Quantised coordinate rings of semisimple groups are unique factorisation domains

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Abstract

We show that the quantum coordinate ring of a semisimple group is a unique factorisation domain in the sense of Chatters and Jordan in the case where the deformation parameter $q$ is a transcendental element.


Key words: Unique factorisation domain, quantum enveloping algebra, quantum coordinate ring.

Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{C}^*$ is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the present authors, together with L Rigal, [11], have shown that many quantum algebras are noetherian UFD. In particular, we have shown that the quantum group $O_q(SL_n)$ is a noetherian UFD.

Let $G$ be a connected simply connected complex semisimple algebraic group. Since in the classical setting it was shown by Popov, [12], that the ring of regular functions on $G$ is a unique factorisation domain, one can ask if a similar result holds for the quantisation

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\( O_q(G) \) of the coordinate ring of \( G \). The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of \( O_q(G) \) that was constructed by Joseph, [8].

1 Quantised enveloping algebras and quantum coordinate rings

1.1 Quantised enveloping algebras

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra of rank \( n \). We denote by \( \pi = \{\alpha_1, \ldots, \alpha_n\} \) the set of simple roots associated to a triangular decomposition \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \). Recall that \( \pi \) is a basis of a euclidean vector space \( E \) over \( \mathbb{R} \), whose inner product is denoted by \((\, , \)\) (\( E \) is usually denoted by \( \mathfrak{h}^*_\mathbb{R} \) in Bourbaki). We denote by \( \mathcal{W} \) the Weyl group of \( \mathfrak{g} \); that is, the subgroup of the orthogonal group of \( E \) generated by the reflections \( s_i := s_{\alpha_i} \), for \( i \in \{1, \ldots, n\} \), with reflecting hyperplanes \( H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\} \), for \( i \in \{1, \ldots, n\} \). If \( w \in \mathcal{W} \), we denote by \( l(w) \) its length. Further, we denote by \( w_0 \) the longest element of \( \mathcal{W} \). Throughout this paper, the Coxeter group \( \mathcal{W} \) will be endowed with the Bruhat order that we denote by \( \leq \). We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by \( R^+ \) the set of positive roots and by \( R \) the set of roots. We set \( Q^+ : = \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n \). We denote by \( \varpi_1, \ldots, \varpi_n \) the fundamental weights, by \( P \) the \( \mathbb{Z} \)-lattice generated by \( \varpi_1, \ldots, \varpi_n \), and by \( P^+ \) the set of dominant weights. In the sequel, \( P \) will always be endowed with the following partial order:

\[
\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.
\]

Finally, we denote by \( A = (a_{ij}) \in M_n(\mathbb{Z}) \) the Cartan matrix associated to these data.

Recall that the scalar product of two roots \((\alpha, \beta)\) is always an integer. As in [1], we assume that the short roots have length \( \sqrt{2} \).

For each \( i \in \{1, \ldots, n\} \), set \( q_i : = q^{(\alpha_i, \alpha_i)} \) and

\[
\begin{bmatrix}
  m \\
  k
\end{bmatrix}_i : = \frac{(q_i - q_i^{-1})(q_i^{m-1} - q_i^{-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1})(q_i^k - q_i^{-k})(q_i - q_i^{-1}) \cdots (q_i^{m-k} - q_i^{k-m})}
\]

for all integers \( 0 \leq k \leq m \). By convention, we have

\[
\begin{bmatrix}
  m \\
  0
\end{bmatrix}_i : = 1.
\]
We will use the definition of the quantised enveloping algebra given in [1, I.6.3, I.6.4].

The quantised enveloping algebra $U_q(g)$ of $g$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j)$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j).$$

Note that $U_q(g)$ is a Hopf algebra; its comultiplication is defined by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

its counit by

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and its antipode by

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.$$

We refer the reader to [1, 7, 8] for more details on this algebra. Further, as usual, we denote by $U_q^+(g)$ the subalgebra of $U_q(g)$ generated by $E_1, \ldots, E_n$ and by $U_q(b^+)$ the subalgebra of $U_q(g)$ generated by $E_1, \ldots, E_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. In a similar manner, $U_q^-(g)$ is the subalgebra of $U_q(g)$ generated by $F_1, \ldots, F_n$ and $U_q(b^-)$ is the subalgebra of $U_q(g)$ generated by $F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$.

