Twisting the quantum grassmannian

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Abstract

In contrast to the classical and semiclassical settings, the Coxeter element $(12\dots n)$ which cycles the columns of an $m\times n$ matrix does not determine an automorphism of the quantum grassmannian. Here, we show that this cycling can be obtained by defining a cocycle twist. A consequence is that the torus invariant prime ideals of the quantum grassmannian are permuted by the action of the Coxeter element $(12\dots n)$; we view this as a quantum analogue of the recent result of Knutson, Lam and Speyer that the Lusztig strata of the classical grassmannian are permuted by $(12\dots n)$.

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1 Introduction

The symmetric group S_n acts on the grassmannian $G(m,n)$ by permuting the columns of an $m\times n$ matrix that determines a point in $G(m,n)$. If one restricts to considering the totally nonnegative grassmannian $G(m,n)^{\rm tnn}$ this is no longer true; however, Postnikov, [14, Remark 3.3], notes that the cycle $c=(12\dots n)$ acts on the totally nonnegative grassmannian. Recently, Knutson, Lam and Speyer, [9], showed that the Lusztig strata of the classical grassmannian are permuted by $(12\dots n)$. In fact this invariance property is even stronger. Indeed, Goodearl and Yakimov, [5], have found a Poisson interpretation of the Lusztig strata: they coincide with the $\mathcal{H}$-orbits of symplectic leaves of $G(m,n)$, where $\mathcal{H}$ is an n-dimensional algebraic torus. Recently Yakimov, [15], showed that the Coxeter element c induces a Poisson automorphism of $G(m,n)$. As a consequence he showed that the $\mathcal{H}$-orbits of symplectic leaves of $G(m,n)$ are permuted by c; this gives a Poisson geometric proof of Knutson, Lam and Speyer result.

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In view of the close connections that have been discovered between totally nonnegative matrices, the standard Poisson matrix variety and quantum matrices, see, for example, [2, 3], and between the totally nonnegative grassmannian and the quantum grassmannian, see, for example, [10], one might expect that the cycle c produces an automorphism of the quantum grassmannian. This is not the case, see Example 3.1 below. With this in mind, one wonders what the analogous result should be. Here, we provide the answer: there is a 2-cocyle which can be used to twist the quantum grassmannian; the resulting twisted algebra is again isomorphic to the quantum grassmannian, and the effect of the twist on a generating quantum minor I is to produce (a scalar multiple of) the quantum minor obtained by letting the cycle c act on the indices of I. A consequence of this result is that the torus invariant prime ideals of the quantum grassmannian are permuted by the cycle $(12 \dots n)$, see Corollary 6.2; we view this as a quantum analogue of the Knutson, Lam and Speyer result.

2 Basic definitions

In this section, we will give the basic definitions of the objects that interest us in this paper and recall several results that we need in our proofs. Throughout, $\mathbb{K}$ will denote a base field, we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$, q will be a non-zero element of $\mathbb{K}$ and m and n denote positive integers with $m < n$. Moreover, we assume that there exists $p \in \mathbb{K}$ such that $p^m = q^2$.

The quantisation of the coordinate ring of the affine variety $M_{m,n}$ of $m \times n$ matrices with entries in $\mathbb{K}$ is denoted $\mathcal{O}_q(M_{m,n})$. It is the $\mathbb{K}$-algebra generated by mn indeterminates X_{ij}, with $1 \le i \le m$ and $1 \le j \le n$, subject to the relations:

$$\begin{array}{ll} X_{ij}X_{il} = qX_{il}X_{ij}, & \text{for } 1 \le i \le m, \text{ and } 1 \le j < l \le n\,; \\ X_{ij}X_{kj} = qX_{kj}X_{ij}, & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j \le n\,; \\ X_{ij}X_{kl} = X_{kl}X_{ij}, & \text{for } 1 \le k < i \le m, \text{ and } 1 \le j < l \le n\,; \\ X_{ij}X_{kl} - X_{kl}X_{ij} = (q-q^{-1})X_{il}X_{kj}, & \text{for } 1 \le i < k \le m, \text{ and } 1 \le j < l \le n. \end{array}$$

An *index pair* is a pair (I, J) such that $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ are subsets with the same cardinality. Hence, an index pair is given by an integer t such that $1 \le t \le m$ and ordered sets $I = \{i_1 < \cdots < i_t\} \subseteq \{1, \dots, m\}$ and $J = \{j_1 < \cdots < j_t\} \subseteq \{1, \dots, n\}$. To any such index pair we associate the quantum minor

$$[I|J] = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)}j_1} \cdots X_{i_{\sigma(t)}j_t}.$$

Definition 2.1 The *quantisation of the coordinate ring of the grassmannian of m-dimensional subspaces of $\mathbb{K}^n$*, denoted by $\mathcal{O}_q(G(m,n))$ and informally referred to as the $(m \times n)$ *quantum grassmannian* is the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by the $m \times m$ quantum minors.

