Weakly multiplicative coactions of quantized function algebras

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Abstract

A condition is identified which guarantees that the coinvariants of a coaction of a Hopf algebra on an algebra form a subalgebra, even though the coaction may fail to be an algebra homomorphism. A Hilbert Theorem (finite generation of the subalgebra of coinvariants) is obtained for such coactions of a cosemisimple Hopf algebra. This is applied for two coactions $\alpha, \beta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}$, where $\mathcal{A}$ is the coordinate algebra of the quantum matrix space associated with the quantized coordinate algebra $\mathcal{O}$ of a classical group, and α, β are quantum analogues of the conjugation action on matrices. Provided that $\mathcal{O}$ is cosemisimple and coquasitriangular, the α-coinvariants and the β-coinvariants form two finitely generated, commutative, graded subalgebras of $\mathcal{A}$, having the same Hilbert series. Consequently, the cocommutative elements and the S^2-cocommutative elements in $\mathcal{O}$ form finitely generated subalgebras. A Hopf algebra monomorphism from the quantum general linear group to Laurent polynomials over the quantum special linear group is found and used to explain the strong relationship between the corepresentation (and coinvariant) theories of these quantum groups.

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Let k be a field. Let A be a k-algebra and $\mathcal{O}$ a Hopf algebra over k and suppose that $\varphi : A \longrightarrow A \otimes \mathcal{O}$ is a right coaction. The *coinvariants* of φ consist of the set $A^{\varphi,\mathcal{O}} := \{c \in A \mid \varphi(c) = c \otimes 1\}$. When φ is also an algebra homomorphism, $A^{\varphi,\mathcal{O}}$ is automatically a subalgebra of A. However, it has recently become apparent that progress can be made even when φ is not an algebra homomorphism, see [6, 7, 8, 9], for example, where examples of such coactions are studied, motivated by seeking quantum versions of results concerning the classical invariant theory of the general linear and special linear groups. In these

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quantum cases, at the outset, it is not even clear that the set of coinvariants forms a subalgebra of A. However, under weaker conditions it is sometimes possible to show that the set of coinvariants does indeed form a subalgebra, and that this subalgebra enjoys desirable properties. The first purpose of this paper is to isolate the key points that make this process work. In particular, we are able to show that, under certain assumptions on φ, when $\mathcal{O}$ is cosemisimple, the set of coinvariants forms a subalgebra which is finitely generated.

Our leading examples for such coactions are associated to quantized algebras of functions on simple Lie groups. Let $\mathcal{O}$ be any of these quantum versions of classical groups (cf. [17]). We consider two right coactions α, β of $\mathcal{O}$ on the coordinate algebra $\mathcal{A}$ of the corresponding quantum matrix space. (In the case of $\mathcal{O}(SL_q(N))$, the coordinate algebra of the quantum special linear group, both of these coactions are possible quantum deformations of the conjugation action of $SL(N)$ on $N \times N$ matrices). As an application of the general considerations mentioned above, we show that provided that $\mathcal{O}$ is cosemisimple and coquasitriangular, each of the α-coinvariants and β-coinvariants forms a finitely generated, graded, commutative subalgebra of $\mathcal{A}$, and these subalgebras have the same Hilbert series. In particular, both the cocommutative elements and the S^2-cocommutative elements (cf. [3]) form a finitely generated subalgebra in $\mathcal{O}$.

The last section clarifies the relation between $\mathcal{O}(SL_q(N))$ and $\mathcal{O}(GL_q(N))$, the coordinate algebra of the quantum general linear group, which has consequences for coinvariant theory. Levasseur and Stafford, [13], have shown that $\mathcal{O}(GL_q(N))$ and $\mathcal{O}(SL_q(N)) \otimes k[z^{\pm 1}]$ are isomorphic as k-algebras, and that ring theoretic properties of $\mathcal{O}(SL_q(N))$ can be derived from this isomorphism. However, the algebra isomorphism $\mathcal{O}(GL_q(N)) \to \mathcal{O}(SL_q(N)) \otimes k[z^{\pm 1}]$ given in [13] is not a morphism of coalgebras. Here we construct a Hopf algebra homomorphism which embeds $\mathcal{O}(GL_q(N))$ into the Hopf algebra $\mathcal{O}(SL_q(N)) \otimes k[z^{\pm 1}]$ in such a way that $\mathcal{O}(SL_q(N)) \otimes k[z^{\pm 1}]$ is a finitely generated free module over the image of $\mathcal{O}(GL_q(N))$. This explains the strong relationship between the corepresentation theories of these quantum groups. To illustrate this, we derive explicitly generating $\mathcal{O}(SL_q(N))$-coinvariants in $\mathcal{O}(M_q(N))$ and $\mathcal{O}(SL_q(N))$ (with respect to the coactions α, β, under the assumption that q is not a root of unity) from the corresponding results for $\mathcal{O}(GL_q(N))$ obtained in [6].

1 Weakly multiplicative coactions

First, we identify a condition that is frequently satisfied by coactions even when they are not algebra maps, and which guarantees that the coinvariants form a subalgebra.

Let A be a k-algebra with unity, let $\mathcal{O}$ be a Hopf algebra over k, and let $\varphi : A \longrightarrow A \otimes \mathcal{O}$ be a right coaction. We say that φ is *left weakly multiplicative* if $\varphi(1) = 1 \otimes 1$, and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $b \in A$, provided that $\varphi(a) = a \otimes 1$. Similarly, φ is *right weakly multiplicative* if $\varphi(1) = 1 \otimes 1$, and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a \in A$, provided that $\varphi(b) = b \otimes 1$. The following trivial lemma shows that it is useful to identify when a right coaction is left or right weakly multiplicative.

Lemma 1.1 *Let A be a k-algebra and let $\mathcal{O}$ be a Hopf algebra over k. If $\varphi : A \longrightarrow A \otimes \mathcal{O}$ is either a left or right weakly multiplicative right coaction then the set of coinvariants $A^{\varphi,\mathcal{O}} = \{c \in A \mid \varphi(c) = c \otimes 1\}$ forms a subalgebra of A.*

Next, we show how to construct some left or right weakly multiplicative right coactions which, in general, are not algebra homomorphisms.

We fix the following notation. Let A be a k-algebra with unity and let $\mathcal{O}$ be a Hopf algebra over k. Denote by m, Δ, S and ε the multiplication, comultiplication, antipode and counit maps of $\mathcal{O}$, respectively.

Suppose that $\lambda : A \longrightarrow \mathcal{O} \otimes A$ is a left coaction and that $\rho : A \longrightarrow A \otimes \mathcal{O}$ is a right coaction, and that both λ and ρ are algebra homomorphisms. Suppose further that λ and ρ commute, in the sense that $(\lambda \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \lambda$.

Define

$$\alpha := (\mathrm{id} \otimes m) \circ \tau_{132} \circ (S \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\lambda \otimes \mathrm{id}) \circ \rho : A \longrightarrow A \otimes \mathcal{O}, \tag{1}$$

and

$$\beta := (\mathrm{id} \otimes m) \circ (\tau \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\lambda \otimes \mathrm{id}) \circ \rho : A \longrightarrow A \otimes \mathcal{O} \tag{2}$$

where τ_{132} is the map that sends $a \otimes b \otimes c$ to $b \otimes c \otimes a$, and τ is the flip $a \otimes b \mapsto b \otimes a$. We will show that α is a right weakly multiplicative coaction, and that β is a left weakly multiplicative coaction.