### 1.2 Representation theory of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra $U_q(g)$ is analogous to the representation theory of the classical enveloping algebra $U(g)$. In this section, we collect the properties that will be needed in the rest of the paper.
As usual, if $M$ is a left $U_q(\mathfrak{g})$-module, we denote its dual by $M^*$. Observe that $M^*$ is a right $U_q(\mathfrak{g})$-module in a natural way. However, by using the antipode of $U_q(\mathfrak{g})$, this right action of $U_q(\mathfrak{g})$ on $M^*$ can be twisted to a left action, so that $M^*$ can be viewed as a left $U_q(\mathfrak{g})$-module.

Let $M$ be a $U_q(\mathfrak{g})$-module and $m \in M$. The element $m$ is said to have weight $\lambda \in P$ if $K_i.m = q^{(\lambda,\alpha_i)}m$ for all $i \in \{1, \ldots, n\}$. For each $\lambda \in P$, set $M_{\lambda} := \{ m \in M \mid K_i.m = q^{(\lambda,\alpha_i)}m \text{ for all } i \in \{1, \ldots, n\} \}$. If $M_{\lambda} \neq 0$ then $M_{\lambda}$ is said to be a weight space of $M$ and $\lambda$ is a weight of $M$.

It is well-known, see, for example [1, 7], that, for each dominant weight $\lambda \in P^+$, there exists a unique (up to isomorphism) simple finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda$ that we denote by $V(\lambda)$. In the following proposition, we collect some well-known properties of the $V(\lambda)$, for $\lambda \in P^+$. We refer the reader to [1, especially I.6.12], [6] and [7] for details and proofs.

**Proposition 1.1** Denote by $\Omega(\lambda)$ the set of those weights $\mu \in P$ such that $V(\lambda)_{\mu} \neq 0$.

1. $V(\lambda) = \bigoplus_{\mu \in \Omega(\lambda)} V(\lambda)_{\mu}$

2. The weights of $V(\lambda)$ are given by Weyl’s character formula. In particular, if $\mu \in \Omega(\lambda)$, then $w\mu \in \Omega(\lambda)$ for all $w \in W$.

3. For all $w \in W$, one has $\dim_{\mathbb{C}} V(\lambda)_{w\lambda} = 1$.

4. $V(\lambda)^* \simeq V(-w_0\lambda)$.

5. The weight $w_0\lambda$ is the unique lowest weight of $V(\lambda)$.

In particular, for all $\mu \in \Omega(\lambda)$, one has $w_0\lambda \leq \mu \leq \lambda$.

For all $w \in W$ and $\lambda \in P^+$, let $u_{w\lambda}$ denote a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Then we denote by $V_w^+(\lambda)$ the Demazure module associated to the pair $\lambda, w$, that is: $V_w^+(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q^+(\mathfrak{b}^+)u_{w\lambda}$. We also set $V_w^-(\lambda) := U_q^-(\mathfrak{g})u_{w\lambda} = U_q^-(\mathfrak{b}^-)u_{w\lambda}$.

(Observable that these definitions are independent of the choice of $u_{w\lambda}$ because of Proposition 1.1 (3).)

The following result may be well-known; however, we have been unable to locate a precise statement.
Proposition 1.2

1. \(V^+_{w_0}(\lambda) = V(\lambda) = V^-_{id}(\lambda)\).

2. For all \(i, j \in \{1, \ldots, n\}\), one has

\[
V^+_{w_0s_i}(\varpi_j) = \begin{cases} \sum_{\mu \in \Omega(\varpi_j) \setminus \{w_0\varpi_j\}} V(\varpi_j)^{\mu} & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}, \end{cases}
\]

and

\[
V^-_{s_i}(\varpi_j) = \begin{cases} \sum_{\mu \in \Omega(\varpi_j) \setminus \{\varpi_j\}} V(\varpi_j)^{\mu} & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}. \end{cases}
\]

Proof. We only prove the assertions corresponding to “positive” Demazure modules, the proof for “negative” Demazure modules is similar.

