PRIME IDEALS IN CERTAIN QUANTUM DETERMINANTAL RINGS

K. R. Goodearl and T. H. Lenagan

Abstract. The ideal $I_1$ generated by the $2 \times 2$ quantum minors in the coordinate algebra of quantum matrices, $\mathcal{O}_q(M_{m,n}(k))$, is investigated. Analogues of the First and Second Fundamental Theorems of Invariant Theory are proved. In particular, it is shown that $I_1$ is a completely prime ideal, that is, $\mathcal{O}_q(M_{m,n}(k))/I_1$ is an integral domain, and that $\mathcal{O}_q(M_{m,n}(k))/I_1$ is the ring of coinvariants of a coaction of $k[x,x^{-1}]$ on $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$, a tensor product of two quantum affine spaces. There is a natural torus action on $\mathcal{O}_q(M_{m,n}(k))/I_1$ induced by an $(m+n)$-torus action on $\mathcal{O}_q(M_{m,n}(k))$. We identify the invariant prime ideals for this action and deduce consequences for the prime spectrum of $\mathcal{O}_q(M_{m,n}(k))/I_1$.

Introduction

Let $k$ be a field and let $q \in k^\times$. The coordinate ring of quantum $m \times n$ matrices, $\mathcal{A} := \mathcal{O}_q(M_{m,n}(k))$, is a deformation of the classical coordinate ring of $m \times n$ matrices, $\mathcal{O}(M_{m,n}(k))$. As such it is a $k$-algebra generated by $mn$ indeterminates $X_{ij}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the relations

\begin{align*}
X_{ij}X_{lj} &= qX_{lj}X_{ij} \quad \text{when } i < l; \\
X_{ij}X_{is} &= qX_{is}X_{ij} \quad \text{when } j < s; \\
X_{is}X_{lj} &= X_{ij}X_{is} \quad \text{when } i < l \text{ and } j < s; \\
X_{ij}X_{ls} - X_{ls}X_{ij} &= (q - q^{-1})X_{is}X_{lj} \quad \text{when } i < l \text{ and } j < s.
\end{align*}

In some references (e.g., [6, §3.5]), $q$ is replaced by $q^{-1}$. When $q = 1$ we recover $\mathcal{O}(M_{m,n}(k))$, which is the commutative polynomial algebra $k[X_{ij}]$. 

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When \( m = n \), the algebra \( A \) possesses a special element, the quantum determinant, \( D_q \), defined by

\[
D_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)}X_{2,\sigma(2)} \cdots X_{n,\sigma(n)},
\]

where \( l(\sigma) \) denotes the number of inversions in the permutation \( \sigma \). The quantum determinant \( D_q \) is a central element of \( A \) (see, for example, [6, Theorem 4.6.1]), and the localization \( A[D_q^{-1}] \) is the coordinate ring of the quantum general linear group, denoted \( O_q(GL_n(k)) \).

If \( I \subseteq \{1,\ldots,m\} \) and \( J \subseteq \{1,\ldots,n\} \) with \( |I| = |J| = t \), let \( D(I,J) \) denote the \( t \times t \) quantum minor obtained as the quantum determinant of the subalgebra of \( A \) obtained by deleting generators \( X_{ij} \) from the rows outside \( I \) and from the columns outside \( J \). We write \( I_t \) for the ideal generated by the \( (t+1) \times (t+1) \) quantum minors of \( A \). In [3] it is proved that \( A/I_t \) is an integral domain, for each \( 1 \leq t \leq \min\{m,n\} \). Independently, Rigal [7] has shown that \( A/I_1 \) is a domain; he also shows that \( A/I_1 \) is a maximal order in its division ring of fractions.

There is an action of the torus \( \mathcal{H} := (k^\times)^m \times (k^\times)^n \) by \( k \)-algebra automorphisms on \( A \) such that

\[
(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \cdot X_{ij} := \alpha_i \beta_j X_{ij}
\]

for all \( i, j \). The ideals \( I_t \) are easily seen to be invariant under \( \mathcal{H} \); so there is an induced action of \( \mathcal{H} \) on the factor algebras \( A/I_t \). In this paper, we study the prime ideal structure in the algebra \( A/I_1 \), paying particular attention to the \( \mathcal{H} \)-invariant prime ideals.

1. Complete primeness of \( I_1 \)

We give a direct derivation of the fact that \( A/I_1 \) is a domain. Although this is already established in both [3] and [7], the proof we give here is so much simpler and more transparent than either of the previous proofs that we think it will be useful to have it in a published form.

The coordinate ring of quantum affine n-space, denoted \( O_q(k^n) \), is defined to be the \( k \)-algebra generated by elements \( y_1, \ldots, y_n \) subject to the relations \( y_i y_j = q y_j y_i \) for each \( 1 \leq i < j \leq n \). It is well known that \( O_q(k^n) \) is an iterated Ore extension, and thus, in particular, \( O_q(k^n) \) is a domain. Our strategy is to produce a homomorphism of \( A \) into \( O_q(k^m) \otimes O_q(k^n) \). This latter algebra can also be presented as an iterated Ore extension and thus is a domain. We show that \( I_1 \) is the kernel of this map and so \( A/I_1 \) is a domain.
1.1. Theorem. The algebra $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ is isomorphic to a subalgebra of the tensor product $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$. In particular, $\mathcal{I}_1$ is a completely prime ideal of $\mathcal{O}_q(M_{m,n}(k))$.