Lemma 1.2 *The maps α and β are right coactions.*

Proof. We do the computations for α. The computations for β are similar. First, to show that α is a coaction, we have to show that $(\mathrm{id} \otimes \Delta) \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \alpha$ and that $(\mathrm{id} \otimes \varepsilon) \circ \alpha = \mathrm{id}_A$. The commutativity assumption on λ and ρ make it possible to use the Sweedler notation. Note that in this notation, the formula for α is $\alpha(a) = \sum_{(a)} a_0 \otimes a_1 S(a_{-1})$. Thus, $(\alpha \otimes \mathrm{id}) \circ \alpha(a) = \sum_{(a)} a_0 \otimes a_1 S(a_{-1}) \otimes a_2 S(a_{-2})$. Now, $(\mathrm{id} \otimes \Delta) \circ \alpha(a) = (\mathrm{id} \otimes \Delta)(\sum_{(a)} a_0 \otimes a_1 S(a_{-1})) = \sum_{(a)} a_0 \otimes \Delta(a_1 S(a_{-1})) = \sum_{(a)} a_0 \otimes \{\Delta(a_1) \cdot \Delta(S(a_{-1}))\} = \sum_{(a)} a_0 \otimes \{(a_1 \otimes a_2) \cdot (S(a_{-1}) \otimes S(a_{-2}))\} = \sum_{(a)} a_0 \otimes a_1 S(a_{-1}) \otimes a_2 S(a_{-2})$, as required.

Next, $(\mathrm{id} \otimes \varepsilon) \circ \alpha(a) = (\mathrm{id} \otimes \varepsilon)(\sum_{(a)} a_0 \otimes a_1 S(a_{-1})) = \sum_{(a)} a_0 \otimes \varepsilon(a_1)\varepsilon(S(a_{-1})) = \sum_{(a)} a_0 \otimes \varepsilon(a_1)\varepsilon(a_{-1}) = \sum_{(a)} \varepsilon(a_{-1}) a_0 \otimes \varepsilon(a_1) = \sum_{(a)} a_0 \otimes \varepsilon(a_1) = \sum_{(a)} a_0 \varepsilon(a_1) = a$, as required. Thus, α is a coaction. □

Proposition 1.3 *The coaction α is right weakly multiplicative and the coaction β is left weakly multiplicative. In particular, the sets of coinvariants $A^{\alpha,\mathcal{O}}$ and $A^{\beta,\mathcal{O}}$ are subalgebras.*

Proof. The proofs are similar to [6, Lemma 2.2] and [9, Proposition 1.1]. We give the proof for α, the proof for β is similar. Let $a, b \in A$ and suppose that $\alpha(b) = b \otimes 1$. Then, $\alpha(ab) = \sum_{(ab)} (ab)_0 \otimes (ab)_1 S((ab)_{-1}) = \sum_{(a)} \sum_{(b)} a_0 b_0 \otimes a_1 b_1 S(b_{-1}) S(a_{-1}) = \sum_{(a)} (a_0 \otimes a_1) \cdot (\sum_{(b)} b_0 \otimes b_1 S(b_{-1})) \cdot (1 \otimes S(a_{-1})) = \sum_{(a)} (a_0 \otimes a_1) \cdot (b \otimes 1) \cdot (1 \otimes S(a_{-1})) = \sum_{(a)} (a_0 \otimes a_1 S(a_{-1})) \cdot (b \otimes 1) = \alpha(a)\alpha(b)$; so that α is right weakly multiplicative. Now, apply Lemma 1.1. □

In [6, 8, 9] coactions that are not algebra homorphisms are constructed in this way, and their coinvariants are calculated. In [6], for example, $A = \mathcal{O} = \mathcal{O}(GL_q(N))$, and α, β are two possible quantum deformations of the conjugation coaction. Explicit generators and the structure of the two subalgebras of coinvariants were determined for the case when q is not a root of unity. The commutativity (and a vector space isomorphism) of these coinvariant subalgebras was explained recently in [3]; it was shown to follow from the fact that $\mathcal{O}(GL_q(N))$ is coquasitriangular, and an interpretation of coinvariants via cocommutativity conditions (see Section 4). However, in [6] (and in [3]) the connection between these two possible quantum deformations α, β of the conjugation coaction is not considered. This is done in the next section of this paper for coquasitriangular Hopf algebras.

2 The coquasitriangular case

Recall, from [12, 10.1.1], that a bialgebra $\mathcal{O}$ is *coquasitriangular* if there exists a bilinear form (called a *universal r-form*) $\mathbf{r} : \mathcal{O} \times \mathcal{O} \longrightarrow k$ such that the following conditions hold (in Sweedler notation):
(i) there exists a bilinear form $\bar{\mathbf{r}} : \mathcal{O} \times \mathcal{O} \longrightarrow k$ satisfying

$$\sum \mathbf{r}(a_1, b_1)\bar{\mathbf{r}}(a_2, b_2) = \sum \bar{\mathbf{r}}(a_1, b_1)\mathbf{r}(a_2, b_2) = \varepsilon(a)\varepsilon(b);$$

that is, $\mathbf{r}$ (considered as a linear form on $\mathcal{O} \otimes \mathcal{O}$) is convolution invertible;
(ii) $ba = \sum \mathbf{r}(a_1, b_1)a_2b_2\bar{\mathbf{r}}(a_3, b_3)$
(iii) $\mathbf{r}(ab, c) = \sum \mathbf{r}(a, c_1)\mathbf{r}(b, c_2)$ and $\mathbf{r}(a, bc) = \sum \mathbf{r}(a_1, c)\mathbf{r}(a_2, b)$
for $a, b, c \in \mathcal{O}$.

Proposition 2.1 *Let $\mathcal{O}$ be a coquasitriangular Hopf algebra. Suppose that A is an algebra and that $\lambda : A \longrightarrow \mathcal{O} \otimes A$ and $\rho : A \longrightarrow A \otimes \mathcal{O}$ are algebra homomorphisms that are left and right coactions, respectively, and that λ and ρ commute. Suppose that α and β are defined as in the previous section. Then α and β are isomorphic coactions.*

Proof. Let $\mathbf{r}$ be a universal r-form for $\mathcal{O}$, with convolution inverse $\bar{\mathbf{r}}$. Define $\psi : A \longrightarrow A$ by $\psi(v) = \sum_{(v)} \mathbf{r}(S(v_{-1}), v_1)v_0$.

First we show that ψ is a vector space isomorphism, by giving an explicit inverse ϕ. Define $\phi : A \longrightarrow A$ by $\phi(v) = \sum_{(v)} \bar{\mathbf{r}}(S(v_{-1}), v_1)v_0$. Then,

$$\begin{aligned}\phi(\psi(v)) &= \phi(\sum_{(v)} \mathbf{r}(S(v_{-1}), v_1)v_0) = \sum_{(v)} \mathbf{r}(S(v_{-2}), v_2)\bar{\mathbf{r}}(S(v_{-1}), v_1)v_0 \\ &= \sum_{(v)} \varepsilon(S(v_{-1}))v_0\varepsilon(v_1) = v,\end{aligned}$$

and, similarly, one checks that $\psi(\phi(v)) = v$; so that ψ and ϕ are inverse linear transformations.