Since \(w_0\lambda\) is the lowest weight of \(V(\lambda)\), we have \(U_q^+(g)u_{w_0}\lambda = V(\lambda)\); that is, \(V^+_{w_0}(\lambda) = V(\lambda)\). This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let \(i, j \in \{1, \ldots, n\}\) with \(i \neq j\). Then \(s_i(\varpi_j) = \varpi_j\). Hence, in this case, one has:

\[
V^+_{w_0s_i}(\varpi_j) = U_q^+(g)u_{w_0s_i}\varpi_j = U_q^+(g)u_{w_0}\varpi_j = V^+_{w_0}(\varpi_j) = V(\varpi_j).
\]

Next, let \(j \in \{1, \ldots, n\}\). Then \(s_j(\varpi_j) = \varpi_j - \alpha_j\). Let \(\mu \in \Omega(\varpi_j)\) with \(\mu \neq w_0\varpi_j\), and let \(m \in V(\varpi_j)_\mu\) be any nonzero element. It follows from the first assertion that there exists \(x \in U_q^+(g)\) such that \(m = x.u_{w_0}\varpi_j\). The element \(x\) can be written as a linear combination of products \(E_{i_1}\ldots E_{i_k}\), with \(k \in \mathbb{N}^*\) and \(i_1, \ldots, i_k \in \{1, \ldots, n\}\). Naturally, one can assume that \(E_{i_1}\ldots E_{i_k}.u_{w_0}\varpi_j \neq 0\) for each such product. Let \(E_{i_1}\ldots E_{i_k}\) be one of these products. Since \(w_0\pi = -\pi\), there exists \(l \in \{1, \ldots, n\}\) such that \(w_0\alpha_{i_k} = -\alpha_l\). We will prove that \(l = j\). Indeed, assume that \(l \neq j\). Since \(E_{i_k}.u_{w_0}\varpi_j\) is a nonzero vector of \(V(\varpi_j)\) of weight \(w_0\varpi_j + \alpha_{i_k}\), we get that

\[
w_0\varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).
\]

Then, we deduce from Proposition 1.1 that

\[
s_l w_0 (w_0\varpi_j + \alpha_{i_k}) \in \Omega(\varpi_j),
\]

that is,

\[
s_l \varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

Further, since we have assumed that \(l \neq j\), we get \(s_l \varpi_j = \varpi_j\), so that

\[
\varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

This contradicts the fact that \(\varpi_j\) is the highest weight of \(V(\varpi_j)\).
Thus, we have just proved that \( w_0 \alpha_{ik} = -\alpha_j \) for all products \( E_{i_1} \ldots E_{i_k} \) that appear in \( x \). Now, observe that \( E_{i_k}.u_{w_0 \varpi_j} \) is a nonzero vector of \( V(\varpi_j) \) of weight \( w_0(\varpi_j - \alpha_j) = w_0 s_j \varpi_j \). Since \( \dim \, V(\varpi_j)_{w_0 s_j \varpi_j} = 1 \), we get that \( E_{i_k}.u_{w_0 \varpi_j} = a u_{w_0 s_j \varpi_j} \) for a certain nonzero complex number \( a \). Hence we get that
\[
m = x.u_{w_0 \varpi_j} = \sum \bullet E_{i_1} \ldots E_{i_k}.u_{w_0 \varpi_j} = y.u_{w_0 s_j \varpi_j},
\]
where \( \bullet \) denote some nonzero complex numbers and \( y \in U^+_q(g) \). Thus \( m \in V^+_{w_0 s_j}(\varpi_j) \). This shows that
\[
\bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{ w_0 \varpi_j \}} V(\varpi_j)_\mu \subseteq V^+_{w_0 s_j}(\varpi_j).
\]
As the reverse inclusion is trivial, this finishes the proof. □

### 1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let \( G \) be a connected, simply connected, semisimple algebraic group over \( \mathbb{C} \) with Lie algebra \( \text{Lie}(G) = g \). Since \( U_q(g) \) is a Hopf algebra, one can define its Hopf dual \( U_q(g)^\ast \) (see [8, 1.4]) via
\[
U_q(g)^\ast := \{ f \in \text{Hom}_\mathbb{C}(U_q(g), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension} \}.
\]

The quantised coordinate ring \( O_q(G) \) of \( G \) is the subalgebra of \( U_q(g)^\ast \) generated by the coordinate functions \( c_{\lambda, \xi, v}^\lambda \) for all \( \lambda \in P^+ \), \( \xi \in V(\lambda)^\ast \) and \( v \in V(\lambda) \), where \( c_{\xi, v}^\lambda \) is the element of \( U_q(g)^\ast \) defined by
\[
c_{\xi, v}^\lambda(u) := \xi(uv) \text{ for all } u \in U_q(g),
\]
see, for example, [8, Chapter 9]. As usual, if \( \xi \in V(\lambda)_\eta^\ast \) and \( v \in V(\lambda)_\mu \), we write \( c_{\eta, \mu}^\lambda \) instead of \( c_{\xi, v}^\lambda \). Naturally, this leads to some ambiguity. However, when \( \mu \in W.\lambda \) and \( \eta \in W.(-w_0 \lambda) \), then \( \dim \, V(\lambda)_\mu = 1 = \dim \, V(\lambda)_\eta^\ast \), so that this ambiguity is very minor.