Proof. Let $\mathcal{O}_q(k^m) = k[y_1, \ldots, y_m]$ and $\mathcal{O}_q(k^n) = k[z_1, \ldots, z_n]$ be the coordinate rings of quantum affine $m$-space and $n$-space, respectively. We define an algebra homomorphism $\theta : \mathcal{A} \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ such that $\theta(X_{ij}) = y_i \otimes z_j$ for all $i, j$. In order that this does extend to a well-defined algebra homomorphism, we must check that the elements $y_i \otimes z_j$ satisfy at least the relations defining $\mathcal{A}$. These are routine verifications; for example, if $i < l$ and $j < s$ then

$$(y_i \otimes z_j)(y_l \otimes z_s) = y_i y_l \otimes z_j z_s = y_i y_l \otimes qz_s z_j = q(y_i \otimes z_s)(y_l \otimes z_j),$$

while

$$(y_i \otimes z_s)(y_i \otimes z_j) = y_i y_i \otimes z_s z_j = q^{-1}y_i y_i \otimes z_s z_j = q^{-1}(y_i \otimes z_s)(y_i \otimes z_j).$$

Thus,

$$(y_i \otimes z_j)(y_l \otimes z_s) - (y_i \otimes z_s)(y_l \otimes z_j) = (q - q^{-1})(y_i \otimes z_s)(y_l \otimes z_j),$$

so that the fourth relation of the introduction holds. One can also obtain $\theta$ as the composition of the comultiplication $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ with the tensor product of the quotient maps from $\mathcal{A}$ to $\mathcal{A}/(X_{ij} \mid i > 1)$ and $\mathcal{A}/(X_{ij} \mid j > 1)$. We shall pursue the latter point of view in the next section.

Thus, there exists a unique $k$-algebra homomorphism

$$\theta : \mathcal{A} \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$$

such that $\theta(X_{ij}) = y_i \otimes z_j$ for all $i, j$. If $i < l$ and $j < s$ then the above calculations also show that $\theta(X_{ij} X_{ls} - qX_{ls} X_{ij}) = 0$; thus $\mathcal{I}_1 \subseteq \ker(\theta)$.

Now, $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ is a domain, since it can be viewed as a (multiparameter) quantum affine $(m + n)$-space with respect to the generators $y_1 \otimes 1, \ldots, y_m \otimes 1, 1 \otimes z_1, \ldots, 1 \otimes z_n$. Hence, $\ker(\theta)$ is a completely prime ideal of $\mathcal{A}$. We show that $\mathcal{I}_1 = \ker(\theta)$, so that $\mathcal{I}_1$ is completely prime. It remains to show that the induced map $\overline{\theta} : \mathcal{A}/\mathcal{I}_1 \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ is injective. Let $\mathcal{S}$ denote the set of monomials $X_{i_1j_1} X_{i_2j_2} \cdots X_{i_lj_l}$ in $\mathcal{A}$ such that $i_1 \geq i_2 \geq \cdots \geq i_l$ and $j_1 \leq j_2 \leq \cdots \leq j_l$. (We allow the monomial to be equal to 1 when $l = 0$.) We claim that the set $\overline{\mathcal{S}}$ of images forms a spanning set of $\mathcal{A}/\mathcal{I}_1$. 
It suffices to show that an arbitrary monomial $C$ in $A$ is congruent modulo $I_1$ to a linear combination of monomials from $S$. We proceed by induction on the index sets, where row index sequences $(i_1, i_2, \ldots, i_l)$ are ordered lexicographically with respect to $\geq$, column index sequences $(j_1, j_2, \ldots, j_l)$ are ordered lexicographically with respect to $\leq$, and pairs of sequences are ordered lexicographically.

If the claim fails, then it fails for a monomial $C = X_{i_1j_1} X_{i_2j_2} \cdots X_{i_lj_l}$ whose index set is minimal with respect to the ordering given in the previous paragraph. In particular, $C \notin S$. Let $r$ be the first subindex such that either $i_r < i_{r+1}$ or $j_r > j_{r+1}$.

If $i_r < i_{r+1}$ and $j_r \geq j_{r+1}$ then $C = \lambda C'$, where $\lambda$ is either 1 or $q$ and $C'$ is obtained from $C$ by switching $X_{i_rj_r}$ and $X_{i_{r+1}j_{r+1}}$. However,

$$(i_1, \ldots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \ldots, i_l) < (i_1, i_2, \ldots, i_l)$$

in our ordering, so $C'$ is congruent modulo $I_1$ to a linear combination of elements of $S$. Then $C$ is congruent to such a linear combination, contradicting our assumptions. A similar contradiction occurs if $i_r \leq i_{r+1}$ and $j_r > j_{r+1}$: this time, the row indices might not change, but

$$(j_1, \ldots, j_{r-1}, j_{r+1}, j_r, j_{r+2}, \ldots, j_l) < (j_1, \ldots, j_l),$$

so again we have a contradiction. Therefore, we must either have $i_r < i_{r+1}$ and $j_r < j_{r+1}$ or $i_r > i_{r+1}$ and $j_r > j_{r+1}$.

