Conjugation coinvariants of quantum matrices

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Abstract

A quantum deformation of the classical conjugation action of $GL(N,\mathbb{C})$ on the space of $N \times N$ matrices $M(N,\mathbb{C})$ is defined via a coaction of the quantum general linear group $\mathcal{O}(GL_q(N,\mathbb{C}))$ on the algebra of quantum matrices $\mathcal{O}(M_q(N,\mathbb{C}))$. The coinvariants of this coaction are calculated. In particular, interesting commutative subalgebras of $\mathcal{O}(M_q(N,\mathbb{C}))$ generated by (weighted) sums of principal quantum minors are obtained. For general Hopf algebras, co-commutative elements are characterized as coinvariants with respect to a version of the adjoint coaction.

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1 Introduction

The conjugation $A \to g^{-1}Ag$ of a complex $N \times N$ matrix A by an invertible matrix g induces an action of $GL(N,\mathbb{C})$ on the coordinate algebra $\mathbb{C}[X_{ij}]$ of $N \times N$ matrices. The invariant functions with respect to this action are well-known, they are the elements in the $\mathbb{C}$-subalgebra of $\mathbb{C}[X_{ij}]$ that is generated by the trace functions σ_i, for $i = 1, \ldots, N$, where σ_i is the sum of the $i \times i$ principal minors.

We start this paper by developing a quantum analogue of this conjugation action. The natural way to proceed is to operate at the level of coordinate algebras. At this level the conjugation action of $GL(N,\mathbb{C})$ translates into a coaction of the coordinate algebra of the general linear group, $\mathcal{O}(GL(N,\mathbb{C}))$, on the coordinate algebra of $N \times N$ matrices, $\mathcal{O}(M(N,\mathbb{C}))$ and the problem of finding invariants becomes the problem of finding the coinvariants of this coaction, where a is a *coinvariant* if $a \mapsto a \otimes 1$. In order to quantize the action, we need to operate with the algebra of quantum matrices, $\mathcal{O}(M_q(N,\mathbb{C}))$ and the quantum general linear group, $\mathcal{O}(GL_q(N,\mathbb{C}))$. We start by recalling the definitions of these algebras and fixing our notation, since notation varies within the literature.

The *algebra of $N \times N$ quantum matrices over* $\mathbb{C}$ is the $\mathbb{C}$-algebra generated by N^2 indeterminates x_{ij}, for $i = 1, \ldots, N$, subject to the following relations.

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$$\begin{array}{rcl} x_{ij}x_{il} &=& qx_{il}x_{ij}, \\ x_{ij}x_{kj} &=& qx_{kj}x_{ij}, \\ x_{il}x_{kj} &=& x_{kj}x_{il}, \\ x_{ij}x_{kl} - x_{kl}x_{ij} &=& (q-q^{-1})x_{il}x_{kj}, \end{array}$$

for $1 \le i < k \le N$ and $1 \le j < l \le N$, where $q \in \mathbb{C}^*$.

The algebra $\mathcal{O}(M_q(N,\mathbb{C}))$ can be presented as an iterated Ore extension, and so it is easily seen to be a noetherian domain.

The *quantum determinant*, $\det_q$, is defined by

$$\det_q := \sum_{\sigma \in S_N} (-q)^{l(\sigma)} x_{1,\sigma(1)} \dots x_{N,\sigma(N)}.$$

From [7, Theorem 4.6.1], we know that $\det_q$ is a central element in $\mathcal{O}(M_q(N,\mathbb{C}))$, and so it may be inverted. The resulting algebra is the *quantum general linear group*, $\mathcal{O}(GL_q(N,\mathbb{C}))$, so that

$$\mathcal{O}(GL_q(N,\mathbb{C})) := \mathcal{O}(M_q(N,\mathbb{C}))[\det_q^{-1}].$$

The algebra $\mathcal{O}(GL_q(N,\mathbb{C}))$ is a Hopf algebra, and we denote the *antipode* of this algebra by S. We will often denote the element x_{ij} by u_{ij} when we are considering it as an element of $\mathcal{O}(GL_q(N,\mathbb{C}))$ rather than as an element of $\mathcal{O}(M_q(N,\mathbb{C}))$.

The reader should be aware that in many papers the rôles of q and q^{-1} are interchanged, and so one has to be careful in translating results from one paper to another.

2 Co-commutative elements as coinvariants

Let $\mathcal{A}$ be a Hopf algebra with multiplication μ, comultiplication Δ, counit ε, and antipode S. We define a right coaction α of $\mathcal{A}$ on itself by

$$\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad h \mapsto \sum h_2 \otimes h_3 S(h_1), \tag{1}$$

where we are using Sweedler's notation in the defining formula.

In the present paper we call α the *adjoint coaction*, see Section 7 for further comments on this definition and terminology.

