The main goal of the paper is to establish the existence of tensor product decompositions for those prime ideals \( P \) of the algebra \( A = \mathcal{O}_q(M_n(k)) \) of quantum \( n \times n \) matrices which are invariant under winding automorphisms of \( A \), in the generic case (\( q \) not a root of unity). More specifically, every such \( P \) is the kernel of a map of the form

\[
A \rightarrow A \otimes A \rightarrow A^+ \otimes A^- \rightarrow (A^+/P^+) \otimes (A^-/P^-)
\]

where \( A \rightarrow A \otimes A \) is the comultiplication, \( A^+ \) and \( A^- \) are suitable localized factor algebras of \( A \), and \( P^\pm \) is a prime ideal of \( A^\pm \) invariant under winding automorphisms. Further, the algebras \( A^\pm \), which vary with \( P \), can be chosen so that the correspondence \((P^+, P^-) \mapsto P\) is a bijection. The main theorem is applied, in a sequel to this paper, to completely determine the winding-invariant prime ideals in the generic quantum \( 3 \times 3 \) matrix algebra.

Introduction

This paper represents part of an ongoing project to determine the prime and primitive spectra of the generic quantized coordinate ring of \( n \times n \) matrices, \( \mathcal{O}_q(M_n(k)) \). Here \( k \) is an arbitrary field and \( q \in k^\times \) is a non-root of unity. The current intermediate goal is to determine the prime ideals of \( \mathcal{O}_q(M_n(k)) \) invariant under all winding automorphisms. (See below for a discussion of the relations between these winding-invariant primes and the full prime spectrum of \( \mathcal{O}_q(M_n(k)) \).) Our main result exhibits a bijection between these primes and pairs of winding-invariant primes from certain ‘localized step-triangular factors’ of \( \mathcal{O}_q(M_n(k)) \), namely the algebras

\[
R_r^+ = (\mathcal{O}_q(M_n(k))/\langle X_{ij} \mid j > t \text{ or } i < r_j \rangle)[\overline{X}_{r_1,1}, \ldots, \overline{X}_{r_t,t}]
\]

\[
R_c^- = (\mathcal{O}_q(M_n(k))/\langle X_{ij} \mid i > t \text{ or } j < c_i \rangle)[\overline{X}_{1c_1,1}, \ldots, \overline{X}_{tc_t}]}
\]

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where \( r = (r_1, \ldots, r_t) \) and \( c = (c_1, \ldots, c_t) \) are strictly increasing sequences of integers in the range \( 1, 2, \ldots, n \). In particular, since each \( R_r^+ \) and \( R_c^- \) can be presented as a skew-Laurent extension of a localized factor algebra of \( \mathcal{O}_q(M_{n-1}(k)) \), the above bijection can be used to obtain descriptions (as pullbacks of primes in the algebras \( R_r^+ \otimes R_c^- \)) of the winding-invariant primes of \( \mathcal{O}_q(M_n(k)) \) from those of \( \mathcal{O}_q(M_{n-1}(k)) \). In a sequel [7] to this paper, we follow the route just sketched to develop a complete list, with sets of generators, of the winding-invariant primes in \( \mathcal{O}_q(M_3(k)) \).

The theorem indicated above depends on some detailed structural results concerning \( \mathcal{O}_q(M_n(k)) \) and on some general work with primes in tensor product algebras invariant under group actions. First, we construct a partition of \( \text{spec} \mathcal{O}_q(M_n(k)) \) indexed by pairs \( (r, c) \) as above, together with localized factor algebras \( A_{r,c} \) of \( \mathcal{O}_q(M_n(k)) \), such that the portion of \( \text{spec} \mathcal{O}_q(M_n(k)) \) indexed by \( (r, c) \) is Zariski-homeomorphic to \( \text{spec} A_{r,c} \). We next prove that \( A_{r,c} \) is isomorphic to a subalgebra \( B_{r,c} \) of \( R_r^+ \otimes R_c^- \), identify the structure of \( B_{r,c} \), and show that \( R_r^+ \otimes R_c^- \) is a skew-Laurent extension of \( B_{r,c} \). Finally, with the help of some general work on tensor products, we prove that each winding-invariant prime of \( B_{r,c} \) extends uniquely to a winding-invariant prime of \( R_r^+ \otimes R_c^- \), and that the latter primes can be uniquely expressed in the form \( (P^+ \otimes R_c^-) + (R_r^+ \otimes P^-) \) where \( P^+ \) (respectively, \( P^- \)) is a winding-invariant prime in \( R_r^+ \) (respectively, \( R_c^- \)). We thus conclude that every winding-invariant prime of \( \mathcal{O}_q(M_n(k)) \) can be uniquely expressed as the kernel of a map

\[
\mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \longrightarrow R_r^+ \otimes R_c^- \longrightarrow (R_r^+/P^+) \otimes (R_c^-/P^-),
\]

where the first arrow is comultiplication and the others are tensor products of localization or quotient maps.

