^n$ acts on A in a natural way. The main result of [4] is a natural parameterisation of $H - \mathrm{Spec}(A)$ by means of combinatorial objects called *Cauchon diagrams.* It can be expressed as follows.

Theorem. *There is a natural one-to-one correspondance between $H - \mathrm{Spec}(\mathcal{O}_q(G_{m,n}))$ and the set of Cauchon diagrams fitting in a $m \times (n-m)$ rectangular Young diagram.*

Here, a Cauchon diagram is a Young diagram with at most m rows and $n-m$ columns each box of which is coloured, either black or white, with the following rule: if a box is black, then either all the boxes above it or all the boxes to its left must also be black.

Though the non-commutative algebra A is not associated with any actual geometric space (unless $q = 1$), its algebraic structure keeps track of the geometry of $G_{m,n}$. This is the main idea of the proof. More precisely, we proceed in two steps. First, using the notion of *quantum algebra with a straightening law* as definied in [5], we are able to partition $H - \mathrm{Spec}(A)$ into finitely many pieces S_γ indexed by the maximal quantum minors $\gamma \in \Pi_{m,n}$. In addition, using a non-commutative process of dehomogenisation, we are able to associate to each $\gamma \in \Pi_{m,n}$ an algebra A_γ, which we call the quantum Schubert cell associated to γ (on which H still acts). It turns out that $H - \mathrm{Spec}(A_\gamma)$ is homeomorphic to S_γ. Moreover, A_γ is a quantum polynomial algebra and the Goodearl-Letzter and Cauchon theories apply to it allowing us to provide a parameterisation of $H - \mathrm{Spec}(A_\gamma)$. Glueing together the parameterisations obtained at the local level of each S_γ, we end up with a global description of the set of $H-\mathrm{Spec}(A)$, as desired. It should be stressed at this point that the name *quantum Schubert cells* is justified by the fact that the above partition is reminiscent of the classical decomposition of $G_{m,n}$ by means of Schubert cells.

Actually, the interest of the above theorem goes beyond its original motivation. Indeed, it is strongly linked to recent results in total positivity. More precisely, as shown by Postnikov in [6], the totally nonnegative grassmannian has a decomposition into *totally nonnegative cells* which is completely parallel to our above cell decomposition for $H - \mathrm{Spec}(A)$. More explicitly, Postnikov exhibits a natural parameterisation of the set of totally nonnegative cells in the nonnegative grassmannian by means of Cauchon diagrams (which he calls Le-diagrams).

The strong parallel thus established between the "quantum" picture and the geometry of the "totally nonnegative" world seems deep and significant (and actually extends to other instances). In some sense, the (algebraic) quantum world seems to describe the geometry of the nonnegative one. The more systematic study of this relationship is a very motivating perspective for future work.

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On total positivity in flag varieties

Konstanze Rietsch

Classically, a matrix is called totally positive if all of its entries and all of its minors are totally positive. The definition of total positivity for matrices can be found in work of Schoenberg from the 1930's. This talk was about a modern version of the theory, introduced by Lusztig in the 1990's [3]. Lusztig's theory defines a so-called "totally positive" subset inside any complex reductive algebraic group G and also inside its compact homogeneous spaces, that is *flag varieties* G/B^+ and their further quotients. [In the case of GL_n, the group B^+ may be taken to be the upper-triangular matrices, and the further quotients include, by considering the maximal parabolic subgroups P_k, the important special case of the Grassmannians $Gr_k(n) = GL_n/P_k$.]

Suppose e_i, f_i, h_i are usual Chevalley generators for the Lie algebra $\mathfrak{g}$ of G (we now assume G is semi-simple of rank n). Let U^- be the subgroup of G generated by its simple root subgroups $y_i(a) = \exp(af_i)$, where $a \in \mathbb{C}$. The totally nonnegative part $U^-_{\geq 0}$ was defined by Lusztig to be the semigroup generated by the $y_i(a)$, for $a \geq 0$. If G is of simply laced type and $\mathbb{B}$ is the basis of the coordinate ring $\mathbb{C}[U^-]$ obtained as the classical limit of the dual canonical basis, then Lusztig showed that $u \in U^-_{\geq 0}$ if and only if $b(u) \geq 0$ for all $b \in \mathbb{B}$. An analogous statement for the totally positive part $G_{>0}$ of the whole group says that a totally positive g acts by matrices with strictly positive entries in *any* irreducible representation $V(\lambda)$ of G with respect to its canonical basis $\mathbb{B}_\lambda$.

