PRIME IDEALS INVARIANT UNDER WINDING AUTOMORPHISMS IN QUANTUM MATRICES

K. R. Goodearl and T. H. Lenagan

Abstract. The main goal of the paper is to establish the existence of tensor product decompositions for those prime ideals P of the algebra $A = \mathcal{O}_q(M_n(k))$ of quantum $n \times n$ matrices which are invariant under winding automorphisms of A, in the generic case (q not a root of unity). More specifically, every such P is the kernel of a map of the form

$$A \longrightarrow A \otimes A \longrightarrow A^+ \otimes A^- \longrightarrow (A^+/P^+) \otimes (A^-/P^-)$$

where $A \to A \otimes A$ is the comultiplication, A^+ and A^- are suitable localized factor algebras of A, and $P^\pm$ is a prime ideal of $A^\pm$ invariant under winding automorphisms. Further, the algebras $A^\pm$, which vary with P, can be chosen so that the correspondence $(P^+, P^-) \mapsto P$ is a bijection. The main theorem is applied, in a sequel to this paper, to completely determine the winding-invariant prime ideals in the generic quantum 3×3 matrix algebra.

Introduction

This paper represents part of an ongoing project to determine the prime and primitive spectra of the generic quantized coordinate ring of $n \times n$ matrices, $\mathcal{O}_q(M_n(k))$. Here k is an arbitrary field and $q \in k^\times$ is a non-root of unity. The current intermediate goal is to determine the prime ideals of $\mathcal{O}_q(M_n(k))$ invariant under all winding automorphisms. (See below for a discussion of the relations between these *winding-invariant* primes and the full prime spectrum of $\mathcal{O}_q(M_n(k))$.) Our main result exhibits a bijection between these primes and pairs of winding-invariant primes from certain 'localized step-triangular factors' of $\mathcal{O}_q(M_n(k))$, namely the algebras

$$R^+_{\boldsymbol{r}} = \big(\mathcal{O}_q(M_n(k))/\langle X_{ij} \mid j > t \text{ or } i < r_j\rangle\big)[\overline{X}_{r_1 1}^{-1}, \dots, \overline{X}_{r_t t}^{-1}]$$
$$R^-_{\boldsymbol{c}} = \big(\mathcal{O}_q(M_n(k))/\langle X_{ij} \mid i > t \text{ or } j < c_i\rangle\big)[\overline{X}_{1 c_1}^{-1}, \dots, \overline{X}_{t c_t}^{-1}]$$

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where $\boldsymbol{r} = (r_1, \dots, r_t)$ and $\boldsymbol{c} = (c_1, \dots, c_t)$ are strictly increasing sequences of integers in the range $1, 2, \dots, n$. In particular, since each $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$ can be presented as a skew-Laurent extension of a localized factor algebra of $\mathcal{O}_q(M_{n-1}(k))$, the above bijection can be used to obtain descriptions (as pullbacks of primes in the algebras $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$) of the winding-invariant primes of $\mathcal{O}_q(M_n(k))$ from those of $\mathcal{O}_q(M_{n-1}(k))$. In a sequel [**7**] to this paper, we follow the route just sketched to develop a complete list, with sets of generators, of the winding-invariant primes in $\mathcal{O}_q(M_3(k))$.

The theorem indicated above depends on some detailed structural results concerning $\mathcal{O}_q(M_n(k))$ and on some general work with primes in tensor product algebras invariant under group actions. First, we construct a partition of $\operatorname{spec} \mathcal{O}_q(M_n(k))$ indexed by pairs $(\boldsymbol{r}, \boldsymbol{c})$ as above, together with localized factor algebras $A_{\boldsymbol{r},\boldsymbol{c}}$ of $\mathcal{O}_q(M_n(k))$, such that the portion of $\operatorname{spec} \mathcal{O}_q(M_n(k))$ indexed by $(\boldsymbol{r}, \boldsymbol{c})$ is Zariski-homeomorphic to $\operatorname{spec} A_{\boldsymbol{r},\boldsymbol{c}}$. We next prove that $A_{\boldsymbol{r},\boldsymbol{c}}$ is isomorphic to a subalgebra $B_{\boldsymbol{r},\boldsymbol{c}}$ of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$, identify the structure of $B_{\boldsymbol{r},\boldsymbol{c}}$, and show that $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ is a skew-Laurent extension of $B_{\boldsymbol{r},\boldsymbol{c}}$. Finally, with the help of some general work on tensor products, we prove that each winding-invariant prime of $B_{\boldsymbol{r},\boldsymbol{c}}$ extends uniquely to a winding-invariant prime of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$, and that the latter primes can be uniquely expressed in the form $(P^+ \otimes R^-_{\boldsymbol{c}}) + (R^+_{\boldsymbol{r}} \otimes P^-)$ where P^+ (respectively, P^-) is a winding-invariant prime in $R^+_{\boldsymbol{r}}$ (respectively, $R^-_{\boldsymbol{c}}$). We thus conclude that every winding-invariant prime of $\mathcal{O}_q(M_n(k))$ can be uniquely expressed as the kernel of a map

$$\mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \longrightarrow R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}} \longrightarrow (R^+_{\boldsymbol{r}}/P^+) \otimes (R^-_{\boldsymbol{c}}/P^-),$$

where the first arrow is comultiplication and the others are tensor products of localization or quotient maps.

Algebraic background. The algebra $\mathcal{O}_q(M_n(k))$ has standard generators X_{ij} for $i, j = 1, \dots, n$ and relations which we recall in (5.1)(a), along with the bialgebra structure of this algebra. The latter structure allows us to define *left and right winding automorphisms* corresponding to those characters (k-algebra homomorphisms $\mathcal{O}_q(M_n(k)) \to k$) which are invertible in $\mathcal{O}_q(M_n(k))^*$ with respect to the convolution product (cf. [**1**, (I.9.25)] or [**12**, (1.3.4)] for the Hopf algebra case). It is well known that the collection of left (respectively, right) winding automorphisms of $\mathcal{O}_q(M_n(k))$ forms a group isomorphic to the diagonal subgroup of $GL_n(k)$, whose action on the matrix of generators (X_{ij}) is given by left (respectively, right) multiplication. We combine these actions to obtain an action of the group $H = (k^\times)^n \times (k^\times)^n$ on $\mathcal{O}_q(M_n(k))$ by k-algebra automorphisms satisfying the rule

$$(*) \qquad (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n).X_{ij} = \alpha_i \beta_j X_{ij}.$$

One indication of the extent of the symmetry given by this action is the fact that there are only finitely many (actually, at most 2^{n^2}) primes of $\mathcal{O}_q(M_n(k))$ invariant under H [**9**, (5.7)(i)]. The quoted result also shows that all H-primes of this algebra are prime, and so the H-primes coincide with the winding-invariant primes in $\mathcal{O}_q(M_n(k))$. (Recall that the definition of an *H-prime* ideal is obtained from the standard ideal-theoretic definition of a prime ideal by restricting to H-invariant ideals.)

In [**9**, Theorem 6.6], Letzter and the first author showed that the overall picture of the prime spectrum of an algebra with certain basic features like those of $\mathcal{O}_q(M_n(k))$ is determined to a great extent by the primes invariant under a suitable group action. We quote the improved version of this picture presented in [**1**, Theorem II.2.13]. Let A be a noetherian algebra over an infinite field k, and let $H = (k^\times)^r$ be (the group of k-points of) an algebraic torus acting rationally on A by k-algebra automorphisms. Each H-prime of A is a prime ideal, and $\operatorname{spec} A$ is the disjoint union of the sets

$$\operatorname{spec}_J A := \{P \in \operatorname{spec} A \mid \bigcap_{h \in H} h(P) = J\}$$

as J ranges over the H-primes of A. Further:

1. Let $\mathcal{E}_J$ denote the set of all regular H-eigenvectors in A/J. Then $\mathcal{E}_J$ is a denominator set, and the localization $A_J = (A/J)[\mathcal{E}_J^{-1}]$ is an H-simple ring.
2. $\operatorname{spec}_J A$ is homeomorphic to $\operatorname{spec} A_J$ via localization and contraction.
3. $\operatorname{spec} A_J$ is homeomorphic to $\operatorname{spec} Z(A_J)$ via contraction and extension.
4. $Z(A_J)$ is a Laurent polynomial ring, in at most r indeterminates, over the fixed field $Z(\operatorname{Fract} A/J)^H$.

Under some additional hypotheses, satisfied by $\mathcal{O}_q(M_n(k))$, we also have:

5. The primitive ideals of A are exactly the maximal elements of the sets $\operatorname{spec}_J A$.
6. If k is algebraically closed, then the primitive ideals within each $\operatorname{spec}_J A$ are permuted transitively by H.

Statement (5) was proved in [**9**, Corollary 6.9] (see also [**1**, Theorem II.8.4]). Statement (6) is a consequence of general transitivity theorems for algebraic group actions due to Moeglin-Rentschler [**14**, Théorème 2.12(ii)] and Vonessen [**17**, Theorem 2.2], but the case where the acting group is a torus is much easier (see [**9**, Theorem 6.8] or [**1**, Theorem II.8.14]).

The above results indicate that to draw a complete picture of $\operatorname{spec} \mathcal{O}_q(M_n(k))$, we need to determine the H-primes. That is easy to do in case $n = 2$; the result is recorded, for instance, in [**5**, (3.6)] (see [**1**, Example II.2.14(d)] for more detail). In general, we make the following

Conjecture. *Every H-prime of $\mathcal{O}_q(M_n(k))$ can be generated by a set of quantum minors.*

This conjecture is easily checked in case $n = 2$ using the information above, and we verify it for the case $n = 3$ in [**7**]. Further supporting evidence is provided by recent work of Cauchon, who showed that distinct, comparable H-primes in $\mathcal{O}_q(M_n(k))$ can be distinguished by the quantum minors they contain [**2**, Proposition 6.2.2 and Théorème 6.2.1]. Another source of support for the conjecture is the work of Hodges and Levasseur [**10**, **11**], from which one can deduce that, up to certain localizations, the winding-invariant primes of $\mathcal{O}_q(SL_n(k))$ are generated by quantum minors (cf. [**1**, Corollary II.4.12]). Since $\mathcal{O}_q(GL_n(k))$ is isomorphic to a Laurent polynomial ring over $\mathcal{O}_q(SL_n(k))$ [**13**, Proposition], the above statement also holds in $\mathcal{O}_q(GL_n(k))$. In particular, those H-primes of $\mathcal{O}_q(M_n(k))$ which do not contain the quantum determinant can be generated, up to suitable localizations, by quantum minors.

Geometric background. The classical origins of our main theorem, especially as it applies to winding-invariant primes of $\mathcal{O}_q(M_n(k))$ not containing the quantum determinant, lie in the geometry of 'LU-decompositions' of invertible matrices. For this part of the introduction, let us assume (to avoid complications) that k is algebraically closed. An LU-decomposition of a matrix $X \in GL_n(k)$ is any expression $X = LU$ where L (respectively, U) is a lower (respectively, upper) triangular invertible matrix. It is well known that X has such a decomposition if and only if the principal minors of X (those indexed by rows and columns from an initial segment of $\{1, \dots, n\}$) are all nonzero. The LU-decomposable matrices thus form a dense open subvariety of $GL_n(k)$, known as the *big cell*. We may write the big cell in the form B^+B^- where B^+ (respectively, B^-) is the subgroup of lower (respectively, upper) triangular matrices in $GL_n(k)$, and we have

$$\mathcal{O}(B^+B^-) = \mathcal{O}(GL_n(k))[D^{-1}]$$

where D is the multiplicative subset of $\mathcal{O}(GL_n(k))$ generated by the principal minors. The comorphism of the multiplication map $B^+ \times B^- \to GL_n(k)$ provides an embedding

$$\beta : \mathcal{O}(GL_n(k))[D^{-1}] \longrightarrow \mathcal{O}(B^+) \otimes \mathcal{O}(B^-);$$

the restriction of β to $\mathcal{O}(GL_n(k))$ is just the composition of the comultiplication map $\mathcal{O}(GL_n(k)) \to \mathcal{O}(GL_n(k)) \otimes \mathcal{O}(GL_n(k))$ with the tensor product of the restriction maps $\mathcal{O}(GL_n(k)) \to \mathcal{O}(B^\pm)$. The structure of the image of β is easy to determine, since $B^+ \cap B^-$ is the diagonal subgroup of $GL_n(k)$ and the subgroups $B^\pm$ are semidirect products of their unipotent subgroups with this diagonal subgroup. Namely, the image of β is the subalgebra of $\mathcal{O}(B^+) \otimes \mathcal{O}(B^-)$ generated by (the cosets of) the elements

$$(\dagger) \qquad X_{ij}X_{jj}^{-1} \otimes 1 \qquad\qquad 1 \otimes X_{ij}X_{ii}^{-1} \qquad\qquad (X_{ii} \otimes X_{ii})^{\pm 1}.$$

Further, $\mathcal{O}(B^+) \otimes \mathcal{O}(B^-)$ is a Laurent polynomial ring over the image of β with respect to indeterminates $1 \otimes X_{ii}^{\pm 1}$.

Quantum analogs of the above facts are known, but we have not been able to locate complete statements in the literature. To formulate them, let us write $\mathcal{O}_q(B^+)$ and $\mathcal{O}_q(B^-)$ for the respective quotients of $\mathcal{O}_q(GL_n(k))$ modulo the ideals $\langle X_{ij} \mid i < j \rangle$ and $\langle X_{ij} \mid i > j \rangle$. Then:

(a) The composition of the comultiplication map on $\mathcal{O}_q(GL_n(k))$ with the tensor product of the quotient maps $\mathcal{O}_q(GL_n(k)) \to \mathcal{O}_q(B^\pm)$ yields an embedding $\beta : \mathcal{O}_q(GL_n(k)) \to \mathcal{O}_q(B^+) \otimes \mathcal{O}_q(B^-)$ [**16**, Theorem 8.1.1].

(b) The multiplicative subset D of $\mathcal{O}_q(GL_n(k))$ generated by the principal quantum minors is a denominator set.

(c) β extends to $\mathcal{O}_q(GL_n(k))[D^{-1}]$, and the image of this extension is the subalgebra $\mathcal{B}$ of $\mathcal{O}_q(B^+) \otimes \mathcal{O}_q(B^-)$ generated by (the cosets of) the elements (†).

(d) $\mathcal{O}_q(B^+) \otimes \mathcal{O}_q(B^-)$ is a skew-Laurent extension of $\mathcal{B}$ with respect to the variables $1 \otimes X_{ii}^{\pm 1}$.

These facts will be proved as part of the case $\boldsymbol{r} = \boldsymbol{c} = (1, \dots, n)$ of our work below. To obtain them via existing results in the literature, one first transfers the problem to $\mathcal{O}_q(SL_n(k))$ using the isomorphism $\mathcal{O}_q(GL_n(k)) \cong \mathcal{O}_q(SL_n(k))[z^{\pm 1}]$ established in [**13**, Proposition]; the desired conditions then hold in the generality of $\mathcal{O}_q(G)$, where G is an arbitrary semisimple algebraic group. The isomorphism of the appropriate localization of $\mathcal{O}_q(G)$ with the analog of $\mathcal{B}$ is given in [**3**, Theorem 4.6] and [**12**, Proposition 9.2.14]. It is easy to see that $\mathcal{O}_q(B^+) \otimes \mathcal{O}_q(B^-)$ is a skew-Laurent extension of $\mathcal{B}$ with appropriate variables; this is mentioned for the case where q is a root of unity in [**4**, (4.6)].

The above facts concerning $\mathcal{O}_q(GL_n(k))$ immediately carry over to $\mathcal{O}_q(M_n(k))$, since D is also a denominator set in that algebra and $\mathcal{O}_q(M_n(k))[D^{-1}] = \mathcal{O}_q(GL_n(k))[D^{-1}]$. Whereas in $\mathcal{O}_q(GL_n(k))$ all prime ideals are disjoint from D, that no longer holds in $\mathcal{O}_q(M_n(k))$, and a sequence of modified versions of (a)–(d) are needed to yield information about those H-primes of $\mathcal{O}_q(M_n(k))$ that meet D. These modifications, involving maps

$$\beta_{\boldsymbol{r},\boldsymbol{c}} : \mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \longrightarrow R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}},$$

concern quantum analogs of what can be viewed as LU-decompositions for certain locally closed subsets of $M_n(k)$ of the form $B^+w^+ew^-B^-$ where $w^\pm$ are permutation matrices and $e = e_{11} + \cdots + e_{tt}$ is a diagonal idempotent matrix. We leave the formulations of these geometric facts to the interested reader.

Some notation and conventions. Throughout the paper, let $A = \mathcal{O}_q(M_n(k))$, where we fix a base field k, a positive integer n, and a nonzero scalar $q \in k^\times$. See (5.1)(a) for the basic relations satisfied by the standard generators X_{ij} of $\mathcal{O}_q(M_n(k))$. As mentioned in (5.2), our relations for A differ from those in [**16**] by an interchange of q and q^{-1}. On the other hand, they agree with the relations used in [**2**, **15**]. To match the relations in [**10**, **11**], replace q by q^2.

While our main interest is in the case that q is not a root of unity, some of our results do not require this assumption, and others require only that $q \neq \pm 1$. Thus, we impose no blanket hypotheses on q. To simplify some formulas, write $\widehat{q} = q - q^{-1}$. We also fix the torus $H = (k^\times)^n \times (k^\times)^n$ and its action on A by winding automorphisms as described in $(*)$ above. The algebra A is graded in a natural way by $\mathbb{Z}^{2n} = \mathbb{Z}^n \times \mathbb{Z}^n$, each generator X_{ij} having degree (ϵ_i, ϵ_j) where $\epsilon_1, \dots, \epsilon_n$ is the standard basis for $\mathbb{Z}^n$. We refer to this grading as the *standard grading* on A. (As long as k is infinite, the homogeneous components of A for this grading coincide with the eigenspaces for the action of H.)

As in [**6**], we use the notations $[I \mid J]$ and $[i_1 \cdots i_s \mid j_1 \cdots j_s]$ for quantum minors in $\mathcal{O}_q(M_n(k))$, where I (respectively, $i_1, \dots, i_s$) records the set (respectively, a list) of the corresponding row indices, and similarly for column indices. Recall that $[I \mid J]$ corresponds to the quantum determinant in a subalgebra of $\mathcal{O}_q(M_n(k))$ isomorphic to $\mathcal{O}_q(M_s(k))$; we write $\mathcal{O}_q(M_{I,J}(k))$ for that subalgebra (see (5.1)(c)). We allow the index sets I and J to be empty, following the convention that $[\varnothing \mid \varnothing] = 1$.