Next, we check that ψ intertwines between α and β; that is, $\alpha \circ \psi = (\psi \otimes \mathrm{id}) \circ \beta$. An equivalent condition to (ii) above is the requirement that $\sum \mathbf{r}(a_1, b_1)a_2b_2 = \sum \mathbf{r}(a_2, b_2)b_1a_1$, for $a, b \in \mathcal{O}$, see [12, p332, equation (5)]. Applying this with $a = S(v_{-1})$ and $b = v$, and noting that $\Delta(S(v_{-1})) = \sum S(v_{-1}) \otimes S(v_{-2})$, we obtain the equation $\sum \mathbf{r}(S(v_{-1}), v_1)S(v_{-2})v_2 = \sum \mathbf{r}(S(v_{-2}), v_2)v_1S(v_{-1})$. Thus,

$$\begin{aligned}\alpha(\psi(v)) = \alpha(\sum_{(v)} \mathbf{r}(S(v_{-1}), v_1)v_0) &= \sum_{(v)} \mathbf{r}(S(v_{-2}), v_2)v_0 \otimes v_1S(v_{-1}) \\ &= \sum_{(v)} v_0 \otimes \mathbf{r}(S(v_{-2}), v_2)v_1S(v_{-1}) = \sum_{(v)} v_0 \otimes \mathbf{r}(S(v_{-1}), v_1)S(v_{-2})v_2 \\ &= \sum_{(v)} \mathbf{r}(S(v_{-1}), v_1)v_0 \otimes S(v_{-2})v_2 = (\psi \otimes \mathrm{id})(\beta(v))\end{aligned}$$

for $v \in A$; that is, $\alpha \circ \psi = (\psi \otimes \mathrm{id}) \circ \beta$, as required. □

The above proof shows that $v \in A$ is a β-coinvariant if and only if $\psi(v)$ is an α-coinvariant.

In Section 4, we will see in examples that there is often a natural grading present on the algebra A, and that this induces a grading on the subalgebras of coinvariants for α and β. The maps ψ and ϕ respect these gradings, and so the subalgebras of coinvariants automatically have the same Hilbert series. However, the maps ψ and ϕ are not algebra homomorphisms in general (nor are their restrictions to the coinvariant subalgebras).

3 Coinvariant subalgebras are finitely generated

Next, we identify conditions that will guarantee that the coinvariants of a weakly multiplicative coaction form a finitely generated algebra. It is a well known basic fact in invariant theory that the noetherian property of the commutative polynomial algebra implies that the ring of invariants is finitely generated for linear group actions with an averaging operator, see for example [1, Chapter 1.6]. This argument can be translated to coinvariant theory as follows.

Recall that a Hopf algebra $\mathcal{O}$ is *cosemisimple* if any $\mathcal{O}$-comodule decomposes as the direct sum of irreducible subcomodules, [12, 11.2.1]. An equivalent condition for $\mathcal{O}$ to be cosemisimple is given in terms of Haar functionals.

Definition 3.1 *A linear functional h on $\mathcal{O}$ is* left invariant *if* $(\mathrm{id} \otimes h)\Delta(a) = h(a)1$*, for all $a \in \mathcal{O}$. Similarly, $\mathcal{O}$ is* right invariant *if* $(h \otimes \mathrm{id})\Delta(a) = h(a)1$*, for all $a \in \mathcal{O}$.*

A Hopf algebra $\mathcal{O}$ is *cosemisimple* if and only if there exists a unique left and right invariant linear functional h on $\mathcal{O}$ such that $h(1) = 1$, [12, 11.2.1, Theorem 13]. Such a functional is then called the *Haar functional* of $\mathcal{O}$. In Sweedler notation, the Haar functional satisfies $h(a)1 = \sum_{(a)} a_1h(a_2) = \sum_{(a)} h(a_1)a_2$ for all $a \in \mathcal{O}$. The Haar functional takes over the rôle of the averaging operator mentioned above, as is shown by the following known lemma.

Lemma 3.2 *Let $\mathcal{O}$ be a cosemisimple Hopf algebra, with Haar functional h, and let V be a k-vector space, with a right coaction $\varphi : V \longrightarrow V \otimes \mathcal{O}$. Set $\pi := (\mathrm{id} \otimes h) \circ \varphi : V \longrightarrow V$. Then (i) $\pi(V) \subseteq V^{\varphi,\mathcal{O}}$, and (ii) $\pi(v) = v$ for all $v \in V^{\varphi,\mathcal{O}}$.*

Thus, $\pi := (\mathrm{id} \otimes h) \circ \varphi$ is a projection from V onto $V^{\varphi,\mathcal{O}}$, cf. [12, 11.2.2, Corollary 19] and the comment following that result. The next lemma is known for multiplicative coactions, see [5, Sections 6.2, 6.3]. The weakly multiplicative condition is sufficient to carry through the proof.

Lemma 3.3 *Let A be a k-algebra and let $\mathcal{O}$ be a cosemisimple Hopf algebra over k. Suppose that $\varphi : A \longrightarrow A \otimes \mathcal{O}$ is a left weakly multiplicative right coaction with $B := A^{\varphi,\mathcal{O}}$. Set $\pi := (\mathrm{id} \otimes h) \circ \varphi$. Then π is a left B-module epimorphism from A to B; that is, $\pi(A) = B$ and $\pi(ba) = b\pi(a)$ for all $a \in A$ and $b \in B$.*

The noetherian property is used in the form of the following well-known proposition.

Proposition 3.4 *Suppose that $B = B_0 \oplus B_1 \oplus \ldots$ is an $\mathbb{N}$-graded subring of a right noetherian ring A. Suppose that there exists an epimorphism $\pi : A \longrightarrow B$ of left B-modules. Then there exist homogeneous elements $b_1, \ldots, b_s \in B$, of positive degree, such that $B = \sum_{u \in M} uB_0$, where M is the multiplicative semigroup generated by $1, b_1, \ldots, b_s$. In particular, $B = B_0\langle b_1, \ldots, b_s\rangle$ as a ring.*

The finite generation of coinvariants now follows easily.

Theorem 3.5 *Let $A = A_0 \oplus A_1 \oplus \ldots$ be a right noetherian $\mathbb{N}$-graded k-algebra with $A_0 = k$ and let $\mathcal{O}$ be a cosemisimple Hopf algebra over k. Suppose that $\varphi : A \longrightarrow A \otimes \mathcal{O}$ is a left weakly multiplicative right coaction such that $\varphi(A_i) \subseteq A_i \otimes \mathcal{O}$ for each i. Then $A^{\varphi,\mathcal{O}}$ is a subalgebra of A that is finitely generated as a k-algebra.*

Proof. Note that $B := A^{\varphi,\mathcal{O}}$ is a subalgebra of A, by Lemma 1.1. Set $B_i := B \cap A_i$ and note that $B = B_0 \oplus B_1 \oplus \ldots$, since $\varphi(A_i) \subseteq A_i \otimes \mathcal{O}$. Set $\pi := (\mathrm{id} \otimes h) \circ \varphi$. Then $\pi : A \rightarrow B$ is a left B-module epimorphism, by Lemma 3.3. The result now follows from Proposition 3.4. □

Remark 3.6 The obvious modification of Theorem 3.5 holds for a left noetherian algebra endowed with a right weakly multiplicative right coaction.

Remark 3.7 The referee has pointed out that the results in this section can easily be established in the setting where the Hopf algebra $\mathcal{O}$ is co-Frobenius and A possesses a total integral, by using results from [5, Sections 6.2, 6.3].