In another aspect of this theory Lusztig showed that $U^-_{\geq 0}$ has a decomposition into semi-algebraic cells $U^-_{>0}(w)$ indexed by the Weyl group $W =< s_1, \dots, s_n >$. Namely, given a reduced expression $w = s_{i_1} \dots s_{i_m}$, one sets

$$U^-_{>0}(w) := \{y_{i_1}(a_1) \dots y_{i_m}(a_m) \mid a_j \in \mathbb{R}_{\geq 0}\}.$$

As Lusztig showed, this set is independent of the reduced expression for w. In fact, $U^-_{>0}(w) = U^-_{\geq 0} \cap B^+ \dot{w} B^+$, so that this cell decomposition is induced from the Bruhat decomposition of G.

Note that the flag variety G/B^+ naturally contains a copy of U^-, namely as the "big cell" in its (opposite) Bruhat decomposition – that is, its decomposition into B^--orbits. We reviewed the extension of Lusztig's cell decomposition of $U^-_{\geq 0}$ to its closure in G/B^+ [4, 6], especially with emphasis on the parameterizations of the new cells, obtained jointly with R. Marsh in [4]. These are similar to Lusztig's parameterizations of the $U^-_{>0}(w)$ but involve also factors $\dot{s}_i$ coming from simple reflections. Where these simple reflection factors are placed is governed by combinatorics going back to Deodhar [1] of 'positive subexpressions' (in the terminology of [4]) .

Joint with L. Williams are more recent results on the topology of these cell decompositions, including that $(G/P)_{\geq 0}$ is a CW complex [9] , and that the closures of cells are homotopy equivalent to balls and their boundaries homotopy equivalent to spheres [10].

Finally it was mentioned that a somewhat parallel theory exists, at least in type A, if we replace U^- by a subgroup $X = U^-_f$, the stabilizer in U^- of a standard principal nilpotent f. Explicitly,

$$X = \left\{ x \in U^- \mid x = \begin{pmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1 & 1 & & & \\ a_3 & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & a_1 & 1 & \\ a_{n-1} & \dots & a_3 & a_2 & a_1 & 1 \end{pmatrix} \right\}.$$

The totally nonnegative part $X_{\geq 0}$ of X also has a cell decomposition, in fact it is homeomorphic to $\mathbb{R}^n_{\geq 0}$ [7]. And this cell decomposition extends to the closure of $X_{\geq 0}$ in the flag variety [8] – giving the totally nonnegative part of the *Peterson variety*. Considering the coordinate ring in this case one has $\mathbb{C}[X] = H_*(\Omega SU_n, \mathbb{C})$, see [2, 5], and the role of the dual canonical basis in $\mathbb{C}[U^-]$ within Lusztig's theory of total positivity is taken over by the Schubert basis (see [7] together with [2, 5]).

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Cauchon diagrams for quantised enveloping nilpotent algebras

Antoine Mériaux
(joint work with Gérard Cauchon)

Cauchon diagrams were introduced by G. Cauchon as "admissible diagrams" in [1] and permit a combinatorial description of the H-stratification of K. R. Goodearl and E. S. Letzter ([4]) of some quantum algebras. These diagrams can be obtained using the so-called deleting derivation algorithm and were explicitly computed for the algebra of quantum matrices by G. Cauchon in [2]. They have been used to describe the H-prime spectrum of the quantum grassmannian in [5]. We describe here the results obtained in [3] and [8] regarding Cauchon diagrams of $\mathcal{U}_q^+(\mathfrak{g})$ and more generally of $\mathcal{U}_q^+(w)$.