The symbols $\subset$ and $\subseteq$ will be reserved for proper and arbitrary inclusions, respectively. We write $\sqcup$ to denote a disjoint union.

1. A partition of spec A

We begin by investigating sets $\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A$ of prime ideals defined by certain 'stepwise patterns' of quantum minors corresponding to strictly increasing sequences $\boldsymbol{r}$ and $\boldsymbol{c}$ of row and column indices. Our aim in this section is to show that these sets partition spec A when q is not a root of unity. In fact, as long as $q \neq \pm 1$, these sets at least partition the collection of completely prime ideals of A. (Recall from [**8**, Theorem 3.2] that when q is not a root of unity, all primes of A are completely prime.) We proceed without imposing special hypotheses on q until needed.

1.1. We introduce the following partial ordering $\leq$ on index sets $I, I' \subseteq \{1, \dots, n\}$ of the same cardinality. Write $I = \{i_1 < \cdots < i_l\}$ and $I' = \{i'_1 < \cdots < i'_l\}$; then $I \leq I'$ if and only if $i_s \leq i'_s$ for $s = 1, \dots l$. (This is the same as the 'column ordering' $\leq_c$ used in [**6**].) All order relations among index sets in this paper will refer to the above partial ordering. This includes statements that an index set with a particular property is minimal among index sets with the same cardinality satisfying that property (e.g., the sets $\widetilde{I}$ and $\widetilde{J}$ in Theorem 1.9).

1.2. Let $\boldsymbol{RC}$ denote the set of all pairs $(\boldsymbol{r}, \boldsymbol{c})$ where $\boldsymbol{r}$ and $\boldsymbol{c}$ are strictly increasing sequences in $\{1, \dots, n\}$ of the same length, that is, $\boldsymbol{r} = (r_1, \dots, r_t)$ and $\boldsymbol{c} = (c_1, \dots, c_t)$ in $\mathbb{N}^t$ with $1 \leq r_1 < r_2 < \cdots < r_t \leq n$ and $1 \leq c_1 < c_2 < \cdots < c_t \leq n$. We allow $t = 0$, in which case $\boldsymbol{r}$ and $\boldsymbol{c}$ are empty sequences, denoted either $(\quad)$ or $\varnothing$. When referring to the length t of $\boldsymbol{r}$ and $\boldsymbol{c}$, we write $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}_t$.

For $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}_t$, let $K_{\boldsymbol{r},\boldsymbol{c}}$ be the ideal of A generated by the following set of quantum minors:

$$\bigl\{[I \mid J] \bigm| |I| > t\bigr\} \cup \bigl\{[I \mid J] \bigm| |I| = l \leq t \text{ and } I \not\geq \{r_1, \dots, r_l\}\bigr\} \\ \cup \bigl\{[I \mid J] \bigm| |I| = l \leq t \text{ and } J \not\geq \{c_1, \dots, c_l\}\bigr\}.$$

In particular, $K_{\varnothing,\varnothing} = \langle X_{ij} \mid i, j \in \{1, \dots, n\}\rangle$.

Now set $d_l^{\boldsymbol{r},\boldsymbol{c}} = [r_1 \cdots r_l \mid c_1 \cdots c_l]$ for $l \leq t$, and observe that these quantum minors commute with each other (cf. (5.1)(c)). Let $D_{\boldsymbol{r},\boldsymbol{c}}$ denote the multiplicative subset of A generated by $d_1^{\boldsymbol{r},\boldsymbol{c}}, \dots, d_t^{\boldsymbol{r},\boldsymbol{c}}$. (In particular, $D_{\varnothing,\varnothing} = \{1\}$.) Set $\widetilde{d}_l^{\boldsymbol{r},\boldsymbol{c}} = d_l^{\boldsymbol{r},\boldsymbol{c}} + K_{\boldsymbol{r},\boldsymbol{c}} \in A/K_{\boldsymbol{r},\boldsymbol{c}}$ for $l \leq t$, and let $\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}$ denote the image of $D_{\boldsymbol{r},\boldsymbol{c}}$ in $A/K_{\boldsymbol{r},\boldsymbol{c}}$.

We prove in this section that $\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}$ is a denominator set in $A/K_{\boldsymbol{r},\boldsymbol{c}}$, and that when q is generic, spec A is partitioned by the subsets

$$\begin{aligned} \operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A &:= \{P \in \operatorname{spec} A \mid K_{\boldsymbol{r},\boldsymbol{c}} \subseteq P \text{ and } P \cap D_{\boldsymbol{r},\boldsymbol{c}} = \varnothing\} \\ &\approx \operatorname{spec}\,(A/K_{\boldsymbol{r},\boldsymbol{c}})[\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}^{-1}] \end{aligned}$$

as $(\boldsymbol{r}, \boldsymbol{c})$ ranges over $\boldsymbol{RC}$.

1.3. Lemma. *Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J|$, and set*

$$L = \bigl\langle\ [I' \mid J'] \bigm| |I'| = |I|, \text{ and } I' < I \text{ or } J' < J\ \bigr\rangle.$$

If $r, c \in \{1, \dots, n\}$ *with* $r \le \max(I)$ *or* $c \le \max(J)$, *then*

$$(*) \qquad [I \mid J]X_{rc} - q^{2-\delta(r,I)-\delta(c,J)} X_{rc}[I \mid J] \;\in\; L.$$

Proof. If $r \in I$ and $c \in J$, then $[I \mid J]$ commutes with X_{rc} (cf. Lemma 5.2(a)), and so (*) holds.

Next, suppose that $r \in I$ and $c \notin J$. If $j \in J$ and $j > c$, then $J \sqcup \{c\} \setminus \{j\} < J$, whence $[I \mid J \sqcup \{c\} \setminus \{j\}] \in L$ by definition of L. It follows from Lemma 5.2(b2) that $[I \mid J]X_{rc} - qX_{rc}[I \mid J] \in L$. Thus (*) holds in this case. The case where $r \notin I$ and $c \in J$ is proved similarly, using Lemma 5.2(c2).

Finally, suppose that $r \notin I$ and $c \notin J$, and note that either $r < \max(I)$ or $c < \max(J)$. (This part of the proof is similar to that of Lemma 5.7.) There is no loss of generality in assuming that

$$I \sqcup \{r\} = J \sqcup \{c\} = \{1, \dots, n\},$$

whence either $r < n$ or $c < n$. In the notation of [**16**, (4.3)], $[I \mid J] = A(r\,c)$. Note that $\{1, \dots, n\} \setminus \{i\} < I$ when $i > r$, while $\{1, \dots, n\} \setminus \{j\} < J$ when $j > c$. Hence, $A(i\,j) \in L$ whenever either $i > r$ or $j > c$. Set $D_q = [1 \cdots n \mid 1 \cdots n]$. The basic q-Laplace relations (Corollary 5.5) imply that D_q lies in the ideal generated by all the $A(n\,j)$, and in the ideal generated by all the $A(i\,n)$. Since either $n > r$ or $n > c$, it follows that $D_q \in L$.

We now use the basic q-Laplace relations in the form given in [**16**, Corollary 4.4.4]. The first two relations yield

$$(1) \qquad \sum_{j=1}^{n} (-q)^{j-r} X_{rj} A(r\,j) = \sum_{j=1}^{n} (-q)^{r-j} A(r\,j) X_{rj} = D_q \in L.$$

Since $A(r\,j) \in L$ for $j > c$, we obtain the following congruences, after multiplying the two sums in (1) by $(-q)^{r-c}$ and $(-q)^{c-r}$, respectively:

$$(2) \qquad X_{rc}A(r\,c) \equiv -\sum_{j<c} (-q)^{j-c} X_{rj} A(r\,j) \qquad (\text{mod } L)$$

$$(3) \qquad A(r\,c)X_{rc} \equiv -\sum_{j<c} (-q)^{c-j} A(r\,j) X_{rj} \qquad (\text{mod } L).$$

For any j, the third relation of [**16**, Lemma 5.1.2] implies that

$$(4) \qquad X_{rj}A(r\,j) \equiv A(r\,j)X_{rj} + (1-q^{-2}) \sum_{l<j} (-q)^{j-l} A(r\,l) X_{rl} \qquad (\text{mod } L),$$

since $X_{sj}A(s\,j) \in L$ for $s > r$. Substituting (4) into (2) for all $j < c$, we obtain

$$(5) \qquad \begin{aligned} X_{rc}A(r\,c) &\equiv -\sum_{j<c} (-q)^{j-c} A(r\,j) X_{rj} \\ &\qquad - (1-q^{-2}) \sum_{j<c} \sum_{l<j} (-q)^{2j-l-c} A(r\,l) X_{rl} \qquad (\text{mod } L) \\ &= -\sum_{l<c} \Big[(-q)^{l-c} + (1-q^{-2}) \sum_{l<j<c} (-q)^{2j-l-c} \Big] A(r\,l) X_{rl}. \end{aligned}$$

The expression in square brackets can be simplified as follows:

(6)
$$\begin{aligned}(-q)^{l-c}+(1-q^{-2})\sum_{l<j<c}(-q)^{2j-l-c} &= (-q)^{l-c}\Bigl[1+(1-q^{-2})\sum_{0<m<c-l}(-q)^{2m}\Bigr]\\ &= (-q)^{c-l-2}.\end{aligned}$$

Substituting (6) into (5) and replacing l by j, we obtain

(7)
$$X_{rc}A(r\,c)\equiv-\sum_{j<c}(-q)^{c-j-2}A(r\,j)X_{rj}\qquad(\operatorname{mod} L).$$

Finally, combining (3) with (7), we conclude that

$$[I\mid J]X_{rc}=A(r\,c)X_{rc}\equiv q^2X_{rc}A(r\,c)=q^2X_{rc}[I\mid J]\qquad(\operatorname{mod} L),$$

as desired. □

1.4. Corollary. *Let $I,J\subseteq\{1,\dots,n\}$ with $|I|=|J|$, and set*

$$L=\bigl\langle\,[I'\mid J']\;\big|\;|I'|=|I|,\text{ and }\ I'<I\ \text{ or }\ J'<J\,\bigr\rangle.$$

Then the coset $d=[I\mid J]+L$ generates a denominator set in A/L.

Proof. Set $B=A/L$, and set $x_{ij}=X_{ij}+L$ for all i,j. Lemma 1.3 says that

(1)
$$dx_{ij}=q^{2-\delta(i,I)-\delta(j,J)}x_{ij}d$$

whenever $i\le\max(I)$ or $j\le\max(J)$. Hence, in this case we have $d^rx_{ij}\in Bd^r$ for all $r\ge 0$.

When $i>\max(I)$ and $j>\max(J)$, Lemma 5.7 says that

(2)
$$dx_{ij}-q^2x_{ij}d=e:=(1-q^2)[I\sqcup\{i\}\mid J\sqcup\{j\}]+L.$$

Observe that d and e commute. Hence, it follows from (2) by an easy induction that

(3)
$$\begin{aligned}d^rx_{ij}&=q^{2r}x_{ij}d^r+(q^{2r-2}+\cdots+q^2+1)ed^{r-1}\\&=\bigl[q^{2r}x_{ij}d+(q^{2r-2}+\cdots+q^2+1)e\bigr]d^{r-1}\end{aligned}$$

for all $r>0$. Combining (1) and (3), we see that

(4)
$$d^rx_{ij}\ \in\ Bd^{r-1}\qquad(i,j=1,\dots,n;\ r=1,2,\dots).$$

Since B is spanned by products of the x_{ij}, it follows from (4) that $D:=\{d^r\mid r\ge 0\}$ is a left Ore set in B. Similarly, D is right Ore, and therefore D is a denominator set because B is noetherian. □

1.5. Proposition. *Let* $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}_t$*, and set*

$$L = \big\langle\, [I \mid J] \;\big|\; |I| = l \le t, \text{ and } I < \{r_1, \dots, r_l\} \text{ or } J < \{c_1, \dots, c_l\} \,\big\rangle.$$

Then the image of $D_{\boldsymbol{r},\boldsymbol{c}}$ *in* A/L *is a denominator set. Consequently,* $\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}$ *is a denominator set in* $A/K_{\boldsymbol{r},\boldsymbol{c}}$*.*

Proof. For each $l = 1, \dots, t$, Corollary 1.4 shows that $d_l^{\boldsymbol{r},\boldsymbol{c}} + L$ generates a denominator set in A/L. The proposition follows. □

1.6. In view of Proposition 1.5, we can form Ore localizations

$$A_{\boldsymbol{r},\boldsymbol{c}} = (A/K_{\boldsymbol{r},\boldsymbol{c}})[\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}^{-1}]$$

for $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$. (It will follow from Lemma 2.5 that $A_{\boldsymbol{r},\boldsymbol{c}} \neq 0$.) The localization maps $A \to A/K_{\boldsymbol{r},\boldsymbol{c}} \to A_{\boldsymbol{r},\boldsymbol{c}}$ induce Zariski homeomorphisms

$$\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A \longrightarrow \operatorname{spec} A_{\boldsymbol{r},\boldsymbol{c}}.$$

Lemma 1.7. *Let* $I, J \subseteq \{1, \dots, n\}$ *with* $|I| = |J|$*, and let* P *be an ideal of* A*.*

(a) *Fix* $J_1 \subseteq J$*. If* $[I_1 \mid J_1] \in P$ *for all* $I_1 \subseteq I$ *with* $|I_1| = |J_1|$*, then* $[I \mid J] \in P$*.*

(b) *Fix* $I_1 \subseteq I$*. If* $[I_1 \mid J_1] \in P$ *for all* $J_1 \subseteq J$ *with* $|J_1| = |I_1|$*, then* $[I \mid J] \in P$*.*

Proof. By symmetry (see (5.1)(b)), we need only prove (a). Set $J_2 = J \setminus J_1$. Then Lemma 5.4(a) provides a relation of the form

$$\sum_{I_1 \sqcup I_2 = I} \pm q^{\bullet} [I_1 \mid J_1][I_2 \mid J_2] = \pm q^{\bullet} [I \mid J],$$

where $\pm$ is an unspecified sign and $q^{\bullet}$ stands for an unspecified power of q. Since all the $[I_1 \mid J_1] \in P$ by assumption, it follows that $[I \mid J] \in P$. □

Lemma 1.8. *Let* $I_1, I_2, J_1, J_2 \subseteq \{1, \dots, n\}$ *with* $|I_1| = |J_1|$ *and* $|I_2| = |J_2|$*, and let* P *be a completely prime ideal of* A*. Assume that one of the following conditions* (a), (b), *or* (c) *holds:*

(a) (1) $|I_1 \cap I_2| > |J_1 \cap J_2|$, *and*
(2) $[I_1 \mid J'] \in P$ *whenever* $J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2$ *with* $|J'| = |J_1|$ *but* $J' \neq J_1$*.*

(b) (1) $|I_1 \cap I_2| < |J_1 \cap J_2|$, *and*
(2) $[I' \mid J_1] \in P$ *whenever* $I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2$ *with* $|I'| = |I_1|$ *but* $I' \neq I_1$*.*

(c) (1) $|I_1 \cap I_2| = |J_1 \cap J_2|$ *and* $[I_1 \cup I_2 \mid J_1 \cup J_2] \in P$*, and*
(2) *Either* (a)(2) *or* (b)(2) *holds.*

Then either $[I_1 \mid J_1] \in P$ *or* $[I_2 \mid J_2] \in P$.

Proof. By symmetry, it suffices to prove cases (a) and (c).

(a) Set $V = J_1 \cup J_2 = J \sqcup L$ where $J = J_1 \cap J_2$ and $L = V \setminus J$. Since $|I_1 \cap I_2| > |J|$, we have $|I_1 \cup I_2| < |V|$, and so there exists $U \subseteq \{1, \dots, n\}$ such that $I_1 \cup I_2 \subset U$ and $|U| = |V|$. Also, we have $|I_1| + |I_2| = |J_1| + |J_2| = 2|J| + |L|$. Thus, Lemma 5.6(b) yields a relation of the form

$$(\dagger) \qquad \sum_{L = L' \sqcup L''} \pm q^\bullet [I_1 \mid J \sqcup L'][I_2 \mid J \sqcup L''] = 0.$$

Now for each term in this sum, $J_1 \cap J_2 \subseteq J \sqcup L' \subseteq J_1 \cup J_2$ with $|J \sqcup L'| = |J_1|$. If $L' \neq J_1 \setminus J$, then $J \sqcup L' \neq J_1$, whence $[I_1 \mid J \sqcup L'] \in P$ by hypothesis. Therefore the remaining term in ($\dagger$) must lie in P. This is the term with $L' = J_1 \setminus J$, whence $L'' = J_2 \setminus J$, and so $J \sqcup L' = J_1$ and $J \sqcup L'' = J_2$. Thus

$$\pm q^\bullet [I_1 \mid J_1][I_2 \mid J_2] \in P.$$

Since P is completely prime, either $[I_1 \mid J_1] \in P$ or $[I_2 \mid J_2] \in P$.

(c) By symmetry, we may assume that (a)(2) holds. Again, set $V = J_1 \cup J_2 = J \sqcup L$ where $J = J_1 \cap J_2$ and $L = V \setminus J$. Set $U = I_1 \cup I_2$, and observe that $|I_1| + |I_2| = 2|J| + |L|$. This time, Lemma 5.6(b) provides a relation of the form

$$(\ddagger) \qquad \sum_{L = L' \sqcup L''} \pm q^\bullet [I_1 \mid J \sqcup L'][I_2 \mid J \sqcup L''] = \pm q^\bullet [I_1 \cap I_2 \mid J][U \mid V].$$

Since $[U \mid V] \in P$ by hypothesis, the right hand side of ($\ddagger$) lies in P. Therefore we can proceed as in the proof above. □

1.9. Theorem. *Assume that* $q \neq \pm 1$. *Let* P *be a completely prime ideal of* A, *and let* $t \leq n$ *be maximal such that* P *does not contain all* $t \times t$ *quantum minors. Choose*

$$\widetilde{I} = \{r_1 < \cdots < r_t\} \subseteq \{1, \dots, n\} \qquad \text{and} \qquad \widetilde{J} = \{c_1 < \cdots < c_t\} \subseteq \{1, \dots, n\}$$

with $\widetilde{I}$ *minimal such that some* $[\widetilde{I} \mid *] \notin P$, *and* $\widetilde{J}$ *minimal such that some* $[* \mid \widetilde{J}] \notin P$. *Then*

(a) $[r_1 \cdots r_s \mid c_1 \cdots c_s] \notin P$ *for* $s = 1, \dots, t$. *In particular,* $[\widetilde{I} \mid \widetilde{J}] \notin P$.