Recall that an element h of $\mathcal{A}$ is said to be *co-commutative* if $\tau\Delta(h) = \Delta(h)$, where τ denotes the flip.

Theorem 2.1 *The element h is co-commutative if and only if it is a coinvariant with respect to α.*

Proof. Suppose that h is a coinvariant; that is, $\sum h_2 \otimes h_3 S(h_1) = h \otimes 1$. By applying $\Delta \otimes \mathrm{id}$ to this equality, and then applying the twist $\tau_{(132)}$, we obtain $\sum h_3 \otimes h_4 S(h_1) \otimes h_2 = \sum h_2 \otimes$

$1 \otimes h_1$. Next, by applying $\mathrm{id} \otimes \mu$ to this equality we obtain $\sum h_3 \otimes h_4 S(h_1) h_2 = \sum h_2 \otimes h_1$. Note that the right hand side of this last equality is $\tau\Delta(h)$, whereas the left hand side is

$$\sum h_3 \otimes h_4 S(h_1) h_2 = \sum h_2 \otimes h_3 \varepsilon(h_1) = \sum \varepsilon(h_1) h_2 \otimes h_3 = \sum h_1 \otimes h_2 = \Delta(h),$$

as required.

Conversely, assume that $\Delta(h) = \tau\Delta(h)$; that is, $\sum h_1 \otimes h_2 = \sum h_2 \otimes h_1$. By applying $\Delta \otimes \mathrm{id}$ followed by $S \otimes \mathrm{id} \otimes \mathrm{id}$ we obtain $\sum S(h_1) \otimes h_2 \otimes h_3 = \sum S(h_2) \otimes h_3 \otimes h_1$. Next, by applying the twist $\tau_{(132)}$ to this equation, followed by applying $\mathrm{id} \otimes \mu$, we obtain $\sum h_2 \otimes h_3 S(h_1) = \sum h_3 \otimes h_1 S(h_2)$.

The left hand side of the last equality is $\alpha(h)$, whereas the right hand side is

$$\sum h_3 \otimes h_1 S(h_2) = \sum h_2 \otimes \varepsilon(h_1) = \sum \varepsilon(h_1) h_2 \otimes 1 = h \otimes 1;$$

and so h is a coinvariant.

$\square$

Therefore $\mathcal{A}^{\alpha\text{-co-}\mathcal{A}}$, the set of α-coinvariants is a subalgebra of $\mathcal{A}$. However, $\mathcal{A}$ is not a comodule algebra with respect to α. We have the following product formula.

Lemma 2.2 *For any $a, b \in \mathcal{A}$ we have*

$$\alpha(ab) = \sum (a_2 \otimes a_3) \cdot \alpha(b) \cdot (1 \otimes S(a_1)).$$

In particular, if b is an α-coinvariant, then $\alpha(ab) = \alpha(a)\alpha(b)$.

Proof. Using the fact that $\Delta^{(2)} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ is an algebra homomorphism and that S is an algebra anti-isomorphism we get

$$\begin{aligned} \alpha(ab) &= \sum (ab)_2 \otimes (ab)_3 S((ab)_1) \\ &= \sum \sum a_2 b_2 \otimes a_3 b_3 S(b_1) S(a_1) \\ &= \sum (a_2 \otimes a_3)(\sum b_2 \otimes b_3 S(b_1))(1 \otimes S(a_1)) \end{aligned}$$

which gives the desired formula. Moreover, if $\alpha(b) = b \otimes 1$, then $\alpha(b)$ and $1 \otimes S(a_1)$ commute, and we get the second statement. $\square$

3 The conjugation coaction on quantum matrices

We define a quantum analogue of the conjugation action of the general linear group on the space of $N \times N$ matrices.

It is easy to check (see [2]) that $\mathcal{O}(M_q(N, \mathbb{C}))$ is a left (respectively, right) comodule algebra with respect to the left (respectively, right) comultiplications λ (respectively, ρ) given by the formulae

$$\lambda : \mathcal{O}(M_q(N, \mathbb{C})) \to \mathcal{O}(GL_q(N, \mathbb{C})) \otimes \mathcal{O}(M_q(N, \mathbb{C})), \quad x_{ij} \mapsto \sum_{s=1}^{N} u_{is} \otimes x_{sj}$$

$$\rho : \mathcal{O}(M_q(N,\mathbb{C})) \to \mathcal{O}(M_q(N,\mathbb{C})) \otimes \mathcal{O}(GL_q(N,\mathbb{C})), \quad x_{ij} \mapsto \sum_{s=1}^{N} x_{is} \otimes u_{sj}.$$

These two comultiplications commute in the sense that we have the equality $(\mathrm{id} \otimes \rho) \circ \lambda = (\lambda \otimes \mathrm{id}) \circ \rho$. This equality follows from the coassociativity of the comultiplication Δ on $\mathcal{O}(GL_q(N,\mathbb{C}))$, since the previous maps are identical to the restriction of $(\mathrm{id} \otimes \Delta) \circ \Delta = \Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta$ to $\mathcal{O}(M_q(N,\mathbb{C}))$.