Cauchon diagrams for $\mathcal{U}_q^+(\mathfrak{g})$. Let $\mathfrak{g}$ be a simple Lie algebra of rank n. The algebra $R := \mathcal{U}_q^+(\mathfrak{g})$ is generated by $(E_i)_{1\leq i\leq n}$ subject to quantum Serre relations. We denote by $\Pi := \{\alpha_1, \alpha_2, ..., \alpha_n\}$ a chosen basis of Φ. Such a choice of a basis for Φ induced a decomposition into positive and negative roots ($\Phi = \Phi^+ \sqcup \Phi^-$). Each reduced decomposition $w_0 := s_{i_1}...s_{i_N}$ of the longest longest weyl word of W induces an order on Φ^+:

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), ..., \beta_N = s_{i_1}...s_{i_{N-1}}(\alpha_{i_N}).$$

We construct then the canonical generators of $\mathcal{U}_q^+(\mathfrak{g})$, $X_t := T_{i_1}...T_{i_{t-1}}(E_{i_t})$ (the T_i's are the Lusztig automorphisms of $U_q(\mathfrak{g})$). The ordered monomials in the canonical generators $\{X_1...X_t \mid i_1 \leq ... \leq i_t, t \in \mathbb{N}\}$ form a basis of $\mathcal{U}_q^+(\mathfrak{g})$ (the so-called PBW basis).

The deleting derivation algorithm can be applied on the iterated Ore extension $\mathcal{U}_q^+(\mathfrak{g})$; this algorithm is closely related to the commutation relations between the canonical generators X_i and these relations are not known in general. Lusztig introduced conditions on the reduced decomposition of w_0, more precisely on the order induced on Φ^+ by this decomposition, that permits to obtain the commutation relation between two generators X_i and X_j when β_i and β_j span a so-called "admissible plane". Those commutations permit to determine the implications that Cauchon diagrams must obey. In [8], we prove that conversely Cauchon diagrams are exactly the subsets of $[\![1, N]\!]$ (or equivalently subsets of Φ^+) which satisfy the implications coming from the admissible planes.

In [8], we compute explicitly the implications in every type for a particular reduced decomposition of w_0. We then show that the cardinality of the set of Cauchon diagrams (which is in one to one correspondence with the set of prime ideals invariant under the action of the torus H) is equal to the cardinality of W.

Cauchon diagrams for $\mathcal{U}_q^+(w)$. For a given $w \in W$, a reduced decomposition $w := s_{i_1}...s_{i_t}$ induces an order on a subset of Φ^+:

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), ..., \beta_t = s_{i_1}...s_{i_{t-1}}(\alpha_{i_t}).$$

Set $X_l := T_{i_1}...T_{i_{l-1}}(E_{i_l})$ for $1 \leq l \leq t$. $\mathcal{U}_q^+(w)$ is the subalgebra of $\mathcal{U}_q^+(\mathfrak{g})$ generated by the X_l's (this subalgebra does not depend on the chosen reduced decomposition). One can show that the ordered monomials on the X_l form a basis of $\mathcal{U}_q^+(w)$ as a vector space. A Cauchon diagram in this setting is a subset $\{j_1, j_2, ..., j_r\}$ of $[\![1, t]\!]$, and we show in [3], using the theory of positive subexpressions of [7], that the following map is a bijection :

$$\begin{array}{rcl} \{ \text{ Cauchon diagrams } \} & \rightarrow & w^{\Delta} = \{u \in W | u \leq w\} \\ \Delta = \{j_1, j_2, ..., j_r\} & \mapsto & s_{i_{j_1}} \circ s_{i_{j_2}} \circ ... \circ s_{i_{j_r}}. \end{array}$$

As Cauchon diagrams are in bijection with H-primes ideals of $\mathcal{U}_q^+(w)$, we obtaine, by composing the two bijections, a one-to-one correspondence between H-primes ideals of $\mathcal{U}_q^+(w)$ and $\{u \in W \mid u \leq w\}$.
In type A_n, when $n = m + p$, $w = (s_p \circ s_{p-1} \circ ... \circ s_1) \circ (s_{p+1} \circ ... \circ s_2) \circ ... \circ (s_{p+m} \circ ... \circ s_{m+1})$, we have $\mathcal{U}_q^+(w) = O_q(\mathcal{M}_{p,m})$. As we know from [2] that, in this case, Cauchon diagrams are unions of truncated columns and truncated rows (ℷ-diagrams), the bijection $\Delta \leftrightarrow w^{\Delta}$ is a generalisation of a Postinikov's result with ℷ-diagrams in [9].

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Total positivity for loop groups

Thomas Lam

In my talk I discussed joint work with Pavlo Pylyavskyy developing a theory of total positivity for the loop groups $GL_n(\mathbb{R}((t)))$ and $GL_n(\mathbb{R}[t,t^{-1}])$. Our work is motivated by the theory of total positivity for the nonsingular $n \times n$ matrices $GL_n(\mathbb{R})$ and the theory of totally positive functions.