(b) $[I \mid J] \in P$ *whenever* $|I| = s \leq t$ *and either* $I \not\geq \{r_1, \dots, r_s\}$ *or* $J \not\geq \{c_1, \dots, c_s\}$. *In particular, it follows that* $\widetilde{I}$ *and* $\widetilde{J}$ *are unique.*

Proof. Since the theorem holds trivially when $t = 0$, we may assume that $t > 0$. By assumption, there exists J_0 such that $[\widetilde{I} \mid J_0] \notin P$. We first claim that

(1) $[I \mid J] \in P$ whenever $|I| = s \leq t$ and $I < \{r_1, \dots, r_s\}$.

Suppose not, so that some $[I \mid J] \notin P$ where $|I| = s \leq t$ and $I < \{r_1, \dots, r_s\}$. We may assume that s is minimal for this, and that with I fixed, $|J \cap J_0|$ is maximal. Note from the minimality of $\widetilde{I}$ that $s < t$.

Write $I = \{i_1 < \cdots < i_s\}$. There is some $b \le s$ such that $i_l = r_l$ for $l < b$ while $i_b < r_b$, whence $\{i_1, \ldots, i_b\} < \{r_1, \ldots, r_b\}$. Since $[I \mid J] \notin P$, Lemma 1.7(b) implies that some $[i_1 \cdots i_b \mid *] \notin P$. Hence, the minimality of s implies that $b = s$. Thus, we have $i_l = r_l$ for $l < s$ while $i_s < r_s$. In particular, $r_{s-1} = i_{s-1} < i_s < r_s$.

Assume that $|J \cap J_0| \ge s-1$. In this case, we will apply Lemma 1.8 with

$$I_1 = \widetilde{I} \qquad I_2 = I \qquad J_1 = J_0 \qquad J_2 = J.$$

Note that $|I_1 \cap I_2| = s-1$ and $[I_1 \mid J_1], [I_2 \mid J_2] \notin P$. When $I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2$ with $|I'| = t$ and $I' \ne I_1$, we have $I' = \widetilde{I} \cup \{i_s\} \setminus \{r_l\}$ for some $l \ge s$. Then since $r_{s-1} < i_s < r_s \le r_l$, we have $I' < \widetilde{I}$, and so $[I' \mid J_1] \in P$ by the minimality of $\widetilde{I}$. Further, if $|J_1 \cap J_2| = s-1$, then $[I_1 \cup I_2 \mid J_1 \cup J_2] \in P$ because P contains all $(t+1)\times(t+1)$ quantum minors. Therefore by Lemma 1.8(b), if $|J \cap J_0| \ge s$, or by Lemma 1.8(c), if $|J \cap J_0| = s-1$, we have either $[I_1 \mid J_1] \in P$ or $[I_2 \mid J_2] \in P$, giving us a contradiction. Therefore $|J \cap J_0| < s-1$.

Next, we will apply Lemma 1.8 with the roles of I_1, I_2 and J_1, J_2 reversed, that is, with

$$I_1 = I \qquad I_2 = \widetilde{I} \qquad J_1 = J \qquad J_2 = J_0.$$

When $J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2$ with $|J'| = s$ and $J' \ne J_1$, we must have $|J' \cap J_0| = |J' \cap J_2| > |J \cap J_0|$. By the maximality of $|J \cap J_0|$, we obtain $[I \mid J'] \in P$ in this case. But then Lemma 1.8(a) leads to the same contradiction.

Therefore (1) holds. By symmetry, we must also have

(2) $[I \mid J] \in P$ whenever $|I| = s \le t$ and $J < \{c_1, \ldots, c_s\}$.

We now proceed by induction on $s = 1, \ldots, t$ to verify the following properties:

(P_s) $[r_1 \cdots r_s \mid c_1 \cdots c_s] \notin P$;

(Q_s) $[I \mid J] \in P$ whenever $|I| = s$ and either $I \not\ge \{r_1, \ldots, r_s\}$ or $J \not\ge \{c_1, \ldots, c_s\}$.

The theorem will then be established.

To start, note that we cannot have all $[r_1 \mid *] \in P$ or all $[* \mid c_1] \in P$, by Lemma 1.7. Choose i, j such that $[r_1 \mid j], [i \mid c_1] \notin P$. If $j = c_1$ or $i = r_1$, then $[r_1 \mid c_1] \notin P$. Otherwise, in view of (1) and (2) we must have $j > c_1$ and $i > r_1$. Hence, because of the assumption that $q \ne \pm 1$, we have

$$[r_1 \mid c_1][i \mid j] - [i \mid j][r_1 \mid c_1] = (q - q^{-1})[r_1 \mid j][i \mid c_1] \notin P,$$

which implies that $[r_1 \mid c_1] \notin P$. Therefore (P_1) holds. Property (Q_1) is immediate from (1) and (2).

Now let $1 < s \le t$ and assume that (P_a) and (Q_a) hold for all $a < s$. By Lemma 1.7, there exist $j_1 < \cdots < j_s$ such that

$$[r_1 \cdots r_s \mid j_1 \cdots j_s] \notin P,$$

and we may assume that $\{j_1,\dots,j_s\}$ is minimal for this. Likewise, there exist $i_1 < \cdots < i_s$ such that $[i_1\cdots i_s \mid c_1\cdots c_s] \notin P$ and such that $\{i_1,\dots,i_s\}$ is minimal for this. We cannot have $\{i_1,\dots,i_{s-1}\} \not\geq \{r_1,\dots,r_{s-1}\}$, since then (Q_{s-1}) would imply that all $[i_1\cdots i_{s-1} \mid *] \in P$, whence Lemma 1.7(b) would imply that all $[i_1\cdots i_s \mid *] \in P$. Therefore $\{i_1,\dots,i_{s-1}\} \geq \{r_1,\dots,r_{s-1}\}$, and similarly $\{j_1,\dots,j_{s-1}\} \geq \{c_1,\dots,c_{s-1}\}$.

Suppose there exists $b < s$ such that $j_l = c_l$ for $l < b$ while $j_b > c_b$. We will apply Lemma 1.8 with

$$I_1 = \{r_1,\dots,r_s\} \qquad I_2 = \{r_1,\dots,r_b\}$$
$$J_1 = \{j_1,\dots,j_s\} \qquad J_2 = \{c_1,\dots,c_b\}.$$

Observe that $|I_1 \cap I_2| = b > |J_1 \cap J_2|$. If $J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2$ with $|J'| = s$ and $J' \neq J_1$, then $J' = J_1 \cup \{c_b\} \setminus \{j_d\}$ for some $d \geq b$. In this case, $J' < \{j_1,\dots,j_s\}$, whence $[I_1 \mid J'] \in P$ by the minimality of $\{j_1,\dots,j_s\}$. Therefore Lemma 1.8(a) implies that either $[I_1 \mid J_1] \in P$ or $[I_2 \mid J_2] \in P$. But $[I_1 \mid J_1] \notin P$ by choice of J_1, and $[I_2 \mid J_2] \notin P$ by (P_b), so we have a contradiction. Therefore $j_l = c_l$ for all $l < s$. Similarly, $i_l = r_l$ for all $l < s$.

If $j_s = c_s$, then (P_s) holds. If $j_s < c_s$, then $\{j_1,\dots,j_s\} < \{c_1,\dots,c_s\}$, which would imply $[r_1\cdots r_s \mid j_1\cdots j_s] \in P$ by (2), contradicting our assumptions. Therefore we may assume that $j_s > c_s$. Likewise, we may assume that $i_s > r_s$.

Set $U = \{r_1,\dots,r_s,i_s\}$ and $V = \{c_1,\dots,c_s,j_s\}$. By Lemma 5.3(d), we have

$$[i_1\cdots i_s \mid j_1\cdots j_s][r_1\cdots r_s \mid c_1\cdots c_s] - [r_1\cdots r_s \mid c_1\cdots c_s][i_1\cdots i_s \mid j_1\cdots j_s] = (q^{-1}-q)[r_1\cdots r_s \mid j_1\cdots j_s][i_1\cdots i_s \mid c_1\cdots c_s].$$

Since neither of the factors $[r_1\cdots r_s \mid j_1\cdots j_s]$ and $[i_1\cdots i_s \mid c_1\cdots c_s]$ is in P, it follows that $[r_1\cdots r_s \mid c_1\cdots c_s]$ cannot be in P. This establishes property (P_s).

Finally, suppose that (Q_s) fails. By symmetry, we may assume that $[I \mid J] \notin P$ for some I, J with $|I| = s$ and $I \not\geq \{r_1,\dots,r_s\}$. We may also assume that I is minimal for this, and that with I fixed, J is minimal.

Write $I = \{i_1 < \cdots < i_s\}$ and $J = \{j_1 < \cdots < j_s\}$. If $\{i_1,\dots,i_{s-1}\} \not\geq \{r_1,\dots,r_{s-1}\}$, then by (Q_{s-1}) we would have all $[i_1\cdots i_{s-1} \mid *] \in P$, whence Lemma 1.7(b) would imply that all $[I \mid *] \in P$, contradicting our choice of I. Thus $\{i_1,\dots,i_{s-1}\} \geq \{r_1,\dots,r_{s-1}\}$, and similarly $\{j_1,\dots,j_{s-1}\} \geq \{c_1,\dots,c_{s-1}\}$. Since $I \not\geq \{r_1,\dots,r_s\}$, we must also have $i_s < r_s$. Note that $r_{s-1} \leq i_{s-1} < i_s < r_s$, and so $i_s \notin \{r_1,\dots,r_{s-1}\}$. Further, $\{r_1,\dots,r_{s-1},i_s\} < \{r_1,\dots,r_s\}$, and so all $[r_1\cdots r_{s-1}i_s \mid *] \in P$ by (1), whence $I \neq \{r_1,\dots,r_{s-1},i_s\}$. Therefore there is some $b < s$ such that $i_l = r_l$ for $l < b$ while $i_b > r_b$.

Suppose there exists $d \leq b$ such that $j_m = c_m$ for $m < d$ while $j_d > c_d$. We will apply Lemma 1.8 with

$$I_1 = I = \{i_1,\dots,i_s\} \qquad I_2 = \{r_1,\dots,r_d\}$$
$$J_1 = J = \{j_1,\dots,j_s\} \qquad J_2 = \{c_1,\dots,c_d\}.$$

Note that $|I_1 \cap I_2| \geq d-1 = |J_1 \cap J_2|$, and that $|I_1 \cap I_2| = |J_1 \cap J_2|$ only when $d = b$. Since $r_{b-1} < r_b < i_b$, we have $\{r_1,\dots,r_b,i_b,\dots,i_{s-1}\} < I$, and so all $[r_1\cdots r_b i_b\cdots i_{s-1} \mid *] \in P$

by the minimality of I. Then Lemma 1.7(b) implies that all $[r_1 \cdots r_b i_b \cdots i_s \mid *] \in P$. In particular, when $d = b$ we find that $[I_1 \cup I_2 \mid J_1 \cup J_2] \in P$.

If $J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2$ with $|J'| = s$ and $J' \neq J_1$, then $J' = J \cup \{c_d\} \setminus \{j_p\}$ for some $p \geq d$. In this case, $J' < J$, and so $[I_1 \mid J'] \in P$ by the minimality of J. Hence, case (a) of Lemma 1.8 (if $d > b$) or case (c) (if $d = b$) implies that either $[I_1 \mid J_1] \in P$ or $[I_2 \mid J_2] \in P$. But $[I_1 \mid J_1] \notin P$ by assumption, and $[I_2 \mid J_2] \notin P$ by (P_d), so we have a contradiction. Therefore $j_m = c_m$ for all $m \leq b$.

We will conclude by applying Lemma 1.8 with

$$I_1 = I = \{i_1, \dots, i_s\} \qquad I_2 = \{r_1, \dots, r_b\}$$
$$J_1 = J = \{j_1, \dots, j_s\} \qquad J_2 = \{c_1, \dots, c_b\}.$$

Note that $i_{b-1} = r_{b-1} < r_b < i_b$ implies $r_b \notin I$, and so $|I_1 \cap I_2| < |J_1 \cap J_2|$. If $I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2$ with $|I'| = s$ and $I' \neq I_1$, then $I' = I \cup \{r_b\} \setminus \{i_p\}$ for some $p \geq b$. In this case, $I' < I$, whence $[I' \mid J_1] \in P$ by the minimality of I. Thus Lemma 1.8(b) implies that either $[I_1 \mid J_1] \in P$ or $[I_2 \mid J_2] \in P$, and again we have reached a contradiction.

Therefore (Q_s) must hold, which establishes our induction step. □

1.10. Corollary. *Assume that $q \neq \pm 1$. Given any completely prime ideal $P \in \operatorname{spec} A$, there is a unique pair $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$ such that $K_{\boldsymbol{r},\boldsymbol{c}} \subseteq P$ and $P \cap D_{\boldsymbol{r},\boldsymbol{c}} = \varnothing$.*

Thus, if q is not a root of unity,

$$\operatorname{spec} A = \bigsqcup_{(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{RC}} \operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A.$$

Proof. Let $t \leq n$ be maximal such that P does not contain all $t \times t$ quantum minors, let $\{r_1 < \cdots < r_t\}$ and $\{c_1 < \cdots < c_t\}$ be as in Theorem 1.9, and set $\boldsymbol{r} = (r_1, \dots, r_t)$ and $\boldsymbol{c} = (c_1, \dots, c_t)$. The theorem implies that $K_{\boldsymbol{r},\boldsymbol{c}} \subseteq P$ and that $d_s^{\boldsymbol{r},\boldsymbol{c}} \notin P$ for $s = 1, \dots, t$. Since P is completely prime, it follows that $P \cap D_{\boldsymbol{r},\boldsymbol{c}} = \varnothing$.

Now suppose that we also have $(\boldsymbol{r}', \boldsymbol{c}') \in \boldsymbol{RC}_{t'}$ for some t' such that $K_{\boldsymbol{r}',\boldsymbol{c}'} \subseteq P$ and $P \cap D_{\boldsymbol{r}',\boldsymbol{c}'} = \varnothing$. Then P contains all $(t'+1) \times (t'+1)$ quantum minors but not all $t' \times t'$ quantum minors, whence $t' = t$. Moreover, we have $d_t^{\boldsymbol{r},\boldsymbol{c}} \notin K_{\boldsymbol{r}',\boldsymbol{c}'}$ and $d_t^{\boldsymbol{r}',\boldsymbol{c}'} \notin K_{\boldsymbol{r},\boldsymbol{c}}$. The first relation implies that $\boldsymbol{r} \geq \boldsymbol{r}'$ and $\boldsymbol{c} \geq \boldsymbol{c}'$, and the second relation yields the reverse inequalities. (Here we have transferred the relation $\leq$ in (1.1) from index sets to sequences in the obvious manner.) Therefore $\boldsymbol{r}' = \boldsymbol{r}$ and $\boldsymbol{c}' = \boldsymbol{c}$. □

2. Structure of $A_{\boldsymbol{r},\boldsymbol{c}}$

The purpose of this section is to develop a structure theorem for the localizations $A_{\boldsymbol{r},\boldsymbol{c}}$. We introduce localized factor algebras $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$ of A (patterned after quantized coordinate rings of groups of triangular matrices) together with subalgebras $B^+_{\boldsymbol{r}} \subset R^+_{\boldsymbol{r}}$ and $B^-_{\boldsymbol{c}} \subset R^-_{\boldsymbol{c}}$ (patterned after quantized coordinate rings of unipotent groups of triangular matrices), and we show that $A_{\boldsymbol{r},\boldsymbol{c}}$ is isomorphic to an algebra $B_{\boldsymbol{r},\boldsymbol{c}}$ trapped between $B^+_{\boldsymbol{r}} \otimes B^-_{\boldsymbol{c}}$ and $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$. More precisely, we prove that $B_{\boldsymbol{r},\boldsymbol{c}}$ is a skew-Laurent extension of $B^+_{\boldsymbol{r}} \otimes B^-_{\boldsymbol{c}}$ (this is a key ingredient in establishing that $A_{\boldsymbol{r},\boldsymbol{c}} \cong B_{\boldsymbol{r},\boldsymbol{c}}$), and that $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ is a skew-Laurent extension of $B_{\boldsymbol{r},\boldsymbol{c}}$.

2.1. Fix $t \in \{0,1,\dots,n\}$ and $(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{RC}_t$ throughout the section. Set

$$R^+_{\boldsymbol{r},0} = A/\langle X_{ij} \mid j > t \text{ or } i < r_j\rangle \qquad \text{and} \qquad R^-_{\boldsymbol{c},0} = A/\langle X_{ij} \mid i > t \text{ or } j < c_i\rangle.$$

Write Y_{ij} and Z_{ij} for the images of X_{ij} in $R^+_{\boldsymbol{r},0}$ and $R^-_{\boldsymbol{c},0}$, respectively. Note that these algebras are iterated skew polynomial extensions of k, hence noetherian domains, the natural indeterminates for these iterated skew polynomial structures being those Y_{ij} and Z_{ij} which are nonzero. These indeterminates, when recorded within an $n \times n$ matrix, display 'stairstep' patterns – for example, if $n = 4$ and $\boldsymbol{r} = (1,2,4)$, the Y_{ij} may be displayed as follows:

$$\begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ Y_{21} & Y_{22} & 0 & 0 \\ Y_{31} & Y_{32} & 0 & 0 \\ Y_{41} & Y_{42} & Y_{43} & 0 \end{bmatrix}.$$

Since all the information is recorded in the placement of the zero and nonzero positions within this matrix, a convenient abbreviation for this example is to write

$$R^+_{(1,2,4),0} = k \begin{bmatrix} + & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & 0 \end{bmatrix}.$$

Observe that the $Y_{r_s s}$ are regular normal elements in $R^+_{\boldsymbol{r},0}$, and that the Z_{sc_s} are regular normal elements in $R^-_{\boldsymbol{c},0}$. More precisely,

$$Y_{r_s s} Y_{ij} = \begin{cases} q^{-1} Y_{ij} Y_{r_s s} & (i = r_s,\ j \neq s) \\ q Y_{ij} Y_{r_s s} & (i \neq r_s,\ j = s) \\ Y_{ij} Y_{r_s s} & (i \neq r_s,\ j \neq s) \end{cases}$$

$$Z_{sc_s} Z_{lm} = \begin{cases} q Z_{lm} Z_{sc_s} & (l = s,\ m \neq c_s) \\ q^{-1} Z_{lm} Z_{sc_s} & (l \neq s,\ m = c_s) \\ Z_{lm} Z_{sc_s} & (l \neq s,\ m \neq c_s). \end{cases}$$

(For instance, the first relation above holds when $j > s$ because $Y_{ij} = 0$ in that case. To verify the third relation, observe that $Y_{r_s j} = 0$ if $j > s$, while $Y_{is} = 0$ if $i < r_s$.) In particular, the $Y_{r_s s}$ commute with each other, and the Z_{sc_s} commute with each other.