Suppose that $i_r < i_{r+1}$ and $j_r < j_{r+1}$. In this case, we have

$$X_{i_rj_r} X_{i_{r+1}j_{r+1}} - q X_{i_{r+1}j_r} X_{i_rj_{r+1}} \in I_1,$$

so that $C - qC' \in I_1$, where

$$C' = X_{i_1j_1} \cdots X_{i_{r-1}j_{r-1}} X_{i_rj_{r+1}} X_{i_{r+1}j_r} X_{i_{r+2}j_{r+2}} \cdots X_{i_lj_l}.$$ 

We obtain a contradiction as above.

The final case is $i_r > i_{r+1}$ and $j_r > j_{r+1}$, where we have

$$X_{i_rj_r} X_{i_{r+1}j_{r+1}} - q^{-1} X_{i_{r+1}j_r} X_{i_rj_{r+1}} \in I_1.$$

Thus, $C - q^{-1}C' \in I_1$, where

$$C' = X_{i_1j_1} \cdots X_{i_{r-1}j_{r-1}} X_{i_rj_{r+1}} X_{i_{r+1}j_r} X_{i_{r+2}j_{r+2}} \cdots X_{i_lj_l}.$$
and once again we reach a contradiction. This finishes the proof of the claim and establishes that $\mathcal{S}$ spans $A/I_1$.

Now, observe that in $O_q(k^m) \otimes O_q(k^n)$ we have

$$\theta(X_{i_1j_1} X_{i_2j_2} \cdots X_{i_lj_l}) = y_{i_1} y_{i_2} \cdots y_{i_l} \otimes z_{j_1} z_{j_2} \cdots z_{j_l}.$$ 

The monomials $y_{i_1} y_{i_2} \cdots y_{i_l}$ with $i_1 \geq i_2 \geq \cdots \geq i_l$ are linearly independent over $k$, and, likewise, the monomials $z_{j_1} z_{j_2} \cdots z_{j_l}$ with $j_1 \leq j_2 \leq \cdots \leq j_l$ are linearly independent over $k$. Hence, $\theta$ maps $\mathcal{S}$ bijectively to a linearly independent set in $O_q(k^m) \otimes O_q(k^n)$, so that $\mathcal{S}$ is a linearly independent set in $A/I_1$. Therefore, the map $\overline{\theta} : A/I_1 \to O_q(k^m) \otimes O_q(k^n)$ maps the $k$-basis $\mathcal{S}$ bijectively onto a linearly independent set, so that $\overline{\theta}$ is injective. □

2. Coinvariants

Theorem 1.1 has an invariant theoretic interpretation, which we discuss in this section. First, we outline what happens in the classical ($q = 1$) case.

2.1. Let $M_{u,v}(k)$ denote the algebraic variety of $u \times v$ matrices over $k$. Fix positive integers $m,n$ and $t \leq \min\{m,n\}$. The general linear group $GL_t(k)$ acts on $M_{m,t}(k) \times M_{t,n}(k)$ via

$$g \cdot (A, B) := (Ag^{-1}, gB).$$

Matrix multiplication yields a map

$$\mu : M_{m,t}(k) \times M_{t,n}(k) \to M_{m,n}(k),$$

the image of which is the variety of $m \times n$ matrices with rank at most $t$, and there is an induced map

$$\mu_* : \mathcal{O}(M_{m,n}(k)) \to \mathcal{O}(M_{m,t}(k) \times M_{t,n}(k)) = \mathcal{O}(M_{m,t}(k)) \otimes \mathcal{O}(M_{t,n}(k)).$$

The First Fundamental Theorem of invariant theory identifies the fixed ring of the coordinate ring $\mathcal{O}(M_{m,t}(k) \times M_{t,n}(k))$ under the induced action of $GL_t(k)$ as precisely the image of $\mu_*$. The Second Fundamental Theorem states that the kernel of $\mu_*$ is $\mathcal{I}_t$, the ideal generated by the $(t+1) \times (t+1)$ minors of $\mathcal{O}(M_{m,n}(k))$, so that the coordinate ring of the variety of $m \times n$ matrices of rank at most $t$ is $\mathcal{O}(M_{m,n}(k))/\mathcal{I}_t$. As a consequence, since this variety is irreducible, the ideal $\mathcal{I}_t$ is a prime ideal of $\mathcal{O}(M_{m,n}(k))$. 