A matrix $X \in GL_n(\mathbb{R})$ is *totally nonnegative* if every minor of it is nonnegative. A classical theorem of Loewner and Whitney states that the totally nonnegative nonsingular matrices $GL_n(\mathbb{R})_{\geq 0}$ is exactly the semigroup generated by the diagonal matrices with positive entries, together with the Chevalley generators $e_i(a)$ and $f_i(a)$ with nonnegative parameters $a \geq 0$.

A totally positive function is a formal power series $a(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ such that the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally nonnegative. The Edrei-Thoma theorem states that every totally positive function is of the form

$$a(t) = e^{\gamma t} \prod_{i=1}^{\infty} (1+\alpha_i t)/(1-\beta_i t)$$

where $\gamma + \sum_i \alpha_i + \beta_i < \infty$ and all the parameters are nonnegative. Furthermore, the characters of the infinite symmetric group S_∞ are classified by normalized totally positive functions.

Total positivity for the formal loop group $GL_n(\mathbb{R}((t)))$ is a common generalization of these two notions of total positivity. In my talk I discussed generalizations of the Edrei-Thoma theorem to the loop group setting, and in addition I discussed infinite products

$$e_{i_1}(a_1)\, e_{i_2}(a_2)\, e_{i_3}(a_3) \cdots$$

of Chevalley generators. These infinite products lead to an interesting partial order on the maximal chains in the weak order of the affine symmetric group.

Weak splittings of surjective Poisson submersions

Milen Yakimov

Assume that $p\colon (M,\Pi) \to (N,\pi)$ is a surjective Poisson submersion of Poisson manifolds. In many cases we understand the geometry of (M,Π) and would like to to study (N,π). Very rarely we could find a Poisson section of p. We do this in a weak sense based on the following notion, see [1, 5].

Definition A submanifold X of a Poisson manifold (M,Π) is called a Poisson–Dirac submanifold admitting a Dirac projection if there exits a subbundle $E \subset T_X M$ such that $E \oplus TX = T_X M$ and $\Pi \in \Gamma(X, \wedge^2 TX \oplus \wedge^2 E)$.

In this setting the projection of $\Pi|_X \in \Gamma(X, \wedge^2 T_X M)$ onto $\Gamma(X, \wedge^2 TX)$ is a Poisson structure. Its symplectic leaves of are the connected components of the intersections of symplectic leaves of (M,π) with X.

Definition Assume that (M,Π) and (N,π) are Poisson manifolds, X is a Poisson submanifold of (N,π), and that $p\colon (M,\Pi) \to (N,\pi)$ is a surjective Poisson submersion. A *weak section* of p over X is a smooth map $i\colon X \to M$ such that $p \circ i = \mathrm{Id}_X$ and $i(X)$ is a Poisson–Dirac submanifold of (M,Π) (admitting a Dirac projection) with induced Poisson structure $i_*(\pi|_X)$.

Fix a Manin triple $(\mathfrak{d}, \mathfrak{g}_+, \mathfrak{g}_-)$ where $\mathfrak{d}$ is a quadratic Lie algebra with (a fixed) nondegenerate invariant symmetric bilinear form $\langle .,. \rangle$ and $\mathfrak{g}_\pm$ are two Lagrangian subalgebras such that $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces. They induce the r-matrix $r = \frac{1}{2} \sum \xi_j \wedge x_j$ (where $\{\xi_j\}$ and $\{x_j\}$ are dual bases of $\mathfrak{g}_-$ and $\mathfrak{g}_+$) and the Poisson structure $\pi' = R(r) + L(r)$ on D called Heisenberg double structure. ($R(.)$ and $L(.)$ refer to the right and left invariant vector fields on D as usual.) Let $N_\pm$ and $\mathfrak{n}_\pm$ denote the normalizers $\mathfrak{g}_\pm$ in D and $\mathfrak{g}$, respectively. Set $H = N(\mathfrak{g}_+) \cap N(\mathfrak{g}_-)$. The Poisson structure π' can be pushed down to Poisson structures on π_H on D/H and π_{N_+} on $D/N(\mathfrak{g}_+)$. The latter is important to study in relation to the varieties of Lagrangian subalgebras [2]. We have the canonical surjective Poisson submersion $\eta\colon (D/H, \pi_H) \to (D/N_+, \pi_{N_+})$.