A maximal (that is, $m \times m$) quantum minor in $\mathcal{O}_q(M_{m,n})$ corresponds to an index pair $[\{1,\dots,m\}|J]$ with $J = \{j_1,\dots,j_m\} \subseteq \{1,\dots,n\}$. We call such J *index sets* and denote the corresponding minor by $[J]$ or $[j_1,\dots,j_m]$ in what follows. Thus, such a $[J]$ is a generator of $\mathcal{O}_q(G(m,n))$.

When writing down an $m \times m$ quantum minor in $\mathcal{O}_q(G(m,n))$, we will use the convention that if a column index j is greater than n then j is to be read as $j - n$. For example, in $\mathcal{O}_q(G(2,4))$ the minor specified by $[45]$ is the quantum minor $[14]$. In order to stress this point, we will use the convention that given any integer j then $\widetilde{j}$ is the integer in the set $\{1,\dots,n\}$ that is congruent to j modulo n.

A quantum minor $[\widetilde{i}, \widetilde{i+1}, \dots, \widetilde{i+m-1}]$ is said to be a *consecutive quantum minor* of $\mathcal{O}_q(G(m,n))$. Recalling the convention above, we see that there are four consecutive minors in $\mathcal{O}_q(G(2,4))$: they are $[12], [23], [34]$ and $[\widetilde{4}\,\widetilde{5}] = [14]$. More generally, $\mathcal{O}_q(G(m,n))$ has n consecutive minors.

Two maximal quantum minors $[I]$ and $[J]$ are said to *quasi-commute* if there is an integer c such that $[I][J] = q^c[J][I]$. Recall that an element u of a ring R is said to be a *normal element* if $uR = Ru$, in which case uR is a two-sided ideal. The following lemma, first obtained in [8, Lemma 3.7], shows that consecutive quantum minors quasi-commute with all maximal quantum minors.

Lemma 2.2 *Let $[\widetilde{i}, \widetilde{i+1}, \dots, \widetilde{i+m-1}]$ be a consecutive quantum minor in the quantum grassmannian $\mathcal{O}_q(G(m,n))$. Then $[\widetilde{i}, \widetilde{i+1}, \dots, \widetilde{i+m-1}]$ quasi-commutes with each of the generating quantum minors of $\mathcal{O}_q(G(m,n))$. In particular, each consecutive quantum minor is a normal element of $\mathcal{O}_q(G(m,n))$.* □

A consequence of this result is that the powers of a consecutive quantum minor form an Ore set in the noetherian domain $\mathcal{O}_q(G(m,n))$; and so it is possible to invert the consecutive quantum minor in a localisation.

In order to facilitate computations, we need a version of the Quantum Muir's Law of Extensible Minors. This result was first obtained by Krob and Leclerc, [8, Theorem 3.4], with a proof involving quasi-determinants. The version below, which is sufficient for our needs, is taken from [12, Proposition 1.3], and is adapted for use in the quantum grassmannian.

Proposition 2.3 *Let I_s, J_s, for $1 \leq s \leq d$, be m-element subsets of $\{1, \ldots, n\}$ and let $c_s \in \mathbb{K}$ be such that $\sum_{s=1}^{d} c_s[I_s][J_s] = 0$ in $\mathcal{O}_q(G(m,n))$. Suppose that P is a subset of $\{1, \ldots, n\}$ such that $(\cup_{s=1}^{d} I_s) \cup (\cup_{s=1}^{d} J_s) \subseteq P$ and let $\overline{P}$ denote $\{1, \ldots, n\} \backslash P$. Then*

$$\sum_{s=1}^{d} c_s[I_s \sqcup \overline{P}][J_s \sqcup \overline{P}] = 0.$$

holds in $\mathcal{O}_q(G(m', n))$, where $m' = m + \#\overline{P}$. $\square$

This result is used, for example, when it is necessary to write down a commutation relation between two maximal quantum minors $[I]$ and $[J]$, say. The usefulness of the result is that one may delete the common members of the index pairs I and J to establish the commutation relation.

3 Cycling does not induce an automorphism

In contrast to the classical and semiclassical settings, the cycle $(12 \ldots n)$ does not act as an automorphism on the quantum grassmannian. We show this here by considering $\mathcal{O}_q(G(2,4))$.

First, we summarize the commutation relations and the quantum Plücker relation for $\mathcal{O}_q(G(2,4))$; which can easily be obtained from the defining relations of quantum matrices.

$$[ij][ik] = q[ik][ij], \quad [ik][jk] = q[jk][ik], \quad \text{for } i < j < k$$

and

$$[14]\,[23] = [23]\,[14]\,, \quad [12][34] = q^2[34][12], \quad [13]\,[24] = [24]\,[13] + \left(q - q^{-1}\right)[14]\,[23]\,.$$

There is also a quantum Plücker relation $[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0$. This quantum Plücker relation may be rewritten as $[34]\,[12] - q^{-1}\,[24]\,[13] + q^{-2}\,[23]\,[14] = 0$ and one can also check that $[13]\,[24] = q^2\,[24]\,[13] + \left(q^{-1} - q\right)[12]\,[34]$.

Example 3.1 Let $\theta[ij] := [i+1, j+1]$, with the convention that $\theta(4) = 1$; that is, we work modulo 4 and θ is cycling the indices of quantum minors:

$$\theta[ij] = [\widetilde{c(i)}, \widetilde{c(j)}],$$

where c denotes the cycle (1234).