2.2. We now proceed to explain the connection between Theorem 1.1 and the above invariant theoretic point of view.

The analog of $\mu_s$ is the $k$-algebra homomorphism

$$\theta_t : O_q(M_{m,n}(k)) \rightarrow O_q(M_{m,t}(k)) \otimes O_q(M_{t,n}(k))$$

induced from the comultiplication on $O_q(M_{m,n}(k))$, that is,

$$\theta_t(X_{ij}) = \sum_{l=1}^{t} X_{il} \otimes X_{lj}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. The comultiplications on $O_q(M_{m,t}(k))$ and $O_q(M_{t,n}(k))$ yield $k$-algebra homomorphisms

$$\rho_t : O_q(M_{m,t}(k)) \rightarrow O_q(M_{m,t}(k)) \otimes O_q(M_t(k)) \rightarrow O_q(M_{m,t}(k)) \otimes O_q(GL_t(k))$$

$$\lambda_t : O_q(M_{t,n}(k)) \rightarrow O_q(M_t(k)) \otimes O_q(M_{t,n}(k)) \rightarrow O_q(GL_t(k)) \otimes O_q(M_{t,n}(k))$$

which make $O_q(M_{m,t}(k))$ into a right $O_q(GL_t(k))$-comodule and $O_q(M_{t,n}(k))$ into a left $O_q(GL_t(k))$-comodule. Since $O_q(GL_t(k))$ is a Hopf algebra, the right comodule $O_q(M_{m,t}(k))$ becomes a left $O_q(GL_t(k))$-comodule on composing $\rho_t$ with $1 \otimes S$ followed by the flip (where $S$ denotes the antipode). Finally, the tensor product of the two left $O_q(GL_t(k))$-comodules $O_q(M_{m,t}(k))$ and $O_q(M_{t,n}(k))$ becomes a left $O_q(GL_t(k))$-comodule via the multiplication map on $O_q(GL_t(k))$. This comodule structure map,

$$\gamma_t : O_q(M_{m,t}(k)) \otimes O_q(M_{t,n}(k)) \rightarrow O_q(GL_t(k)) \otimes O_q(M_{m,t}(k)) \otimes O_q(M_{t,n}(k)),$$

can be described (using the Sweedler summation notation) as follows:

$$\gamma_t(a \otimes b) = \sum_{(a)} \sum_{(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$$

where $\rho_t(a) = \sum_{(a)} a_0 \otimes a_1$ and $\lambda_t(b) = \sum_{(b)} b_{-1} \otimes b_0$ for $a \in O_q(M_{m,t}(k))$ and $b \in O_q(M_{t,n}(k))$. Note that for $t > 1$, the map $\gamma_t$ is not an algebra homomorphism, since neither the antipode nor the multiplication map on $O_q(GL_t(k))$ is an algebra homomorphism. On the other hand, $\gamma_1$ is a $k$-algebra homomorphism.
Recall that the coinvariants of the coaction $\gamma_t$ are the elements $x$ in the tensor product $O_q(M_{m,t}(k)) \otimes O_q(M_{t,n}(k))$ such that $\gamma_t(x) = 1 \otimes x$. Quantum analogs of the First and Second Fundamental Theorems would be the following:

**Conjecture 1.** The set of coinvariants of $\gamma_t$ equals the image of $\theta_t$.

**Conjecture 2.** The kernel of $\theta_t$ is the ideal $I_t$.

We have proved Conjecture 2 in [3, Proposition 2.4] (essentially; the cited result covers the case $m = n$, and the general case follows easily by the method of [3, Corollary 2.6]). However, Conjecture 1 is open at present. Here we shall establish it in the case $t = 1$.

**2.3.** Note that $O_q(M_{m,1}(k))$ and $O_q(M_{1,n}(k))$ are quantum affine spaces on generators $X_{11}, X_{12}, \ldots, X_{m1}$ and $X_{11}, X_{12}, \ldots, X_{1n}$, respectively. In studying the case $t = 1$, it is convenient to replace $O_q(M_{m,1}(k))$ and $O_q(M_{1,n}(k))$ by $O_q(k^m) = k[y_1, \ldots, y_m]$ and $O_q(k^n) = k[z_1, \ldots, z_n]$, respectively. Then $\theta_1$ becomes the $k$-algebra homomorphism

$$\theta : O_q(M_{m,n}(k)) \to O_q(k^m) \otimes O_q(k^n), \quad X_{ij} \mapsto y_i \otimes z_j$$

used in the proof of Theorem 1.1. Next, the (quantum) coordinate ring of $1 \times 1$ matrices is just a polynomial ring $k[x]$, and the (quantum) coordinate ring of the $1 \times 1$ general linear group is the localization $k[x, x^{-1}]$. Thus, in the present case the coaction $\gamma_1$ becomes the $k$-algebra homomorphism

$$\gamma : O_q(k^m) \otimes O_q(k^n) \to k[x^{\pm 1}] \otimes O_q(k^m) \otimes O_q(k^n),$$

$$y_i \otimes 1 \mapsto x^{-1} \otimes y_i \otimes 1, \quad 1 \otimes z_j \mapsto x \otimes 1 \otimes z_j.$$

**2.4. Theorem.** The set of coinvariants of $\gamma$ is exactly the image of the algebra $O_q(M_{m,n}(k))$ in $O_q(k^m) \otimes O_q(k^n)$ under $\theta$.