4 Coinvariants of quantum groups in FRT-bialgebras

Examples of coactions α, β studied in Section 1 are naturally associated to quantized algebras of functions on the simple Lie groups $SL(N)$, $O(N)$, $Sp(N)$. Let us briefly recall their construction due to Faddeev-Reshetikhin-Takhtajan [17]; see [12, Chapter 9] for a detailed presentation.

Take the free associative algebra $\mathbb{C}\langle x_{ij}\rangle$ in N^2 generators x_{ij}, $i, j = 1, \ldots, N$. Fix an $N^2 \times N^2$ matrix R with complex entries R_{ij}^{mn}, $i, j, m, n = 1, \ldots, N$. Let $\mathcal{A}(R)$ be the quotient algebra of $\mathbb{C}\langle x_{ij}\rangle$ modulo the two-sided ideal generated by

$$\sum_{k,l} R_{kl}^{ji} x_{km} x_{ln} - \sum_{k,l} x_{ik} x_{jl} R_{mn}^{lk}, \quad i, j, m, n = 1, \ldots, N. \tag{3}$$

Since this ideal is generated by homogeneous elements (with respect to the standard grading of the free algebra), the algebra $\mathcal{A}(R)$ is graded, the generators $x_{ij} \in \mathcal{A}(R)$ having degree 1. There is a unique bialgebra structure on $\mathcal{A}(R)$ such that

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \tag{4}$$

where $\delta_{ii} = 1$, and $\delta_{ij} = 0$ for $i \neq j$. Note that the comultiplication and the counit respect the grading. The bialgebra $\mathcal{A}(R)$ is called the *coordinate algebra of the quantum matrix space* associated with R, or briefly, the *FRT-bialgebra*. It is shown in [2] that if R is lower (or upper) triangular, then $\mathcal{A}(R)$ is a noetherian algebra.

For the rest of this section we assume that $q \in \mathbb{C}^*$ is not a root of unity, and that R is the R-matrix of the vector representation of one of the Drinfeld-Jimbo algebras $U_q(\mathfrak{sl}_N)$, $U_q(\mathfrak{so}_N)$ with $N = 2n$, $U_{q^{1/2}}(\mathfrak{so}_N)$ with $N = 2n+1$, $U_q(\mathfrak{sp}_N)$ with $N = 2n$. See [12, 9.2, 9.3] for the explicit formulae for R; for the case of $\mathfrak{sl}_N$, the relations (3) will be given in Section 5, (9). For each of these cases $\mathcal{A} := \mathcal{A}(R)$ contains a distinguished central group-like element Q (for the case of $\mathfrak{sl}_N$, Q is the quantum determinant, and Q is a quadratic element in the other cases, given explicitly in [12, 9.3.1]), such that the quotient bialgebra $\mathcal{O} := \mathcal{A}/\langle Q - 1\rangle$ has a unique Hopf algebra structure. The Hopf algebra $\mathcal{O}$ is denoted by $\mathcal{O}(SL_q(N))$, $\mathcal{O}(O_q(N))$, $\mathcal{O}(Sp_q(N))$ in the respective cases, and is called the *coordinate algebra of the quantum special linear group, orthogonal group*, and *symplectic group*, respectively. Furthermore, $\mathcal{O}(O_q(N))$ contains a group-like element D (similar to the quantum determinant). The quotient Hopf algebra $\mathcal{O}(O_q(N))/\langle D - 1\rangle$ is denoted by $\mathcal{O}(SO_q(N))$, and is called the *coordinate algebra of the quantum special orthogonal group* (cf. [11]).

From now on $\mathcal{O}$ stands for any of $\mathcal{O}(SL_q(N))$, $\mathcal{O}(O_q(N))$, $\mathcal{O}(SO_q(N))$, $\mathcal{O}(Sp_q(N))$, and $\pi : \mathcal{A} \to \mathcal{O}$ denotes the natural homomorphism from the corresponding FRT-bialgebra. If q is transcendental over $\mathbb{Q}$, then $\mathcal{O}$ is cosemisimple by [11]. The cosemisimplicity of $\mathcal{O}(SL_q(N))$ is known under the weaker assumption that q is not a root of unity, see [18] (or [15] combined with our Section 5). It is well known that the bialgebras $\mathcal{A}$, $\mathcal{O}(SL_q(N))$, $\mathcal{O}(O_q(N))$, $\mathcal{O}(Sp_q(N))$ are coquasitriangular (for all q), see [12, 10.1.2].

Set

$$\lambda := (\pi \otimes \mathrm{id}) \circ \Delta : \mathcal{A} \to \mathcal{O} \otimes \mathcal{A}, \quad \rho := (\mathrm{id} \otimes \pi) \circ \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}.$$

Clearly λ is a left coaction, ρ is a right coaction, and both λ and ρ are algebra homomorphisms. Moreover, λ and ρ commute by the coassociativity of Δ. So we are in the setup of Section 1, and the formulae (1), (2) of Section 1 define the right coactions α and β of $\mathcal{O}$ on $\mathcal{A}$. First we observe that the coinvariants can be interpreted independently from the coactions α, β as follows.

Proposition 4.1 *For $f \in \mathcal{A}$, with $\Delta(f) = \sum f_1 \otimes f_2$ we have*

(i) f is an α-coinvariant if and only if $\sum f_1 \otimes \pi(f_2) = \sum f_2 \otimes \pi(f_1)$.

(ii) f is a β-coinvariant if and only if $\sum f_1 \otimes f_2 = \sum f_2 \otimes S^2(\pi(f_1))$.

Proof. This is a straightforward modification of the proof of [6, Theorem 2.1] and [4, Lemma 1.1]. □

As an application of our results, properties of the subalgebras of α-coinvariants and β-coinvariants in $\mathcal{A}$ are summarized in the following theorem.

Theorem 4.2 *Let $\mathcal{O}$ be any of $\mathcal{O}(SL_q(N))$, $\mathcal{O}(O_q(N))$, $\mathcal{O}(Sp_q(N))$, and let $\mathcal{A}$ be the corresponding FRT-bialgebra. Then*

(i) the sets of coinvariants $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ are graded subalgebras of $\mathcal{A}$, having the same Hilbert series;

(ii) the algebras $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ are commutative.

Assume in addition that $\mathcal{O}$ is cosemisimple (this holds if q is transcendental over $\mathbb{Q}$, or if $\mathcal{O} = \mathcal{O}(SL_q(N))$ and q is not a root of unity). Then

(iii) $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ are finitely generated subalgebras of $\mathcal{A}$.

Proof. (i) The coinvariants form a subalgebra by Proposition 1.3. This is a graded subalgebra, since the comultiplication on $\mathcal{A}$ is homogeneous. Since $\mathcal{O}$ is coquasitriangular, α and β are isomorphic coactions by Proposition 2.1; the map ψ in its proof is homogeneous on $\mathcal{A}$, hence the restrictions α_d, β_d of α, β to the homogeneous component $\mathcal{A}_d$ of $\mathcal{A}$ are also isomorphic coactions. It follows that the dimension of the space of α-coinvariants in $\mathcal{A}_d$ equals the dimension of the space of β-coinvariants in $\mathcal{A}_d$. In other words, the graded algebras $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ have the same Hilbert series.