In the classical case, θ induces an isomorphism, and this is also the case in the Poisson setting, [15].

However, θ does not induce an automorphism of $\mathcal{O}_q(G(2,4))$, since, for example, the quantum Plücker relation is not preserved: if we assume that θ induces an automorphism then we calculate

$$0 = \theta(0) = \theta([12][34] - q[13][24] + q^2[14][23]) = [23][14] - q[24][13] + q^2[12][34].$$

However, one can check that $[23][14]-q[24][13]+q^2[12][34] \neq 0$. For, suppose that $[23][14]-q[24][13]+q^2[12][34] = 0$, then $[23][14]-q[24][13]+q^4[34][12] = 0$. However, from the second version of the quantum Plücker relation, we know that $[14][23] - q[24][13] + q^2[34][12] = 0$. Subtract one of these equations from the other and note that $[14][23] = [23][14]$ to obtain $(q^4 - q^2)[34][12] = 0$, a contradiction, provided that $q^2 \neq 1$.

4 Dehomogenisation at a consecutive minor

Explicit calculations in the quantum grassmannian can be difficult due to the awkward defining relations (quantum Plücker relations). For this reason, it is often useful to transfer to an overring where the defining relations are simpler. This can be achieved by localising at any consecutive quantum minor, and this leads to consideration of the noncommutative dehomogenisation isomorphism for an arbitrary consecutive quantum minor.

Set $M_\alpha := \{\widetilde{\alpha}, \widetilde{\alpha+1}, \dots, \widetilde{\alpha+m-1}\}$ in $\mathcal{O}_q(G(m,n))$. Now, $[M_\alpha]$ is a normal element, by Lemma 2.2; and so we may form the localisation $\mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$. In $\mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$ set

$$x_{ij} := [M_\alpha \cup \{j + \widetilde{\alpha+m} - 1\} \backslash \{\widetilde{\alpha+m} - i\}][M_\alpha]^{-1}.$$

Theorem 4.1 *The subalgebra $\mathbb{K}[x_{ij}]$ of $\mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$ is a q-quantum matrix algebra; that is, $\mathbb{K}[x_{ij}]$ is isomorphic to $\mathcal{O}_q(M_{m,n-m})$ by an isomorphism that send x_{ij} to X_{ij}. Moreover there is an isomorphism*

$$\phi_\alpha : \mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}] \longrightarrow \mathbb{K}[x_{ij}][y_\alpha^{\pm 1}; \sigma_\alpha].$$

where σ_α is the automorphism of the quantum matrix algebra $\mathbb{K}[x_{ij}]$ defined by $\sigma_\alpha(x_{ij})M_\alpha = M_\alpha x_{ij}$. Under this isomorphism, $y_\alpha = \phi_\alpha(M_\alpha)$.

Proof: The fact that $\mathbb{K}[x_{ij}]$ is a quantum matrix algebra is established in [13, Theorem 3.2]. The inclusion $\rho_\alpha : \mathbb{K}[x_{ij}] \longrightarrow \mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$ extends to a homomorphism $\rho_\alpha : \mathbb{K}[x_{ij}][y_\alpha^{\pm 1}; \sigma_\alpha] \longrightarrow \mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$, by the universal property of skew polynomial extensions. The fact that the extension ρ_α is an isomorphism follows from [13, Lemma 3.1] and the dehomogenisation isomorphism [6, Lemma 3.1]. Now, set $\phi_\alpha = \rho_\alpha^{-1}$. $\square$

Next, we need to calculate the effect of ϕ_α on generating quantum minors of $\mathcal{O}_q(G(m,n))$.

Let I be an m-element subset of $\{1,\dots,n\}$. For a fixed α, set $I_r := I \cap M_\alpha$ and $I_c := I \backslash I_r$; so that $I = I_r \sqcup I_c$ (the notation is chosen because I_r will give information about the row set of the image of $[I]$ and I_c will give information about the column set).

To simplify the notation somewhat, if N is a subset of integers, and i is an integer, then $i + N = \{i + k \mid k \in N\}$.

Corollary 4.2 *Let $[I]$ be a generating quantum minor of $\mathcal{O}_q(G(m,n))$. Then*

$$\phi_\alpha([I]) = [(\alpha+m) - (M_\alpha \backslash I_r) \mid I_c - (\alpha+m-1)]y_\alpha.$$

Proof: By using [13, Proposition 4.3], we see that for a quantum minor $[I|J]$ of the quantum matrix algebra $\mathbb{K}[x_{ij}]$

$$\rho_\alpha([I|J]) = [M_\alpha \backslash ((\alpha+m) - I) \sqcup ((\alpha+m-1)+J)][M_\alpha]^{-1}.$$

As $\phi_\alpha = \rho_\alpha^{-1}$, the claim will be established once we show that

$$\rho_\alpha([(\alpha+m) - (M_\alpha \backslash I_r) \mid I_c - (\alpha+m-1)] \cdot y_\alpha) = [I].$$

Now,

$$\begin{aligned}
&\rho_\alpha([(\alpha+m) - (M_\alpha \backslash I_r) \mid I_c - (\alpha+m-1)] \cdot y_\alpha) = \\
&\quad [M_\alpha \backslash ((\alpha+m) - ((\alpha+m) - M_\alpha \backslash I_r) \sqcup ((\alpha+m-1) + (I_c - (\alpha+m-1)))][M_\alpha]^{-1} \cdot [M_\alpha] \\
&\quad = \quad [M_\alpha \backslash (M_\alpha \backslash I_r) \sqcup I_c] = [I_r \sqcup I_c] = [I],
\end{aligned}$$

as required. □

We shall need to use the isomorphisms ϕ_α and ρ_α of Theorem 4.1 in the two cases $\alpha = 1$ and $\alpha = 2$. The next two results record the action of σ_1 and σ_2.