Theorem 0.1. *Assume that for a given $d \in D$ such that $dHd^{-1} \subset N_-$ there exists a subgroup Q of D satisfying the conditions*

$$\mathfrak{n}_- = \mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{n}_+) + \mathfrak{n}_- \cap \mathrm{Ad}_d(\mathfrak{q}) \quad \text{and} \tag{1}$$

$$Q \cap N_+ = H, \ \mathfrak{n}_+ + \mathfrak{q} = \mathfrak{n}_+ + \mathfrak{q}^\perp + \mathrm{Ad}_d^{-1}(\mathfrak{n}_-) = \mathfrak{g}. \tag{2}$$

Then the smooth map $i\colon G_d \cdot dN_+ \to D/H$ defined by $i(gdN_+) = gdH$ for $g \in G_d := N_- \cap dQd^{-1}$ is a weak section of the surjective Poisson submersion $\eta\colon (D/H, \pi_H) \to (D/N_+, \pi_{N_+})$ over $G_d \cdot dN_+$.

This is a far reaching generalization of the construction of [3] for flag varieties. It is applicable in a many different cases, including double flag varieties [4].

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Geometric construction of cluster categories

Karin Baur
(joint work with Robert Marsh)

Cluster categories provide a categorification of the theory of cluster algebras. They have been introduced a few years ago independently by two sets of authors.

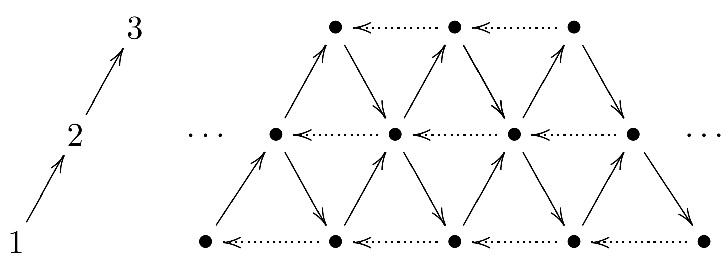
(i) Buan, Marsh, Reineke, Reiten and Todorov have defined cluster categories for Dynkin types A, D, E[1] using representation theory of finite dimensional algebras, cf. [BMRRT06].

(ii) Caldero, Chapoton and Schiffler have constructed cluster categories of Dynkin type A using a category of diagonals in a polygon. Schiffler has later given a model for type D by using arcs in a punctured polygon, cf. [CCS06] and [S06].

Approach of Buan et al. Let Q be a simply-laced Dynkin quiver, kQ be the path algebra of Q. Let $\mathcal{D} = D^b(kQ)$ be the bounded derived category of finitely generated kQ-modules. By a result of Happel, $\mathcal{D}$ is triangulated ([H88]). We denote the shift functor by [1]. In order to study the category $\mathcal{D}$, it is very useful to consider its Auslander-Reiten quiver AR($\mathcal{D}$) as it carries a lot of information about $\mathcal{D}$. It is defined as follows: The vertices of AR($\mathcal{D}$) are the isomorphism classes of indecomposable objects of $\mathcal{D}$; there are k arrows from X to Y (X, Y two representatives) if the space of irreducible maps between them has dimension k. $\mathcal{D}$ and thus AR($\mathcal{D}$) are equipped with the Auslander-Reiten translate τ (for details, see [ASS06]). Unless specified otherwise we will restrict to the case A_n. The reader should keep in mind that the results from the $\mathcal{D}$ point of view hold in the general set-up.

It turns out that AR($\mathcal{D}$) is described by the quiver $\mathbb{Z}Q$, which is formed by a $\mathbb{Z}$-strip of copies of Q. The arrows of $\mathbb{Z}Q$ are $(n, i) \to (n, j)$ and $(n, j) \to (n+1, i)$ for every arrow $i \to j$ in Q. In addition, there is a map $\tau : Q \to Q$, $(n, i) \mapsto (n-1, i)$ on $\mathbb{Z}Q$. The pair ($\mathbb{Z}Q$, τ) is a stable translation quiver (as defined by Riedtmann, cf. [R90]).