It is well-known that \( O_q(G) \) is a noetherian domain and a Hopf-subalgebra of \( U_q(g)^\ast \), see [1, 8]. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, [1, 1.9.25] or [8, 1.3.5]), and then to obtain an action of the torus \( \mathcal{H} := (\mathbb{C}^*)^{2n} \) on \( O_q(G) \) (see [2, 5.2]). More precisely, observe that the torus \( H := (\mathbb{C}^*)^n \) can be identified with \( \text{Hom}(P, \mathbb{C}^*) \) via:
\[
h(\lambda) = h_1^\lambda \ldots h_n^\lambda,
\]
where \( h = (h_1, \ldots, h_n) \in H \) and \( \lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n \) with \( \lambda_1, \ldots, \lambda_n \in \mathbb{Z} \). Then, it is known (see [5, 3.3] or [1, I.1.18]) that the torus \( H \) acts rationally by \( \mathbb{C} \)-algebra automorphisms on \( O_q(G) \) via:

\[
g \cdot c^\lambda_{\xi,v} = g_1(\mu)g_2(\eta)c^\lambda_{\xi,v},
\]

for all \( g = (g_1, g_2) \in H = H \times H, \lambda \in P^+, \xi \in V(\lambda)^*_\mu \) and \( v \in V(\lambda)_\eta \).

(We refer the reader to [1, II.2.6] for the definition of a rational action.)

As usual, we denote by Spec\((O_q(G))\) the set of prime ideals in \( O_q(G) \). Recall that Joseph has proved [9] that every prime in \( O_q(G) \) is completely prime.

Since \( H \) acts by automorphisms on \( O_q(G) \), this induces an action of \( H \) on the prime spectrum of \( O_q(G) \). As usual, we denote by \( \mathcal{H}\text{-Spec}(O_q(G)) \) the set of those prime ideals of \( O_q(G) \) that are \( \mathcal{H} \)-invariant. This is a finite set since Brown and Goodearl [2, Section 5] (see also [1, II.4]) have shown using previous results of Joseph that

\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W\},
\]

where

\[
Q^+_{w_+} := \langle c^\lambda_{\xi,v} \mid \lambda \in P^+, \xi \in (V^+_\mu(\lambda))_\mu^* \subseteq V(\lambda)^*_\mu \rangle,
\]

\[
Q^-_{w_-} := \langle c^\lambda_{\xi,v} \mid \lambda \in P^+, \xi \in (V^-_{w_0}(\lambda))_\mu^* \subseteq V(\lambda)^*_\mu \rangle,
\]

and

\[
Q_{w_+,w_-} := Q^+_{w_+} + Q^-_{w_-}.
\]

Since \( q \) is transcendental, it follows from [10, Théorème 3] that it is enough to consider the fundamental weights in the definition of \( Q^+_{w_+} \) and \( Q^-_{w_-} \). More precisely, we deduce from [10, Théorème 3] the following result.

**Theorem 1.3 (Joseph)**

\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W\},
\]

where

\[
Q^+_{w_+} := \langle c^\varpi_j \mid j \in \{1, \ldots, n\}, \varpi \in V(\varpi_j)^* \rangle\text{ and } Q^-_{w_-} := \langle c^\varpi_j \mid j \in \{1, \ldots, n\}, \varpi \in (V^-_{w_0}(\varpi_j))_\mu^* \subseteq V(\varpi_j)^* \rangle,
\]

and

\[
Q_{w_+,w_-} := Q^+_{w_+} + Q^-_{w_-}.
\]

Moreover the prime ideals \( Q_{w_+,w_-} \), for \( (w_+, w_-) \in W \times W \), are pairwise distinct.
2 $O_q(G)$ is a noetherian UFD.