Due to the normality of the $Y_{r_s s}$ and the Z_{sc_s}, we can form Ore localizations

$$R^+_{\boldsymbol{r}} = R^+_{\boldsymbol{r},0}[Y^{-1}_{r_1 1}, Y^{-1}_{r_2 2}, \dots, Y^{-1}_{r_t t}] \qquad \text{and} \qquad R^-_{\boldsymbol{c}} = R^-_{\boldsymbol{c},0}[Z^{-1}_{1c_1}, Z^{-1}_{2c_2}, \dots, Z^{-1}_{tc_t}].$$

These algebras are noetherian domains, and they may be viewed as quantized coordinate rings of certain locally closed subvarieties of $M_n(k)$. Extending the abbreviated description given for the example above, we display the following abbreviation for $R^+_{\boldsymbol{r}}$ in that case:

$$R^+_{(1,2,4)} = k \begin{bmatrix} \pm & 0 & 0 & 0 \\ + & \pm & 0 & 0 \\ + & + & 0 & 0 \\ + & + & \pm & 0 \end{bmatrix}.$$

Observe that the standard $\mathbb{Z}^{2n}$-grading on A induces $\mathbb{Z}^{2n}$-gradings on $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$, which we also refer to as *standard*.

2.2. Let $\pi^+_{\boldsymbol{r},0} : A \to R^+_{\boldsymbol{r},0}$ and $\pi^-_{\boldsymbol{c},0} : A \to R^-_{\boldsymbol{c},0}$ be the quotient maps, and define

$$\beta_{\boldsymbol{r},\boldsymbol{c}} : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi^+_{\boldsymbol{r},0} \otimes \pi^-_{\boldsymbol{c},0}} R^+_{\boldsymbol{r},0} \otimes R^-_{\boldsymbol{c},0} \xrightarrow{\subseteq} R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}\,.$$

Observe that

$$\beta_{\boldsymbol{r},\boldsymbol{c}}(X_{ij}) = \sum_{l \le t,\, r_l \le i,\, c_l \le j} Y_{il} \otimes Z_{lj}$$

for all i, j. In particular, $\beta_{\boldsymbol{r},\boldsymbol{c}}(X_{ij}) = 0$ when $i < r_1$ or $j < c_1$.

2.3. Lemma. $K_{\boldsymbol{r},\boldsymbol{c}} \subseteq \ker(\beta_{\boldsymbol{r},\boldsymbol{c}})$.

Remark. We conjecture that $\ker(\beta_{\boldsymbol{r},\boldsymbol{c}}) = K_{\boldsymbol{r},\boldsymbol{c}}$.

Proof. Since $X_{ij} \in \ker(\pi^+_{\boldsymbol{r},0})$ for $j > t$, all $(t+1) \times (t+1)$ and larger quantum minors lie in $\ker(\pi^+_{\boldsymbol{r},0})$. In view of the rule for comultiplication of quantum minors (see (5.1)(d)), it follows that all $(t+1) \times (t+1)$ and larger quantum minors lie in $\ker(\beta_{\boldsymbol{r},\boldsymbol{c}})$.

Consider an index set I with $|I| = l \le t$ and $I \not\ge \{r_1, \dots, r_l\}$. Write $I = \{i_1 < \cdots < i_l\}$; then $i_m < r_m$ for some $m \le l$. Hence, $Y_{i_s j} = 0$ for all $s \le m$ and $j \ge m$. This implies that $\pi^+_{\boldsymbol{r},0}[i_1 \cdots i_m \mid M] = 0$ for all M with $|M| = m$, whence $\pi^+_{\boldsymbol{r},0}[I \mid K] = 0$ for all K with $|K| = l$ (cf. Lemmas 1.7 or 5.4). Therefore $\beta_{\boldsymbol{r},\boldsymbol{c}}[I \mid J] = 0$ for all J with $|J| = l$.

Likewise, $\beta_{\boldsymbol{r},\boldsymbol{c}}[I \mid J] = 0$ whenever $|I| = l \le t$ and $J \not\ge \{c_1, \dots, c_l\}$. $\square$

2.4. Let $B_{\boldsymbol{r},\boldsymbol{c}}$ denote the k-subalgebra of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ generated by the set

$$\{Y_{il} \otimes Z_{lj} \mid l \le t,\ i \ge r_l,\ j \ge c_l\} \cup \{Y^{-1}_{r_l l} \otimes Z^{-1}_{l c_l} \mid l \le t\}.$$

We may also express $B_{\boldsymbol{r},\boldsymbol{c}}$ as the subalgebra of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ generated by

$$\{Y_{il}Y^{-1}_{r_l l} \otimes 1 \mid l \le t,\ i > r_l\} \cup \{1 \otimes Z_{lj}Z^{-1}_{l c_l} \mid l \le t,\ j > c_l\} \cup \{(Y_{r_l l} \otimes Z_{l c_l})^{\pm 1} \mid l \le t\}.$$

Note that $\beta_{\boldsymbol{r},\boldsymbol{c}}(X_{ij}) \in B_{\boldsymbol{r},\boldsymbol{c}}$ for all i, j, so that $\beta_{\boldsymbol{r},\boldsymbol{c}}(A) \subseteq B_{\boldsymbol{r},\boldsymbol{c}}$.

Let $l \le t$. Since $Y_{r_s j} = 0$ for $s \le l$ and $j > s$, we have

$$\pi^+_{\boldsymbol{r},0}[r_1 \cdots r_l \mid K] = \begin{cases} Y_{r_1 1} Y_{r_2 2} \cdots Y_{r_l l} & (K = \{1, \dots, l\}) \\ 0 & (K \ne \{1, \dots, l\}). \end{cases}$$

Similarly, $\pi^-_{\boldsymbol{c},0}[1 \cdots l \mid c_1 \cdots c_l] = Z_{1c_1} Z_{2c_2} \cdots Z_{l c_l}$, and therefore

$$\beta_{\boldsymbol{r},\boldsymbol{c}}(d^{\boldsymbol{r},\boldsymbol{c}}_l) = (Y_{r_1 1} \otimes Z_{1 c_1})(Y_{r_2 2} \otimes Z_{2 c_2}) \cdots (Y_{r_l l} \otimes Z_{l c_l}).$$

In particular, $\beta_{\boldsymbol{r},\boldsymbol{c}}(d^{\boldsymbol{r},\boldsymbol{c}}_l)$ is invertible in $B_{\boldsymbol{r},\boldsymbol{c}}$.

2.5. Lemma. *The map $\beta_{\boldsymbol{r},\boldsymbol{c}}$ induces a surjective k-algebra homomorphism*

$$\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}} : A_{\boldsymbol{r},\boldsymbol{c}} \longrightarrow B_{\boldsymbol{r},\boldsymbol{c}}.$$

Remark. We shall prove later (Theorem 2.11) that $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$ is an isomorphism. Note that the lemma already implies that $A_{\boldsymbol{r},\boldsymbol{c}}$ is nonzero.

Proof. We have $K_{\boldsymbol{r},\boldsymbol{c}} \subseteq \ker(\beta_{\boldsymbol{r},\boldsymbol{c}})$ by Lemma 2.3 and $\beta_{\boldsymbol{r},\boldsymbol{c}}(A) \subseteq B_{\boldsymbol{r},\boldsymbol{c}}$ by (2.4), and so $\beta_{\boldsymbol{r},\boldsymbol{c}}$ induces a homomorphism $\beta'_{\boldsymbol{r},\boldsymbol{c}} : A/K_{\boldsymbol{r},\boldsymbol{c}} \to B_{\boldsymbol{r},\boldsymbol{c}}$. It also follows from (2.4) that $\beta'_{\boldsymbol{r},\boldsymbol{c}}$ sends the elements of $\widetilde{D}_{\boldsymbol{r},\boldsymbol{c}}$ to units of $B_{\boldsymbol{r},\boldsymbol{c}}$, and therefore $\beta'_{\boldsymbol{r},\boldsymbol{c}}$ does induce a homomorphism $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}} : A_{\boldsymbol{r},\boldsymbol{c}} \to B_{\boldsymbol{r},\boldsymbol{c}}$.

Set $E = \widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}(A_{\boldsymbol{r},\boldsymbol{c}})$; we must show that $E = B_{\boldsymbol{r},\boldsymbol{c}}$. Note that

$$(Y_{r_l l} \otimes Z_{lc_l})^{\pm 1} = \beta_{\boldsymbol{r},\boldsymbol{c}}(d_l^{\boldsymbol{r},\boldsymbol{c}})^{\pm 1} \beta_{\boldsymbol{r},\boldsymbol{c}}(d_{l-1}^{\boldsymbol{r},\boldsymbol{c}})^{\mp 1} \in E$$

for $l \le t$, where $d_0^{\boldsymbol{r},\boldsymbol{c}} = 1$. It remains to show that $Y_{il} \otimes Z_{lj} \in E$ for all i, l, j.

As in (2.4), we see that

$$\begin{aligned}\beta_{\boldsymbol{r},\boldsymbol{c}}\big([r_1 \cdots r_{l-1} i \mid c_1 \cdots c_l]\big) &= (Y_{r_1 1} \otimes Z_{1c_1}) \cdots (Y_{r_{l-1}, l-1} \otimes Z_{l-1, c_{l-1}})(Y_{il} \otimes Z_{lc_l}) \\ &= \beta_{\boldsymbol{r},\boldsymbol{c}}(d_{l-1}^{\boldsymbol{r},\boldsymbol{c}})(Y_{il} \otimes Z_{lc_l})\end{aligned}$$

for $l \le t$ and $i \ge r_l$, whence $Y_{il} \otimes Z_{lc_l} \in E$. Similarly, $Y_{r_l l} \otimes Z_{lj} \in E$ for $j \ge c_l$, and therefore

$$Y_{il} \otimes Z_{lj} = q^{1-\delta(j,c_l)} (Y_{r_l l}^{-1} \otimes Z_{lc_l}^{-1})(Y_{r_l l} \otimes Z_{lj})(Y_{il} \otimes Z_{lc_l}) \in E,$$

as desired. □

2.6. Lemma. *$R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$ is a skew-Laurent extension of $B_{\boldsymbol{r},\boldsymbol{c}}$ of the form*

$$R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^- = B_{\boldsymbol{r},\boldsymbol{c}}[1 \otimes Z_{1c_1}^{\pm 1}, \ldots, 1 \otimes Z_{tc_t}^{\pm 1}; \tau_1, \ldots, \tau_t]$$

for some $\tau_1, \ldots, \tau_t \in \{1\}^{2n} \times H$.

Proof. First observe that there exist $\sigma_1, \ldots, \sigma_t \in H$ such that $Z_{lc_l} r = \sigma_l(r) Z_{lc_l}$ for $l \le t$ and $r \in R_{\boldsymbol{c},0}^-$. This relation extends to $r \in R_{\boldsymbol{c}}^-$, and so

$$(1 \otimes Z_{lc_l}) w = (1 \times \sigma_l)(w)(1 \otimes Z_{lc_l})$$

for $l \le t$ and $w \in R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$. In particular, if $\tau_l = (1, \ldots, 1, \sigma_l)$ then $1 \otimes Z_{lc_l}$ is τ_l-normal with respect to $B_{\boldsymbol{r},\boldsymbol{c}}$.

The standard $\mathbb{Z}^{2n}$-gradings on $R_{\boldsymbol{r}}^+$ and $R_{\boldsymbol{c}}^-$ (cf. (2.1)) induce a $\mathbb{Z}^{4n}$-grading on $R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$. With respect to this grading, $B_{\boldsymbol{r},\boldsymbol{c}}$ is a homogeneous subalgebra of $R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$, and its homogeneous components have degrees of the form $(*, b, b, *)$. On the other hand, $1 \otimes Z_{lc_l}$ has degree $(0, 0, \epsilon_l, *)$, so the monomials

$$(1 \otimes Z_{1c_1})^{m_1} (1 \otimes Z_{2c_2})^{m_2} \cdots (1 \otimes Z_{tc_t})^{m_t}$$

have degrees $(0, 0, m_1\epsilon_1 + \cdots + m_t\epsilon_t, *)$. It follows that these monomials are left (or right) linearly independent over $B_{\boldsymbol{r},\boldsymbol{c}}$. Hence, the subalgebra

$$C = \sum_{m_1,\dots,m_t \in \mathbb{Z}} B_{\boldsymbol{r},\boldsymbol{c}}(1 \otimes Z_{1c_1})^{m_1}(1 \otimes Z_{2c_2})^{m_2} \cdots (1 \otimes Z_{tc_t})^{m_t}$$

of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ is a skew-Laurent extension of $B_{\boldsymbol{r},\boldsymbol{c}}$ of the desired form.

It remains to show that $C = R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$. First note that $Y_{il} \otimes 1 = (Y_{il} \otimes Z_{lc_l})(1 \otimes Z^{-1}_{lc_l}) \in C$ for $l \le t$ and $i \ge r_l$, and that

$$Y^{-1}_{r_l l} \otimes 1 = (Y^{-1}_{r_l l} \otimes Z^{-1}_{lc_l})(1 \otimes Z_{lc_l}) \in C$$

for $l \le t$. On the other hand,

$$1 \otimes Z_{lj} = (Y_{r_l l} \otimes Z_{lj})(Y^{-1}_{r_l l} \otimes Z^{-1}_{lc_l})(1 \otimes Z_{lc_l}) \in C$$

for $l \le t$ and $j \ge c_l$, and $1 \otimes Z^{-1}_{lc_l} \in C$ for $l \le t$. Therefore $C = R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$. □

2.7. Let $B^+_{\boldsymbol{r}}$ and $B^-_{\boldsymbol{c}}$ denote the subalgebras of $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$ generated by the respective subsets

$$\{Y_{il}Y^{-1}_{r_l l} \mid l \le t,\ i > r_l\} \qquad \text{and} \qquad \{Z_{lj}Z^{-1}_{lc_l} \mid l \le t,\ j > c_l\}.$$

The algebra $B^+_{\boldsymbol{r}}$, for instance, may be viewed as a quantized coordinate ring of the variety

$$\{(a_{ij}) \in M_n(k) \mid a_{ij} = 0 \text{ when } j > t \text{ or } i < r_j, \text{ and } a_{r_j j} = 1 \text{ for } j \le t\}.$$

(The factor algebra $R^+_{\boldsymbol{r}}/\langle Y_{r_1 1} - 1, \ldots, Y_{r_t t} - 1\rangle$, which one might expect to appear in the above role, is inappropriate because it collapses to $k[Y^{\pm 1}_{r_1 1}, \ldots, Y^{\pm 1}_{r_t t}]$ when $q \ne 1$.)

As noted in (2.4), $B_{\boldsymbol{r},\boldsymbol{c}}$ is generated by its subalgebra $B^+_{\boldsymbol{r}} \otimes B^-_{\boldsymbol{c}}$ together with the elements $(Y_{r_l l} \otimes Z_{lc_l})^{\pm 1}$ for $l \le t$. In fact:

Lemma. *$B_{\boldsymbol{r},\boldsymbol{c}}$ is a skew-Laurent extension of $B^+_{\boldsymbol{r}} \otimes B^-_{\boldsymbol{c}}$ of the form*

$$B_{\boldsymbol{r},\boldsymbol{c}} = (B^+_{\boldsymbol{r}} \otimes B^-_{\boldsymbol{c}})[(Y_{r_1 1} \otimes Z_{1c_1})^{\pm 1}, \ldots, (Y_{r_t t} \otimes Z_{tc_t})^{\pm 1}; \eta_1, \ldots, \eta_t]$$

for some $\eta_1, \ldots, \eta_t \in H \times H$.

Proof. This is proved in the same manner as Lemma 2.6. □

2.8. We use the above structure of $B_{\boldsymbol{r},\boldsymbol{c}}$ in constructing a homomorphism $B_{\boldsymbol{r},\boldsymbol{c}} \to A_{\boldsymbol{r},\boldsymbol{c}}$ which will be the inverse of $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$. To begin the construction, we will define suitable homomorphisms from $B^+_{\boldsymbol{r}}$ and $B^-_{\boldsymbol{c}}$ to $A_{\boldsymbol{r},\boldsymbol{c}}$ whose images centralize each other. For that purpose, we need to know the defining relations for $B^+_{\boldsymbol{r}}$ and $B^-_{\boldsymbol{c}}$, as well as certain commutation relations in $A_{\boldsymbol{r},\boldsymbol{c}}$.