(ii) Both $\mathcal{A}$ and $\mathcal{O}$ are coquasitriangular bialgebras. Moreover, by [12, 10.1.2, Theorem 9] we know that there is a universal r-form $\mathbf{r}$ on $\mathcal{A}$ which induces a universal r-form on $\mathcal{O}$ denoted by the same symbol $\mathbf{r}$. Thus we have

$$\mathbf{r}(x, y) = \mathbf{r}(\pi(x), \pi(y)) \text{ for } x, y \in \mathcal{A}. \tag{5}$$

Assume that a, b are γ-coinvariants, where γ is either α or β. Let T denote the identity operator on $\mathcal{O}$ when $\gamma = \alpha$, and the automorphism S^2 of $\mathcal{O}$ when $\gamma = \beta$. Note that

$$\mathbf{r}(T(x), T(y)) = \mathbf{r}(x, y) \quad \text{and} \quad \bar{\mathbf{r}}(T(x), T(y)) = \bar{\mathbf{r}}(x, y) \quad \text{for all} \quad x, y \in \mathcal{O} \tag{6}$$

by [12, 10.1.1, Proposition 2.(v)]. Then we have

$$\sum a_1 \otimes \pi(a_2) = \sum a_2 \otimes T(\pi(a_1)) \quad \text{and} \quad \sum b_1 \otimes \pi(b_2) = \sum b_2 \otimes T(\pi(b_1))$$

by Proposition 4.1. Applying $(\pi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})$ to these equalities we get

$$\sum \pi(a_1) \otimes a_2 \otimes \pi(a_3) = \sum \pi(a_2) \otimes a_3 \otimes T(\pi(a_1)) \tag{7}$$

$$\sum \pi(b_1) \otimes b_2 \otimes \pi(b_3) = \sum \pi(b_2) \otimes b_3 \otimes T(\pi(b_1)). \tag{8}$$

We have

$$ba = \sum \mathbf{r}(a_1, b_1) a_2 b_2 \bar{\mathbf{r}}(a_3, b_3)$$

by condition (ii) in the defining properties of $\mathbf{r}$ (see Section 2). By (5), (7), (8), (6), and the defining property (i) of $\mathbf{r}, \bar{\mathbf{r}}$ in Section 2, the right hand side of the above equality equals

$$\begin{aligned}\sum \mathbf{r}(\pi(a_1), \pi(b_1)) a_2 b_2 \bar{\mathbf{r}}(\pi(a_3), \pi(b_3)) &= \sum \mathbf{r}(\pi(a_2), \pi(b_2)) a_3 b_3 \bar{\mathbf{r}}(T(\pi(a_1)), T(\pi(b_1)) \\ &= \sum \mathbf{r}((a_2, b_2) a_3 b_3 \bar{\mathbf{r}}(a_1, b_1) \\ &= \sum \varepsilon(a_1)\varepsilon(b_1) a_2 b_2 = ab,\end{aligned}$$

thus $ab = ba$ for all γ-coinvariants a, b.

(iii) As we noted above, $\mathcal{A}$ is both left and right Noetherian by [2], since the R-matrix is triangular. The homogeneous components of $\mathcal{A}$ are subcomodules with respect to α and β. Hence we may apply Theorem 3.5 to conclude that $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ are finitely generated subalgebras.

□

Remark 4.3 If $\mathcal{O} = \mathcal{O}(SO_q(N))$ with q transcendental over $\mathbb{Q}$, then the α-coinvariants and the β-coinvariants form finitely generated graded subalgebras in $\mathcal{A}$ by the same arguments as above.

Remark 4.4 The above theorem was motivated by the main result of [6]. In that paper $\mathcal{O} = \mathcal{O}(GL_q(N))$, the coordinate algebra of the quantum general linear group, $\mathcal{A} = \mathcal{O}(M_q(N))$, the coordinate algebra of $N \times N$ quantum matrices, and π is the natural embedding of $\mathcal{O}(M_q(N))$ into $\mathcal{O}(GL_q(N))$. The main result of [6] is that if q is not a root of unity, then $\mathcal{A}^{\alpha,\mathcal{O}}$ and $\mathcal{A}^{\beta,\mathcal{O}}$ are N-variable commutative polynomial algebras, with explicitly given generators. (This result implies the case of $\mathcal{O}(SL_q(N))$ of Theorem 4.2, as we shall show in Theorem 5.5.) Moreover, α and β extend to coactions of the Hopf algebra $\mathcal{O}(GL_q(N))$ on itself, and the subalgebras of coinvariants can be explicitly described. It was observed later in [3] that the commutativity of these coinvariant subalgebras follows from the coquasitriangularity of $\mathcal{O}(GL_q(N))$, and the interpretation of the coinvariants in terms of certain cocommutativity conditions. The proof of statement (ii) in Theorem 4.2 is a modification of the proof of [3, Proposition 2.1].

Remark 4.5 Theorem 3.5 can be applied to conclude the finite generation property of various other subalgebras of coinvariants arising in this context. For example, the algebra $\mathcal{A}^{\lambda,\mathcal{O}}$ of λ-coinvariants and the algebra $\mathcal{A}^{\rho,\mathcal{O}}$ of ρ-coinvariants are also finitely generated, provided that $\mathcal{O}$ is cosemisimple. See also the problem studied in [9].

There are unique right coactions $\bar{\alpha}$, $\bar{\beta}$ of the Hopf algebra $\mathcal{O}$ on itself such that π intertwines between α and $\bar{\alpha}$, and π intertwines between β and $\bar{\beta}$. In Sweedler's notation we have

$$\bar{\beta}(x) = \sum x_2 \otimes S(x_1)x_2 \quad \text{and} \quad \bar{\alpha}(x) = \sum x_2 \otimes x_3 S(x_1).$$

Since $\bar{\beta}$ is the formal dual of the adjoint action (see [14, p.36]), it is called the *adjoint coaction* of $\mathcal{O}$; the version $\bar{\alpha}$ was introduced in [6]. Recall that $x \in \mathcal{O}$ is *cocommutative* if $\sum x_1 \otimes x_2 = \sum x_2 \otimes x_1$, Following [3], we call $x \in \mathcal{O}$ S^2-*cocommutative* if $\sum x_1 \otimes x_2 = \sum x_2 \otimes S^2(x_1)$. The S^2-cocommutative elements coincide with the $\bar{\beta}$-coinvariants (see [4, Lemma 1.1], whereas the cocommutative elements coincide with the $\bar{\alpha}$-coinvariants (see [6, Theorem 2.1]).

Corollary 4.6 *Let $\mathcal{O}$ be any of $\mathcal{O}(SL_q(N))$, $\mathcal{O}(O_q(N))$, $\mathcal{O}(SO_q(N))$, $\mathcal{O}(Sp_q(N))$, and assume that $\mathcal{O}$ is cosemisimple (this holds if q is transcendental over $\mathbb{Q}$, or if $\mathcal{O} = \mathcal{O}(SL_q(N))$ and q is not a root of unity). Then each of the cocommutative elements and the S^2-cocommutative elements in $\mathcal{O}$ forms a finitely generated subalgebra.*

Proof. Since $\mathcal{O}$ is cosemisimple, the subspace of α-coinvariants (respectively β-coinvariants) in $\mathcal{A}$ is mapped onto the subspace of $\bar{\alpha}$-coinvariants (respectively $\bar{\beta}$-coinvariants) under the morphism π of $\mathcal{O}$-comodules. Moreover, π is an algebra homomorphism. So the statement follows from Theorem 4.2 and Remark 4.3. □

Remark 4.7 It was proved in [3, Theorem 2.2] that the coquasitriangularity of $\mathcal{O}$ implies that the two subalgebras in Corollary 4.6 are commutative. A vector space isomorphism between them was also established there.