**Algebraic background.** The algebra \( \mathcal{O}_q(M_n(k)) \) has standard generators \( X_{ij} \) for \( i, j = 1, \ldots, n \) and relations which we recall in (5.1)(a), along with the bialgebra structure of this algebra. The latter structure allows us to define left and right winding automorphisms corresponding to those characters \( k \)-algebra homomorphisms \( \mathcal{O}_q(M_n(k)) \rightarrow k \) which are invertible in \( \mathcal{O}_q(M_n(k))^\times \) with respect to the convolution product (cf. [1, (I.9.25)] or [12, (1.3.4)] for the Hopf algebra case). It is well known that the collection of left (respectively, right) winding automorphisms of \( \mathcal{O}_q(M_n(k)) \) forms a group isomorphic to the diagonal subgroup of \( GL_n(k) \), whose action on the matrix of generators \( X_{ij} \) is given by left (respectively, right) multiplication. We combine these actions to obtain an action of the group \( H = (k^\times)^n \times (k^\times)^n \) on \( \mathcal{O}_q(M_n(k)) \) by \( k \)-algebra automorphisms satisfying the rule

\[
(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n).X_{ij} = \alpha_i \beta_j X_{ij}.
\]

One indication of the extent of the symmetry given by this action is the fact that there are only finitely many (actually, at most \( 2^{n^2} \)) primes of \( \mathcal{O}_q(M_n(k)) \) invariant under \( H \) [9, (5.7)(i)]. The quoted result also shows that all \( H \)-primes of this algebra are prime, and so the \( H \)-primes coincide with the winding-invariant primes in \( \mathcal{O}_q(M_n(k)) \). (Recall that the definition of an \( H \)-prime ideal is obtained from the standard ideal-theoretic definition of a prime ideal by restricting to \( H \)-invariant ideals.)
In [9, Theorem 6.6], Letzter and the first author showed that the overall picture of the prime spectrum of an algebra with certain basic features like those of \( O_q(M_n(k)) \) is determined to a great extent by the primes invariant under a suitable group action. We quote the improved version of this picture presented in [1, Theorem II.2.13]. Let \( A \) be a noetherian algebra over an infinite field \( k \), and let \( H = (k^\times)^r \) be (the group of \( k \)-points of) an algebraic torus acting rationally on \( A \) by \( k \)-algebra automorphisms. Each \( H \)-prime of \( A \) is a prime ideal, and \( \text{spec} A \) is the disjoint union of the sets

\[
\text{spec}_J A := \{ P \in \text{spec} A \mid \bigcap_{h \in H} h(P) = J \}
\]

as \( J \) ranges over the \( H \)-primes of \( A \). Further:

1. Let \( \mathcal{E}_J \) denote the set of all regular \( H \)-eigenvectors in \( A/J \). Then \( \mathcal{E}_J \) is a denominator set, and the localization \( A_J = (A/J)[\mathcal{E}_J^{-1}] \) is an \( H \)-simple ring.
2. \( \text{spec}_J A \) is homeomorphic to \( \text{spec} A_J \) via localization and contraction.
3. \( \text{spec} A_J \) is homeomorphic to \( \text{spec} Z(A_J) \) via contraction and extension.
4. \( Z(A_J) \) is a Laurent polynomial ring, in at most \( r \) indeterminates, over the fixed field \( Z(\text{Fract} A/J)^H \).

Under some additional hypotheses, satisfied by \( O_q(M_n(k)) \), we also have:

5. The primitive ideals of \( A \) are exactly the maximal elements of the sets \( \text{spec}_J A \).
6. If \( k \) is algebraically closed, then the primitive ideals within each \( \text{spec}_J A \) are permuted transitively by \( H \).

Statement (5) was proved in [9, Corollary 6.9] (see also [1, Theorem II.8.4]). Statement (6) is a consequence of general transitivity theorems for algebraic group actions due to Moeglin-Rentschler [14, Théorème 2.12(ii)] and Vonessen [17, Theorem 2.2], but the case where the acting group is a torus is much easier (see [9, Theorem 6.8] or [1, Theorem II.8.14]).

The above results indicate that to draw a complete picture of \( \text{spec} O_q(M_n(k)) \), we need to determine the \( H \)-primes. That is easy to do in case \( n = 2 \); the result is recorded, for instance, in [5, (3.6)] (see [1, Example II.2.14(d)] for more detail). In general, we make the following

**Conjecture.** Every \( H \)-prime of \( O_q(M_n(k)) \) can be generated by a set of quantum minors.

This conjecture is easily checked in case \( n = 2 \) using the information above, and we verify it for the case \( n = 3 \) in [7]. Further supporting evidence is provided by recent work of Cauchon, who showed that distinct, comparable \( H \)-primes in \( O_q(M_n(k)) \) can be distinguished by the quantum minors they contain [2, Proposition 6.2.2 and Théorème 6.2.1]. Another source of support for the conjecture is the work of Hodges and Levasseur [10, 11], from which one can deduce that, up to certain localizations, the winding-invariant primes of \( O_q(SL_n(k)) \) are generated by quantum minors (cf. [1, Corollary II.4.12]). Since \( O_q(GL_n(k)) \) is isomorphic to a Laurent polynomial ring over \( O_q(SL_n(k)) \) [13, Proposition], the above statement also holds in \( O_q(GL_n(k)) \). In particular, those \( H \)-primes of \( O_q(M_n(k)) \) which do not contain the quantum determinant can be generated, up to suitable localizations, by quantum minors.
Geometric background. The classical origins of our main theorem, especially as it applies to winding-invariant primes of $O_q(M_n(k))$ not containing the quantum determinant, lie in the geometry of ‘LU-decompositions’ of invertible matrices. For this part of the introduction, let us assume (to avoid complications) that $k$ is algebraically closed. An LU-decomposition of a matrix $X \in GL_n(k)$ is any expression $X = LU$ where $L$ (respectively, $U$) is a lower (respectively, upper) triangular invertible matrix. It is well known that $X$ has such a decomposition if and only if the principal minors of $X$ are nonzero. The LU-decomposable matrices thus form a dense open subvariety of $GL_n(k)$, known as the big cell. We may write the big cell in the form $B^+B^-$ where $B^+$ (respectively, $B^-$) is the subgroup of lower (respectively, upper) triangular matrices in $GL_n(k)$, and we have