Lemma 4.3 *For $1 \le i \le m$ and $1 \le j \le n-m$*

$$\sigma_1(x_{ij}) = qx_{ij}.$$

Consequently, $y_1 x_{ij} = qx_{ij}y_1$ for $1 \le i \le m$ and $1 \le j \le n-m$.

Proof: In order to calculate the commutation relation between x_{ij} and y_1, we need to consider the commutation relation between x_{ij} and M_1. This will be the same as the commutation relation between $x_{ij}M_1$ and M_1. Set $N := \{1,\dots,m\}\backslash\{m+1-i\}$. Then $x_{ij}M_1 = [N \cup \{j+m\}]$ and $M_1 = [N \cup \{m+1-i\}]$. Note that $m+1-i < j+m$; so that $[m+1-i][j+m] = q[j+m][m+1-i]$ in $\mathcal{O}_q(G(1,n))$. By using Proposition 2.3, it follows that $M_1(x_{ij}M_1) = q(x_{ij}M_1)M_1$. Hence, $M_1x_{ij} = qx_{ij}M_1$, and so $\sigma_1(x_{ij}) = qx_{ij}$ and $y_1x_{ij} = qx_{ij}y_1$, as claimed. □

Lemma 4.4 *For $1 \leq i \leq m$ and $1 \leq j < n - m$*

$$\sigma_2(x_{ij}) = qx_{ij}$$

while $\sigma_2(x_{i,n-m}) = q^{-1}x_{i,n-m}$. Consequently, $y_2 x_{ij} = qx_{ij}y_2$ for $1 \leq i \leq m$ and $1 \leq j < n - m$ while $y_2 x_{i,n-m} = q^{-1}x_{i,n-m}y_2$.

Proof: When $j < n - m$, the calculations are similar to those in the proof of the previous result and so are omitted.

Set $N := \{2, \ldots, m+1\} \backslash \{m+2-i\}$. Then, $x_{i,n-m}M_2 = [N \cup \{1\}]$ and $M_2 = [N \cup \{m+2-i\}]$. Now, $1 < m+2-i$; so that $[1][m+2-i] = q[m+2-i][1]$ in $\mathcal{O}_q(G(1,n))$. By using Proposition 2.3, it follows that $(x_{i,n-m}M_2)M_2 = qM_2(x_{i,n-m}M_2)$. Hence, $x_{i,n-m}M_2 = qM_2x_{i,n-m}$, and so $\sigma_2(x_{i,n-m}) = q^{-1}x_{i,n-m}$ and $y_2x_{i,n-m} = q^{-1}x_{i,n-m}y_2$, as claimed. □

5 Twisting by a 2-cocycle

Given a $\mathbb{K}$-algebra A that is graded by a semigroup, one can twist the multiplication in A by using a cocycle to produce a new multiplication. We only need to deal with $\mathbb{Z}^n$-graded algebras; so restrict our discussion to this case.

Definition 5.1 A 2*-cocycle (with values in $\mathbb{K}^*$)* on $\mathbb{Z}^n$ is a map $c : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{K}^*$ such that

$$c(s, t + u)c(t, u) = c(s, t)c(s + t, u)$$

for all $s, t, u \in \mathbb{Z}^n$.

Given a $\mathbb{Z}^n$-graded $\mathbb{K}$-algebra A if a is a homogeneous element in A_s, for $s \in \mathbb{Z}^n$, then we set $\text{content}(a) := s$.

Given a $\mathbb{Z}^n$-graded $\mathbb{K}$-algebra A and a 2-cocycle c on $\mathbb{Z}^n$, one can define a new $\mathbb{K}$-algebra $T(A)$ in the following way. As a graded vector space, A and $T(A)$ are isomorphic via an isomorphism $a \mapsto a'$. The multiplication in $T(A)$ is given by

$$a'b' := c(s, t)(ab)'$$

for homogeneous elements $a, b \in A$ with content s and t, respectively. The defining condition of a 2-cocycle is precisely the condition needed to ensure that this multiplication is associative. We refer to $T(A)$ as the *twist of A by c*, and the map $a \mapsto a'$ is the *twist map*.

The property of being an integral domain is preserved under twists, as the next lemma shows.