As an example take Q to be the quiver $1 \to 2 \to 3$. Then $\mathbb{Z}Q$ is the infinite quiver to the right, a copy of Q is to the left to indicate how the copies of Q are lined up. The dotted lines show τ.

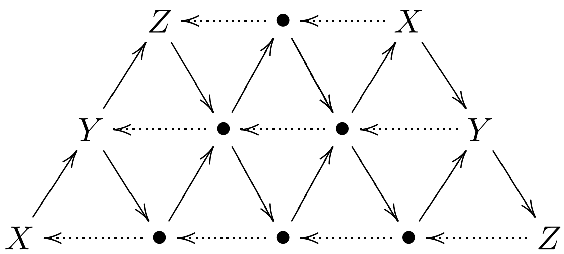


[1]their set-up works more generally for hereditary algebras.

Happel has proved ([H88]) that $\mathbb{Z}Q$ is $\mathrm{AR}(\mathcal{D})$. In particular, it is independent of the orientation of Q.

Using the Auslander-Reiten translate τ on $\mathcal{D}$ and [1] we define $F = \tau^{-1}[1]$. Then the cluster category $\mathcal{C}$ of type A_n is the orbit category $\mathcal{D}/F$. Its objects are the F-orbits of objects in $\mathcal{D}$ and its morphisms are $\mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(F^i X, Y)$ ($\tilde{X}$ in the orbit of X, $\tilde{Y}$ in the orbit of Y). The sum is non-zero for only finitely many i.

The Auslander-Reiten quiver $\mathrm{AR}(\mathcal{C})$ is a finite region consisting of one triangle of vertices and an extra copy of Q to the right of it:



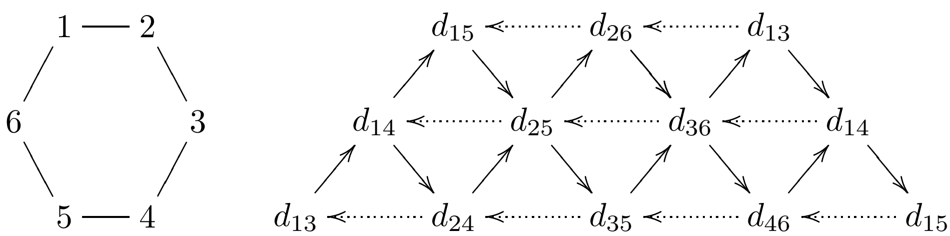
The vertices X, Y and Z are repeated to show how the quiver wraps around. The region indicated by filled dots reminds us of the module category kQ-mod whose Auslander-Reiten quiver has such a triangular shape (but τ is not the same!).

The cluster category has very nice properties: it is Krull-Schmidt ([BMRRT06]), triangulated and Calabi-Yau of dimension 2 ([K05]).

Cluster categories are an example of a categorification, as the following result of Buan et al. shows: There is a bijection between the cluster variables of type A_n and the isomorphism classes of indecomposable objects of $\mathcal{C}$. Under this correspondence, a cluster corresponds to a tilting object for $\mathcal{C}$ ([BMRRT06]).

Approach of Caldero et al. In [CCS06], Caldero et al. have shown that the Auslander-Reiten quiver $\mathrm{AR}(\mathcal{C}_{\mathrm{A}_{n-1}})$ is the quiver $\Gamma(n)$ of diagonals in an $n+2$-gon: The cluster category can be constructed by diagonals in a polygon with $n+2$ vertices; they give rise to a translation quiver which is isomorphic to $\mathrm{AR}(\mathcal{C}_{\mathrm{A}_{n-1}})$. This quiver is defined as follows. Let Π be an $n+2$-gon and label its vertices clock-wise by $1, 2, \ldots, n+2$. The vertices of $\Gamma(n)$ are the diagonals d_{ij}, $|i-j| > 2$, (counting modulo $n+2$). There are arrows $d_{ij} \to d_{i,j+1}$ and $d_{ij} \to d_{i+1,j}$ whenever the image is also a diagonal. Furthermore, $\tau : d_{ij} \to d_{i-1,j-1}$.