Now set

$$\alpha_q : \mathcal{O}(M_q(N,\mathbb{C})) \to \mathcal{O}(M_q(N,\mathbb{C})) \otimes \mathcal{O}(GL_q(N,\mathbb{C})),$$

$$\alpha_q := (\mathrm{id} \otimes \mu) \circ (\rho \otimes \mathrm{id}) \circ \tau \circ (S \otimes \mathrm{id}) \circ \lambda. \tag{2}$$

(Here S is the antipode and μ is the multiplication in $\mathcal{O}(GL_q(N,\mathbb{C}))$, and τ is the flip.)

In the special case $q = 1$, it is obvious that α_1 is the comorphism induced by

$$M(N,\mathbb{C}) \times GL(N,\mathbb{C}) \to M(N,\mathbb{C}), \quad (A, g) \mapsto g^{-1}Ag.$$

The coaction α_q defined in (2) coincides with the restriction to $\mathcal{O}(M_q(N,\mathbb{C}))$ of the adjoint coaction (1) of the Hopf algebra $\mathcal{O}(GL_q(N,\mathbb{C}))$ on itself; that is, using Sweedler's notation

$$\alpha_q(u) \mapsto \sum u_2 \otimes u_3 S(u_1), \quad u \in \mathcal{O}(GL_q(N,\mathbb{C})).$$

On the generators of $\mathcal{O}(M_q(N,\mathbb{C}))$ the coaction α_q can be given explicitly as

$$\alpha_q(x_{ij}) = \sum_{m,s=1}^{N} x_{ms} \otimes u_{sj} S(u_{im}).$$

In the sequel α_q will denote both the coactions of $\mathcal{O}(GL_q(N,\mathbb{C}))$ on itself and on $\mathcal{O}(M_q(N,\mathbb{C}))$, and we will omit the index q and write α for α_q, provided that no confusion is likely to arise.

As we noted in Section 2, $\mathcal{O}(M_q(N,\mathbb{C}))$ is not a comodule algebra with respect to α. However, as an immediate corollary of Theorem 2.1 or Lemma 2.2 we get the following.

Proposition 3.1 *The set of coinvariants $\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))}$ is a subalgebra of $\mathcal{O}(M_q(N,\mathbb{C}))$.*

4 Construction of coinvariants

Fix an integer t with $1 \leq t \leq N$. Let I and J be subsets of $\{1, \ldots, N\}$ with $|I| = |J| = t$. The subalgebra of $\mathcal{O}(M_q(N,\mathbb{C}))$ generated by x_{ij} with $i \in I$ and $j \in J$ can be regarded as an algebra of $t \times t$ quantum matrices, and so we can calculate its quantum determinant -

this is a $t \times t$ *quantum minor* and we denote it by $[I|J]$. Recall that comultiplication of quantum minors is given by the rule

$$\Delta([I|J]) = \sum_{|K|=t} [I|K] \otimes [K|J],$$

see [6, (1.9)], so that

$$\alpha([I|J]) = \sum_{|K|=|M|=t} [M|K] \otimes [K|J]S([I|M]).$$

The quantum minor $[I|I]$ is said to be a *principal quantum minor*. We denote the sum of all the principal quantum minors of a given size i by σ_i. Note that $\sigma_1 = x_{11} + \cdots + x_{NN}$ and that $\sigma_N = \det_q$. We denote the subalgebra of $\mathcal{O}(M_q(N,\mathbb{C}))$ generated by all of the σ_i by $\mathbb{C}\langle \sigma_1, \dots, \sigma_N \rangle$.

Proposition 4.1

$$\mathbb{C}\langle \sigma_1, \dots, \sigma_N \rangle \subseteq \mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))}.$$

Proof. By Corollary 3.1, it is enough to show that each σ_i is a coinvariant.

$$\begin{aligned}
\alpha(\sigma_i) &= \alpha(\sum_{|I|=i} [I|I]) \\
&= \sum_{|I|=i} \left\{ \sum_{|K|=|M|=i} [M|K] \otimes [K|I]S([I|M]) \right\} \\
&= \sum_{|K|=|M|=i} [M|K] \otimes \left\{ \sum_{|I|=i} [K|I]S([I|M]) \right\} \\
&= \sum_{|K|=|M|=i} [M|K] \otimes \varepsilon([K|M]) \\
&= \sum_{|K|=i} [K|K] \otimes 1 \\
&= \sigma_i \otimes 1
\end{aligned}$$

where the antepenultimate equality follows from $\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon$, cf. [5, 1.5.2]. (Alternatively, one can check easily that σ_i is co-commutative, and refer to Theorem 2.1.) □

Our aim is to show that, in the case that q is not a root of unity, $\mathbb{C}\langle \sigma_1, \dots, \sigma_N \rangle = \mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))}$ and that this algebra is a commutative polynomial algebra in the σ_i, $i = 1 \dots, N$. In order to do this, we need to recall the character theory of corepresentations of cosemisimple Hopf algebras. This is done in the next section.