Set $y_{ij} = Y_{ij}Y_{r_jj}^{-1}$ for $j \le t$ and $i > r_j$. In view of the basic commutation relations satisfied by the Y_{lm}, it is easily checked that the y_{ij} satisfy the following relations:

$$
\begin{aligned}
&y_{ij}y_{im} = q y_{im}y_{ij} && (j < m)\\
&y_{ij}y_{lj} = q y_{lj}y_{ij} && (i < l)\\
&y_{ij}y_{lm} = y_{lm}y_{ij} && (i < l,\ j > m)\\
&y_{ij}y_{lm} = \begin{cases} y_{lm}y_{ij} & (i < r_m)\\ q^{-1}y_{lm}y_{ij} + \widehat{q}y_{lj} & (i = r_m)\\ y_{lm}y_{ij} + \widehat{q}y_{im}y_{lj} & (i > r_m)\end{cases} && (i < l,\ j < m).
\end{aligned}
\tag{1}
$$

Lemma. *The relations* (1) *are defining relations for the elements y_{ij} generating the algebra $B_{\boldsymbol{r}}^+$.*

Proof. Let S be the k-algebra presented by generators s_{ij} for $j \le t$ and $i > r_j$ satisfying the analogs of (1). Then there is a k-algebra homomorphism $\phi : S \to B_{\boldsymbol{r}}^+$ such that $\phi(s_{ij}) = y_{ij}$ for all i, j.

In $R_{\boldsymbol{r}}^+$, list the Y_{ij} lexicographically, and observe that the ordered monomials in the Y_{ij} are linearly independent. This remains true for ordered monomials in which we allow the Y_{r_jj} to have negative exponents. Since the Y_{r_jj} commute up to scalars with the Y_{lm} (recall (2.1)), it follows that the ordered monomials in the y_{ij} are linearly independent, and so these monomials form a basis for $B_{\boldsymbol{r}}^+$. On the other hand, there are sufficient commutation relations for the s_{ij} to show that the ordered monomials in the s_{ij} span S. Hence, ϕ maps a spanning subset of S to a basis for $B_{\boldsymbol{r}}^+$. Therefore ϕ is an isomorphism, and the lemma is proved. □

2.9. Lemma. *Let $M = [I \sqcup \{a\} \mid J]$ and $N = [I' \mid J']$ be quantum minors in A with $I \subset I'$ and $J \subseteq J'$. Assume that $a \notin I'$ and that $b = \max(I') \notin I$.*

(a) *If $a > b$, then $MN = q^{-1}NM$.*

(b) *If $a < b$, then*

$$MN - q^{-1}NM \equiv \widehat{q}(-q)^{|(I'\setminus I)\cap(a,b)|}[I \sqcup \{b\} \mid J][I' \sqcup \{a\} \setminus \{b\} \mid J']$$

modulo the ideal $L := \big\langle\, [I' \sqcup \{a\} \setminus \{i'\} \mid J'] \;\big|\; i' \in I' \cap (a,b) \,\big\rangle$.

Proof. (a) Expand M using the q-Laplace relation of Corollary 5.5(b2) with $r = a$. Since $(a,n] \cap I = \varnothing$, we get

$$M = \sum_{j\in J} (-q)^{|(j,n]\cap J|}[I \mid J \setminus \{j\}]X_{aj}. \tag{1}$$

Note that all the $[I \mid J \setminus \{j\}]$ commute with N. Lemma 5.2(c1) implies that $X_{aj}N = q^{-1}NX_{aj}$ for all $j \in J$, and thus it follows from (1) that $MN = q^{-1}NM$.

(b) Set $\alpha = |(a,n] \cap I|$, and again expand M using Corollary 5.5(b2) with $r = a$. This time, we get

$$(-q)^\alpha M = \sum_{j \in J} (-q)^{|(j,n] \cap J|} [I \mid J \setminus \{j\}] X_{aj}. \tag{2}$$

For all $j \in J$, Lemma 5.2(c2) implies that

$$NX_{aj} - qX_{aj}N = \widehat{q} \sum_{\substack{i' \in I' \\ i' > a}} (-q)^{|I' \cap [a,i']|} X_{i'j} [I' \sqcup \{a\} \setminus \{i'\} \mid J'],$$

whence

$$X_{aj}N - q^{-1}NX_{aj} \equiv \widehat{q}(-q)^\beta X_{bj} [I' \sqcup \{a\} \setminus \{b\} \mid J'] \qquad (\mathrm{mod}\ L), \tag{3}$$

where $\beta = |I' \cap [a,b]| - 1 = |I' \cap (a,b)|$. Note that $\beta - \alpha = |(I' \setminus I) \cap (a,b)|$.

Combining (2) and (3), we obtain

$$MN \equiv \sum_{j \in J} (-q)^{|(j,n] \cap J| - \alpha} [I \mid J \setminus \{j\}] \Big(q^{-1}NX_{aj} + \widehat{q}(-q)^\beta X_{bj} [I' \sqcup \{a\} \setminus \{b\} \mid J'] \Big) \tag{4}$$

modulo L. Since all the $[I \mid J \setminus \{j\}]$ commute with N, it follows from (2) that

$$\sum_{j \in J} (-q)^{|(j,n] \cap J| - \alpha} [I \mid J \setminus \{j\}] q^{-1} N X_{aj} = q^{-1} NM. \tag{5}$$

Further, an application of Corollary 5.5(b2) with $r = b$ yields

$$\sum_{j \in J} (-q)^{|(j,n] \cap J|} [I \mid J \setminus \{j\}] X_{bj} = [I \sqcup \{b\} \mid J]. \tag{6}$$

Combining (4), (5), and (6), we complete the proof. □

2.10. Corollary. *Let $M = [I \mid J \sqcup \{a\}]$ and $N = [I' \mid J']$ be quantum minors in A with $I \subseteq I'$ and $J \subset J'$. Assume that $a \notin J'$ and that $b = \max(J') \notin J$.*

(a) *If $a > b$, then $MN = q^{-1}NM$.*

(b) *If $a < b$, then*

$$MN - q^{-1}NM \equiv \widehat{q}(-q)^{|(J' \setminus J) \cap (a,b)|} [I \mid J \sqcup \{b\}][I' \mid J' \sqcup \{a\} \setminus \{b\}]$$

modulo the ideal $L := \big\langle [I' \mid J' \sqcup \{a\} \setminus \{j'\}] \bigm| j' \in J' \cap (a,b) \big\rangle$.

Proof. This follows from Lemma 2.9 by symmetry. □

2.11. Theorem. *The map* $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}} : A_{\boldsymbol{r},\boldsymbol{c}} \longrightarrow B_{\boldsymbol{r},\boldsymbol{c}}$ *is an isomorphism.*

Proof. Recall the notation $\widetilde{d}_l^{\boldsymbol{r},\boldsymbol{c}} = d_l^{\boldsymbol{r},\boldsymbol{c}} + K_{\boldsymbol{r},\boldsymbol{c}}$ from (1.2). Similarly, we shall use tildes to denote other cosets in $A/K_{\boldsymbol{r},\boldsymbol{c}}$. To abbreviate the relation of congruence modulo $K_{\boldsymbol{r},\boldsymbol{c}}$, we adopt the notation $\equiv_{\boldsymbol{r},\boldsymbol{c}}$. We shall use the same symbols for elements of $A/K_{\boldsymbol{r},\boldsymbol{c}}$ and their images in the localization $A_{\boldsymbol{r},\boldsymbol{c}}$, which does not cause problems as long as we only transfer equations from $A/K_{\boldsymbol{r},\boldsymbol{c}}$ to $A_{\boldsymbol{r},\boldsymbol{c}}$ and not in the reverse direction.

We proceed to construct, in several steps, a k-algebra homomorphism $\phi : B_{\boldsymbol{r},\boldsymbol{c}} \to A_{\boldsymbol{r},\boldsymbol{c}}$ that will be an inverse for $\beta_{\boldsymbol{r},\boldsymbol{c}}$. The construction of ϕ is based on the skew-Laurent structure of $B_{\boldsymbol{r},\boldsymbol{c}}$ given in Lemma 2.7. The first ingredient will be a homomorphism from $B_{\boldsymbol{r}}^+$ to $A_{\boldsymbol{r},\boldsymbol{c}}$. To describe it, set

$$\begin{aligned} u_{ij} &= [r_1 \cdots r_{j-1} i \mid c_1 \cdots c_j] \in A && (j \le t,\ i \ge r_j) \\ v_{ij} &= \widetilde{u}_{ij} (\widetilde{d}_j^{\boldsymbol{r},\boldsymbol{c}})^{-1} \in A_{\boldsymbol{r},\boldsymbol{c}} && (j \le t,\ i > r_j) \\ y_{ij} &= Y_{ij} Y_{r_j j}^{-1} \in R_{\boldsymbol{r}}^+ \qquad \text{(as in (2.8))} && (j \le t,\ i > r_j). \end{aligned}$$

(While the given expressions for v_{ij} and y_{ij} also make sense for $j \le t$ and $i = r_j$, they yield $v_{r_j j} = 1 \in A_{\boldsymbol{r},\boldsymbol{c}}$ and $y_{r_j j} = 1 \in R_{\boldsymbol{r}}^+$. It is more convenient for the proof to exclude these possibilities.) Recall that the y_{ij} generate $B_{\boldsymbol{r}}^+$.

Claim 1: There exists a homomorphism $\phi^+ : B_{\boldsymbol{r}}^+ \to A_{\boldsymbol{r},\boldsymbol{c}}$ such that $\phi^+(y_{ij}) = v_{ij}$ for $j \le t$ and $i > r_j$.

To prove this, we must show that the v_{ij} satisfy the analogs of the relations (2.8)(1), i.e., the corresponding equations with all y's replaced by v's. We first check that the u_{ij} satisfy the relations (1) below. The first relation follows from Lemma 2.9(a); for the second, observe that $\{r_1, \dots, r_{j-1}, i\} \subset \{r_1, \dots, r_{m-1}, l\}$ and $\{c_1, \dots, c_j\} \subset \{c_1, \dots, c_m\}$ in that case.

$$(1) \qquad u_{lm} u_{ij} = \begin{cases} q^{-1} u_{ij} u_{lm} & (i < l,\ j \ge m) \\ u_{ij} u_{lm} & (i \le l,\ j < m,\ i \in \{r_j, \dots, r_{m-1}, l\}). \end{cases}$$

Furthermore, when $i < l$ and $j < m$ but $i \notin \{r_j, \dots, r_{m-1}\}$, Lemma 2.9(b) implies that

$$(2) \qquad u_{ij} u_{lm} - q^{-1} u_{lm} u_{ij} \equiv_{\boldsymbol{r},\boldsymbol{c}} \begin{cases} \widehat{q} u_{lj} u_{im} & (i \ge r_m) \\ 0 & (i < r_m). \end{cases}$$

Since $u_{r_l l} = d_l^{\boldsymbol{r},\boldsymbol{c}}$ for $l \le t$, the relations (1) and (2) yield commutation relations for the $d_l^{\boldsymbol{r},\boldsymbol{c}}$ and u_{ij}, which we combine in the following form:

$$(3) \qquad \begin{aligned} d_l^{\boldsymbol{r},\boldsymbol{c}} u_{ij} &= u_{ij} d_l^{\boldsymbol{r},\boldsymbol{c}} && \begin{cases} (l < j), \text{ or} \\ (l \ge j \text{ and } i \in \{r_j, \dots, r_l\}) \end{cases} \\ d_l^{\boldsymbol{r},\boldsymbol{c}} u_{ij} &\equiv_{\boldsymbol{r},\boldsymbol{c}} q u_{ij} d_l^{\boldsymbol{r},\boldsymbol{c}} && (l \ge j \text{ and } i \notin \{r_j, \dots, r_l\}). \end{aligned}$$

(Note that when $l < j$, we have $i \ge r_j > r_l$.)

It follows from (1), (2), and (3) that the elements $v_{ij} \in A_{\boldsymbol{r},\boldsymbol{c}}$ indeed satisfy the analogs of the relations (2.8)(1), as desired. This establishes Claim 1.

Next, set

$$\begin{aligned} w_{lm} &= [r_1 \cdots r_l \mid c_1 \cdots c_{l-1} m] \in A && (l \le t,\ m \ge c_l) \\ t_{lm} &= \widetilde{w}_{lm} (\widetilde{d}_l^{\boldsymbol{r},\boldsymbol{c}})^{-1} \in A_{\boldsymbol{r},\boldsymbol{c}} && (l \le t,\ m > c_l) \\ z_{lm} &= Z_{lm} Z_{lc_l}^{-1} \in R_{\boldsymbol{c}}^- && (l \le t,\ m > c_l), \end{aligned}$$

and recall that the z_{lm} generate $B_{\boldsymbol{c}}^-$.

Claim 2. There exists a homomorphism $\phi^- : B_{\boldsymbol{c}}^- \to A_{\boldsymbol{r},\boldsymbol{c}}$ such that $\phi^-(z_{lm}) = t_{lm}$ for $l \le t$ and $m > c_l$.

This follows from Claim 1 by symmetry.

Claim 3. Each v_{ij} commutes with each t_{lm}.

We first collect the following commutation relations between the u_{ij} and the w_{lm}:

$$(4) \qquad \begin{aligned} u_{ij} w_{lm} &= w_{lm} u_{ij} && \begin{cases} (j = l), \text{ or} \\ (j < l \text{ and } i \in \{r_{j+1}, \dots, r_l\}), \text{ or} \\ (j > l \text{ and } m \in \{c_{l+1}, \dots, c_j\}) \end{cases} \\ u_{ij} w_{lm} &\equiv_{\boldsymbol{r},\boldsymbol{c}} \begin{cases} q^{-1} w_{lm} u_{ij} & (j < l \text{ and } i \notin \{r_{j+1}, \dots, r_l\}) \\ q w_{lm} u_{ij} & (j > l \text{ and } m \notin \{c_{l+1}, \dots, c_j\}). \end{cases} \end{aligned}$$

The first equation in (4) follows from Lemma 5.3(c). The next two equations hold because $\{r_1, \dots, r_{j-1}, i\} \subset \{r_1, \dots, r_l\}$ and $\{c_1, \dots, c_j\} \subset \{c_1, \dots, c_{l-1}, m\}$ in the first case, while those inclusions are reversed in the second case. Finally, the last two relations follow from Lemma 2.9 and Corollary 2.10.

Commutation relations between the $d_l^{\boldsymbol{r},\boldsymbol{c}}$ and the u_{ij} are given in (3), and, by symmetry, we also have

$$(5) \qquad \begin{aligned} d_j^{\boldsymbol{r},\boldsymbol{c}} w_{lm} &= w_{lm} d_j^{\boldsymbol{r},\boldsymbol{c}} && \begin{cases} (j < l), \text{ or} \\ (j \ge l \text{ and } m \in \{c_l, \dots, c_j\}) \end{cases} \\ d_j^{\boldsymbol{r},\boldsymbol{c}} w_{lm} &\equiv_{\boldsymbol{r},\boldsymbol{c}} q w_{lm} d_j^{\boldsymbol{r},\boldsymbol{c}} && (j \ge l \text{ and } m \notin \{c_l, \dots, c_j\}). \end{aligned}$$

It follows from (3), (4), and (5) that v_{ij} indeed commutes with t_{lm}, and Claim 3 is proved.

Combining Claims 1, 2, and 3, we see that there exists a homomorphism

$$\phi : B_{\boldsymbol{r}}^+ \otimes B_{\boldsymbol{c}}^- \to A_{\boldsymbol{r},\boldsymbol{c}}$$

such that $\phi(y_{ij} \otimes 1) = v_{ij}$ and $\phi(1 \otimes z_{lm}) = t_{lm}$ for all i, j, l, m.

Claim 4. ϕ extends to a homomorphism $B_{\boldsymbol{r},\boldsymbol{c}} \to A_{\boldsymbol{r},\boldsymbol{c}}$ such that $\phi(Y_{r_s s} \otimes Z_{sc_s}) = \widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}} (\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1}$ for $s \le t$.

In view of the relations given in (2.1), we see that

$$
(6)\qquad
\begin{aligned}
Y_{r_s s} y_{ij} Y_{r_s s}^{-1} &= \begin{cases} q^{-1} y_{ij} & (i = r_s) \\ q y_{ij} & (j = s) \\ y_{ij} & (i \neq r_s,\ j \neq s) \end{cases} \\
Z_{s c_s} z_{lm} Z_{s c_s}^{-1} &= \begin{cases} q z_{lm} & (l = s) \\ q^{-1} z_{lm} & (m = c_s) \\ z_{lm} & (l \neq s,\ m \neq c_s). \end{cases}
\end{aligned}
$$

On the other hand, it follows from (3) and (5) that the units $\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1}$ in $A_{\boldsymbol{r},\boldsymbol{c}}$ normalize the elements v_{ij} and t_{lm} in exactly the same way, that is,

$$
(7)\qquad
\begin{aligned}
\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1} v_{ij} \widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}})^{-1} &= \begin{cases} q^{-1} v_{ij} & (i = r_s) \\ q v_{ij} & (j = s) \\ v_{ij} & (i \neq r_s,\ j \neq s) \end{cases} \\
\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1} t_{lm} \widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}})^{-1} &= \begin{cases} q t_{lm} & (l = s) \\ q^{-1} t_{lm} & (m = c_s) \\ t_{lm} & (l \neq s,\ m \neq c_s). \end{cases}
\end{aligned}
$$

Claim 4 follows from (6), (7), and Lemma 2.7.

Claim 5. $\phi = \widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}^{-1}$.

Since $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$ is surjective (Lemma 2.5), it is enough to show that $\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$ is the identity on $A_{\boldsymbol{r},\boldsymbol{c}}$. First note, using (2.4), that

$$\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1}) = \phi(Y_{r_s s} \otimes Z_{s c_s}) = \widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_{s-1}^{\boldsymbol{r},\boldsymbol{c}})^{-1}$$

for all s, whence $\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}) = \widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}}$ for all s. Next, for $j \leq t$ and $i > r_j$ we have

$$
\begin{aligned}
\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}(\widetilde{u}_{ij}) &= \phi\big((Y_{r_1 1} \otimes Z_{1c_1}) \cdots (Y_{r_{j-1},j-1} \otimes Z_{j-1,c_{j-1}})(Y_{ij} \otimes Z_{jc_j})\big) \\
&= \phi\big((y_{ij} \otimes 1)(Y_{r_1 1} \otimes Z_{1c_1}) \cdots (Y_{r_j j} \otimes Z_{jc_j})\big) \\
&= \phi\big((y_{ij} \otimes 1)\beta_{\boldsymbol{r},\boldsymbol{c}}(\widetilde{d}_j^{\boldsymbol{r},\boldsymbol{c}})\big) = v_{ij}\widetilde{d}_j^{\boldsymbol{r},\boldsymbol{c}} = \widetilde{u}_{ij}.
\end{aligned}
$$

By symmetry, $\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}(\widetilde{w}_{lm}) = \widetilde{w}_{lm}$ for $l \leq t$ and $m > c_l$. Therefore $\phi\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$ at least equals the identity on the subalgebra C of $A_{\boldsymbol{r},\boldsymbol{c}}$ generated by (the image of) the set

$$\{(\widetilde{d}_s^{\boldsymbol{r},\boldsymbol{c}})^{\pm 1} \mid s \leq t\} \cup \{\widetilde{u}_{ij} \mid j \leq t,\ i > r_j\} \cup \{\widetilde{w}_{lm} \mid l \leq t,\ m > c_l\}.$$

To finish the proof, we just need to show that $C = A_{\boldsymbol{r},\boldsymbol{c}}$, that is, that $\widetilde{X}_{ij} \in C$ for all i, j. This is clear in case $i < r_1$ or $j < c_1$, since in those cases $\widetilde{X}_{ij} = 0$. We also have

$\widetilde{X}_{ic_1} = \widetilde{u}_{i1} \in C$ for $i \geq r_1$ and $\widetilde{X}_{r_1 j} = \widetilde{w}_{1j} \in C$ for $j \geq c_1$. Hence, $\widetilde{X}_{ij} \in C$ whenever $i \leq r_1$ or $j \leq c_1$.