**Proof.** Clearly $\gamma \theta(X_{ij}) = 1 \otimes y_i \otimes z_j = 1 \otimes \theta(X_{ij})$ for all $i, j$. Since $\theta$ and $\gamma$ are $k$-algebra homomorphisms, it follows that the image of $\theta$ is contained in the coinvariants of $\gamma$.

The algebra $O_q(k^m) \otimes O_q(k^n)$ has a basis consisting of pure tensors $Y \otimes Z$ where $Y$ is an ordered monomial in the $y_i$ and $Z$ is an ordered monomial in the $z_j$. Note that $\gamma(Y \otimes Z) = x^{s-r} \otimes Y \otimes Z$ where $r$ and $s$ are the total degrees of $Y$ and $Z$, respectively. Hence, the images $\gamma(Y \otimes Z)$ are $k$-linearly independent, and a linear combination $\sum_{l=1}^d \alpha_l Y_l \otimes Z_l$ of distinct monomial tensors is a coinvariant for $\gamma$ if and only if each $Y_l \otimes Z_l$ is a coinvariant.
Thus, we need only consider a single term
\[ Y \otimes Z = y_1 y_2 \cdots y_r \otimes z_1 z_2 \cdots z_s. \]
If \( Y \otimes Z \) is a coinvariant, then because \( \gamma(Y \otimes Z) = x^{s-r} \otimes Y \otimes Z \) we must have \( r = s \). Therefore
\[ Y \otimes Z = \theta(X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_r j_r}), \]
which shows that \( Y \otimes Z \) is in the image of \( \theta \), as desired.

3. \( \mathcal{H} \)-invariant prime ideals of \( \mathcal{O}_q(M_{m,n}(k))/I_1 \)

Under the mild assumption that our ground field \( k \) is infinite, we identify the \( \mathcal{H} \)-invariant prime ideals of the domain \( \mathcal{A}/I_1 = \mathcal{O}_q(M_{m,n}(k))/I_1 \). (Recall that \( \mathcal{H} \) denotes the torus \( (k^\times)^m \times (k^\times)^n \), acting on \( \mathcal{A} \) as described in the introduction.) This identifies the minimal elements in a stratification of \( \text{spec} \mathcal{A}/I_1 \), and yields a description of this prime spectrum as a finite disjoint union of commutative schemes corresponding to Laurent polynomial rings.

3.1. Let \( \mathcal{H} \) be a group acting as automorphisms on a ring \( \mathcal{A} \). We refer the reader to [1] for the definition of the \( \mathcal{H} \)-stratification of \( \text{spec} \mathcal{A} \), and here recall only that the \( \mathcal{H} \)-stratum of \( \text{spec} \mathcal{A} \) corresponding to an \( \mathcal{H} \)-prime ideal \( J \) is the set
\[ \text{spec}_J \mathcal{A} := \{ P \in \text{spec} \mathcal{A} \mid (P : H) = J \}. \]

In the case of the algebra \( \mathcal{A}/I_1 \), we shall (assuming \( k \) infinite) identify the \( \mathcal{H} \)-prime ideals – they turn out to be the same as the \( \mathcal{H} \)-invariant primes – and thus pin down the minimum elements of the \( \mathcal{H} \)-strata. Further, we shall see that each \( \mathcal{H} \)-stratum of \( \text{spec} \mathcal{A}/I_1 \) is homeomorphic to the spectrum of a Laurent polynomial ring over an algebraic extension of \( k \). This pattern is also known to hold for \( \text{spec} \mathcal{A} \) itself (at least when \( q \) is not a root of unity), but there the \( \mathcal{H} \)-prime ideals have not yet been completely identified.

3.2. It turns out that if a generator \( X_{ij} \) lies in an \( \mathcal{H} \)-prime ideal \( P \) of \( \mathcal{A} \) containing \( I_1 \), then either all the generators from the same row, or all the generators from the same column must also lie in \( P \). This leads us to make the following definition.

For subsets \( I \subseteq \{1, \ldots, m\} \) and \( J \subseteq \{1, \ldots, n\} \), set
\[ P(I, J) := I_1 + \langle X_{ij} \mid i \in I \rangle + \langle X_{ij} \mid j \in J \rangle. \]
Obviously, \( P(I, J) \) is an \( \mathcal{H} \)-invariant ideal of \( \mathcal{A} \). We shall show that \( P(I, J) \) is (completely) prime, and hence \( \mathcal{H} \)-prime.
Lemma. The factor algebra $\mathcal{A}/P(I,J)$ is isomorphic to $O_q(M_{m',n'}(k))/I_1'$, where $m' = m - |I|$ and $n' = n - |J|$, and $I_1'$ is the ideal generated by the $2 \times 2$ quantum minors of $O_q(M_{m',n'}(k))$. Hence, $P(I,J)$ is a completely prime ideal of $\mathcal{A}$.

Proof. The second statement follows immediately from the first statement and Theorem 1.1.