5 Corepresentation theory

We recall some basic facts about the corepresentation theory of the quantum general linear group. These can be found, for example, in the book [3, Part III., 11.5.].

By a *corepresentation* of a Hopf algebra $\mathcal{A}$ on the vector space V we mean a right $\mathcal{A}$-comodule $T : V \to V \otimes \mathcal{A}$. If V is finite dimensional, then we may define the character χ_T as the element of $\mathcal{A}$ obtained by summing the diagonal elements of the *matrix corepresentation* belonging to T with respect to an arbitrary basis of V. Given a Hopf algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ we have that $T\big|_{\mathcal{B}} := (\mathrm{id}_V \otimes \pi) \circ T$ is a corepresentation of B on V. If V is finite dimensional, then obviously we have $\chi_{T|_{\mathcal{B}}} = \pi(\chi_T)$.

If $q \in \mathbb{C}^*$ is not a root of unity (or if $q = 1$), then $\mathcal{O}(GL_q(N, \mathbb{C}))$ is a cosemisimple Hopf algebra; that is, any corepresentation decomposes into a direct sum of irreducible corepresentations. The irreducible corepresentations are indexed by $P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N\}$, the set of integral dominant weights of the Lie algebra $gl(N, \mathbb{C})$. For $\lambda \in P_+$ denote by T_q^λ the associated finite dimensional irreducible corepresentation of $\mathcal{O}(GL_q(N, \mathbb{C}))$; for the definition of T_q^λ see [3, page 439]. In the case that q is not a root of unity or $q = 1$ the T_q^λ ($\lambda \in P_+$) provide a complete list of pairwise non-isomorphic irreducible corepresentations of $\mathcal{O}(GL_q(N, \mathbb{C}))$.

Let D be the diagonal subgroup of $GL(N, \mathbb{C})$. Its coordinate Hopf algebra $\mathcal{O}(D)$ is the commutative algebra $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_N, t_N^{-1}]$ of Laurent polynomials with comultiplication $\Delta(t_i) = t_i \otimes t_i$ and counit $\varepsilon(t_i) = 1$. There exists a surjective Hopf algebra homomorphism $\pi_D : \mathcal{O}(GL_q(N, \mathbb{C})) \to \mathcal{O}(D)$ determined by $\pi_D(u_{ij}) = \delta_{ij} t_i$, $i, j = 1, \ldots, N$. Therefore we say that D is a quantum subgroup of $GL_q(N, \mathbb{C})$, called the *diagonal subgroup*.

The character of $T_q^\lambda\big|_D$ does not depend on q. Moreover,

$$\pi_D(\chi_{T_q^\lambda}) = S_\lambda(t_1, \ldots, t_N) \in \mathbb{Z}[t_1, \ldots, t_N],$$

where $S_\lambda(t_1, \ldots, t_N) \cdot (t_1 \cdots t_N)^{-\lambda_N}$ is the Schur polynomial belonging to the partition $(\lambda_1 - \lambda_N, \lambda_2 - \lambda_N, \ldots, \lambda_{N-1} - \lambda_N, 0)$ (see [4]).

Note that $\mathcal{O}(M_q(N, \mathbb{C})) = \bigoplus_{d \in \mathbb{N}_0} \mathcal{O}(M_q(N, \mathbb{C}))_d$ is an algebra graded by the semigroup of non-negative integers, and the corepresentation α_q is homogeneous; that is, α decomposes as $\alpha_q = \bigoplus_{d \in \mathbb{N}_0} \alpha_q^d$, where $\alpha_q^d : \mathcal{O}(M_q(N, \mathbb{C}))_d \to \mathcal{O}(M_q(N, \mathbb{C}))_d \otimes \mathcal{O}(GL_q(N, \mathbb{C}))$ is the restriction of α_q to the degree d component.