Remark 4.8 More precise information on the subalgebra of S^2-cocommutative elements in $\mathcal{O}$ can be derived from the representation theory of the corresponding quantized enveloping algebra. We plan to return to this in a subsequent paper.

5 $\mathcal{O}(GL_q(N))$ is a central extension of $\mathcal{O}(SL_q(N))$

Over an algebraically closed base field, any element of the general linear group $GL(N)$ can be written as the product of a scalar matrix and an element from $SL(N)$. This and the centrality of scalar matrices imply a strong relationship between the representation theories of $GL(N)$ and $SL(N)$. Our aim is to establish such a link between the corepresentation theories of $\mathcal{O}(GL_q(N))$ and $\mathcal{O}(SL_q(N))$. By a *corepresentation* we mean a right coaction of a Hopf algebra on a vector space.

We work now over an arbitrary base field k, and q is a non-zero element of k. We fix N for the rest of this section, and to simplify notation, we write $\mathcal{O}(GL_q)$ and $\mathcal{O}(SL_q)$ instead of $\mathcal{O}(GL_q(N))$ and $\mathcal{O}(SL_q(N))$. Recall that the FRT-bialgebra of $\mathcal{O}(SL_q)$ is $\mathcal{O}(M_q)$, the *coordinate algebra of quantum $N \times N$ matrices*, which is the k-algebra generated by N^2 indeterminates x_{ij}, for $i, j = 1, \dots, N$, subject to the following relations.

$$\begin{aligned} x_{ij}x_{il} = qx_{il}x_{ij},\ x_{ij}x_{kj} = qx_{kj}x_{ij},\ x_{il}x_{kj} = x_{kj}x_{il},\\ x_{ij}x_{kl} - x_{kl}x_{ij} = (q-q^{-1})x_{il}x_{kj}, \end{aligned} \tag{9}$$

for $1 \leq i < k \leq N$ and $1 \leq j < l \leq N$. The algebra $\mathcal{O}(M_q)$ is an iterated Ore extension, and so a noetherian domain. The *quantum determinant*, $\det_q$, is the element

$$\det_q := \sum_{\sigma \in S_N} (-q)^{l(\sigma)} x_{1,\sigma(1)} \dots x_{N,\sigma(N)}.$$

The element $\det_q$ is central in $\mathcal{O}(M_q)$ (see, for example, [16, Theorem 4.6.1]), and by adjoining its inverse we get the *coordinate algebra of the quantum general linear group*

$$\mathcal{O}(GL_q) := \mathcal{O}(M_q)[\det_q^{-1}].$$

The algebra $\mathcal{O}(GL_q)$ is a Hopf algebra, the *comultiplication* Δ being given by $\Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj}$. The ideal generated by $(\det_q - 1)$ is a Hopf ideal of $\mathcal{O}(GL_q)$. The *coordinate algebra of the quantum special linear group* $\mathcal{O}(SL_q)$ is defined as the quotient Hopf algebra of $\mathcal{O}(GL_q)$ modulo the ideal generated by $(\det_q - 1)$. Denote by π the natural homomorphism $\mathcal{O}(GL_q) \to \mathcal{O}(SL_q)$, and set $y_{ij} := \pi(x_{ij})$.

The coordinate algebra of the multiplicative group of k is the algebra of Laurent polynomials $k[z, z^{-1}] = k[z^{\pm 1}]$. It is another Hopf algebra homomorphic image of $\mathcal{O}(GL_q)$: the homomorphism $\pi_Z : \mathcal{O}(GL_q) \to k[z^{\pm 1}]$ is given by $\pi_Z(x_{ii}) = z$ for $i = 1, \dots, N$ and $\pi_Z(x_{ij}) = 0$ if $i \neq j$. This can be paraphrased by saying that the multiplicative group of k is a *quantum subgroup* of the quantum general linear group. Moreover, it is a *central quantum subgroup* in the following sense:

$$\tau \circ (\pi_Z \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \pi_Z) \circ \Delta \tag{10}$$

where $\tau : k[z^{\pm 1}] \otimes \mathcal{O}(GL_q) \to \mathcal{O}(GL_q) \otimes k[z^{\pm 1}]$ is the flip $\tau(a \otimes b) = b \otimes a$.

Consider $\eta : k[z^{\pm 1}] \to k[\zeta \mid \zeta^N = 1]$, the natural homomorphism with $\eta(z) = \zeta$ and kernel $\langle z^N - 1 \rangle$. It follows from $\pi_Z(\det_q) = z^N$ that $\eta \circ \pi_Z$ factors through π, so we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(GL_q) & \xrightarrow{\pi_Z} & k[z^{\pm 1}] \\ \downarrow \pi & & \downarrow \eta \\ \mathcal{O}(SL_q) & \xrightarrow{\pi_\zeta} & k[\zeta \mid \zeta^N = 1] \end{array} \tag{11}$$

of Hopf algebra homomorphisms. The map π_ζ is given explicitly by $\pi_\zeta(y_{ii}) = \zeta$ for $i = 1, \ldots, N$ and $\pi_\zeta(y_{ij}) = 0$ for $i \neq j$. Equation (10) implies

$$\tau \circ (\pi_\zeta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \pi_\zeta) \circ \Delta. \tag{12}$$

The relevance of the properties (10) and (12) is shown by Proposition 5.1 below. We state it in a general form. Let $\kappa : \mathcal{O} \to \mathcal{C}$ be a homomorphism of Hopf algebras such that

$$\tau \circ (\mathrm{id} \otimes \kappa) \circ \Delta_{\mathcal{O}} = (\kappa \otimes \mathrm{id}) \circ \Delta_{\mathcal{O}}.$$

Assume that $\mathcal{C}$ is cosemisimple. Then $\mathcal{C}$ decomposes as a direct sum $\bigoplus_{\varphi \in \Lambda} \mathcal{C}(\varphi)$ of simple subcoalgebras, where Λ is a complete list of irreducible corepresentations of $\mathcal{C}$, and $\mathcal{C}(\varphi)$ is the coefficient space of φ (cf. [10]). Any $\mathcal{O}$-corepresentation $\psi : V \to V \otimes \mathcal{O}$ 'restricts' to a $\mathcal{C}$-corepresentation $(\mathrm{id} \otimes \kappa) \circ \psi : V \to V \otimes \mathcal{C}$. Therefore V decomposes as $\bigoplus_{\varphi \in \Lambda} V_\varphi$, where

$$V_\varphi = \{v \in V \mid (\mathrm{id} \otimes \kappa) \circ \psi(v) \in V \otimes \mathcal{C}(\varphi)\}.$$

In other words, V_φ is the sum of simple $\mathcal{C}$-subcomodules of V on which the $\mathcal{C}$-corepresentation is isomorphic to φ. In particular, $\mathcal{O} = \bigoplus_{\varphi \in \Lambda} \mathcal{O}_\varphi$, where

$$\mathcal{O}_\varphi = \{f \in \mathcal{O} \mid (\mathrm{id} \otimes \kappa) \circ \Delta(f) \in \mathcal{O} \otimes \mathcal{C}(\varphi)\}.$$

The proof of the following proposition is standard.