As an example, for $\mathcal{C}_{\mathrm{A}_3}$, we consider a hexagon and draw $\Gamma(4,1)$:



It clearly is the same quiver as AR($\mathcal{C}$) for $\mathcal{C}$ of type A_3. Let us note that a cluster corresponds to a triangulation of the polygon: one can show that tilting objects correspond to maximal collections of non-crossing diagonals.

m-cluster categories and diagonals. Keller has has defined the m-cluster category $\mathcal{C}^m$ of type A_n as the orbit category $\mathcal{D}/F_m$ for $F_m = \tau^{-1}[m]$. Its definition is analoguous to the definition of $\mathcal{C} = \mathcal{D}/F$. The Auslander-Reiten quiver of $\mathcal{C}^m$ is now covered by m triangular shapes and an extra strip. If m is odd, this is a Möbius strip, if m is even, it has cylindrical shape. By [BMRRT06], $\mathcal{C}^m$ is Krull-Schmidt; it is triangulated and Calabi-Yau of dimension m ([K05]).

It is possible to construct the m-cluster category via diagonals in a polygon: in joint work with R. Marsh ([BM08]) we have shown that the Auslander-Reiten quiver of $\mathcal{C}^m$ is the quiver $\Gamma(n,m)$ of m-diagonals in an $nm+2$-gon. An m-diagonal is a diagonal dividing Π into an $mj+2$-gon and $m(n-j)+2$-gon for some j. The arrows of the quiver $\Gamma(m,n)$ are $d_{ij} \to d_{i,j+m}$ and $d_{ij} \to d_{i+m,j}$ whenever the image is an m-diagonal. The translation on $\Gamma(m,n)$ is $\tau_m : d_{ij} \to d_{i-m,j-m}$.
Let us finish with a few remarks:

(1) We can define the m-cluster categories in a similar way using m-arcs in a punctured n-gon, [BM07].
(2) An alternative way to obtain the Auslander-Reiten quiver of $\mathcal{C}^m$ (types A_n and D_n) is to consider the m-th power of the translation quiver $\Gamma(n,1)$ in type A_n (and its counterpart in type D_n), cf. [BM08] and [BM07].
(3) It is still open whether there is a way to translate m-cluster categories into the cluster algebra setting.
(4) It is an open problem how to model type E. We propose an approach of gluing together three discs in an appropriate way. This allows us, to obtain cluster categories for the hereditary algebras associated to the quivers $T_{p,q,r}$ with three legs of lengths $p-1$, $q-1$ and $r-1$, respectively. This new model has the advantage to give a unified approach for all simply-laced cases.

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Conclusion: The matrix case.

Stéphane Launois

(joint work with Ken Goodearl, Tom Lenagan)

In recent publications, the same combinatorial description has arisen for three separate objects of interest: nonnegative cells in the real grassmannian [Pos, Wil]; torus orbits of symplectic leaves in the classical grassmannian [BGY, GoYa]; and, torus invariant prime ideals in the quantum grassmannian [LLR]. The aim of this mini-workshop was to explore the reasons for this coincidence in the grassmannian in particular, and to explore similar ideas in more general flag varieties.

In the matrix case, one has the following result.

Theorem. **[Goodearl-Launois-Lenagan]** Let $\mathcal{F}$ be a family of minors in the coordinate ring of $M_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $O_q(M_{m,p}(\mathbb{C}))$. Then the following are equivalent:

(1) The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.
(2) $\mathcal{F}$ is the set of minors that vanish on the closure of a torus-orbit of symplectic leaves in $M_{m,p}(\mathbb{C})$.
(3) $\mathcal{F}_q$ is the set of quantum minors that belong to torus-invariant prime in $O_q(M_{m,p}(\mathbb{C}))$.

The proof of this result relies on an algorithm, called the *restoration algorithm*, that was first developed for use in quantum matrices [Lau2].

The sets of minors that vanish on the closure of a torus-orbit of symplectic leaves in $M_{m,p}(\mathbb{C})$ can be explicitely described thanks to results of Fulton [Ful] and Brown-Goodearl-Yakimov [BGY]. So, as a consequence of the previous result, the sets of minors that defined non-empty totally nonnegative cells are explicitely described.

On the other hand, when the deformation parameter q is transcendental over the rationals, then the torus-invariant primes in $O_q(M_{m,p}(\mathbb{C}))$ are generated by quantum minors [Lau1], and so we deduce from the above result explicit generating sets of quantum minors for the torus-invariant prime ideals of $O_q(M_{m,p}(\mathbb{C}))$.

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