Lemma 5.2 *Let A be a $\mathbb{Z}^n$-graded $\mathbb{K}$-algebra that is an integral domain, and let c be a 2-cocycle on $\mathbb{Z}^n$. Then $T(A)$ is an integral domain.*

Proof: We may view A as graded by $\mathbb{Z}^n$, which can be made into a totally ordered group; then $T(A)$ is graded by the same totally ordered group. In order to see that the product of two nonzero elements a', b' of $T(A)$ is nonzero, it suffices to show that the product of their highest terms is nonzero. Hence, we may assume that a, b are homogeneous elements. In this case, $a'b'$ is a nonzero scalar multiple of $(ab)'$ and $ab \neq 0$, since A is a domain. Hence, $T(A)$ is a domain, as required. $\square$

Our aim is to twist the quantum grassmannian $\mathcal{O}_q(G(m,n))$ by a suitable 2-cocycle in such a way that the effect of the twist map is to cycle the indices of the generating quantum minors. There is a technical problem associated with this attempt, in that the defining relations for the quantum grassmannian (quantum Plücker relations) are complicated to deal with. We avoid the problem by using the notion of noncommutative dehomogenisation introduced earlier.

Let the standard basis of $\mathbb{Z}^n$ be denoted by $\{\epsilon(1), \dots, \epsilon(n)\}$, and let $(s_1, \dots, s_n)$ denote the element $s_1\epsilon(1) + \cdots + s_n\epsilon(n)$.

The quantum grassmannian $\mathcal{O}_q(G(m,n))$ has a natural grading by $\mathbb{Z}^n$ determined by the *content* of a generating quantum minor, where content$([I]) := \sum_{i \in I} \epsilon(i)$.

Note that M_α is a homogeneous element of $\mathcal{O}_q(G(m,n))$ and so the $\mathbb{Z}^n$-grading of $\mathcal{O}_q(G(m,n))$ extends in a natural way to $\mathcal{O}_q(G(m,n))[[M_\alpha]^{-1}]$ and hence to $\mathbb{K}[x_{ij}][y_\alpha^{\pm 1}; \sigma_\alpha]$ by using the dehomogenisation isomorphism of Theorem 4.1.

Lemma 5.3 *Let $p = q^{2/m}$. The map $c : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{K}^*$ defined by*

$$c((s_1, \dots, s_n), (t_1, \dots, t_n)) := \prod_{j \neq n} p^{s_n t_j}.$$

is a 2-cocycle.

Proof: Set $s = (s_1, \dots, s_n)$, $t = (t_1, \dots, t_n)$ and $u = (u_1, \dots, u_n)$. We have to check that

$$c(s, t+u)c(t, u) = c(s, t)c(s+t, u).$$

The proof is routine, one checks that each side is equal to

$$\prod_{j \neq n} p^{s_n t_j + s_n u_j + t_n u_j}.$$

$\square$

Now, we look at the effect of twisting the algebra $A := \mathbb{K}[x_{ij}][y_1^{\pm 1}; \sigma_1]$ by using the 2-cocycle c. Write y and σ for y_1 and σ_1, respectively.

We denote by $T(A)$ the twist of A by using the 2-cocyle c; so that if a, b are homogeneous elements with content $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_n)$, respectively, then

$$a'b' := c((s_1, \dots, s_n), (t_1, \dots, t_n))(ab)'.$$

Now, we are in the case that $\alpha = 1$, so that

$$x_{ij} = [\{1, \dots, m\} \cup \{j + m\} \backslash \{m + 1 - i\}][1, \dots, m]^{-1}.$$

Note that the content of x_{ij} is $\epsilon(j + m) - \epsilon(m + 1 - i)$ and that the content of y is $\epsilon(1) + \dots + \epsilon(m)$.

As A is generated by the homogeneous elements x_{ij} and y, the twisted algebra $T(A)$ is generated by the homogeneous elements x'_{ij} and y'. Our first aim is to describe the commutation relations satisfied by these elements.

We will often abuse notation by writing $c(a, b)$ instead of $c(\text{content}(a), \text{content}(b))$ for homogeneous elements $a, b \in A$.

Note that the value taken by c on a pair of elements from the set $\{x_{ij}, y\}$ is often equal to $p^0 = 1$. In fact, the only possibilities for a value other than p^0 occur in the cases when $\epsilon(n)$ occurs in the content of the first argument in c. This can only occur for $x_{i,n-m}$ and we check that

$$c(x_{i,n-m}, x_{l,n-m}) = p^{-1}, \qquad\qquad c(x_{i,n-m}, y) = p^m = q^2.$$

while $c(x_{i,n-m}, x_{l,j}) = 1$ for $j < n - m$ and $c(y, x_{ij}) = 1$ for all i, j. These observations make the calculation of the twisted product on pairs from the set $\{x'_{ij}, y'\}$ very easy.