$$O(B^+B^-) = O(GL_n(k))[D^{-1}]$$

where $D$ is the multiplicative subset of $O(GL_n(k))$ generated by the principal minors. The comorphism of the multiplication map $B^+ \times B^- \rightarrow GL_n(k)$ provides an embedding

$$\beta : O(GL_n(k))[D^{-1}] \rightarrow O(B^+) \otimes O(B^-);$$

the restriction of $\beta$ to $O(GL_n(k))$ is just the composition of the comultiplication map $O(GL_n(k)) \rightarrow O(GL_n(k)) \otimes O(GL_n(k))$ with the tensor product of the restriction maps $O(GL_n(k)) \rightarrow O(B^\pm)$. The structure of the image of $\beta$ is easy to determine, since $B^+ \cap B^-$ is the diagonal subgroup of $GL_n(k)$ and the subgroups $B^\pm$ are semidirect products of their unipotent subgroups with this diagonal subgroup. Namely, the image of $\beta$ is the subalgebra of $O(B^+) \otimes O(B^-)$ generated by (the cosets of) the elements

$$(\dagger) \quad X_{ij}X_{ji}^{-1} \otimes 1 \quad 1 \otimes X_{ij}X_{ii}^{-1} \quad (X_{ii} \otimes X_{ii})^{\pm1}.$$ 

Further, $O(B^+) \otimes O(B^-)$ is a Laurent polynomial ring over the image of $\beta$ with respect to indeterminates $1 \otimes X_{ii}^{\pm1}$.

Quantum analogs of the above facts are known, but we have not been able to locate complete statements in the literature. To formulate them, let us write $O_q(GL_n(k))$ modulo the ideals $\langle X_{ij} \mid i < j \rangle$ and $\langle X_{ij} \mid i > j \rangle$. Then:

(a) The composition of the comultiplication map on $O_q(GL_n(k))$ with the tensor product of the quotient maps $O_q(GL_n(k)) \rightarrow O_q(B^\pm)$ yields an embedding $\beta : O_q(GL_n(k)) \rightarrow O_q(B^+) \otimes O_q(B^-)$ [16, Theorem 8.1.1].

(b) The multiplicative subset $D$ of $O_q(GL_n(k))$ generated by the principal quantum minors is a denominator set.

(c) $\beta$ extends to $O_q(GL_n(k))[D^{-1}]$, and the image of this extension is the subalgebra $B$ of $O_q(B^+) \otimes O_q(B^-)$ generated by (the cosets of) the elements $(\dagger)$.

(d) $O_q(B^+) \otimes O_q(B^-)$ is a skew-Laurent extension of $B$ with respect to the variables $1 \otimes X_{ii}^{\pm1}$.
These facts will be proved as part of the case \( r = e = (1, \ldots, n) \) of our work below. To obtain them via existing results in the literature, one first transfers the problem to \( \mathcal{O}_q(SL_n(k)) \) using the isomorphism \( \mathcal{O}_q(GL_n(k)) \cong \mathcal{O}_q(SL_n(k))[z^{\pm 1}] \) established in [13, Proposition]; the desired conditions then hold in the generality of \( \mathcal{O}_q(G) \), where \( G \) is an arbitrary semisimple algebraic group. The isomorphism of the appropriate localization of \( \mathcal{O}_q(G) \) with the analog of \( \mathcal{B} \) is given in [3, Theorem 4.6] and [12, Proposition 9.2.14]. It is easy to see that \( \mathcal{O}_q(B^+) \otimes \mathcal{O}_q(B^-) \) is a skew-Laurent extension of \( \mathcal{B} \) with appropriate variables; this is mentioned for the case where \( q \) is a root of unity in [4, (4.6)].

The above facts concerning \( \mathcal{O}_q(GL_n(k)) \) immediately carry over to \( \mathcal{O}_q(M_n(k)) \), since \( D \) is also a denominator set in that algebra and \( \mathcal{O}_q(M_n(k))[D^{-1}] = \mathcal{O}_q(GL_n(k))[D^{-1}] \). Whereas in \( \mathcal{O}_q(GL_n(k)) \) all prime ideals are disjoint from \( D \), that no longer holds in \( \mathcal{O}_q(M_n(k)) \), and a sequence of modified versions of (a)–(d) are needed to yield information about those \( H \)-primes of \( \mathcal{O}_q(M_n(k)) \) that meet \( D \). These modifications, involving maps

\[
\beta_{r,e} : \mathcal{O}_q(M_n(k)) \rightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \rightarrow R_r^+ \otimes R_e^- ,
\]

concern quantum analogs of what can be viewed as LU-decompositions for certain locally closed subsets of \( M_n(k) \) of the form \( B^+ w^+ e w^- B^- \) where \( w^\pm \) are permutation matrices and \( e = e_{11} + \cdots + e_{tt} \) is a diagonal idempotent matrix. We leave the formulations of these geometric facts to the interested reader.

**Some notation and conventions.** Throughout the paper, let \( A = \mathcal{O}_q(M_n(k)) \), where we fix a base field \( k \), a positive integer \( n \), and a nonzero scalar \( q \in k^\times \). See (5.1)(a) for the basic relations satisfied by the standard generators \( X_{ij} \) of \( \mathcal{O}_q(M_n(k)) \). As mentioned in (5.2), our relations for \( A \) differ from those in [16] by an interchange of \( q \) and \( q^{-1} \). On the other hand, they agree with the relations used in [2, 15]. To match the relations in [10, 11], replace \( q \) by \( q^2 \).