Now let $1 < l \leq t$ and assume that $\widetilde{X}_{ij} \in C$ whenever $i \leq r_{l-1}$ or $j \leq c_{l-1}$. For $r_{l-1} < i \leq t$ and $c_{l-1} < j \leq c_l$, it follows from Corollary 5.5(b1) that

$$(8) \quad (-q)^{l-1}[r_1 \cdots r_{l-1} i \mid c_1 \cdots c_{l-1} j] = (-q)^{l-1} X_{ij} [r_1 \cdots r_{l-1} \mid c_1 \cdots c_{l-1}] + \sum_{s=1}^{l-1} (-q)^{s-1} X_{ic_s} [r_1 \cdots r_{l-1} \mid c_1 \cdots \widehat{c_s} \cdots c_{l-1} j].$$

Most of the cosets of the factors in (8) can be seen to lie in C right away. For instance, the coset of the left hand side is zero if either $i < r_l$ or $j < c_l$, and it equals $\widetilde{u}_{il}$ if $i \geq r_l$ and $j = c_l$. For $s < l$, we have $\widetilde{X}_{ic_s} \in C$ by the inductive hypothesis, and similarly the coset $[r_1 \cdots r_{l-1} \mid c_1 \cdots \widehat{c_s} \cdots c_{l-1} j] + K_{\boldsymbol{r},\boldsymbol{c}}$ is in C since it is a linear combination of products of elements $\widetilde{X}_{ab}$ with $a \leq r_{l-1}$. Consequently, we obtain from (8) that

$$X_{ij}[r_1 \cdots r_{l-1} \mid c_1 \cdots c_{l-1}] + K_{\boldsymbol{r},\boldsymbol{c}} \in C.$$

In other words, $\widetilde{X}_{ij} \widetilde{d^{\boldsymbol{r},\boldsymbol{c}}_{l-1}} \in C$, whence $\widetilde{X}_{ij} \in C$. Thus, $\widetilde{X}_{ij} \in C$ whenever $j \leq c_l$. By symmetry, $\widetilde{X}_{ij} \in C$ whenever $i \leq r_l$.

The above induction proves that $\widetilde{X}_{ij} \in C$ whenever $i \leq r_t$ or $j \leq c_t$. If there exist indices $i > r_t$ and $j > c_t$, we have

$$(9) \quad (-q)^{t}[r_1 \cdots r_t i \mid c_1 \cdots c_t j] = (-q)^{t} X_{ij} [r_1 \cdots r_t \mid c_1 \cdots c_t] + \sum_{s=1}^{t} (-q)^{s-1} X_{ic_s} [r_1 \cdots r_t \mid c_1 \cdots \widehat{c_s} \cdots c_t j]$$

by Corollary 5.5(b1), from which we see as above that $\widetilde{X}_{ij} \in C$. (Note that the left hand side of (9) necessarily lies in $K_{\boldsymbol{r},\boldsymbol{c}}$ because it involves a $(t+1) \times (t+1)$ quantum minor.) Therefore all $\widetilde{X}_{ij} \in C$, and the proof is complete. $\square$

3. Tensor product decompositions of H-primes

Throughout this section, we assume that q is not a root of unity; we shall place reminders of this hypothesis in the relevant results. Thus, by [**8**, Theorem 3.2], all primes of A are completely prime. Since this property survives in factors and localizations, all primes in the algebras $A_{\boldsymbol{r},\boldsymbol{c}}$, $R^+_{\boldsymbol{r}}$, and $R^-_{\boldsymbol{c}}$ are completely prime, and also in $B_{\boldsymbol{r},\boldsymbol{c}}$ because of Theorem 2.11. We have already observed that the algebras $R^+_{\boldsymbol{r},0}$ and $R^-_{\boldsymbol{c},0}$ are iterated skew polynomial algebras over k, and so is their tensor product. The iterated skew polynomial structure of $R^+_{\boldsymbol{r},0} \otimes R^-_{\boldsymbol{c},0}$ is easily seen to satisfy the hypotheses of [**8**, Theorem 2.3], and thus all its primes are completely prime. Consequently, all primes in the localizations $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ are completely prime.

3.1. In order to deal with H-primes in tensor products, we need the following rationality property. Suppose that S is a noetherian k-algebra and that G is a group acting on S by k-algebra automorphisms. We say that a G-prime P of S is *strongly G-rational* provided the algebra $Z(\operatorname{Fract} S/P)^G$, the fixed ring of the center of the Goldie quotient ring of S/P under the induced G-action, equals k.

By [**9**, (5.7)(i)] (cf. [**1**, Theorem II.5.14]) and [**1**, Corollary II.6.5], A has only finitely many H-primes, and they are all completely prime and strongly H-rational. These properties carry over to $A_{\boldsymbol{r},\boldsymbol{c}}$, $R^+_{\boldsymbol{r}}$, and $R^-_{\boldsymbol{c}}$. Analogous results [**1**, Theorems II.5.12 and II.6.4] imply that $R^+_{\boldsymbol{r},0} \otimes R^-_{\boldsymbol{c},0}$ has only finitely many $(H \times H)$-primes, and they are all completely prime and strongly $(H \times H)$-rational. These properties now carry over to $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$.

Identify H with the subgroup

$$\widetilde{H} = \big((k^\times)^n \times \{1\}^n\big) \times \big(\{1\}^n \times (k^\times)^n\big) \lhd H \times H$$

in the obvious way. With this identification, $\beta_{\boldsymbol{r},\boldsymbol{c}}$ and $\widetilde{\beta}_{\boldsymbol{r},\boldsymbol{c}}$ are H-equivariant. In particular, it follows that $B_{\boldsymbol{r},\boldsymbol{c}}$ has only finitely many $\widetilde{H}$-primes, and they are all completely prime and strongly $\widetilde{H}$-rational.

3.2. Lemma. *For $i = 1, 2$, let A_i be a k-algebra, H_i a group acting on A_i by k-algebra automorphisms, and P_i an H_i-prime ideal of A_i. Set $P = (P_1 \otimes A_2) + (A_1 \otimes P_2)$, and let $H_1 \times H_2$ act on $A_1 \otimes A_2$ in the natural manner.*

(a) *If each A_i/P_i is H_i-simple and $Z(A_1/P_1)^{H_1} = k$, then $(A_1 \otimes A_2)/P$ is $(H_1 \times H_2)$-simple.*

(b) *If each A_i is noetherian and P_1 is strongly H_1-rational, then P is an $(H_1 \times H_2)$-prime ideal of $A_1 \otimes A_2$. Moreover, P is the only $(H_1 \times H_2)$-prime ideal of $A_1 \otimes A_2$ that contracts to $P_1 \otimes 1$ in $A_1 \otimes 1$ and to $1 \otimes P_2$ in $1 \otimes A_2$.*

(c) *If each A_i is noetherian and each P_i is strongly H_i-rational, then P is strongly $(H_1 \times H_2)$-rational.*

Proof. Since $(A_1 \otimes A_2)/P \cong (A_1/P_1) \otimes (A_2/P_2)$, there is no loss of generality in assuming that each $P_i = 0$.

(a) This is a standard shortest length argument. Let I be a nonzero $(H_1 \times H_2)$-ideal of $A_1 \otimes A_2$, and let m be the shortest length for nonzero elements of I (as sums of pure tensors). Choose a nonzero element

$$x = b_1 \otimes c_1 + \cdots + b_m \otimes c_m \in I$$

of length m, where the $b_j \in A_1$ and $c_j \in A_2$, and note that the c_j are linearly independent over k. Now the set

$$\{b \in A_1 \mid (b \otimes c_1 + A_1 \otimes c_2 + \cdots + A_1 \otimes c_m) \cap I \neq \varnothing\}$$

is a nonzero H_1-ideal of A_1, and so it equals A_1. Thus, without loss of generality, $b_1 = 1$. For any $a \in A_1$, we now have $x(a \otimes 1) - (a \otimes 1)x$ in I with length less than m, whence

$x(a\otimes 1)-(a\otimes 1)x=0$ and so $b_ja=ab_j$ for all j. For any $h\in H_1$, we have $(h,1)(x)-x$ in I with length less than m, whence $(h,1)(x)-x=0$ and so $h(b_j)=b_j$ for all j. Therefore all b_j lie in $Z(A_1)^{H_1}=k$. It follows that $x\in 1\otimes A_2$, whence $m=1$ and $x=1\otimes c_1$. Consequently, the set $\{c\in A_2\mid 1\otimes c\in I\}$ is a nonzero H_2-ideal of A_2, and so it equals A_2. Therefore $1\otimes 1\in I$, proving that $I=A_1\otimes A_2$.

(b) Each A_i is an H_i-prime noetherian ring, and so is semiprime. Let $\mathcal{C}_i$ be the set of regular elements in A_i, and note that the set

$$\mathcal{C}=\{c_1\otimes c_2\mid c_i\in\mathcal{C}_i\}$$

is an $(H_1\times H_2)$-stable denominator set in $A_1\otimes A_2$, consisting of regular elements. Each $A_i\mathcal{C}_i^{-1}$ is H_i-simple artinian, and $Z(A_1\mathcal{C}_1^{-1})^{H_1}=k$ by our hypothesis on P_1. Thus by part (a), the localization

$$(A_1\otimes A_2)\mathcal{C}^{-1}=A_1\mathcal{C}_1^{-1}\otimes A_2\mathcal{C}_2^{-1}$$

is $(H_1\times H_2)$-simple. It follows that each nonzero $(H_1\times H_2)$-ideal of $A_1\otimes A_2$ meets $\mathcal{C}$; in particular, $A_1\otimes A_2$ is an $(H_1\times H_2)$-prime ring.

Now let Q be any $(H_1\times H_2)$-prime ideal of $A_1\otimes A_2$ that contracts to zero in both $A_1\otimes 1$ and $1\otimes A_2$. Then Q is a semiprime ideal, disjoint from both $\mathcal{C}_1\otimes 1$ and $1\otimes\mathcal{C}_2$. If some prime ideal Q_0 minimal over Q meets $\mathcal{C}_1\otimes 1$, then $h(Q_0)$ meets $\mathcal{C}_1\otimes 1$ for all $h\in H_1\times H_2$. But since Q is a finite intersection of some of the $h(Q_0)$, it would follow that Q meets $\mathcal{C}_1\otimes 1$, a contradiction. Therefore $\mathcal{C}_1\otimes 1$ is disjoint from all primes minimal over Q, whence $\mathcal{C}_1\otimes 1$ is regular modulo Q. Likewise, $1\otimes\mathcal{C}_2$ is regular modulo Q. It follows that $\mathcal{C}$ is disjoint from Q, and therefore $Q=0$.

(c) After localization, we can assume that each A_i is H_i-simple artinian. By part (a), $A_1\otimes A_2$ is now $(H_1\times H_2)$-simple. Consider an element u in $Z(\mathrm{Fract}(A_1\otimes A_2))^{H_1\times H_2}$. The set $\{a\in A_1\otimes A_2\mid au\in A_1\otimes A_2\}$ is a nonzero $(H_1\times H_2)$-ideal of $A_1\otimes A_2$, and so it equals $A_1\otimes A_2$. Therefore $u\in A_1\otimes A_2$. Now write $u=v_1\otimes w_1+\cdots+v_t\otimes w_t$ for some $v_j\in A_1$ and some linearly independent $w_j\in A_2$. Since u is fixed by $H_1\times 1$ and commutes with $A_1\otimes 1$, we see that all $v_j\in Z(A_1)^{H_1}=k$. Hence, $u=1\otimes w$ for some $w\in A_2$. But then $w\in Z(A_2)^{H_2}=k$, and therefore $u\in k$. □

3.3. Proposition. *For $i=1,2$, let A_i be a noetherian k-algebra and H_i a group acting on A_i by k-algebra automorphisms. Assume that all H_1-primes of A_1 are strongly H_1-rational. Then the rule $(P_1,P_2)\mapsto(P_1\otimes A_2)+(A_1\otimes P_2)$ provides a bijection*

$$(H_1\text{-spec}\,A_1)\times(H_2\text{-spec}\,A_2)\longrightarrow(H_1\times H_2)\text{-spec}(A_1\otimes A_2).$$

Proof. Lemma 3.2(b) shows that the given rule maps $(H_1\text{-spec}\,A_1)\times(H_2\text{-spec}\,A_2)$ to $(H_1\times H_2)\text{-spec}(A_1\otimes A_2)$.

Now consider an $(H_1\times H_2)$-prime P in $A_1\otimes A_2$. Let P_1 and P_2 be the inverse images of P under the natural maps $A_i\to A_1\otimes A_2$. Then each P_i is an H_i-ideal of A_i, and $(P_1\otimes A_2)+(A_1\otimes P_2)\subseteq P$. There are H_1-primes $Q_1,\dots,Q_t$ in A_1, containing P_1, such that $Q_1Q_2\cdots Q_t\subseteq P_1$. Then the $Q_i\otimes A_2$ are $(H_1\times H_2)$-ideals of $A_1\otimes A_2$ such that

$$(Q_1\otimes A_2)(Q_2\otimes A_2)\cdots(Q_t\otimes A_2)\subseteq P_1\otimes A_2\subseteq P.$$

Consequently, some $Q_j \otimes A_2 \subseteq P$, whence $Q_j \subseteq P_1$, and so $Q_j = P_1$. This shows that P_1 is H_1-prime. Similarly, P_2 is H_2-prime.

By Lemma 3.2(b), $(P_1 \otimes A_2) + (A_1 \otimes P_2)$ is an $(H_1 \times H_2)$-prime of $A_1 \otimes A_2$, and it is the only $(H_1 \times H_2)$-prime of $A_1 \otimes A_2$ that contracts to $P_1 \otimes 1$ in $A_1 \otimes 1$ and to $1 \otimes P_2$ in $1 \otimes A_2$. Therefore $P = (P_1 \otimes A_2) + (A_1 \otimes P_2)$. It is clear that P_1 and P_2 are unique, since P_i equals the inverse image of $(P_1 \otimes A_2) + (A_1 \otimes P_2)$ under the natural map $A_i \to A_1 \otimes A_2$. □

3.4. Lemma. [q not a root of unity] *Let $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$. If $\widetilde{P}$ is an $\widetilde{H}$-prime of $B_{\boldsymbol{r},\boldsymbol{c}}$, then there exists a unique $(H \times H)$-prime Q of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ such that $Q \cap B_{\boldsymbol{r},\boldsymbol{c}} = \widetilde{P}$.*

Proof. Since $B_{\boldsymbol{r},\boldsymbol{c}}$ has only finitely many $\widetilde{H}$-primes and $H \times H$ just permutes them, the $(H \times H)$-orbit of $\widetilde{P}$ in $\operatorname{spec} B_{\boldsymbol{r},\boldsymbol{c}}$ is finite. Since $\widetilde{P}$ is prime, it now follows from [**1**, Proposition II.2.9] that $\widetilde{P}$ must be invariant under $H \times H$. In view of Lemma 2.6, $Q = \widetilde{P}(R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}})$ is an $(H \times H)$-invariant prime of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ such that $Q \cap B_{\boldsymbol{r},\boldsymbol{c}} = \widetilde{P}$. It remains to show that if Q' is any $(H \times H)$-prime of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ that contracts to $\widetilde{P}$, then $Q' = Q$. Note that $Q' \supseteq Q$ by definition of Q.

Set $G = (k^\times)^n$, let $\phi : G \to \big(\{1\}^n \times (k^\times)^n\big) \times \big((k^\times)^n \times \{1\}^n\big) \subset H \times H$ be the homomorphism given by the rule

$$\phi(\alpha_1, \dots, \alpha_n) = (1, \dots, 1, \alpha_1^{-1}, \dots, \alpha_n^{-1}, \alpha_1, \dots, \alpha_n, 1, \dots, 1),$$

and use ϕ to pull back the action of $H \times H$ on $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ to an action of G. With respect to this G-action, $B_{\boldsymbol{r},\boldsymbol{c}}$ is generated by fixed elements, and each of the elements $1 \otimes Z_{sc_s}$ is a G-eigenvector with eigenvalue equal to the projection $(\alpha_1, \dots, \alpha_n) \mapsto \alpha_s$. In view of Lemma 2.6 and the fact that k is infinite, it follows that the G-eigenspaces of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ are the subspaces $B_{\boldsymbol{r},\boldsymbol{c}}\big(1 \otimes (Z_{1c_1}^{m_1} Z_{2c_2}^{m_2} \cdots Z_{tc_1}^{m_t})\big)$ for $(m_1, \dots, m_t) \in \mathbb{Z}^t$. Consequently, any G-eigenvector in $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$ has the form du where $d \in B_{\boldsymbol{r},\boldsymbol{c}}$ and u is a unit. If $du \in Q'$, then $d \in Q' \cap B_{\boldsymbol{r},\boldsymbol{c}} = \widetilde{P}$, whence $du \in Q$. Since Q' is G-invariant, we conclude that $Q' = Q$, as desired. □

3.5. Set $H\text{-}\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A = (H\text{-}\operatorname{spec} A) \cap (\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A)$ for $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$. These sets partition $H\text{-}\operatorname{spec} A$ because of Corollary 1.10.