In this section, we prove that $O_q(G)$ is a noetherian UFD (We refer the reader to [11, Section 1] for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal $\mathcal{H}$-eigenvectors $r_1, \ldots, r_k$ of $O_q(G)$ such that each $\langle r_i \rangle$ is (completely) prime, and that each nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains one of the $r_i$. This property may be thought of as a “weak factoriality” result: $O_q(G)$ is an $\mathcal{H}$-UFD in the terminology of [11].

2. Secondly, by using the $H$-stratification theory of Goodearl and Letzter (see [1, II]), we show that the localisation of $O_q(G)$ with respect to the multiplicative system generated by the $r_i$ is a noetherian UFD.

3. Finally, we use a noncommutative analogue of Nagata’s Lemma (see [11, Proposition 1.6]) to prove that $O_q(G)$ itself is a noetherian UFD.

2.1 $O_q(G)$ is an $\mathcal{H}$-UFD

This aim of this section is two-fold. First, we show that for each $i \in \{1, \ldots, n\}$, the ideal generated by the normal element $c_{w_0, w}^{v_i}$ or $c_{w_0, w}^{-v_i}$ is (completely) prime and then we prove that every nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains either one of the $c_{w_0, w}^{v_i}$ or one of the $c_{w_0, w}^{v_i}$.

**Lemma 2.1** Let $i \in \{1, \ldots, n\}$. Then $Q_{w_0, s_i, w_0} = \langle c_{w_0, w_0, w}^{v_i} \rangle$ and $Q_{w_0, s_i, w_0} = \langle c_{w_0, w_0, w}^{-v_i} \rangle$.

**Proof.** Recall that

$$Q_{w_0, s_i, w_0} = Q^+_{w_0} + Q^-_{w_0},$$

where

$$Q^+_{w_0} = \langle c_{\xi, v}^{v_i} \rangle \mid j \in \{1, \ldots, n\}, v \in V(w_j)_{w_0, w}, \xi \in (V^+_{w_0}(w_j))^+ \subseteq V(w_j)^*,$$

$$Q^-_{s_i, w_0} = \langle c_{\xi, v}^{-v_i} \rangle \mid j \in \{1, \ldots, n\}, v \in V(w_j)_{w_0, w}, \xi \in (V^-_{w_0}(w_j))^+ \subseteq V(w_j)^*.$$  

Next, it follows from Proposition 1.2(1) that $V^+_{w_0}(w_j) = V(w_j)$ for all $j$, so that $Q^+_{w_0} = (0)$. Also, we deduce from Proposition 1.2(2) that $V^-_{w_0}(w_j) = V(w_j)$ if $j \neq i$, and $V^-_{s_i}(w_j) = \oplus_{\mu \in \Omega(w_i) \setminus \{w_i\}} V(w_i, \mu)$. Hence,

$$Q^-_{s_i, w_0} = \langle c_{\xi, v}^{-v_i} \rangle \mid v \in V(w_i)_{w_0, w}, \xi \in V(w_i)^*_{w_0, w}.$$
that is, \( Q_{s_1 w_0} = \langle c_{w_1, u_0 w_0}^i \rangle \). Therefore \( Q_{w_0, s_1 w_0} = Q_{w_0}^+ + Q_{s_1 w_0}^{-} = \langle c_{w_1, u_0 w_0}^i \rangle \), as desired.

The second claim of the lemma is obtained in the same way. \( \square \)

Now observe that, in [8], Joseph uses slightly different conventions for the dual \( M^* \) of a left \( U_q(\mathfrak{g}) \)-module. Indeed, it is mentioned in [8, 9.1] that the dual \( M^* \) is viewed with its natural right \( U_q(\mathfrak{g}) \)-module structure. As a consequence, Joseph’s convention for the weights of the dual \( L(\lambda)^* \) of \( L(\lambda) \), for \( \lambda \in P^+ \), is not exactly the same as our convention. In particular, the elements \( c_{w_1, u_0 w_1}^i \) and \( c_{w_0, u_0 w_1}^i \), \( i \in \{1, \ldots, n\} \), that appear in [8, Corollary 9.1.4], correspond to the elements \( c_{w_1, u_0 w_1}^i \) and \( c_{w_0, w_1}^i \) in our notation. With this in mind, it follows from [8, Corollary 9.1.4] that the elements \( c_{w_1, u_0 w_1}^i \) and \( c_{w_0, w_1}^i \), for \( i \in \{1, \ldots, n\} \), are normal in \( O_q(G) \). Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata’s Lemma in order to prove that \( O_q(G) \) is a noetherian UFD.