While our main interest is in the case that \( q \) is not a root of unity, some of our results do not require this assumption, and others require only that \( q \neq \pm 1 \). Thus, we impose no blanket hypotheses on \( q \). To simplify some formulas, write \( \hat{q} = q - q^{-1} \). We also fix the torus \( H = (k^\times)^n \times (k^\times)^n \) and its action on \( A \) by winding automorphisms as described in (5*) above. The algebra \( A \) is graded in a natural way by \( \mathbb{Z}^{2n} = \mathbb{Z}^n \times \mathbb{Z}^n \), each generator \( X_{ij} \) having degree \( (\epsilon_i, \epsilon_j) \) where \( \epsilon_1, \ldots, \epsilon_n \) is the standard basis for \( \mathbb{Z}^n \). We refer to this grading as the **standard grading** on \( A \). (As long as \( k \) is infinite, the homogeneous components of \( A \) for this grading coincide with the eigenspaces for the action of \( H \).)

As in [6], we use the notations \([I \mid J]\) and \([i_1 \cdots i_s \mid j_1 \cdots j_s]\) for quantum minors in \( \mathcal{O}_q(M_n(k)) \), where \( I \) (respectively, \( i_1, \ldots, i_s \)) records the set (respectively, a list) of the corresponding row indices, and similarly for column indices. Recall that \([I \mid J]\) corresponds to the quantum determinant in a subalgebra of \( \mathcal{O}_q(M_n(k)) \) isomorphic to \( \mathcal{O}_q(M_s(k)) \); we write \( \mathcal{O}_q(M_I,J(k)) \) for that subalgebra (see (5.1)(c)). We allow the index sets \( I \) and \( J \) to be empty, following the convention that \( [\emptyset \mid \emptyset] = 1 \).

The symbols \( \subset \) and \( \subseteq \) will be reserved for proper and arbitrary inclusions, respectively. We write \( \sqcup \) to denote a disjoint union.
1. A partition of \( \text{spec} \, A \)

We begin by investigating sets \( \text{spec}_{r,c} \, A \) of prime ideals defined by certain ‘stepwise patterns’ of quantum minors corresponding to strictly increasing sequences \( r \) and \( c \) of row and column indices. Our aim in this section is to show that these sets partition \( \text{spec} \, A \) when \( q \) is not a root of unity. In fact, as long as \( q \neq \pm 1 \), these sets at least partition the collection of completely prime ideals of \( A \). (Recall from [8, Theorem 3.2] that when \( q \) is not a root of unity, all primes of \( A \) are completely prime.) We proceed without imposing special hypotheses on \( q \) until needed.

1.1. We introduce the following partial ordering \( \leq \) on index sets \( I, I' \subseteq \{1, \ldots, n\} \) of the same cardinality. Write \( I = \{i_1 < \cdots < i_l\} \) and \( I' = \{i'_1 < \cdots < i'_l\} \); then \( I \leq I' \) if and only if \( i_s \leq i'_s \) for \( s = 1, \ldots, l \). (This is the same as the ‘column ordering’ \( \leq_c \) used in [6].) All order relations among index sets in this paper will refer to the above partial ordering. This includes statements that an index set with a particular property is minimal among index sets with the same cardinality satisfying that property (e.g., the sets \( \bar{I} \) and \( \bar{J} \) in Theorem 1.9).

1.2. Let \( RC \) denote the set of all pairs \( (r, c) \) where \( r \) and \( c \) are strictly increasing sequences in \( \{1, \ldots, n\} \) of the same length, that is, \( r = (r_1, \ldots, r_t) \) and \( c = (c_1, \ldots, c_t) \) in \( \mathbb{N}^t \) with \( 1 \leq r_1 < r_2 < \cdots < r_t \leq n \) and \( 1 \leq c_1 < c_2 < \cdots < c_t \leq n \). We allow \( t = 0 \), in which case \( r \) and \( c \) are empty sequences, denoted either ( ) or \( \emptyset \). When referring to the length \( t \) of \( r \) and \( c \), we write \( (r, c) \in RC_t \).

For \( (r, c) \in RC_t \), let \( K_{r,c} \) be the ideal of \( A \) generated by the following set of quantum minors:

\[
\text{spec}_{r,c} \, A := \{ P \in \text{spec} \, A \mid K_{r,c} \subseteq P \text{ and } P \cap D_{r,c} = \emptyset \}
\]

\[
\approx \text{spec}(A/K_{r,c})[\bar{D}^{-1}_{r,c}]
\]

as \( (r, c) \) ranges over \( RC \).

1.3. Lemma. Let \( I, J \subseteq \{1, \ldots, n\} \) with \( |I| = |J| \), and set

\[
L = \langle \{ [I' \mid J'] \mid |I'| = |I|, \text{ and } I' < I \text{ or } J' < J \rangle.
\]
If \( r, c \in \{1, \ldots, n\} \) with \( r \leq \max(I) \) or \( c \leq \max(J) \), then

\[ (*) \quad [I \mid J]X_{rc} - q^{2-\delta(r,1) - \delta(c,J)}X_{rc}[I \mid J] \in L. \]

**Proof.** If \( r \in I \) and \( c \in J \), then \([I \mid J]\) commutes with \( X_{rc} \) (cf. Lemma 5.2(a)), and so \((*)\) holds.