Lemma 5.1 *For any $q \in \mathbb{C}^*$ and $d \in \mathbb{N}_0$ the $\mathcal{O}(D)$-corepresentation $\alpha_q^d\big|_D$ is equivalent to the $\mathcal{O}(D)$-corepresentation $\alpha_1^d\big|_D$.*

Proof. Since

$$\mathrm{id} \otimes \pi_D : \mathcal{O}(GL_q(N, \mathbb{C})) \otimes \mathcal{O}(GL_q(N, \mathbb{C})) \to \mathcal{O}(GL_q(N, \mathbb{C})) \otimes \mathcal{O}(D)$$

is an algebra homomorphism, it follows from Lemma 2.2 that

$$\alpha_q\big|_D(ab) = \sum (a_2 \otimes \pi_D(a_3)) \cdot \alpha_q\big|_D(b) \cdot (1 \otimes \pi_D(S(a_1))).$$

Now $1 \otimes \pi_D(S(a_1))$ is a central element in $\mathcal{O}(GL_q(N, \mathbb{C})) \otimes \mathcal{O}(D)$, so

$$\alpha_q\big|_D(ab) = (\sum a_2 \otimes \pi_D(a_3)\pi_D(S(a_1))) \cdot \alpha_q\big|_D(b) = \alpha_q\big|_D(a) \cdot \alpha_q\big|_D(b).$$

So $\mathcal{O}(M_q(N,\mathbb{C}))$ becomes a right $\mathcal{O}(D)$-comodule algebra under the coaction $\alpha_q\big|_D$ (though it was not necessarily an $\mathcal{O}(GL_q(N,\mathbb{C}))$-comodule algebra under α_q). For an index set $I=(i_1,j_1,\dots,i_d,j_d)\in\{1,\dots,N\}^{2d}$ set $x^I:=x_{i_1j_1}\cdots x_{i_dj_d}$ and $t^I:=t_{i_1}^{-1}t_{j_1}\cdots t_{i_d}^{-1}t_{j_d}$. By the definition of α we have that $\alpha\big|_D(x_{ij})=x_{ij}\otimes t_i^{-1}t_j$, hence $\alpha\big|_D(x^I)=x^I\otimes t^I$. For each d there exists a subset $\mathcal{I}$ of $\{1,\dots,N\}^{2d}$, independent of q, such that $\{x^I\mid I\in\mathcal{I}\}$ is a basis of $\mathcal{O}(M_q(N,\mathbb{C}))_d$. Hence, $\mathcal{O}(M_q(N,\mathbb{C}))_d$ is the direct sum of the one dimensional $\mathcal{O}(D)$-sub-comodules $\mathbb{C}x^I$ $(I\in\mathcal{I})$, where the corepresentation of $\mathcal{O}(D)$ on $\mathbb{C}x^I$ is given by the character t^I. $\square$

Denote by $m_d(\lambda)$ the multiplicity of T_1^λ in the $\mathcal{O}(GL(N,\mathbb{C}))$-corepresentation $\mathcal{O}(M(N,\mathbb{C}))_d$ under α_1, that is,

$$\alpha_1^d\cong\bigoplus_{\lambda\in P_+}m_d(\lambda)T_1^\lambda,$$

where $mT:=T\oplus\cdots\oplus T$ (m direct summands) for a corepresentation T and a non-negative integer m. Then passing to characters we have

$$\pi_D(\chi_{\alpha_1^d})=\sum_{\lambda\in P_+}m_d(\lambda)S_\lambda(t_1,\dots,t_N),$$

and as an immediate corollary of Lemma 5.1 we get the following:

Corollary 5.2 *For any $q\in\mathbb{C}^*$ and $d\in\mathbb{N}_0$ we have*

$$\pi_D(\chi_{\alpha_q^d})=\sum_{\lambda\in P_+}m_d(\lambda)S_\lambda(t_1,\dots,t_N).$$

If q is not a root of unity, then $\mathcal{O}(GL_q(N,\mathbb{C}))$ is cosemisimple, and the decomposition of the character of α_q^d into the sum of characters of irreducible corepresentations implies the corresponding decomposition of the corepresentation itself, hence we get:

Corollary 5.3 *If $q\in\mathbb{C}^*$ is not a root of unity (or if $q=1$) then for any $d\in\mathbb{N}_0$ we have*

$$\alpha_q^d\cong\bigoplus_{\lambda\in P_+}m_d(\lambda)T_q^\lambda.$$

Since the corepresentation α_q is homogeneous, $\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))}$ is a graded subalgebra of $\mathcal{O}(M_q(N,\mathbb{C}))$. Its *Hilbert series* $H(\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))};t)\in\mathbb{Z}[[t]]$ is the formal power series where the coefficient of t^d is the dimension of the degree d component.

Proposition 5.4 *If $q\in\mathbb{C}^*$ is not a root of unity (or if $q=1$), then*

$$H(\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))};t)=\frac{1}{\prod_{i=1}^N(1-t^i)}.$$

Proof. Since $\mathcal{O}(GL_q(N,\mathbb{C}))$ is cosemisimple, by the definitions of Hilbert series and coinvariants the dth coefficient of $H(\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))};t)$ is the multiplicity of the trivial corepresentation $T_q^{(0,\dots,0)}$ in $\mathcal{O}(M_q(N,\mathbb{C}))_d$. By Corollary 5.3 this multiplicity does not depend on q. In the case $q=1$ the algebra $\mathcal{O}(M(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL(N,\mathbb{C}))}$ coincides with the algebra of polynomial invariants of $N\times N$ matrices with respect to the conjugation action of the general linear group. It is well known that the Hilbert series of this algebra is $\prod_{i=1}^{N}(1-t^i)^{-1}$. $\square$