Set $I' := \{1, \ldots, m\} \setminus I$, and $J' := \{1, \ldots, n\} \setminus J$, and let $\mathcal{A}'$ be the $k$-subalgebra of $\mathcal{A}$ generated by the $X_{ij}$ for $i \in I'$ and $j \in J'$. Note that $\mathcal{A}'$ is isomorphic to $O_q(M_{m',n'}(k))$. Let $I_1'$ be the ideal of $\mathcal{A}'$ generated by the $2 \times 2$ quantum minors of $\mathcal{A}'$; that is, those for which both row indices are in $I'$ and both column indices are in $J'$. Obviously, $I_1' \subseteq \mathcal{A}' \cap I_1$, so that the inclusion $\mathcal{A}' \hookrightarrow \mathcal{A}$ induces a $k$-algebra homomorphism $f : \mathcal{A}'/I_1' \to \mathcal{A}/P(I,J)$. It suffices to show that $f$ is an isomorphism.

The factor $\mathcal{A}/P(I,J)$ is generated by the cosets of those $X_{ij}$ with $i \in I'$ and $j \in J'$, since $X_{ij} \in P(I,J)$ whenever $i \in I$ or $j \in J$. These cosets are all in the image of $f$; so $f$ is surjective.

Observe that there exists a $k$-algebra homomorphism $g : \mathcal{A} \to \mathcal{A}'$ such that $g(X_{ij}) = X_{ij}$ when $i \in I'$ and $j \in J'$, and $g(X_{ij}) = 0$ otherwise. To see this, note that the only problematic relations are those of the form $X_{ij}X_{ls} - X_{ls}X_{ij} = (q - q^{-1})X_{ls}X_{ij}$ for $i < l$ and $j < s$. However, if $i \notin I'$ then $X_{ij}$ and $X_{ls}$ both map to zero, and the relation maps to 0 = 0. Likewise, this happens in all cases except when $i, l \in I'$ and $j, s \in J'$: in this case, the relation above maps to a relation in $\mathcal{A}'$.

Consider a $2 \times 2$ quantum minor in $\mathcal{A}$ of the form $D = X_{ij}X_{ls} - qX_{ls}X_{ij}$ where $i < l$ and $j < s$. If $i \notin I'$ then both $X_{ij}$ and $X_{ls}$ are in $\ker(g)$, so that $D \in \ker(g)$. Likewise, $g(D) = 0$ when $l \notin I'$, or $j \notin J'$, or $s \notin J'$. On the other hand, $g(D) = D$ when $i, l \in I'$ and $j, s \in J'$. Further, $g(X_{ij}) = 0$ when $i \in I$ or $j \in J$. Hence, $g(P(I,J)) \subseteq I_1'$.

Therefore, $g$ induces a $k$-algebra homomorphism $\overline{g} : \mathcal{A}/P(I,J) \to \mathcal{A}'/I_1'$. Both of these algebras are generated by the cosets corresponding to those $X_{ij}$ such that $i \in I'$ and $j \in J'$. It follows that both $f\overline{g}$ and $\overline{g}f$ are identity maps, since both $f$ and $\overline{g}$ preserve these cosets. Hence, $f$ is an isomorphism. \[\square\]

Somewhat surprisingly, the $P(I,J)$ turn out to be the only $H$-prime ideals of $\mathcal{A}$ that contain $I_1$. The following lemma will be helpful in establishing this fact.

3.3. Lemma. Let $i, s \in \{1, \ldots, m\}$ and $j, t \in \{1, \ldots, n\}$. Then there exist scalars $\alpha \in \{1, q^{\pm 1}, q^{\pm 2}\}$ and $\beta \in \{1, q^{\pm 1}\}$ such that $X_{ij}X_{st} - \alpha X_{st}X_{ij}$ and


\(X_{ij}X_{st} - \beta X_{it}X_{sj}\) lie in \(\mathcal{I}_1\). In particular, the cosets \(X_{ij} + \mathcal{I}_1\) are all normal elements of \(\mathcal{A}/\mathcal{I}_1\).

**Proof.** If \(i = s\), then in view of the relations in \(\mathcal{A}\) we can take \(\alpha = \beta\) to be \(q\), 1, or \(q^{-1}\) (depending on whether \(j < t\) or \(j = t\) or \(j > t\)). Similarly, if \(j = t\), we can take \(\alpha \in \{1, q^{\pm 1}\}\) and \(\beta = 1\).

If \(i < s\) and \(j > t\), or if \(i > s\) and \(j < t\), then \(X_{ij}\) and \(X_{st}\) commute, and we can take \(\alpha = 1\). On the other hand, one of \(X_{it}X_{sj} - q^{\pm 1}X_{ij}X_{st}\) is a 2 \times 2 quantum minor, and so we can take \(\beta\) to be \(q\) or \(q^{-1}\).