Next, suppose that \( r \in I \) and \( c \notin J \). If \( j \in J \) and \( j > c \), then \( J \cup \{c\} \setminus \{j\} < J \), whence \([I \mid J \cup \{c\} \setminus \{j\}] \in L \) by definition of \( L \). It follows from Lemma 5.2(b2) that \([I \mid J]X_{rc} - qX_{rc}[I \mid J] \in L \). Thus \((*)\) holds in this case. The case where \( r \notin I \) and \( c \in J \) is proved similarly, using Lemma 5.2(c2).

Finally, suppose that \( r \notin I \) and \( c \notin J \), and note that either \( r < \max(I) \) or \( c < \max(J) \).

(This part of the proof is similar to that of Lemma 5.7.) There is no loss of generality in assuming that

\[
I \cup \{r\} = J \cup \{c\} = \{1, \ldots, n\},
\]

whence either \( r < n \) or \( c < n \). In the notation of [16, (4.3)], \([I \mid J] = A(rc)\). Note that \( \{1, \ldots, n\} \setminus \{i\} < I \) when \( i > r \), while \( \{1, \ldots, n\} \setminus \{j\} < J \) when \( j > c \). Hence, \( A(i,j) \in L \) whenever either \( i > r \) or \( j > c \). Set \( D_q = [1 \cdots n \mid 1 \cdots n] \). The basic q-Laplace relations (Corollary 5.5) imply that \( D_q \) lies in the ideal generated by all the \( A(n,j) \), and in the ideal generated by all the \( A(i,n) \). Since either \( n > r \) or \( n > c \), it follows that \( D_q \in L \).

We now use the basic q-Laplace relations in the form given in [16, Corollary 4.4.4]. The first two relations yield

\[
(1) \quad \sum_{j=1}^{n} (-q)^{j-r} X_{rj}A(rj) = \sum_{j=1}^{n} (-q)^{r-j} A(rj)X_{rj} = D_q \in L.
\]

Since \( A(rj) \in L \) for \( j > c \), we obtain the following congruences, after multiplying the two sums in (1) by \(-q^{r-c}\) and \(-q^{c-r}\), respectively:

\[
(2) \quad X_{rc}A(rc) \equiv -\sum_{j < c} (-q)^{j-c} X_{rj}A(rj) \quad (\text{mod } L)
\]

\[
(3) \quad A(rc)X_{rc} \equiv -\sum_{j < c} (-q)^{c-j} A(rj)X_{rj} \quad (\text{mod } L).
\]

For any \( j \), the third relation of [16, Lemma 5.1.2] implies that

\[
(4) \quad X_{rj}A(rj) \equiv A(rj)X_{rj} + (1-q^{-2}) \sum_{l<j} (-q)^{j-l} A(rl)X_{rl} \quad (\text{mod } L),
\]

since \( X_{s,j}A(s,j) \in L \) for \( s > r \). Substituting (4) into (2) for all \( j < c \), we obtain

\[
X_{rc}A(rc) \equiv -\sum_{j < c} (-q)^{j-c} A(rj)X_{rj}
\]

\[
- (1-q^{-2}) \sum_{j < c} \sum_{l < j} (-q)^{2j-l-c} A(rl)X_{rl} \quad (\text{mod } L)
\]

\[
= -\sum_{l < c} \left[ (-q)^{l-c} + (1-q^{-2}) \sum_{l < j < c} (-q)^{2j-l-c} \right] A(rl)X_{rl}.
\]
The expression in square brackets can be simplified as follows:

\[
(6) \quad (-q)^{l-c} + (1 - q^{-2}) \sum_{l<j<c} (-q)^{2j-l-c} = (-q)^{l-c} \left[ 1 + (1 - q^{-2}) \sum_{0<m<c-l} (-q)^{2m} \right] = (-q)^{c-l-2}.
\]

Substituting (6) into (5) and replacing \( l \) by \( j \), we obtain

\[
(7) \quad X_{rc}A(rc) = - \sum_{j<c} (-q)^{c-j-2} A(rj)X_{rj} \pmod{L}.
\]

Finally, combining (3) with (7), we conclude that

\[
[I \mid J]X_{rc} = A(rc)X_{rc} \equiv q^2 X_{rc}A(rc) = q^2 X_{rc}[I \mid J] \pmod{L},
\]
as desired. \( \square \)

1.4. Corollary. Let \( I, J \subseteq \{1, \ldots, n\} \) with \( |I| = |J| \), and set

\[
L = \langle \{ I' \mid J' \} \mid |I'| = |I|, \text{ and } I' < I \text{ or } J' < J \rangle.
\]

Then the coset \( d = [I \mid J] + L \) generates a denominator set in \( A/L \).

Proof. Set \( B = A/L \), and set \( x_{ij} = X_{ij} + L \) for all \( i,j \). Lemma 1.3 says that

\[
(1) \quad dx_{ij} = q^{2-\delta(i,I)-\delta(j,J)}x_{ij}d
\]
whenever \( i \leq \max(I) \) or \( j \leq \max(J) \). Hence, in this case we have \( d^r x_{ij} \in Bd^r \) for all \( r \geq 0 \).