6 Main results

In this section we show that the coinvariants σ_i generate the algebra of coinvariants for the conjugation coaction, and that this algebra of coinvariants is in fact a commutative polynomial algebra in the σ_i. First, some simplifying notation. We set $\mathcal{A} := \mathcal{O}(M_q(N,\mathbb{C}))$, $\mathcal{B} := \mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))}$ and $\mathcal{C} := \mathbb{C}\langle\sigma_1,\dots,\sigma_N\rangle$. Thus, $\mathcal{C}\subseteq\mathcal{B}\subseteq\mathcal{A}$. The first thing to notice is that these are inclusions of graded algebras. This is because the coinvariants σ_i are homogeneous elements of degree i in $\mathcal{O}(M_q(N,\mathbb{C}))$, and so $\mathbb{C}\langle\sigma_1,\dots,\sigma_N\rangle$ inherits a graded algebra structure from $\mathcal{O}(M_q(N,\mathbb{C}))$. The coaction α sends the subspace $\mathcal{O}(M_q(N,\mathbb{C}))_d$ of elements of $\mathcal{O}(M_q(N,\mathbb{C}))$ that have degree d into $\mathcal{O}(M_q(N,\mathbb{C}))_d\otimes\mathcal{O}(GL_q(N,\mathbb{C}))$, and so $\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))}$ has a graded algebra structure with respect to which $\mathbb{C}\langle\sigma_1,\dots,\sigma_N\rangle$ is a graded subalgebra.

We define a partial order on $\mathbb{Z}[[t]]$ in the following way. Let $f=\sum f_it^i$ and $g=\sum g_it^i$ be elements of $\mathbb{Z}[[t]]$. Then, set $f\le g$ if and only if $f_i\le g_i$ for each i.

If B is a graded subalgebra of a graded algebra A then $H(B,t)\le H(A,t)$, with equality if and only if $B=A$. Similarly, if I is a homogeneous ideal of A then $H(A/I,t)\le H(A,t)$, with equality if and only if $I=0$.

Theorem 6.1 *If $q\in\mathbb{C}^*$ is not a root of unity (or if $q=1$), then*

$$\mathcal{O}(M_q(N,\mathbb{C}))^{\alpha\text{-co-}\mathcal{O}(GL_q(N,\mathbb{C}))}=\mathbb{C}\langle\sigma_1,\dots,\sigma_N\rangle$$

and this algebra of coinvariants is a commutative polynomial algebra of dimension N.

Proof. The above inclusions immediately yield that

$$H(\mathcal{C},t)\le H(\mathcal{B},t)=\frac{1}{\prod_{i=1}^{N}(1-t^i)},$$

where the equality comes from Proposition 5.4.

The elements x_{ij}, $i\ne j$, generate an ideal I of $\mathcal{A}=\mathcal{O}(M_q(N,\mathbb{C}))$ such that the factor algebra $\mathcal{D}:=\mathcal{A}/I$ is a commutative polynomial algebra in N indeterminates, the images of $x_{11},\dots,x_{NN}$. Let us set X_i to be the image of x_{ii} and consider $\mathcal{D}$ to be graded with X_i given degree 1. Thus, the natural map π of $\mathcal{A}$ onto $\mathcal{D}$ is a graded homomorphism. Let ϕ be the restriction of this map to the graded subalgebra $\mathcal{C}$. Note that $\phi(\sigma_i)=s_i$, where s_i is the ith elementary symmetric polynomial in the X_j. Then $\mathrm{Im}(\phi)=\mathbb{C}[s_1,\dots,s_N]$. It is well-known that $H(\mathbb{C}[s_1,\dots,s_N],t)=\prod_{i=1}^{N}(1-t^i)^{-1}$, so this epimorphism yields

$$\prod_{i=1}^{N}(1-t^i)^{-1}=H(\mathcal{C}/\ker(\phi),t)\le H(\mathcal{C},t).$$

Together with the earlier inequality, this gives

$$H(\mathcal{C}/\ker(\phi), t) = H(\mathcal{C}, t) = H(\mathcal{B}, t) = \prod_{i=1}^{N}(1-t^i)^{-1}$$

which forces $\mathcal{C} = \mathcal{B}$, as required, and also forces ϕ to be an isomorphism. In particular, $\mathcal{B} = \mathcal{C} \cong \mathbb{C}[s_1, \ldots, s_N]$ is a commutative polynomial algebra. $\square$

As applications of this result, we have the following description of the co-commutative elements of $\mathcal{O}(M_q(N,\mathbb{C}))$ and $\mathcal{O}(GL_q(N,\mathbb{C}))$ which we have not been able to obtain directly.