Now suppose that \(i < s\) and \(j < t\). Then \(X_{ij}X_{st} - qX_{it}X_{sj}\) is a quantum minor, and we can take \(\beta = q\). But \(X_{st}X_{ij} - q^{-1}X_{it}X_{sj}\) is also a quantum minor, so we have \(X_{ij}X_{st} \equiv qX_{it}X_{sj} \equiv q^2 X_{st}X_{ij} \pmod{\mathcal{I}_1}\), and hence we can take \(\alpha = q^2\).

The remaining case follows from the previous one by exchanging \((i, j)\) and \((s, t)\), and then the final statement of the lemma is clear. \(\square\)

**3.4. Proposition.** Assume that \(k\) is an infinite field. Then the \(\mathcal{H}\)-prime ideals of \(\mathcal{O}_q(M_{m,n}(k))\) that contain \(\mathcal{I}_1\) are precisely the ideals \(P(I, J)\).

**Proof.** By Lemma 3.2, we know that the ideals \(P(I, J)\) are \(\mathcal{H}\)-prime. Consider an arbitrary \(\mathcal{H}\)-prime ideal \(P\) of \(\mathcal{A}\) that contains \(\mathcal{I}_1\). If all of the \(X_{ij}\) are in \(P\) then \(P\) must be the maximal ideal generated by the \(X_{ij}\). In that case, \(P = P(I, J)\), where \(I = \{1, \ldots, m\}\) and \(J = \{1, \ldots, n\}\). Hence, we may assume that not all \(X_{ij}\) are in \(P\). Set

\[
I = \{i \in \{1, \ldots, m\} \mid X_{ij} \in P \text{ for all } j\}
\]

\[
J = \{j \in \{1, \ldots, n\} \mid X_{ij} \in P \text{ for all } i\}.
\]

We first show that \(X_{ij} \in P\) if and only if \(i \not\in I\) or \(j \not\in J\). Certainly, if \(i \in I\) or \(j \in J\) then \(X_{ij} \in P\), by the definition of \(I\) and \(J\). Suppose that there exists an \(X_{ij} \in P\) such that \(i \not\in I\) and \(j \not\in J\). Then there exists an index \(s \neq i\) such that \(X_{sj} \not\in P\) and also there exists an index \(t \neq j\) such that \(X_{it} \not\in P\). By Lemma 3.3, there is a nonzero scalar \(\beta \in k\) such that \(X_{ij}X_{st} - \beta X_{it}X_{sj} \in P\). Thus, \(X_{ij} \in P\) would imply that \(X_{it}X_{sj} \in P\). However, \(X_{it}\) and \(X_{sj}\) are \(\mathcal{H}\)-eigenvectors which, by Lemma 3.3, are normal modulo \(P\). Hence, because \(P\) is \(\mathcal{H}\)-prime, \(X_{it}X_{sj} \in P\) would imply \(X_{it} \in P\) or \(X_{sj} \in P\), contradicting the choices of \(s\) and \(t\). Thus, we have established that \(X_{ij} \in P\) if and only if \(i \in I\) or \(j \in J\). Now \(P(I, J) \subseteq P\), and we need to establish equality.

Set \(B := \mathcal{A}/P(I, J)\) and \(\overline{P} = P/P(I, J)\), and note that \(B\) is a domain by Lemma 3.2. Write \(Y_{ij}\) for the image of \(X_{ij}\) in \(B\). The claim just established
implies that $Y_{ij} \notin \overline{P}$ if $i \notin I$ or $j \notin J$. Recall from Lemma 3.3 that the $Y_{ij}$ scalar-commute among themselves.

Now, $I \neq \{1, \ldots, m\}$ and $J \neq \{1, \ldots, n\}$, since not all of the $X_{ij}$ are in $P$. Let $s \in \{1, \ldots, m\} \setminus I$ and $t \in \{1, \ldots, n\} \setminus J$ be minimal, and consider the localization $C := B[Y_{st}^{-1}]$. Since $Y_{st} \notin \overline{P}$ there is an embedding of $B$ into $C$, and $PC$ is an $H$-prime ideal of $C$ such that $PC \cap B = P$.

Note that $Y_{ij} = 0$ if $i < s$ or $j < t$. If $i > s$ and $j > t$, then we have $Y_{st}Y_{ij} - qY_{sj}Y_{it} = 0$, so that $Y_{ij} = q^{-1}Y_{st}Y_{ij}$ in $C$. Hence, $C$ is generated as an algebra by $Y_{st}^{\pm 1}$ together with $Y_{sj}$ for $j > t$ and $Y_{it}$ for $i > s$. Thus, $C$ is a homomorphic image of a localized multiparameter quantum affine space $O_\lambda(k^r)[z_i^{-1}]$, for $r = m - s + n - t + 1$ and for a suitable parameter matrix $\lambda$.