When \( i > \max(I) \) and \( j > \max(J) \), Lemma 5.7 says that

\[
(2) \quad dx_{ij} - q^2 x_{ij}d = e := (1 - q^2)[I \cup \{i\} \mid J \cup \{j\}] + L.
\]

Observe that \( d \) and \( e \) commute. Hence, it follows from (2) by an easy induction that

\[
(3) \quad d^r x_{ij} \in q^{2r} x_{ij}d^r + (q^{2r-2} + \cdots + q^2 + 1)e d^{r-1}
\]
\[
= [q^{2r} x_{ij}d + (q^{2r-2} + \cdots + q^2 + 1)e] d^{r-1}
\]

for all \( r > 0 \). Combining (1) and (3), we see that

\[
(4) \quad d^r x_{ij} \in Bd^r \quad (i,j = 1, \ldots, n; \ r = 1,2,\ldots).
\]

Since \( B \) is spanned by products of the \( x_{ij} \), it follows from (4) that \( D := \{d^r \mid r \geq 0\} \) is a left Ore set in \( B \). Similarly, \( D \) is right Ore, and therefore \( D \) is a denominator set because \( B \) is noetherian. \( \square \)
1.5. Proposition. Let \((r, c) \in RC\), and set

\[ L = \langle [I | J] \mid |I| = l \leq t, \text{ and } I < \{r_1, \ldots, r_t\} \text{ or } J < \{c_1, \ldots, c_l\} \rangle.\]

Then the image of \(D_{r,c}\) in \(A/L\) is a denominator set. Consequently, \(\tilde{D}_{r,c}\) is a denominator set in \(A/K_{r,c}\).

Proof. For each \(l = 1, \ldots, t\), Corollary 1.4 shows that \(d_i^{r,c} + L\) generates a denominator set in \(A/L\). The proposition follows. \(\Box\)

1.6. In view of Proposition 1.5, we can form Ore localizations

\[ A_{r,c} = (A/K_{r,c})[\tilde{D}_{r,c}^{-1}] \]

for \((r, c) \in RC\). (It will follow from Lemma 2.5 that \(A_{r,c} \neq 0\).) The localization maps \(A \to A/K_{r,c} \to A_{r,c}\) induce Zariski homeomorphisms

\[ \text{spec}_{r,c} A \longrightarrow \text{spec} A_{r,c}. \]

Lemma 1.7. Let \(I, J \subseteq \{1, \ldots, n\}\) with \(|I| = |J|\), and let \(P\) be an ideal of \(A\).

(a) Fix \(J_1 \subseteq J\). If \([I_1 | J_1] \in P\) for all \(I_1 \subseteq I\) with \(|I_1| = |J_1|\), then \([I | J] \in P\).

(b) Fix \(I_1 \subseteq I\). If \([I_1 | J_1] \in P\) for all \(J_1 \subseteq J\) with \(|J_1| = |I_1|\), then \([I | J] \in P\).

Proof. By symmetry (see (5.1)(b)), we need only prove (a). Set \(J_2 = J \setminus J_1\). Then Lemma 5.4(a) provides a relation of the form

\[ \sum_{I_1 \cup I_2 = I} \pm q^* [I_1 | J_1][I_2 | J_2] = \pm q^* [I | J], \]

where \(\pm\) is an unspecified sign and \(q^*\) stands for an unspecified power of \(q\). Since all the \([I_1 | J_1] \in P\) by assumption, it follows that \([I | J] \in P\). \(\Box\)

Lemma 1.8. Let \(I_1, I_2, J_1, J_2 \subseteq \{1, \ldots, n\}\) with \(|I_1| = |J_1|\) and \(|I_2| = |J_2|\), and let \(P\) be a completely prime ideal of \(A\). Assume that one of the following conditions (a), (b), or (c) holds:

(a) (1) \(|I_1 \cap I_2| > |J_1 \cap J_2|\), and
(2) \([I_1 | J'] \in P\) whenever \(J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2\) with \(|J'| = |I_1|\) but \(J' \neq J_1\).

(b) (1) \(|I_1 \cap I_2| < |J_1 \cap J_2|\), and
(2) \([I' | J_1] \in P\) whenever \(I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2\) with \(|I'| = |I_1|\) but \(I' \neq I_1\).

(c) (1) \(|I_1 \cap I_2| = |J_1 \cap J_2|\) and \([I_1 \cup I_2 | J_1 \cup J_2] \in P\), and
(2) Either (a)(2) or (b)(2) holds.
Then either \([I_1 \mid J_1] \in P\) or \([I_2 \mid J_2] \in P\).

Proof. By symmetry, it suffices to prove cases (a) and (c).

(a) Set \(V = J_1 \cup J_2 = J \cup L\) where \(J = J_1 \cap J_2\) and \(L = V \setminus J\). Since \(|I_1 \cap I_2| > |J|\), we have \(|I_1 \cup I_2| < |V|\), and so there exists \(U \subseteq \{1, \ldots, n\}\) such that \(I_1 \cup I_2 \subseteq U\) and \(|U| = |V|\). Also, we have \(|I_1| + |I_2| = |J| + |J_2| = 2|J| + |L|\). Thus, Lemma 5.6(b) provides a relation of the form

\[
\sum_{L=U \cup L'} \pm q^* [I_1 \mid J \cup L'][I_2 \mid J \cup L''] = 0.
\]

Now for each term in this sum, \(J_1 \cap J_2 \subseteq J \cup L' \subseteq J_1 \cup J_2\) with \(|J \cup L'| = |J_1|\). If \(L' \neq J_1 \setminus J\), then \(J \cup L' \neq J_1\), whence \([I_1 \mid J \cup L'] \in P\) by hypothesis. Therefore the remaining term in (i) must lie in \(P\). This is the term with \(L' = J_1\setminus J\), whence \(L'' = J_2 \setminus J\), and so \(J \cup L' = J_1\) and \(J \cup L'' = J_2\). Thus

\[
\pm q^* [I_1 \mid J_1][I_2 \mid J_2] \in P.
\]

Since \(P\) is completely prime, either \([I_1 \mid J_1] \in P\) or \([I_2 \mid J_2] \in P\).