**Corollary 2.2** The \( 2n \) elements \( c_{w_1, u_0 w_1}^i \) and \( c_{w_0, w_1}^i \), for \( i \in \{1, \ldots, n\} \), are nonzero normal elements of \( O_q(G) \) and they generate pairwise distinct completely prime ideals of \( O_q(G) \).

Since the \( c_{w_1, u_0 w_1}^i \) and \( c_{w_0, w_1}^i \), for \( i \in \{1, \ldots, n\} \), are \( \mathcal{H} \)-eigenvectors of \( O_q(G) \), in order to prove that \( O_q(G) \) is an \( \mathcal{H} \)-UFD in the sense of [11, Definition 2.7], it only remains to prove that every nonzero \( \mathcal{H} \)-invariant prime ideal of \( O_q(G) \) contains either one of the \( c_{w_1, u_0 w_1}^i \) or one of the \( c_{w_0, w_1}^i \). This is what we do next.

**Lemma 2.3** Let \( w = (w_+, w_-) \in W \times W \), with \( w \neq (w_0, w_0) \). Then \( Q_w \) contains either one of the \( c_{w_1, u_0 w_1}^i \), or one of the \( c_{w_0 w_1}^i \).

**Proof.** Since \( w \neq (w_0, w_0) \), either \( w_+ \neq w_0 \), or \( w_- \neq w_0 \). Assume, for instance, that \( w_+ \neq w_0 \), so that there exists \( i \in \{1, \ldots, n\} \) such that \( w_+ \leq w_0 s_i \). One can easily check from the definition of \( Q_w \) that this forces \( c_{w_1, u_0 w_1}^i \in Q_+^{w_+} \), so that

\[
\begin{align*}
c_{w_1, u_0 w_1}^i & \in Q_+^{w_+} \subseteq Q_w,
\end{align*}
\]

as required. \( \square \)

As a consequence of Corollary 2.2 and Lemma 2.3, we get the following result.

**Corollary 2.4** \( O_q(G) \) is an \( \mathcal{H} \)-UFD.

**Proof.** Theorem 1.3 establishes that \( \mathcal{H} \)-Spec\((O_q(G)) = \{Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W\} \). Note that \( Q_{w_+, w_-} = 0 \) precisely when \( w_+ = w_- = w_0 \). Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero \( \mathcal{H} \)-prime ideal of \( O_q(G) \) contains a nonzero \( \mathcal{H} \)-prime of height one that is generated by a normal \( \mathcal{H} \)-eigenvector. Thus, \( O_q(G) \) is an \( \mathcal{H} \)-UFD. \( \square \)
2.2 \( O_q(G) \) is a noetherian UFD.

Set \( T \) to be the localisation of \( O_q(G) \) with respect to the multiplicatively closed set generated by the normal \( \mathcal{H} \)-eigenvectors \( c_{\omega_i, w_0 \omega_i} \) and \( c_{w_0 \omega_i, \omega_i} \), for \( i \in \{1, \ldots, n\} \). Then the rational action of \( \mathcal{H} \) on \( O_q(G) \) extends to an action of \( \mathcal{H} \) on the localisation \( T \) by \( \mathbb{C} \)-algebra automorphisms, since we are localising with respect to \( \mathcal{H} \)-eigenvectors, and this action of \( \mathcal{H} \) on \( T \) is also rational, by using [1, II.2.7]. The following result is a consequence of Corollary 2.4 and [11, Proposition 3.5].

**Proposition 2.5** The ring \( T \) is \( \mathcal{H} \)-simple; that is, the only \( \mathcal{H} \)-ideals of \( T \) are 0 and \( T \).

We are now in position to show that \( O_q(G) \) is a noetherian UFD.

**Theorem 2.6** \( O_q(G) \) is a noetherian UFD.

**Proof.** By [11, Proposition 1.6], it is enough to prove that the localisation \( T \) is a noetherian UFD. Now, as proved in Proposition 2.5, \( T \) is an \( \mathcal{H} \)-simple ring. Thus, using [1, II.3.9], \( T \) is a noetherian UFD, as required. \( \square \)

As a consequence, we deduce from Theorem 2.6 and [4, Theorem 2.4] the following result.

**Corollary 2.7** \( O_q(G) \) is a maximal order.

The fact that \( O_q(G) \) is a maximal order can also be proved directly by using a suitable localisation of \( O_q(G) \), [8, Corollary 9.3.10], which is itself a maximal order.

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