Corollary 6.2 *If $q \in \mathbb{C}^*$ is not a root of unity (or if $q = 1$), then the subalgebra of co-commutive elements of $\mathcal{O}(M_q(N,\mathbb{C}))$ is $\mathbb{C}\langle \sigma_1, \ldots, \sigma_N \rangle$, and, consequently, the co-commutative elements form a commutative polynomial algebra of dimension N.*

Proof. This follows immediately from the previous theorem and Theorem 2.1. $\square$

It is easy to derive from Theorem 6.1 a description of the α-coinvariants (or in other words, the co-commutative elements) in the Hopf algebra $\mathcal{O}(GL_q(N,\mathbb{C}))$:

Corollary 6.3 *If $q \in \mathbb{C}^*$ is not a root of unity (or if $q = 1$), then*

$$\mathcal{O}(GL_q(N,\mathbb{C}))^{\alpha\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))} = \mathbb{C}\langle \sigma_1, \ldots, \sigma_{N-1}, \sigma_N, \sigma_N^{-1} \rangle$$

and this algebra of coinvariants (and hence the subalgebra of co-commutative elements) is a one-variable Laurent polynomial algebra over the commutative polynomial algebra of dimension $N-1$.

Proof. For any $h \in \mathcal{O}(GL_q(N,\mathbb{C}))$ there is an exponent d such that $h\sigma_N^d$ is contained in $\mathcal{O}(M_q(N,\mathbb{C}))$. Since σ_N is a coinvariant, we have by Lemma 2.2 that h is a coinvariant if and only if $h\sigma_N^d$ is a coinvariant. The claim follows from this observation and Theorem 6.1. $\square$

Remark 6.4 We expect that the same results hold also when q is a root of unity. In fact, we can derive the conclusion that $\mathbb{C}\langle \sigma_1, \ldots, \sigma_N \rangle$ is a commutative polynomial subalgebra of the algebra of coinvariants in the general case from our restricted result. However, we cannot at the moment show that these are all of the coinvariants. The cosemisimplicity of $\mathcal{O}(GL_q(N,\mathbb{C}))$ for q generic was used in Corollary 5.3. We would need another argument to derive Proposition 5.4 from Corollary 5.2, when q is a root of unity. Such kind of replacement of the semisimplicity is provided by the theory of tilting modules (or modules with good filtration), when invariants of the classical conjugation action are studied over a positive characteristic base field.

Remark 6.5 In [9], Zhang proves two versions of a Quantum Cayley-Hamilton Theorem. The coefficients that arise in the identity [9, Theorem 2.4] are the σ_i that we have seen are the basic coinvariants for the conjugation coaction α.

Remark 6.6 We plan to consider the problem of determining the coinvariants of several quantum matrices under the conjugation coaction in a later paper.

7 Other versions of the adjoint coaction

There is an element of choice in the definition (1) of the adjoint coaction. In fact, there are four 'obvious' choices for the adjoint coaction of a Hopf algebra $\mathcal{A}$ on itself, two right coactions (α,β) and two left coactions (γ, δ). These are:

(i) $\alpha(h) := \sum h_2 \otimes h_3 S(h_1)$

(ii) $\beta(h) := \sum h_2 \otimes S(h_1) h_3$

(iii) $\gamma(h) := \sum h_1 S(h_3) \otimes h_2$, and

(iv) $\delta(h) := \sum S(h_3) h_1 \otimes h_2$.

Any of them can be used to define a coaction of $\mathcal{O}(GL_q(N,\mathbb{C}))$ on $\mathcal{O}(M_q(N,\mathbb{C}))$, in order to obtain a quantum analogue of the conjugation action of $GL(N,\mathbb{C})$ on $M(N,\mathbb{C})$. We have chosen case (i). The case of the corresponding left coaction (iv) yields no essential difference, because similarly to Theorem 2.1 we have that h is a δ-coinvariant if and only if h is co-commutative, so Theorem 6.1 remains valid if we replace α by δ.

On the other hand, in all the literature we checked the adjoint coaction is defined to be β or γ. It is shown in [1, Lemma 1.1] that $h \in \mathcal{A}$ is a γ-coinvariant if and only if $\Delta(h) = \sum S^2(h_2) \otimes h_1$. Similarly, $h \in \mathcal{A}$ is a β-coinvariant if and only if $\Delta(h) = \sum h_2 \otimes S^2(h_1)$. Since the square of the antipode of $\mathcal{O}(GL_q(N,\mathbb{C}))$ is not the identity if $q^2 \neq 1$, the sets of coinvariants in $\mathcal{O}(M_q(N,\mathbb{C}))$ with respect to β or γ are not the same as for α.