(c) By symmetry, we may assume that (a)(2) holds. Again, set \(V = J_1 \cup J_2 = J \cup L\) where \(J = J_1 \cap J_2\) and \(L = V \setminus J\). Set \(U = I_1 \cup I_2\), and observe that \(|I_1| + |I_2| = 2|J| + |L|\). This time, Lemma 5.6(b) provides a relation of the form

\[
\sum_{L=U \cup L'} \pm q^* [I_1 \mid J \cup L'][I_2 \mid J \cup L''] = \pm q^* [I_1 \cap I_2 \mid J][U \mid V].
\]

Since \([U \mid V] \in P\) by hypothesis, the right hand side of (i) lies in \(P\). Therefore we can proceed as in the proof above. \(\square\)

1.9. Theorem. Assume that \(q \neq \pm 1\). Let \(P\) be a completely prime ideal of \(A\), and let \(t \leq n\) be maximal such that \(P\) does not contain all \(t \times t\) quantum minors. Choose

\[
\tilde{I} = \{r_1 < \cdots < r_t\} \subseteq \{1, \ldots, n\} \quad \text{and} \quad \tilde{J} = \{c_1 < \cdots < c_t\} \subseteq \{1, \ldots, n\}
\]

with \(\tilde{I}\) minimal such that some \([\tilde{I} \mid *] \notin P\), and \(\tilde{J}\) minimal such that some \([* \mid \tilde{J}] \notin P\). Then

(a) \([r_1 \cdots r_s \mid c_1 \cdots c_s] \notin P\) for \(s = 1, \ldots, t\). In particular, \([\tilde{I} \mid \tilde{J}] \notin P\).

(b) \([I \mid J] \in P\) whenever \(|I| = s \leq t\) and either \(I \notin \{r_1, \ldots, r_s\}\) or \(J \notin \{c_1, \ldots, c_s\}\). In particular, it follows that \(\tilde{I}\) and \(\tilde{J}\) are unique.

Proof. Since the theorem holds trivially when \(t = 0\), we may assume that \(t > 0\). By assumption, there exists \(J_0\) such that \([\tilde{I} \mid J_0] \notin P\). We first claim that

1. \([I \mid J] \in P\) whenever \(|I| = s \leq t\) and \(I \notin \{r_1, \ldots, r_s\}\).

Suppose not, so that some \([I \mid J] \notin P\) where \(|I| = s \leq t\) and \(I \notin \{r_1, \ldots, r_s\}\). We may assume that \(s\) is minimal for this, and that with \(I\) fixed, \(|J \cap J_0|\) is maximal. Note from the minimality of \(\tilde{I}\) that \(s < t\).
Write \( I = \{i_1, \ldots, i_s\} \). There is some \( b \leq s \) such that \( i_l = r_l \) for \( l < b \) while \( i_b < r_b \), whence \( \{i_1, \ldots, i_b\} < \{r_1, \ldots, r_b\} \). Since \( [I \mid J] \notin P \), Lemma 1.7(b) implies that some \( [i_1 \cdots i_b \mid \ast] \notin P \). Hence, the minimality of \( s \) implies that \( b = s \). Thus, we have \( i_l = r_l \) for \( l < s \) while \( i_s < r_s \). In particular, \( r_{s-1} = i_{s-1} < i_s < r_s \).

Assume that \( |J \cap J_0| \geq s - 1 \). In this case, we will apply Lemma 1.8 with

\[
I_1 = \tilde{I} \quad I_2 = I \quad J_1 = J_0 \quad J_2 = J.
\]

Note that \( |I_1 \cap I_2| = s - 1 \) and \( [I_1 \mid J_1], [I_2 \mid J_2] \notin P \). When \( I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2 \) with \( |I'| = t \) and \( I' \neq I_1 \), we have \( I' = \tilde{I} \cup \{i_s\} \setminus \{r_l\} \) for some \( l \geq s \). Then since \( r_{s-1} < i_s < r_s \leq r_l \), we have \( I' < \tilde{I} \), and so \( [I' \mid J_1] \in P \) by the minimality of \( \tilde{I} \). Further, if \( |J_1 \cap J_2| = s - 1 \), then \( [I_1 \cup I_2 \mid J_1 \cup J_2] \in P \) because \( P \) contains all \( (t+1) \times (t+1) \) quantum minors. Therefore by Lemma 1.8(b), if \( |J \cap J_0| \geq s \), or by Lemma 1.8(c), if \( |J \cap J_0| = s - 1 \), we have either \( [I_1 \mid J_1] \in P \) or \( [I_2 \mid J_2] \in P \), giving us a contradiction. Therefore \( |J \cap J_0| < s - 1 \).

Next, we will apply Lemma 1.8 with the roles of \( I_1, I_2 \) and \( J_1, J_2 \) reversed, that is, with

\[
I_1 = I \quad I_2 = \tilde{I} \quad J_1 = J \quad J_2 = J_0.
\]

When \( J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2 \) with \( |J'| = s \) and \( J' \neq J_1 \), we must have \( |J' \cap J_0| = |J' \cap J_2| > |J \cap J_0| \). By the maximality of \( |J \cap J_0| \), we obtain \( [I \mid J'] \in P \) in this case. But then Lemma 1.8(a) leads to the same contradiction.

Therefore (1) holds. By symmetry, we must also have

(2) \( [I \mid J] \in P \) whenever \( |I| = s \leq t \) and \( J < \{c_1, \ldots, c_s\} \).

We now proceed by induction on \( s = 1, \ldots, t \) to verify the following properties:

- \((P_s)\) \( [r_1 \cdots r_s \mid c_1 \cdots c_s] \notin P \);
- \((Q_s)\) \( [I \mid J] \in P \) whenever \( |I| = s \) and either \( I \nsubseteq \{r_1, \ldots, r_s\} \) or \( J \nsubseteq \{c_1, \ldots, c_s\} \).