Proposition 5.1 *Any summand V_φ in $V = \bigoplus_{\varphi \in \Lambda} V_\varphi$ is an $\mathcal{O}$-subcomodule, and $\psi(V_\varphi) \subseteq V_\varphi \otimes \mathcal{O}_\varphi$. In particular, $\mathcal{O}_\varphi$ is a subcoalgebra, and V_φ is an $\mathcal{O}_\varphi$-comodule.*

Apply Proposition 5.1 with $\mathcal{O} = \mathcal{O}(GL_q)$, $\mathcal{C} = k[z^{\pm 1}]$, and $\kappa = \pi_Z$. Thus $\mathcal{O}(GL_q)$ decomposes as the direct sum of its subcoalgebras

$$\mathcal{O}(GL_q)_d = \{f \in \mathcal{O}(GL_q) \mid (\mathrm{id} \otimes \pi_Z) \circ \Delta(f) = f \otimes z^d\},$$

where d ranges over the set of integers $\mathbb{Z}$. An arbitrary $\mathcal{O}(GL_q)$-comodule V decomposes as

$$V = \bigoplus_{d \in \mathbb{Z}} V_d, \text{ where } V_d \text{ is an } \mathcal{O}(GL_q)_d\text{-comodule.} \tag{13}$$

Similarly, apply Proposition 5.1 with $\mathcal{O} = \mathcal{O}(SL_q)$, $\mathcal{C} = k[\zeta \mid \zeta^N = 1]$, and $\kappa = \pi_\zeta$. We obtain the decomposition of $\mathcal{O}(SL_q)$ as the direct sum of its subcoalgebras

$$\mathcal{O}(SL_q)_{\bar{d}} = \{f \in \mathcal{O}(SL_q) \mid (\mathrm{id} \otimes \pi_\zeta) \circ \Delta(f) = f \otimes \zeta^d\},$$

where $\bar{d}$ ranges over the set $\mathbb{Z}/(N)$ of residue classes modulo N (and we write $\bar{d}$ for the residue class of $d \in \mathbb{Z}$). Accordingly, any $\mathcal{O}(SL_q)$-comodule V decomposes as $V = \bigoplus_{\bar{d}\in\mathbb{Z}/(N)} V_{\bar{d}}$, where $V_{\bar{d}}$ is an $\mathcal{O}(SL_q)_{\bar{d}}$-comodule.

To see the decomposition $\mathcal{O}(GL_q) = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(GL_q)_d$ more explicitly, note that it is an algebra grading, and for the generators of $\mathcal{O}(GL_q)$ we have $x_{ij} \in \mathcal{O}(GL_q)_1$, $\det_q^{-1} \in \mathcal{O}(GL_q)_{-N}$, hence $\mathcal{O}(GL_q)_d$ is spanned by the elements $w/\det_q^{-r}$, where w is a monomial in the x_{ij} of degree $rN + d$. The space $\mathcal{O}(SL_q)_{\bar{d}}$ is spanned by the monomials in the y_{ij} whose degree is congruent to d modulo N.

Consider the tensor product Hopf algebra

$$\mathcal{B} := \mathcal{O}(SL_q) \otimes k[z^{\pm 1}],$$

together with the Hopf algebra surjections

$$\mathrm{id} \otimes \varepsilon : \mathcal{B} \to \mathcal{O}(SL_q) \quad \text{and} \quad \varepsilon \otimes \mathrm{id} : \mathcal{B} \to k[z^{\pm 1}],$$

where ε denotes the counit map of the appropriate Hopf algebra. The second surjection can be used to define the right coaction $(\mathrm{id}_{\mathcal{B}} \otimes (\varepsilon \otimes \mathrm{id})) \circ \Delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \otimes k[z^{\pm 1}]$, giving the $\mathbb{Z}$-grading

$$\mathcal{B} = \bigoplus_{d\in\mathbb{Z}} \mathcal{B}_d = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}(SL_q) \otimes z^d$$

on $\mathcal{B}$ as in Proposition 5.1. Clearly $\mathcal{B}$ is a graded algebra.

Proposition 5.2 *(i) The map $\iota = (\pi \otimes \pi_Z) \circ \Delta$ is an embedding of the Hopf algebra $\mathcal{O}(GL_q)$ into $\mathcal{O}(SL_q) \otimes k[z^{\pm 1}]$.*

(ii) The restriction π_d of π to $\mathcal{O}(GL_q)_d$ is a coalgebra isomorphism

$$\mathcal{O}(GL_q)_d \cong \mathcal{O}(SL_q)_{\bar{d}}$$

for all $d \in \mathbb{Z}$.

Proof. (i) The centrality condition (10) and the fact that both π and π_Z are homomorphisms of Hopf algebras imply that ι is a Hopf morphism. The only thing left to show is that ι is injective. Assume to the contrary that f is a nonzero element of $\ker(\iota)$. Multiplying f by an appropriate power of $\det_q$ we may assume that $f \in \mathcal{O}(M_q)$. Recall that $\mathcal{B}$ is a $\mathbb{Z}$-graded algebra, and $\mathcal{O}(M_q)$ is also $\mathbb{Z}$-graded, the generators x_{ij} having degree 1. Since $\iota(x_{ij}) = y_{ij} \otimes z$, the restriction of ι to $\mathcal{O}(M_q)$ is homogeneous. So all of the homogeneous components of f are contained in $\ker(\iota)$. We may assume that f itself is homogeneous of degree d. Then $0 = \iota(f) = \pi(f) \otimes z^d$, implying that $\pi(f) = 0$. Thus f is a multiple of $(\det_q - 1)$. This is a contradiction, because no non-zero multiple of $(\det_q - 1)$ in the domain $\mathcal{O}(M_q)$ is homogeneous.

(ii) From now on we shall freely identify $\mathcal{O}(GL_q)$ with $\iota(\mathcal{O}(GL_q))$. Then $\mathcal{O}(GL_q)_d = \iota(\mathcal{O}(GL_q)) \cap \mathcal{B}_d$. The definition of ι implies that π is the restriction to $\mathcal{O}(GL_q)$ of the map $\mathrm{id} \otimes \varepsilon$, and π_Z is the restriction to $\mathcal{O}(GL_q)$ of $\varepsilon \otimes \mathrm{id}$. By (11) we know that π_d is a surjective coalgebra homomorphism. So we only need to show the injectivity. The restriction of $\mathrm{id} \otimes \varepsilon$ to $\mathcal{B}_d$ is clearly a vector space isomorphism $\mathcal{B}_d \to \mathcal{O}(SL_q)$. Since π_d is the restriction of $\mathrm{id} \otimes \varepsilon$ to $\mathcal{O}(GL_q)_d$, it is also injective. □

Remark 5.3 Clearly $\iota(\mathcal{O}(GL_q))$ is the subalgebra of $\mathcal{B}$ generated by $y_{ij} \otimes z$ $(1 \leq i, j \leq N)$ and $1 \otimes z^{-N}$. It is easy to see that $\mathcal{B}$ is a free $\mathcal{O}(GL_q)$-module of rank N generated by the central elements $1 \otimes z^i$, $i = 0, \ldots, N-1$, but we shall not use this fact. As k-algebras, $\mathcal{O}(GL_q)$ and $\mathcal{B}$ are isomorphic by [13], where ring theoretic properties of $\mathcal{O}(SL_q)$ are derived from this isomorphism. However, the algebra isomorphism $\mathcal{O}(GL_q) \to \mathcal{O}(SL_q) \otimes k[z^{\pm 1}]$ given in [13] is not a morphism of coalgebras. To relate the corepresentation theories, we need the Hopf algebra morphism ι.