Lemma 5.4 *(x'_{ij}) is a generic q-quantum matrix; that is, the algebra $\mathbb{K}[x'_{ij}]$ is isomorphic to $\mathcal{O}_q(M_{m,n-m})$. Moreover*

$$y'x'_{ij} = qx'_{ij}y' \quad \text{for} \quad j < n - m, \qquad \text{and} \qquad y'x'_{i,n-m} = q^{-1}x'_{i,n-m}y'.$$

Proof: First, we show that the x'_{ij} satisfy the commutation relations for a q-quantum matrix. The cases where $c(-, -)$ takes value 1 are easy to check, for example, for $i_1 < i_2$ and $j < n - m$,

$$x'_{i_1 j}x'_{i_2 j} = c(x_{i_1 j}, x_{i_2 j})(x_{i_1 j}x_{i_2 j})' = (x_{i_1 j}x_{i_2 j})'$$

while

$$x'_{i_2j}x'_{i_1j} = c(x_{i_2j}, x_{i_1j})(x_{i_2j}x_{i_1j})' = (x_{i_2j}x_{i_1j})' = q^{-1}(x_{i_1j}x_{i_2j})' = q^{-1}x'_{i_1j}x'_{i_2j}$$

and so $x'_{i_1j}x'_{i_2j} = qx'_{i_2j}x'_{i_1j}$, as required.

Also, for $i_1 < i_2$,

$$x'_{i_1(n-m)}x'_{i_2(n-m)} = p^{-1}(x_{i_1(n-m)}x_{i_2(n-m)})'$$

and

$$x'_{i_2(n-m)}x'_{i_1(n-m)} = p^{-1}(x_{i_2(n-m)}x_{i_1(n-m)})'$$

so again the desired q-commutation follows and the column relations are established.

The row computations are similar and so are omitted.

When $i_1 < i_2$ and $j_1 < j_2$, note that $c(x_{i_1j_2}, x_{i_2j_1}) = c(x_{i_2j_1}, x_{i_1j_2}) = 1$; and so

$$x'_{i_1j_2}x'_{i_2j_1} = (x_{i_1j_2}x_{i_2j_1})' = (x_{i_2j_1}x_{i_1j_2})' = x'_{i_2j_1}x'_{i_1j_2},$$

as required.

Continuing with $i_1 < i_2$ and $j_1 < j_2$, note that $c(x_{i_1j_1}, x_{i_2j_2}) = c(x_{i_2j_2}, x_{i_1j_1}) = 1$; and so

$$\begin{aligned} x'_{i_1j_1}x'_{i_2j_2} - x'_{i_2j_2}x'_{i_1j_1} &= (x_{i_1j_1}x_{i_2j_2})' - (x_{i_2j_2}x_{i_1j_1})' = (x_{i_1j_1}x_{i_2j_2} - x_{i_2j_2}x_{i_1j_1})' \\ &= (q - q^{-1})(x_{i_1j_2}x_{i_2j_1})' = (q - q^{-1})x'_{i_1j_2}x'_{i_2j_1}. \end{aligned}$$

This finishes the verification that the x'_{ij} satisfy the commutation relations of $\mathcal{O}_q(M_{m,n-m})$. As a result, there is an epimorphism from $\mathcal{O}_q(M_{m,n-m})$ onto $\mathbb{K}[x'_{ij}]$. If this epimorphism were not an isomorphism then $\text{GKdim}(\mathbb{K}[x'_{ij}]) < \text{GKdim}(\mathcal{O}_q(M_{m,n-m})) = m(n-m)$, by [7, Proposition 3.15], since $\mathcal{O}_q(M_{m,n-m})$ is a domain.

However, any monomial $x'_{i_1j_1}x'_{i_2j_2}\dots x'_{i_tj_t}$ is a nonzero scalar multiple of $(x_{i_1j_1}x_{i_2j_2}\dots x_{i_tj_t})'$; and so a linear combination of such monomials is zero if and only if a corresponding linear combination of monomials in the x_{ij} is zero. It follows that $\text{GKdim}(\mathbb{K}[x'_{ij}]) = \text{GKdim}(\mathbb{K}[x_{ij}]) = m(n-m)$. Thus, $\mathbb{K}[x'_{ij}] \cong \mathcal{O}_q(M_{m,n-m})$.

Now, we calculate how y' commutes with the x'_{ij}.

For $j < n-m$, observe that

$$x'_{ij}y' = c(x_{ij}, y)(x_{ij}y)' = (x_{ij}y)'$$

and so

$$y'x'_{ij} = c(y, x_{ij})(yx_{ij})' = (yx_{ij})' = q(x_{ij}y)' = qx'_{ij}y'.$$

Finally,

$$x'_{i,n-m}y' = c(x_{i,n-m}, y)(x_{i,n-m}y)' = q^2(x_{i,n-m}y)'$$

and so

$$y'x'_{i,n-m} = c(y, x_{i,n-m})(yx_{i,n-m})' = (yx_{i,n-m})' = q(x_{i,n-m}y)' = q^{-1}x'_{i,n-m}y'.$$

□

We now wish to consider the dehomogenisation isomorphism when $\alpha = 2$. In order to avoid a clash of notation, we will write

$$\mathcal{O}_q(G(m,n))[[M_2]^{-1}] \cong \mathbb{K}[z_{ij}][w^{\pm 1}; \phi]$$

where $z_{ij} := [M_2 \cup \{\widetilde{j+m+1}\} \backslash \{m+2-i\}]$ and $M_2 = [2, 3, \ldots, m+1]$.