Theorem. [q not a root of unity] *For each $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$, there is a bijection*

$$(H\text{-}\operatorname{spec} R^+_{\boldsymbol{r}}) \times (H\text{-}\operatorname{spec} R^-_{\boldsymbol{c}}) \longrightarrow H\text{-}\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A$$

given by the rule $(Q^+, Q^-) \mapsto \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}\big((Q^+ \otimes R^-_{\boldsymbol{c}}) + (R^+_{\boldsymbol{r}} \otimes Q^-)\big)$.

Proof. If $Q^+ \in H\text{-}\operatorname{spec} R^+_{\boldsymbol{r}}$ and $Q^- \in H\text{-}\operatorname{spec} R^-_{\boldsymbol{c}}$, then Proposition 3.3 shows that the ideal $Q = (Q^+ \otimes R^-_{\boldsymbol{c}}) + (R^+_{\boldsymbol{r}} \otimes Q^-)$ is an $(H \times H)$-prime of $R^+_{\boldsymbol{r}} \otimes R^-_{\boldsymbol{c}}$. In particular, Q is completely prime, and so $Q \cap B_{\boldsymbol{r},\boldsymbol{c}}$ is an $\widetilde{H}$-prime of $B_{\boldsymbol{r},\boldsymbol{c}}$, whence $\beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(Q) = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(Q \cap B_{\boldsymbol{r},\boldsymbol{c}})$ is an H-prime of A lying in $\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A$. This shows that the given rule does define a map from $(H\text{-}\operatorname{spec} R^+_{\boldsymbol{r}}) \times (H\text{-}\operatorname{spec} R^-_{\boldsymbol{c}})$ to $H\text{-}\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A$.

Now consider an arbitrary H-prime P in $H\text{-}\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A$. Then P induces an H-prime in $A_{\boldsymbol{r},\boldsymbol{c}}$ that contracts to P under the localization map. In view of Theorem 2.11, it follows that $\beta_{\boldsymbol{r},\boldsymbol{c}}(P)$ induces an $\widetilde{H}$-prime $\widetilde{P}$ of $B_{\boldsymbol{r},\boldsymbol{c}}$ such that $\beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(\widetilde{P}) = P$. By Lemma 3.4, there is a unique $(H \times H)$-prime Q of $R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$ such that $Q \cap B_{\boldsymbol{r},\boldsymbol{c}} = \widetilde{P}$. Then Proposition 3.3 implies that $Q = (Q^+ \otimes R_{\boldsymbol{c}}^-) + (R_{\boldsymbol{r}}^+ \otimes Q^-)$ for some H-primes Q^+ in $R_{\boldsymbol{r}}^+$ and Q^- in $R_{\boldsymbol{c}}^-$. Thus,

$$\beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}\big((Q^+ \otimes R_{\boldsymbol{c}}^-) + (R_{\boldsymbol{r}}^+ \otimes Q^-)\big) = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(Q) = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(Q \cap B_{\boldsymbol{r},\boldsymbol{c}}) = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(\widetilde{P}) = P.$$

It remains to show that Q^+ and Q^- are unique. Consider any $T^+ \in H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^+$ and $T^- \in H\text{-}\operatorname{spec} R_{\boldsymbol{c}}^-$ such that $P = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(T)$ where $T = (T^+ \otimes R_{\boldsymbol{c}}^-) + (R_{\boldsymbol{r}}^+ \otimes T^-)$. As in the first paragraph of the proof, T is an $(H \times H)$-prime of $R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$ and $T \cap B_{\boldsymbol{r},\boldsymbol{c}}$ is an $\widetilde{H}$-prime of $B_{\boldsymbol{r},\boldsymbol{c}}$. Since $\beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}(T \cap B_{\boldsymbol{r},\boldsymbol{c}}) = P$, we must have $T \cap B_{\boldsymbol{r},\boldsymbol{c}} = \widetilde{P}$, whence $T = Q$ by the uniqueness of Q. Therefore Proposition 3.3 shows that $T^+ = Q^+$ and $T^- = Q^-$, as desired. □

3.6. Fix $t \in \{0, 1, \dots, n\}$, and let $H\text{-}\operatorname{spec}^{[t]} A$ be the set of those H-primes of A which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. By Corollary 1.10,

$$H\text{-}\operatorname{spec}^{[t]} A = \bigsqcup_{(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{RC}_t} H\text{-}\operatorname{spec}_{\boldsymbol{r},\boldsymbol{c}} A,$$

and consequently Theorem 3.5 implies that

$$|H\text{-}\operatorname{spec}^{[t]} A| = \sum_{(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{RC}_t} |H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^+| \cdot |H\text{-}\operatorname{spec} R_{\boldsymbol{c}}^-|.$$

If $\boldsymbol{R}_t$ denotes the set of sequences $(r_1, \dots, r_t) \in \mathbb{N}^t$ with $1 \le r_1 < \cdots < r_t \le n$, then $\boldsymbol{RC}_t = \boldsymbol{R}_t \times \boldsymbol{R}_t$. For each $\boldsymbol{r} \in \boldsymbol{R}_t$, the automorphism $\boldsymbol{\tau}$ of A discussed in (5.1) induces an isomorphism $\overline{\boldsymbol{\tau}} : R_{\boldsymbol{r}}^+ \to R_{\boldsymbol{r}}^-$. While $\overline{\boldsymbol{\tau}}$ is not H-equivariant, there is an automorphism γ of H, given by $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \mapsto (\beta_1, \dots, \beta_n, \alpha_1, \dots, \alpha_n)$, such that the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\gamma} & H \\ \big\downarrow & & \big\downarrow \\ \operatorname{Aut} R_{\boldsymbol{r}}^+ & \xrightarrow{\overline{\boldsymbol{\tau}}^*} & \operatorname{Aut} R_{\boldsymbol{r}}^- \end{array}$$

(Here the vertical arrows denote the standard actions of H on $R_{\boldsymbol{r}}^+$ and $R_{\boldsymbol{r}}^-$.) Hence, $\overline{\boldsymbol{\tau}}$ provides a bijection of $H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^+$ onto $H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^-$. Therefore

$$|H\text{-}\operatorname{spec}^{[t]} A| = \sum_{\boldsymbol{r},\boldsymbol{c} \in \boldsymbol{R}_t} |H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^+| \cdot |H\text{-}\operatorname{spec} R_{\boldsymbol{c}}^+| = \Bigg(\sum_{\boldsymbol{r} \in \boldsymbol{R}_t} |H\text{-}\operatorname{spec} R_{\boldsymbol{r}}^+| \Bigg)^2,$$

a perfect square. These numbers are known in three cases:

$$|H\text{-}\operatorname{spec}^{[t]} A| = \begin{cases} 1 & (t = 0) \\ (2^n - 1)^2 & (t = 1) \\ (n!)^2 & (t = n). \end{cases}$$

The case when $t = 0$ is trivial, and the case when $t = 1$ is given by [**5**, Corollary 3.5]. For the remaining case, note first that $H\text{-}\operatorname{spec}^{[n]} A \approx H\text{-}\operatorname{spec} \mathcal{O}_q(GL_n(k))$. It can be checked that there is a bijection between $H\text{-}\operatorname{spec} \mathcal{O}_q(GL_n(k))$ and the set of winding-invariant primes of $\mathcal{O}_q(SL_n(k))$ (e.g., see [**1**, Lemma II.5.16]), and it follows from the work of Hodges and Levasseur [**11**] that the latter set is in bijection with the double Weyl group $S_n \times S_n$ (cf. [**1**, Corollary II.4.12]).

3.7. As a corollary of Theorem 3.5, we obtain the following less specific but more digestible result.

Corollary. [q not a root of unity] *Set*

$$R^+ = A/\langle X_{ij} \mid i < j\rangle \qquad \text{and} \qquad R^- = A/\langle X_{ij} \mid i > j\rangle,$$

let $\pi^\pm : A \to R^\pm$ denote the quotient maps, and let β denote the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi^+ \otimes \pi^-} R^+ \otimes R^-.$$

Given any H-prime P in A, there exist H-primes $P^\pm$ in $R^\pm$ such that

$$P = \beta^{-1}\big((P^+ \otimes R^-) + (R^+ \otimes P^-)\big).$$

Proof. By Corollary 1.10 and Theorem 3.5, $P = \beta_{\boldsymbol{r},\boldsymbol{c}}^{-1}\big((Q^+ \otimes R_{\boldsymbol{c}}^-) + (R_{\boldsymbol{r}}^+ \otimes Q^-)\big)$ for some $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}$ and some H-primes Q^+ in $R_{\boldsymbol{r}}^+$ and Q^- in $R_{\boldsymbol{c}}^-$. Observe that $X_{ij} \in \ker \pi_{\boldsymbol{r},0}^+$ when $i < j$, and that $X_{ij} \in \ker \pi_{\boldsymbol{c},0}^-$ when $i > j$. Hence, there are surjective k-algebra homomorphisms $\tau^+ : R^+ \to R_{\boldsymbol{r},0}^+$ and $\tau^- : R^- \to R_{\boldsymbol{c},0}^-$ such that $\tau^+\pi^+ = \pi_{\boldsymbol{r},0}^+$ and $\tau^-\pi^- = \pi_{\boldsymbol{c},0}^-$. Consequently, $(\tau^+ \otimes \tau^-)\beta = \beta_{\boldsymbol{r},\boldsymbol{c}}$ (with the obvious adjustment of codomains).

Next, observe that $Q_0^+ = Q^+ \cap R_{\boldsymbol{r},0}^+$ and $Q_0^- = Q^- \cap R_{\boldsymbol{c},0}^-$ are H-primes of $R_{\boldsymbol{r},0}^+$ and $R_{\boldsymbol{c},0}^-$, respectively, whence the ideals $P^\pm = (\tau^\pm)^{-1}(Q_0^\pm)$ are H-primes in $R^\pm$. Finally,

$$P = \big((\tau^+ \otimes \tau^-)\beta\big)^{-1}\big((Q_0^+ \otimes R_{\boldsymbol{c},0}^-) + (R_{\boldsymbol{r},0}^+ \otimes Q_0^-)\big) = \beta^{-1}\big((P^+ \otimes R^-) + (R^+ \otimes P^-)\big),$$

as desired. □

4. Illustration: $\mathcal{O}_q(M_2(k))$

Theorem 3.5 opens a potential route to computing the H-primes of A in the generic case: If we can find all the H-primes in each $R_{\boldsymbol{r}}^+$ and $R_{\boldsymbol{c}}^-$, we immediately obtain descriptions of all the H-primes in A. Since these descriptions would be in terms of pullbacks of H-primes from the algebras $R_{\boldsymbol{r}}^+ \otimes R_{\boldsymbol{c}}^-$, it would still remain to find generating sets for these ideals.

To illustrate the procedure, we sketch the case where $n = 2$, for which $H\text{-}\operatorname{spec} A$ is already known. In [**7**], we use the above process to compute $H\text{-}\operatorname{spec} A$ when $n = 3$.

4.1. Assume that q is not a root of unity, and fix $n = 2$. There are only four choices for $\boldsymbol{r}$ and $\boldsymbol{c}$, namely $\varnothing$, (1), (2), and $(1,2)$. The corresponding algebras $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$ are

$$\begin{aligned} R^+_\varnothing &= A/\langle X_{11}, X_{12}, X_{21}, X_{22}\rangle = k & R^-_\varnothing &= k \\ R^+_{(1)} &= \big(A/\langle X_{12}, X_{22}\rangle\big)[X_{11}^{-1}] = k\langle Y_{11}^{\pm1}, Y_{21}\rangle & R^-_{(1)} &= k\langle Z_{11}^{\pm1}, Z_{12}\rangle \\ R^+_{(2)} &= \big(A/\langle X_{11}, X_{12}, X_{22}\rangle\big)[X_{21}^{-1}] = k[Y_{21}^{\pm1}] & R^-_{(2)} &= k[Z_{12}^{\pm1}] \\ R^+_{(1,2)} &= \big(A/\langle X_{12}\rangle\big)[X_{11}^{-1}, X_{22}^{-1}] = k\langle Y_{11}^{\pm1}, Y_{21}, Y_{22}^{\pm1}\rangle & R^-_{(1,2)} &= k\langle Z_{11}^{\pm1}, Z_{12}, Z_{22}^{\pm1}\rangle. \end{aligned}$$

The H-primes in these algebras are easily computed:

$$\begin{aligned} H\text{-}\operatorname{spec} R^+_\varnothing &= \big\{\langle 0\rangle\big\} & H\text{-}\operatorname{spec} R^-_\varnothing &= \big\{\langle 0\rangle\big\} \\ H\text{-}\operatorname{spec} R^+_{(1)} &= \big\{\langle 0\rangle,\ \langle Y_{21}\rangle\big\} & H\text{-}\operatorname{spec} R^-_{(1)} &= \big\{\langle 0\rangle,\ \langle Z_{12}\rangle\big\} \\ H\text{-}\operatorname{spec} R^+_{(2)} &= \big\{\langle 0\rangle\big\} & H\text{-}\operatorname{spec} R^-_{(2)} &= \big\{\langle 0\rangle\big\} \\ H\text{-}\operatorname{spec} R^+_{(1,2)} &= \big\{\langle 0\rangle,\ \langle Y_{21}\rangle\big\} & H\text{-}\operatorname{spec} R^-_{(1,2)} &= \big\{\langle 0\rangle,\ \langle Z_{12}\rangle\big\}. \end{aligned}$$

The only choice for $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}_0$ is $\boldsymbol{r} = \boldsymbol{c} = \varnothing$. In this case, the only H-primes in $R^+_{\boldsymbol{r}}$ and $R^-_{\boldsymbol{c}}$ are the zero ideals, and $\beta^{-1}_{\varnothing,\varnothing}(\langle 0\rangle) = \langle X_{11}, X_{12}, X_{21}, X_{22}\rangle$, the augmentation ideal of A. We record this H-prime using the symbol

$$\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}$$

to denote the generating set $\{X_{11}, X_{12}, X_{21}, X_{22}\}$, the bullet in position (i, j) being a marker for the element X_{ij}.

Corresponding to the four pairs $(\boldsymbol{r}, \boldsymbol{c}) \in \boldsymbol{RC}_1$, there are nine H-primes in A of the form

$$\beta^{-1}_{\boldsymbol{r},\boldsymbol{c}}\big((Q^+ \otimes R^-_{\boldsymbol{c}}) + (R^+_{\boldsymbol{r}} \otimes Q^-)\big)$$

where Q^+ is an H-prime in $R^+_{\boldsymbol{r}}$ and Q^- is an H-prime in $R^-_{\boldsymbol{c}}$. We can record generating sets for these ideals as follows, continuing the notation introduced in the previous paragraph; here $\circ$ is a placeholder and $\square$ denotes the 2×2 quantum determinant.

		$R^-_{(1)}$		$R^-_{(2)}$
		$\langle 0\rangle$	$\langle Z_{12}\rangle$	$\langle 0\rangle$
$R^+_{(1)}$	$\langle 0\rangle$	$\square$	$\begin{smallmatrix}\circ & \bullet \\ \circ & \bullet\end{smallmatrix}$	$\begin{smallmatrix}\bullet & \circ \\ \bullet & \circ\end{smallmatrix}$
	$\langle Y_{21}\rangle$	$\begin{smallmatrix}\circ & \circ \\ \bullet & \bullet\end{smallmatrix}$	$\begin{smallmatrix}\circ & \bullet \\ \bullet & \bullet\end{smallmatrix}$	$\begin{smallmatrix}\bullet & \circ \\ \bullet & \bullet\end{smallmatrix}$
$R^+_{(2)}$	$\langle 0\rangle$	$\begin{smallmatrix}\bullet & \bullet \\ \circ & \circ\end{smallmatrix}$	$\begin{smallmatrix}\bullet & \bullet \\ \circ & \bullet\end{smallmatrix}$	$\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}$

(See, e.g., [**5**, Theorem 1.1] for a proof that the quantum determinant generates the kernel of $\beta_{(1),(1)}$. We leave it to the reader to check that the other H-primes are generated as indicated.)

Finally, the only choice for $(\boldsymbol{r},\boldsymbol{c}) \in \boldsymbol{RC}_2$ is $\boldsymbol{r} = \boldsymbol{c} = (1,2)$, and there are four H-primes in A of the form

$$\beta^{-1}_{(1,2),(1,2)}\big((Q^+ \otimes R^-_{(1,2)}) + (R^+_{(1,2)} \otimes Q^-)\big)$$

with $Q^\pm \in H\text{-}\operatorname{spec} R^\pm_{(1,2)}$. We record generating sets for these ideals as follows:

	$\langle 0\rangle$	$\langle Z_{12}\rangle$
$\langle 0\rangle$	○○ ○○	○● ○○
$\langle Y_{21}\rangle$	○○ ●○	○● ●○

We now conclude from Theorem 3.5 that we have found all the H-primes of $A = \mathcal{O}_q(M_2(k))$. There are 14 in total, which we can display as follows:

●● ●●	□	○● ○●	●○ ●○	○○ ○○	○● ○○
	○○ ●●	○● ●●	●○ ●●	○○ ●○	○● ●○
	●● ○○	●● ○●	●● ●○		

For a display showing the inclusions among these ideals, see [**5**, (3.6)].

5. Appendix. Relations in $\mathcal{O}_q(M_n(k))$

The proofs in this paper rely on a number of relations among the generators and quantum minors in quantum matrix algebras. We record and/or derive those relations in this appendix. Throughout, let $A = \mathcal{O}_q(M_n(k))$ with k an arbitrary field and $q \in k^\times$ an arbitrary nonzero scalar.

5.1. (a) We present the algebra A with generators X_{ij} for $i,j = 1,\dots,n$ and relations

$$\begin{aligned} X_{ij}X_{lj} &= qX_{lj}X_{ij} && (i<l)\\ X_{ij}X_{im} &= qX_{im}X_{ij} && (j<m)\\ X_{ij}X_{lm} &= X_{lm}X_{ij} && (i<l,\ j>m)\\ X_{ij}X_{lm} - X_{lm}X_{ij} &= (q-q^{-1})X_{im}X_{lj} && (i<l,\ j<m). \end{aligned}$$

As is well known, A is in fact a bialgebra, with comultiplication $\Delta : A \to A\otimes A$ and counit $\epsilon : A \to k$ such that

$$\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj} \qquad \text{and} \qquad \epsilon(X_{ij}) = \delta_{ij}$$

for all i, j.