For $i = 1, \ldots, N$ consider the weighted trace functions

$$\tau_i := \sum_I q^{-2|I|}[I|I],$$

$$\rho_i := \sum_I q^{2|I|}[I|I].$$

Here, and elsewhere, all summations are over index sets of a fixed size (eg. i in the case of τ_i). In exponents, we will use $|I|$ to denote the sum of the entries of the index set I.

First, we need to calculate the effect of S^2 on quantum minors.

Lemma 7.1 *The action of S^2 on quantum minors is given by*

$$S^2([I|J]) = q^{2(|I|-|J|)}[I|J]$$

and, consequently,

$$[I|J] = S^2(q^{2(|J|-|I|)}[I|J]).$$

*Proof.*This follows from the formula $S^2((u_{ij})_{i,j=1}^N) = Q \cdot (u_{ij})_{i,j=1}^N \cdot Q^{-1}$, where Q is the diagonal matrix with diagonal entries $1, q^2, \ldots, q^{2(N-1)}$ (see for example [8, Theorem 4]). $\square$

Proposition 7.2 *The weighted sum of principal quantum minors τ_i is a coinvariant for the coaction β, and ρ_i is a coinvariant for γ.*

Proof. Note that

$$\beta([I|J]) = \sum_{K,M} [K|M] \otimes S([I|K])[M|J],$$

hence

$$\begin{aligned}
\beta(\tau_i) &= \sum_I \left\{ \sum_{K,M} q^{-2|I|}[K|M] \otimes S([I|K])[M|I] \right\} \\
&= \sum_{K,M} [K|M] \otimes \left\{ \sum_I q^{-2|I|} S([I|K])[M|I] \right\} \\
&= \sum_{K,M} [K|M] \otimes \left\{ \sum_I q^{-2|I|} S([I|K]) S^2(q^{2(|I|-|M|)}[M|I]) \right\} \\
&= \sum_{K,M} [K|M] \otimes \left\{ q^{-2|M|} S\left(\sum_I S([M|I])[I|K] \right) \right\} \\
&= \sum_{K,M} q^{-2|M|}[K|M] \otimes S(\varepsilon([M|K])) \\
&= \sum_M q^{-2|M|}[M|M] \otimes 1 \\
&= \tau_i \otimes 1.
\end{aligned}$$

Similar calculation shows that ρ_i is a coinvariant for γ. □

Finally, we give the counterpart of Theorem 6.1 for the coactions β and γ.

Theorem 7.3 *If $q \in \mathbb{C}^*$ is not a root of unity, then*

$$\mathcal{O}(M_q(N,\mathbb{C}))^{\beta\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))} = \mathbb{C}\langle \tau_1, \ldots, \tau_N \rangle,$$

$$\mathcal{O}(M_q(N,\mathbb{C}))^{\gamma\text{-}co\text{-}\mathcal{O}(GL_q(N,\mathbb{C}))} = \mathbb{C}\langle \rho_1, \ldots, \rho_N \rangle,$$

and both algebras of coinvariants are commutative polynomial algebras of dimension N.

Proof. We deal with β, the case of γ is the same. Similarly to the proof of Lemma 5.1 it is possible to show that we have the isomorphism of corepresentations $\beta_q^d\big|_D \cong \alpha_q^d\big|_D$. Therefore the algebra of β-coinvariants has the same Hilbert series as the commutative polynomial algebra with generators of degree $1, 2, \ldots, N$. Using this one concludes the result from Proposition 7.2 in the same way as in the proof of Theorem 6.1. □

References

[1] M. Cohen and S. Zhu, Invariants of the adjoint coaction and Yetter-Drinfeld categories, *J. Pure Appl. Alg.* 159 (2001), 149–171.

[2] K. R. Goodearl, T. H. Lenagan and L. Rigal, The first fundamental theorem of coinvariant theory for the quantum general linear group, *Publ. Res. Inst. Math. Sci., Kyoto Univ.* **36**, No. 2 (2000), 269–296.

[3] A. Klimyk and K. Schmüdgen, "Quantum Groups and Their Representations", Springer-Verlag, Berlin, Heidelberg, New York, 1997.

[4] I. G. Macdonald, "Symmetric Functions and Hall Polynomials", 2nd Ed., Clarendon Press, Oxford, 1995.

[5] S. Montgomery, "Hopf algebras and their actions on rings", American Mathematical Society, Providence, 1993.

[6] M. Noumi, H. Yamada, and K. Mimachi, Finite dimensional representations of the quantum group $GL_q(n;\mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, Japanese J. Math. **19** (1993), 31-80.

[7] B. Parshall and J. Wang. *Quantum linear groups.* Mem. Amer. Math. Soc **89** (1991), no.439.

[8] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, Quantization of Lie groups and Lie algebras, (Russian) *Algebra i Analiz* **1** (1989), 178–206.

[9] J. J. Zhang, The quantum Cayley-Hamilton Theorem, *J. Pure and App. Alg.* **129**, (1998), 101-109.

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