Theorem 5.5

$$T(\mathbb{K}[x_{ij}][y^{\pm 1}; \sigma]) \quad \cong \quad \mathbb{K}[z_{ij}][w^{\pm 1}; \phi]$$

via a map $\theta : T(\mathbb{K}[x_{ij}][y^{\pm 1}; \sigma]) \longrightarrow \mathbb{K}[z_{ij}][w^{\pm 1}; \phi]$ *that sends* x'_{ij} *to* z_{ij} *and* y *to* w.

Proof: From Lemma 4.4 and Lemma 5.4, we see that the commutation relations among the $\{x'_{ij}, y'_1\}$ of $T(\mathbb{K}[x_{ij}][y_1^{\pm 1}; \sigma_1])$ are the same as the corresponding commutation relations among the generating set $\{z'_{ij}, y'_2\}$ of $\mathbb{K}[z_{ij}][y_2^{\pm 1}; \sigma_2]$.

Thus, we may define a homomorphism from $T(\mathbb{K}[x_{ij}][y^{\pm 1}; \sigma])$ to $\mathbb{K}[z_{ij}][w^{\pm 1}; \phi]$ by sending x'_{ij} to z'_{ij} and y to w. This homomorphism is an epimorphism, since the generators of $\mathbb{K}[z_{ij}][w^{\pm 1}; \phi]$ are in the image. Finally, the two algebras have the same Gelfand-Kirillov dimension, $m(n-m)+1$; so this epimorphism between two domains must also be a monomorphism, by [7, Proposition 3.15]. □

We may identify $\mathcal{O}_q(G(m,n))$ as a subalgebra of $\mathbb{K}[x_{ij}][y_1^{\pm 1}; \sigma_1]$ via the dehomogenisation isomorphism $\mathcal{O}_q(G(m,n))[[M_1]^{-1}] \cong \mathbb{K}[x_{ij}][y_1^{\pm 1}; \sigma_1]$ and identify another copy of $\mathcal{O}_q(G(m,n))$ with a subalgebra of $\mathbb{K}[z_{ij}][y_2^{\pm 1}; \sigma_2]$ via the isomorphism $\mathcal{O}_q(G(m,n))[[M_2]^{-1}] \cong \mathbb{K}[z_{ij}][y_2^{\pm 1}; \sigma_2]$. Our next aim is to show that the image of the first copy of $\mathcal{O}_q(G(m,n))$ under the map $\theta \circ T$ is the second copy of $\mathcal{O}_q(G(m,n))$. In order to do this, we need to track the image of a generating quantum minor through the sequence of maps

$$\mathcal{O}_q(G(m,n)) \xrightarrow{\phi_1} \mathbb{K}[x_{ij}][y_1^{\pm 1}; \sigma_1] \xrightarrow{T} \mathbb{K}[x'_{ij}][y_1'^{\pm 1}] \xrightarrow{\theta} \mathbb{K}[z_{ij}][y_2^{\pm 1}; \sigma_2] \xrightarrow{\rho_2} \mathcal{O}_q(G(m,n))[[M_2]^{-1}]$$

First, we record the effect of the twist map on quantum minors. We need to consider quantum minors in each of the quantum matrix algebras $\mathbb{K}[x_{ij}]$ and $\mathbb{K}[x'_{ij}]$; so for a given row set I and column set J we will denote the corresponding quantum minors by $[I|J]_x$ and $[I|J]_{x'}$, respectively.

Lemma 5.6 *Let* $[I|J]_x$ *be a quantum minor of the quantum matrix algebra* $\mathbb{K}[x_{ij}]$ *in the previous theorem. Then the image of* $[I|J]_x$ *under the twist map is* $[I|J]_{x'}$.

Proof: This proof is a routine calculation, using induction on the size of the quantum minor and quantum Laplace expansions, noting that each $c(-,-)$ that occurs takes value 1. $\square$

Lemma 5.7 *$c([I|J],y)=1$ when $n-m\notin J$ and $c([I|J],y)=q^2$ when $n-m\in J$*

Proof: This follows from the fact that $\epsilon(n)$ appears (with nonzero coefficients) in content$([I|J])$ if and only if $n-m\in J$ by [13, Proposition 4.3]. $\square$

As before, for a given row set I and column set J we will denote the corresponding quantum minors of the various quantum matrix algebras by $[I|J]_x$, $[I|J]_{x'}$ and $[I|J]_z$, respectively.

Lemma 5.8 *Let $I=[i_1,\dots,i_m]$ be a generating quantum minor of $\mathcal{O}_q(G(m,n))$. Then*

$$\rho_2\circ\theta\circ T\circ\phi_1([I])=\begin{cases}[i_1+1,\dots,i_m+1] & \text{if } i_m\neq n\\ q^{-2}[1,i_1+1,\dots,i_{m-1}+1] & \text{if } i_m=n\end{cases}$$

Proof: Note that

$$\phi_1(I)=[(m+1)-M_1\backslash I_r|I_c-m]_x y_1;$$

and so

$$T\circ\phi_1(I)=([(m+1)-M_1\backslash I_r|I_c-m]_x y_1)'=C^{-1}[(m+1)M_1\backslash I_r|I_c-m]_{x'}y_1'$$

where $C:=c([(m+1)-M_1\backslash I_r|I_c-m]_x,y)$ and note that $C=1$ if $n-m\notin I_c-m$ (and so if $n\notin I$), while $C=q^2$ if $n-m\in I_c-m$ (and so if $n\in I$).