(b) The algebra A possesses various symmetries. In particular, it supports a k-algebra automorphism $\boldsymbol{\tau}$ such that $\boldsymbol{\tau}(X_{ij}) = X_{ji}$ for all i, j [**16**, Proposition 3.7.1]. Pairs of results which imply each other just through applications of $\boldsymbol{\tau}$ will be simply referred to as "symmetric".

(c) Given $U, V \subseteq \{1, \dots, n\}$ with $|U| = |V| = t$, let $\mathcal{O}_q(M_{U,V}(k))$ denote the k-subalgebra of A generated by those X_{ij} with $i \in U$ and $j \in V$. There is a natural isomorphism $\mathcal{O}_q(M_t(k)) \to \mathcal{O}_q(M_{U,V}(k))$, which sends the quantum determinant of $\mathcal{O}_q(M_t(k))$ to the quantum minor in A involving rows from U and columns from V. As in [**6**], we denote this quantum minor by $[U \mid V]$, or in the form $[u_1 \cdots u_t \mid v_1 \cdots v_t]$ if we wish to list the elements of U and V.

We shall also use the isomorphism $\mathcal{O}_q(M_t(k)) \to \mathcal{O}_q(M_{U,V}(k))$ to simplify various proofs, since it will allow us to work in smaller quantum matrix algebras than A on occasion. For example, since the quantum determinant in $\mathcal{O}_q(M_t(k))$ is central (e.g., [**16**, Theorem 4.6.1]), the quantum minor $[U \mid V]$ commutes with X_{ij} for all $i \in U$ and $j \in V$. Consequently,

$$[U \mid V][I \mid J] = [I \mid J][U \mid V] \qquad\qquad (I \subseteq U,\ J \subseteq V).$$

(d) Recall from [**15**, Equation 1.9] the comultiplication rule for quantum minors:

$$\Delta\big([I \mid J]\big) = \sum_{|K|=|I|} [I \mid K] \otimes [K \mid J].$$

5.2. We next restate some identities from [**16**], given there for generators and maximal minors, in a form that applies to minors of arbitrary size. Note the difference between our choice of relations for A (see (5.1)(a)) and that in [**16**, p. 37]. Because of this, we must interchange q and q^{-1} whenever carrying over a formula from [**16**].

Lemma. *Let $r, c \in \{1, \dots, n\}$ and $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| \geq 1$.*

(a) *If $r \in I$ and $c \in J$, then $X_{rc}[I \mid J] = [I \mid J]X_{rc}$.*

(b) *If $r \in I$ and $c \notin J$, set $J^+ = J \sqcup \{c\}$. Then*

$$X_{rc}[I \mid J] - q^{-1}[I \mid J]X_{rc} = (q^{-1} - q) \sum_{\substack{j \in J \\ j > c}} (-q)^{-|J \cap [c,j]|} [I \mid J^+ \setminus \{j\}] X_{rj} \tag{1}$$

$$[I \mid J]X_{rc} - qX_{rc}[I \mid J] = (q - q^{-1}) \sum_{\substack{j \in J \\ j > c}} (-q)^{|J \cap [c,j]|} X_{rj} [I \mid J^+ \setminus \{j\}] \tag{2}$$

(c) *If $r \notin I$ and $c \in J$, set $I^+ = I \sqcup \{r\}$. Then*

$$X_{rc}[I \mid J] - q^{-1}[I \mid J]X_{rc} = (q^{-1} - q) \sum_{\substack{i \in I \\ i > r}} (-q)^{-|I \cap [r,i]|} [I^+ \setminus \{i\} \mid J] X_{ic} \tag{1}$$

$$[I \mid J]X_{rc} - qX_{rc}[I \mid J] = (q - q^{-1}) \sum_{\substack{i \in I \\ i > r}} (-q)^{|I \cap [r,i]|} X_{ic}[I^+ \setminus \{i\} \mid J] \tag{2}$$

Proof. Part (a) is clear. To obtain part (b), we may work in $\mathcal{O}_q(M_{I \sqcup \{r'\}, J^+}(k))$ for some $r' \notin I$, and so there is no loss of generality in assuming that $I \subset J^+ = \{1, \dots, n\}$. The desired relations then follow from the second cases of parts (1) and (2) of [**16**, Lemma 4.5.1]. Part (c) is symmetric to (b). □

5.3. Lemma. *Let $U, V \subseteq \{1, \dots, n\}$ with $|U| = |V|$, and let $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Set $U_s = U \setminus \{u_s\}$ and $V_s = V \setminus \{v_s\}$ for $s = 1, 2$.*

(a) *If $u_1 < u_2$, then $[U_1 \mid V_1][U_2 \mid V_1] = q^{-1}[U_2 \mid V_1][U_1 \mid V_1]$.*

(b) *If $v_1 < v_2$, then $[U_1 \mid V_1][U_1 \mid V_2] = q^{-1}[U_1 \mid V_2][U_1 \mid V_1]$.*

(c) *If $u_1 < u_2$ and $v_1 > v_2$, then $[U_1 \mid V_1][U_2 \mid V_2] = [U_2 \mid V_2][U_1 \mid V_1]$.*

(d) *If $u_1 < u_2$ and $v_1 < v_2$, then*

$$[U_1 \mid V_1][U_2 \mid V_2] - [U_2 \mid V_2][U_1 \mid V_1] = (q^{-1} - q)[U_2 \mid V_1][U_1 \mid V_2].$$

Proof. Since we may work in $\mathcal{O}_q(M_{U,V}(k))$, there is no loss of generality in assuming that $U = V = \{1, \dots, n\}$. The result then follows from [**16**, Theorem 5.2.1]. □

5.4. We also require the form of the *q-Laplace relations* given in [**15**]. For index sets I and J, set

$$\ell(I; J) = \big|\{(i,j) \in I \times J \mid i > j\}\big|.$$

Lemma. (q-Laplace relations) *Let $I, J \subseteq \{1, \dots, n\}$.*

(a) *If $J_1, J_2 \subseteq \{1, \dots, n\}$ with $|J_1| + |J_2| = |I|$, then*

$$\sum_{\substack{I_1 \sqcup I_2 = I \\ |I_\nu| = |J_\nu|}} (-q)^{\ell(I_1;I_2)} [I_1 \mid J_1][I_2 \mid J_2] = \begin{cases} (-q)^{\ell(J_1;J_2)} [I \mid J_1 \sqcup J_2] & (J_1 \cap J_2 = \varnothing) \\ 0 & (J_1 \cap J_2 \neq \varnothing). \end{cases}$$

(b) *If $I_1, I_2 \subseteq \{1, \dots, n\}$ with $|I_1| + |I_2| = |J|$, then*

$$\sum_{\substack{J_1 \sqcup J_2 = J \\ |J_\nu| = |I_\nu|}} (-q)^{\ell(J_1;J_2)} [I_1 \mid J_1][I_2 \mid J_2] = \begin{cases} (-q)^{\ell(I_1;I_2)} [I_1 \sqcup I_2 \mid J] & (I_1 \cap I_2 = \varnothing) \\ 0 & (I_1 \cap I_2 \neq \varnothing). \end{cases}$$

Proof. [**15**, Proposition 1.1]. □

5.5. The q-Laplace relations simplify somewhat when one of the index sets is a singleton, as follows.

Corollary. *Let $r, c \in \{1, \dots, n\}$ and $I, J \subseteq \{1, \dots, n\}$.*

(a) *If $|I| = |J| + 1$, then*

$$(1) \qquad \sum_{i \in I} (-q)^{|[1,i) \cap I|} X_{ic} [I \setminus \{i\} \mid J] = \begin{cases} (-q)^{|[1,c) \cap J|} [I \mid J \sqcup \{c\}] & (c \notin J) \\ 0 & (c \in J) \end{cases}$$

$$(2) \qquad \sum_{i \in I} (-q)^{|(i,n] \cap I|} [I \setminus \{i\} \mid J] X_{ic} = \begin{cases} (-q)^{|(c,n] \cap J|} [I \mid J \sqcup \{c\}] & (c \notin J) \\ 0 & (c \in J). \end{cases}$$

(b) *If $|J| = |I| + 1$, then*

$$(1) \qquad \sum_{j \in J} (-q)^{|[1,j) \cap J|} X_{rj} [I \mid J \setminus \{j\}] = \begin{cases} (-q)^{|[1,r) \cap I|} [I \sqcup \{r\} \mid J] & (r \notin I) \\ 0 & (r \in I) \end{cases}$$

$$(2) \qquad \sum_{j \in J} (-q)^{|(j,n] \cap J|} [I \mid J \setminus \{j\}] X_{rj} = \begin{cases} (-q)^{|(r,n] \cap I|} [I \sqcup \{r\} \mid J] & (r \notin I) \\ 0 & (r \in I). \end{cases}$$

Proof. (a) For the first case, fix $J_1 = \{c\}$ and $J_2 = J$. We will use Lemma 5.4(a), which involves a sum over $I_1 \sqcup I_2 = I$ with $|I_1| = 1$; thus $I_1 = \{i\}$ and $I_2 = I \setminus \{i\}$ for some $i \in I$. In that case, $\ell(I_1; I_2) = |[1,i) \cap I|$ and $\ell(J_1; J_2) = |[1,c) \cap J|$. Thus, formula (1) follows directly from Lemma 5.4(a). Formula (2) follows similarly, where this time we fix $J_1 = J$ and $J_2 = \{c\}$.

(b) These follow from (a) by symmetry. □

5.6. Lemma. *Let $U, V \subseteq \{1, \dots, n\}$ with $|U| = |V|$.*

(a) *Let $U = I \sqcup K$, and let $J_1, J_2 \subseteq V$ such that $|J_1| + |J_2| = 2|I| + |K|$. Then*

$$\sum_{\substack{K = K' \sqcup K'' \\ |K'| = |J_1| - |I|}} (-q)^{\ell(I;K') + \ell(K';K'' \sqcup I)} [I \sqcup K' \mid J_1][K'' \sqcup I \mid J_2]$$

$$= \begin{cases} (-q)^{\ell(J_1 \cap J_2; J_1 \setminus J_2) + \ell(J_1 \setminus J_2; J_2)} [I \mid J_1 \cap J_2][U \mid V] & (|J_1 \cap J_2| = |I|) \\ 0 & (|J_1 \cap J_2| > |I|). \end{cases}$$

(b) *Let $V = J \sqcup L$, and let $I_1, I_2 \subseteq U$ such that $|I_1| + |I_2| = 2|J| + |L|$. Then*

$$\sum_{\substack{L = L' \sqcup L'' \\ |L'| = |I_1| - |J|}} (-q)^{\ell(J;L') + \ell(L';L'' \sqcup J)} [I_1 \mid J \sqcup L'][I_2 \mid J \sqcup L'']$$

$$= \begin{cases} (-q)^{\ell(I_1 \cap I_2; I_1 \setminus I_2) + \ell(I_1 \setminus I_2; I_2)} [I_1 \cap I_2 \mid J][U \mid V] & (|I_1 \cap I_2| = |J|) \\ 0 & (|I_1 \cap I_2| > |J|). \end{cases}$$

Proof. By symmetry, we need only prove (a). Note that

$$|J_1| + |J_2| = 2|I| + |K| = |I| + |U| \geq |I| + |J_1 \cup J_2| = |I| + |J_1| + |J_2| - |J_1 \cap J_2|,$$

whence $|J_1 \cap J_2| \geq |I|$. Let (1) denote the left hand side of the formula to be established, and (2) the first choice on the right hand side.

Expand each term $(-q)^{\ell(I;K')}[I \sqcup K' \mid J_1]$ using Lemma 5.4(b) and insert into (1). Thus, (1) equals

$$\sum_{\substack{K=K'\sqcup K'' \\ J_1=J_1'\sqcup J_1''}} (-q)^{\ell(J_1';J_1'')+\ell(K';K''\sqcup I)}[I \mid J_1'][K' \mid J_1''][K''\sqcup I \mid J_2]. \tag{3}$$

We next claim that the sum

$$\sum_{\substack{U=I_1\sqcup I_2 \\ J_1=J_1'\sqcup J_1''}} (-q)^{\ell(I_1;I_2)+\ell(J_1';J_1'')}[I \mid J_1'][I_1 \mid J_1''][I_2 \mid J_2] \tag{4}$$

equals (3). For I_1 as in (4), we have $|I|+|I_1| = |J_1'|+|J_1''| = |J_1|$, and so Lemma 5.4(b) gives

$$\sum_{J_1=J_1'\sqcup J_1''} (-q)^{\ell(J_1';J_1'')}[I \mid J_1'][I_1 \mid J_1''] = 0 \qquad (I_1 \nsubseteq K), \tag{5}$$

because $I \cap I_1 \neq \varnothing$ in this case. On the other hand, for fixed J_1', J_1'' as in (4), we have

$$\sum_{\substack{U=I_1\sqcup I_2 \\ I_1\subseteq K}} (-q)^{\ell(I_1;I_2)}[I_1 \mid J_1''][I_2 \mid J_2] = \sum_{K=K'\sqcup K''} (-q)^{\ell(K';K''\sqcup I)}[K' \mid J_1''][K''\sqcup I \mid J_2]. \tag{6}$$

It follows from (5) and (6) that (4) = (3) as claimed, and thus (1) = (4).

For J_1'' as in (4), we have $|J_1''| = |K'| = |K|-|K''| = |U|-|K''\sqcup I| = |V|-|J_2|$. Hence, Lemma 5.4(a) says that

$$\sum_{U=I_1\sqcup I_2} (-q)^{\ell(I_1;I_2)}[I_1 \mid J_1''][I_2 \mid J_2] = \begin{cases} (-q)^{\ell(J_1'';J_2)}[U \mid V] & (J_1''\cap J_2=\varnothing) \\ 0 & (J_1''\cap J_2\neq\varnothing). \end{cases} \tag{7}$$

Substituting (7) into (4), it follows that (1) is equal to the sum

$$\sum_{J_1=J_1'\sqcup J_1''} (-q)^{\ell(J_1';J_1'')+\ell(J_1'';J_2)}[I \mid J_1']d(J_1''), \tag{8}$$

where $d(J_1'') = [U \mid V]$ if J_1'' and J_2 are disjoint, but $d(J_1'') = 0$ otherwise.

If $|J_1 \cap J_2| > |I|$, then since any $|J_1'| = |I|$, we see that $J_1 \cap J_2 \nsubseteq J_1'$ and so $J_1'' \cap J_2 \neq \varnothing$. Thus in this case all $d(J_1'') = 0$, and so (1) = (8) = 0.

Finally, suppose that $|J_1 \cap J_2| = |I|$. Then the only time J_1'' and J_2 can be disjoint is when $J_1' = J_1 \cap J_2$, and therefore (1) = (8) = (2) in this case. □

5.7. Lemma. *Let* $r, c \in \{1, \dots, n\}$ *and* $I, J \subseteq \{1, \dots, n\}$ *with* $|I| = |J| \geq 1$. *If* $r > \max(I)$ *and* $c > \max(J)$, *then*

$$[I \mid J]X_{rc} - q^2 X_{rc}[I \mid J] = (1 - q^2)[I \sqcup \{r\} \mid J \sqcup \{c\}].$$

Proof. Since we may work in $\mathcal{O}_q(M_{I\sqcup\{r\},J\sqcup\{c\}}(k))$, it suffices to consider the case that $r = c = n$ and $I = J = \{1, \dots, n-1\}$. Now $[I \mid J] = A(n\,n)$ in the notation of [**16**, (4.3)]. Set $D_q = [I \sqcup \{r\} \mid J \sqcup \{c\}] = [1 \cdots n \mid 1 \cdots n]$.

The first two q-Laplace relations in [**16**, Corollary 4.4.4] yield

$$\sum_{j=1}^{n} (-q)^{j-n} X_{nj} A(n\,j) = \sum_{j=1}^{n} (-q)^{n-j} A(n\,j) X_{nj} = D_q. \tag{1}$$

Solving for $X_{nn}A(n\,n)$ and $A(n\,n)X_{nn}$, we obtain

$$X_{nn}A(n\,n) = D_q - \sum_{j<n} (-q)^{j-n} X_{nj} A(n\,j) \tag{2}$$

$$A(n\,n)X_{nn} = D_q - \sum_{j<n} (-q)^{n-j} A(n\,j) X_{nj}. \tag{3}$$

For any j, the third relation of [**16**, Lemma 5.1.2] implies that

$$X_{nj}A(n\,j) = A(n\,j)X_{nj} + (1 - q^{-2}) \sum_{l<j} (-q)^{j-l} A(n\,l) X_{nl}. \tag{4}$$

Substituting (4) into (2) for all $j < n$, we obtain

$$\begin{aligned} X_{nn}A(n\,n) &= D_q - \sum_{j<n} (-q)^{j-n} A(n\,j) X_{nj} \\ &\quad - (1 - q^{-2}) \sum_{j<n} \sum_{l<j} (-q)^{2j-l-n} A(n\,l) X_{nl} \\ &= D_q - \sum_{l<n} \left[(-q)^{l-n} + (1 - q^{-2}) \sum_{l<j<n} (-q)^{2j-l-n} \right] A(n\,l) X_{nl}. \end{aligned} \tag{5}$$

The expression in square brackets can be simplified as follows:

$$\begin{aligned} (-q)^{l-n} + (1 - q^{-2}) \sum_{l<j<n} (-q)^{2j-l-n} &= (-q)^{l-n} \left[1 + (1 - q^{-2}) \sum_{0<m<n-l} (-q)^{2m} \right] \\ &= (-q)^{n-l-2}. \end{aligned} \tag{6}$$

Substituting (6) into (5) and replacing l by j, we obtain

$$X_{nn}A(n\,n) = D_q - \sum_{j<n} (-q)^{n-j-2} A(n\,j) X_{nj}. \tag{7}$$

Finally, combining (3) with (7), we conclude that

$$A(n\,n)X_{nn} - q^2 X_{nn} A(n\,n) = (1 - q^2) D_q,$$

as desired. $\square$

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Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
E-mail address: `goodearl@math.ucsb.edu`

Department of Mathematics, J.C.M.B., Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland
E-mail address: `tom@maths.ed.ac.uk`