The standard action of the torus $\mathcal{H}_r := (k^\times)^r$ on $O_\lambda(k^r)$ has 1-dimensional eigenspaces generated by individual monomials (here, we use the fact that $k$ is infinite). Therefore, the same holds for $C$. Hence, any nonzero $\mathcal{H}_r$-invariant ideal of $C$ contains a monomial, and so any nonzero $\mathcal{H}_r$-prime ideal of $C$ must contain one of $Y_{s+1,t}, \ldots, Y_{nt}, Y_{s,t+1}, \ldots, Y_{sn}$. Since $PC$ contains none of these elements, to show that $PC = 0$ it suffices to establish that $PC$ is $\mathcal{H}_r$-prime. But $PC$ is already $\mathcal{H}$-prime, so it is enough to see that the $\mathcal{H}_r$-invariant ideals of $C$ are the same as the $\mathcal{H}$-invariant ideals. This will follow from showing that the images of $\mathcal{H}$ and $\mathcal{H}_r$ in aut $C$ coincide.

Since the $Y_{ij}$ are $\mathcal{H}$-eigenvectors, it is clear that the image of $\mathcal{H}$ is contained in the image of $\mathcal{H}_r$. The reverse inclusion amounts to the following statement:

(*) Given any $\alpha_s, \ldots, \alpha_m, \beta_{t+1}, \ldots, \beta_n \in k^\times$, there exists $h \in \mathcal{H}$ such that $h(Y_{it}) = \alpha_iY_{it}$ for $i = s, \ldots, m$ and $h(Y_{sj}) = \beta_jY_{sj}$ for $j = t + 1, \ldots, n$.

Now, there exists $h_1 \in \mathcal{H}$ such that $h_1(X_{ij}) = X_{ij}$ for all $i, j$ with $i < s$, and $h_1(X_{ij}) = \alpha_iX_{ij}$ for all $i, j$ with $i \geq s$. Also, there exists $h_2 \in \mathcal{H}$ such that $h_2(X_{ij}) = X_{ij}$ for all $i, j$ with $j \leq t$ and $h_2(X_{ij}) = \alpha_i^{-1}\beta_jX_{ij}$ for all $i, j$ with $j > t$. Setting $h = h_1h_2$ gives the desired element of $\mathcal{H}$, establishing (*).

Therefore, $PC = 0$, and so $P = 0$. This means that $P = P(I, J)$. □

3.5. Corollary. If the field $k$ is infinite, then $O_q(M_{m,n}(k))/I_1$ has precisely $(2^n - 1)(2^m - 1) + 1$ distinct $\mathcal{H}$-prime ideals, all of which are completely prime. Further, each $\mathcal{H}$-stratum of spec $O_q(M_{m,n}(k))/I_1$ is homeomorphic to the prime spectrum of a Laurent polynomial ring over an algebraic field extension of $k$.

Proof. The first statement is clear from Proposition 3.4. The second statement is not actually a corollary of the Proposition, but is included to fill
3.6. In particular, the corollary above explains why in the algebra $O_q(M_2(k))$ there are precisely $10 = (2^2 - 1)^2 + 1$ distinct $H$-primes which contain the quantum determinant. This fact was known previously by direct enumeration of these primes. The remaining $H$-primes correspond to $H$-primes of $O_q(GL_2(k))$; there are 4 of these, as has long been known. We can display the lattice of $H$-prime ideals of $O_q(M_2(k))$ as in the diagram below, where the symbols $\bullet$ and $\circ$ stand for generators $X_{ij}$ which are or are not included in a given prime, and $\Box$ stands for the $2 \times 2$ quantum determinant. For example, $(\bullet \bullet)$ stands for the ideal $(X_{12}, X_{21})$, and $(\Box)$ stands for the ideal $(X_{11}X_{22} - qX_{12}X_{21})$.

The corresponding $H$-strata in spec $O_q(M_2(k))$ can be easily calculated. For instance, if $q$ is not a root of unity, the strata corresponding to $(\circ \circ)$ and
are 2-dimensional, the strata corresponding to (♦♦), (♦♣), (♣♦), and (♣♣) are all 1-dimensional, and the remaining 8 strata are singletons.

3.7. We close with some remarks concerning catenarity. (Recall that the prime spectrum of a ring \( A \) is catenary provided that for any comparable primes \( P \subset Q \) in spec \( A \), all saturated chains of primes from \( P \) to \( Q \) have the same length.) It is conjectured that spec \( \mathcal{O}_q(M_{m,n}(k)) \) is catenary. In [2, Theorem 1.6], we showed that catenarity holds for any affine, noetherian, Auslander-Gorenstein, Cohen-Macaulay algebra \( A \) with finite Gelfand-Kirillov dimension, provided spec \( A \) has normal separation. All hypotheses but the last are known to hold for the algebra \( A = \mathcal{O}_q(M_{m,n}(k)) \). We can, at least, say that the portion of spec \( A \) above \( \mathcal{I}_1 \) – that is, spec \( A/\mathcal{I}_1 \) – is catenary: In view of Lemma 3.3, \( A/\mathcal{I}_1 \) is a homomorphic image of a multi-parameter quantum affine space \( \mathcal{O}_\lambda(k^{n^2}) \), and spec \( \mathcal{O}_\lambda(k^{n^2}) \) is catenary by [2, Theorem 2.6].

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Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
E-mail address: goodearl@math.ucsb.edu

Department of Mathematics, J.C.M.B., Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland
E-mail address: tom@maths.ed.ac.uk