For an $\mathcal{O}(GL_q)$-corepresentation $\varphi : V \to V \otimes \mathcal{O}(GL_q)$ denote by $\overline{\varphi} = (\mathrm{id} \otimes \pi) \circ \varphi$ the restriction to $\mathcal{O}(SL_q)$. Observe that if φ is indecomposable, then there is a $d \in \mathbb{Z}$ such that $V = V_d$, that is, the coefficient space of φ is contained in $\mathcal{O}(GL_q)_d$. Therefore Proposition 5.2 (ii) immediately implies the following:

- φ is indecomposable if and only if $\overline{\varphi}$ is indecomposable.
- φ is irreducible if and only if $\overline{\varphi}$ is irreducible.
- φ is cosemisimple if and only if $\overline{\varphi}$ is cosemisimple.

The last statement above shows that $\mathcal{O}(GL_q)$ is cosemisimple if and only if $\mathcal{O}(SL_q)$ is cosemisimple. Actually, both of them are known to be cosemisimple if and only if q is not a root of unity, see [15], [11], [18].

Another consequence of Proposition 5.2 (ii) is that any $\mathcal{O}(SL_q)$-corepresentation can be lifted to an $\mathcal{O}(GL_q)$-corepresentation (in several ways). Indeed, it is sufficient to deal with indecomposable corepresentations. Let $\psi : W \to W \otimes \mathcal{O}(SL_q)$ be an indecomposable corepresentation whose coefficient space is contained in $\mathcal{O}(SL_q)_e$, where $e \in \mathbb{Z}/(N)$. Take an arbitrary $d \in \mathbb{Z}$ representing the residue class e, and define the $\mathcal{O}(GL_q)_d$-coaction φ by $\varphi = (\mathrm{id} \otimes \pi_d^{-1}) \circ \psi$. Then φ is an $\mathcal{O}(GL_q)$-corepresentation with $\overline{\varphi} = \psi$.

Next we draw a consequence on $\mathcal{O}(SL_q)$-coinvariants, extending to the quantum case the well known relation between relative $GL(N)$-invariants and absolute $SL(N)$-invariants.

Proposition 5.4 *Let φ be an $\mathcal{O}(GL_q)$-corepresentation on V, and let $\overline{\varphi} := (\mathrm{id} \otimes \pi) \circ \varphi$ be its restriction to $\mathcal{O}(SL_q)$. Then the subspace of $\mathcal{O}(SL_q)$-coinvariants in V equals*

$$\bigoplus_{j \in \mathbb{Z}} \{v \in V \mid \varphi(v) = v \otimes \det_q^j\}.$$

In particular, if the coefficient space of φ is contained in $\mathcal{O}(GL_q)_0$, then $v \in V$ is an $\mathcal{O}(SL_q)$-coinvariant if and only if v is an $\mathcal{O}(GL_q)$-coinvariant.

Proof. Consider the decomposition $V = \bigoplus_{d\in\mathbb{Z}} V_d$ from (13). Obviously each summand V_d is an $\mathcal{O}(SL_q)$-subcomodule (with respect to $\overline{\varphi}$). Therefore $v \in V$ is an $\mathcal{O}(SL_q)$-coinvariant if and only if all components of v are $\mathcal{O}(SL_q)$-coinvariants. Thus we may restrict to the case when $V = V_d$ for some d. If there is a non-zero $\mathcal{O}(SL_q)$-coinvariant v in V_d, then $d = jN$ for an integer j, because $1 \in \mathcal{O}(SL_q)_{\bar{0}}$. Assume that this is the case. We have $\pi_d(\det_q^j) = 1$, and π_d is a coalgebra isomorphism by Proposition 5.2 (ii). So the coefficient space (cf. [10]) of the $\mathcal{O}(GL_q)$-comodule generated by v is spanned by $\det_q^j$. Since $\det_q^j$ is group-like, we have $\varphi(v) = v \otimes \det_q^j$. □

For the rest of the paper we assume that $k = \mathbb{C}$, the field of complex numbers. As an application of Proposition 5.4, we deduce from the results of [6] an explicit description of the subsets of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(M_q)$ and $\mathcal{O}(SL_q)$ with respect to the coactions α, β defined in Section 4.

Fix an integer t with $1 \leq t \leq N$. Let I and J be subsets of $\{1, \ldots, N\}$ with $|I| = |J| = t$. The subalgebra of $\mathcal{O}(M_q)$ generated by x_{ij} with $i \in I$ and $j \in J$ can be regarded as an algebra of $t \times t$ quantum matrices, and so we can calculate its quantum determinant - this is a $t \times t$ *quantum minor* and we denote it by $[I|J]$. The quantum minor $[I|I]$ is said to be a *principal quantum minor*. We denote the sum of all the principal quantum minors of a given size i by σ_i. Note that $\sigma_1 = x_{11} + \cdots + x_{NN}$ and that $\sigma_N = \det_q$. Consider also the *weighted sums of principal minors* $\tau_i := \sum_I q^{-2w(I)}[I|I]$, $i = 1, \ldots, N$ (here $w(I)$ denotes the sum of the elements of I, and the summation ranges over all subsets I of size i).

Theorem 5.5 *Assume that $q \in \mathbb{C}$ is not a root of unity.*

(i) The subset of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(M_q)$ with respect to α is an N-variable commutative polynomial subalgebra of $\mathcal{O}(M_q)$ generated by σ_i, $i = 1, \ldots, N$.

(ii) The subset of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(SL_q)$ with respect to α is an $(N-1)$-variable commutative polynomial subalgebra of $\mathcal{O}(SL_q)$ generated by $\pi(\sigma_i)$, $i = 1, \ldots, N-1$.

(iii) The subset of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(M_q)$ with respect to β is an N-variable commutative polynomial subalgebra of $\mathcal{O}(M_q)$ generated by τ_i, $i = 1, \ldots, N$.

(iv) The subset of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(SL_q)$ with respect to β is an $(N-1)$-variable commutative polynomial subalgebra of $\mathcal{O}(SL_q)$ generated by $\pi(\tau_i)$, $i = 1, \ldots, N-1$.

Proof. The coefficient space of both α and β is contained in $\mathcal{O}(GL_q)_0$, so the subsets of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(M_q)$ are the same as for the corresponding $\mathcal{O}(GL_q)$-coaction by Proposition 5.4. Hence (i) and (iii) follow from the main result of [6], describing the spaces of $\mathcal{O}(GL_q)$-coinvariants. The statements (ii) and (iv) are immediate consequences of (i) and (iii). Indeed, since $\mathcal{O}(SL_q)$ is cosemisimple by our assumption on q, the subsets of $\mathcal{O}(SL_q)$-coinvariants in $\mathcal{O}(M_q)$ are mapped by π (which is a morphism of comodules) onto the corresponding subsets of coinvariants in $\mathcal{O}(SL_q)$. So the only thing left to show is that $\pi(\sigma_i)$, $i = 1, \ldots, N-1$ are algebraically independent, and $\pi(\tau_i)$, $i = 1, \ldots, N-1$ are algebraically independent. This follows from the observation that $\ker(\pi) \cap \mathcal{O}(M_q)^{\alpha,\mathcal{O}} = \mathcal{O}(M_q)^{\alpha,\mathcal{O}}(\det_q - 1)$ and $\ker(\pi) \cap \mathcal{O}(M_q)^{\beta,\mathcal{O}} = (\det_q - 1)\mathcal{O}(M_q)^{\beta,\mathcal{O}}$ by Lemma 3.3. □

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