Thus,

$$\begin{aligned}\theta\circ T\circ\phi_1(I) &= C^{-1}[(m+1)-M_1\backslash I_r|I_c-m]_z w\\ &= C^{-1}[(m+2)-M_2\backslash(I_r+1)|(I_c+1)-(m+1)]_z w\end{aligned}$$

Finally,

$$\begin{aligned}\rho_2\circ\theta\circ T\circ\phi_1(I) &= C^{-1}\rho_2([(m+2)-M_2\backslash(I_r+1)|(I_c+1)-(m+1)]_z w)\\ &= C^{-1}[(I_r+1)\sqcup(I_c+1)]=C^{-1}[I+1]\end{aligned}$$

and the result follows. Note that the last equality is obtained by the same calculation as in the proof of Corollary 4.2. $\square$

We can now reach our conclusion.

Theorem 5.9

$$T(\mathcal{O}_q(G(m,n)))\cong\mathcal{O}_q(G(m,n))$$

via a map θ that sends $[i_1,\dots,i_m]'$ to $[i_1+1,\dots,i_m+1]$, for $i_m<n$, and $[i_1,\dots,i_{m-1},n]$ is sent to $q^{-2}[1,i_1+1,\dots,i_{m-1}+1]$.

Proof: This follows immediately from the previous lemma. $\square$

6 Twisting the $\mathcal{H}$-prime spectrum

In this section we assume that q is a not a root of unity, in order that we know that the prime ideals of $\mathcal{O}_q(G(m,n))$ are completely prime, see [10, Theorem 5.2].

The natural $\mathbb{Z}^n$-grading on $\mathcal{O}_q(G(m,n))$ induces a rational action of the algebraic torus $\mathcal{H} := (\mathbb{K}^*)^n$ on $\mathcal{O}_q(G(m,n))$ by $\mathbb{K}$-automorphisms via

$$(h_1, \ldots, h_n).[i_1, \ldots, i_m] = h_{i_1} \cdots h_{i_m}[i_1, \ldots, i_m],$$

(see [1, Lemma II.2.11] for more details). In this setting, the homogeneous prime ideals of $\mathcal{O}_q(G(m,n))$ are exactly those primes that are invariant under this torus action. Hence homogeneous primes are also called *$\mathcal{H}$-primes*, and the set $\mathcal{H}-\mathrm{Spec}(\mathcal{O}_q(G(m,n)))$ of all $\mathcal{H}$-primes of $\mathcal{O}_q(G(m,n))$ is called the *$\mathcal{H}$-prime spectrum* of $\mathcal{O}_q(G(m,n))$. It was proved in [10] that this set is finite, and its cardinality was computed. The importance of the $\mathcal{H}$-prime spectrum was pointed out by Goodearl and Letzter who proved that the $\mathcal{H}$-prime spectrum parametrizes a natural stratification of the prime spectrum of $\mathcal{O}_q(G(m,n))$.

Theorem 6.1 *Suppose that q is not a root of unity. Let P be an $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G(m,n))$. Then $T(P) := \{p' \mid p \in P\}$ is an $\mathcal{H}$-prime ideal of $T(\mathcal{O}_q(G(m,n)))$.*

Proof: The algebra $\mathcal{O}_q(G(m,n))/P$ inherits a $\mathbb{Z}^n$-grading, as P is homogeneous; and so we can form the twisted algebra $T(\mathcal{O}_q(G(m,n))/P)$. It then follows that $T(\mathcal{O}_q(G(m,n))/P) \cong T(\mathcal{O}_q(G(m,n)))/T(P)$. Hence, it is enough to show that $T(\mathcal{O}_q(G(m,n))/P)$ is a domain and this follows from Lemma 5.2.

Corollary 6.2 *Suppose that q is not a root of unity. Then*

$$\theta(T(\mathcal{H}-\mathrm{Spec}(\mathcal{O}_q(G(m,n))))) = \mathcal{H}-\mathrm{Spec}(\mathcal{O}_q(G(m,n))),$$

where θ is the isomorphism defined in Theorem 5.9.

Proof: If P, Q are two distinct $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G(m,n))$ then $T(P)$ and $T(Q)$ are distinct $\mathcal{H}$-prime ideals of $T(\mathcal{O}_q(G(m,n)))$; and so their images under the isomorphism θ are distinct $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G(m,n))$. As the set of $\mathcal{H}$-prime ideals is finite, this establishes the claim. $\square$

It follows that if P is an $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G(m,n))$ then a quantum minor $[i_1, \ldots, i_m]$ is in P if and only if the quantum minor $[i_1+1, \ldots, i_m+1]$ is in $\theta(T(P))$, where $i_m+1 := 1$ if $i_m = n$. In other words, the sets of quantum minors that are in $\mathcal{H}$-prime ideals are permuted by $\theta \circ T$.

Note that in [10], it was shown that each $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G(2,4))$ is generated by the quantum minors that it contains, and it was conjectured that this holds in any $\mathcal{O}_q(G(m,n))$.

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