The theorem will then be established.

To start, note that we cannot have all \( [r_1 \mid \ast] \in P \) or all \( [\ast \mid c_1] \in P \), by Lemma 1.7. Choose \( i, j \) such that \( [r_1 \mid j] \notin P \). If \( j = c_1 \) or \( i = r_1 \), then \( [r_1 \mid c_1] \notin P \). Otherwise, in view of (1) and (2) we must have \( j > c_1 \) and \( i > r_1 \). Hence, because of the assumption that \( q \neq \pm 1 \), we have

\[ [r_1 \mid c_1][i \mid j] - [i \mid j][r_1 \mid c_1] = (q - q^{-1})[r_1 \mid j][i \mid c_1] \notin P, \]

which implies that \( [r_1 \mid c_1] \notin P \). Therefore \((P_1)\) holds. Property \((Q_1)\) is immediate from (1) and (2).

Now let \( 1 < s \leq t \) and assume that \((P_a)\) and \((Q_a)\) hold for all \( a < s \). By Lemma 1.7, there exist \( j_1 < \cdots < j_s \) such that

\[ [r_1 \cdots r_s \mid j_1 \cdots j_s] \notin P, \]
and we may assume that \( j_1, \ldots, j_s \) is minimal for this. Likewise, there exist \( i_1 < \cdots < i_s \) such that \([i_1 \cdots i_s | c_1 \cdots c_s] \notin P\) and such that \( \{i_1, \ldots, i_s\} \) is minimal for this. We cannot have \( \{i_1, \ldots, i_{s-1}\} \notin \{r_1, \ldots, r_{s-1}\} \), since then \((Q_{s-1})\) would imply that all \([i_1 \cdots i_{s-1} | *] \in P\), whence Lemma 1.7(b) would imply that all \([i_1 \cdots i_s | *] \in P\). Therefore \( \{i_1, \ldots, i_{s-1}\} \geq \{r_1, \ldots, r_{s-1}\} \), and similarly \( \{j_1, \ldots, j_{s-1}\} \geq \{c_1, \ldots, c_{s-1}\} \).

Suppose there exists \( b < s \) such that \( j_i = c_i \) for \( l < b \) while \( j_b > c_b \). We will apply Lemma 1.8 with

\[
I_1 = \{r_1, \ldots, r_s\} \quad J_1 = \{j_1, \ldots, j_s\} \quad I_2 = \{r_1, \ldots, r_b\} \quad J_2 = \{c_1, \ldots, c_b\}.
\]

Observe that \(|I_1 \cap I_2| = b > |J_1 \cap J_2|\). If \( J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2 \) with \(|J'| = s\) and \( J' \neq J_1 \), then \( J' = J_1 \cup \{c_b\} \setminus \{j_d\} \) for some \( d \geq b \). In this case, \( J' < \{j_1, \ldots, j_s\} \), whence \([I_1 | J'] \notin P\) by the minimality of \( \{j_1, \ldots, j_s\} \). Therefore Lemma 1.8(a) implies that either \([I_1 | J_1] \notin P\) or \([I_2 | J_2] \notin P\). But \([I_1 | J_1] \notin P\) by choice of \( J_1 \), and \([I_2 | J_2] \notin P\) by \((P_b)\), so we have a contradiction. Therefore \( j_1 = c_1 \) for all \( l < s \). Similarly, \( j_l = r_l \) for all \( l < s \).

If \( j_s = c_s \), then \((P_s)\) holds. If \( j_s < c_s \), then \( \{j_1, \ldots, j_s\} < \{c_1, \ldots, c_s\} \), which would imply \([r_1 \cdots r_s | j_1 \cdots j_s] \in P\) by \((2)\), contradicting our assumptions. Therefore we may assume that \( j_s > c_s \). Likewise, we may assume that \( i_s > r_s \).

Set \( U = \{r_1, \ldots, r_s, i_s\} \) and \( V = \{c_1, \ldots, c_s, j_s\} \). By Lemma 5.3(d), we have

\[
[i_1 \cdots i_s | j_1 \cdots j_s][r_1 \cdots r_s | c_1 \cdots c_s] - [r_1 \cdots r_s | c_1 \cdots c_s][i_1 \cdots i_s | j_1 \cdots j_s] = (q^{-1} - q)[r_1 \cdots r_s | j_1 \cdots j_s][i_1 \cdots i_s | c_1 \cdots c_s].
\]

Since neither of the factors \([r_1 \cdots r_s | j_1 \cdots j_s] \) and \([i_1 \cdots i_s | c_1 \cdots c_s] \) is in \( P \), it follows that \([r_1 \cdots r_s | c_1 \cdots c_s] \) cannot be in \( P \). This establishes property \((P_s)\).

Finally, suppose that \((Q_s)\) fails. By symmetry, we may assume that \([I | J] \notin P\) for some \( I, J \) with \(|I| = s\) and \( I \notin \{r_1, \ldots, r_s\} \). We may also assume that \( I \) is minimal for this, and that with \( I \) fixed, \( J \) is minimal.

Write \( I = \{i_1 < \cdots < i_s\} \) and \( J = \{j_1 < \cdots < j_s\} \). If \( \{i_1, \ldots, i_{s-1}\} \notin \{r_1, \ldots, r_{s-1}\} \), then by \((Q_{s-1})\) we would have all \([i_1 \cdots i_{s-1} | *] \in P\), whence Lemma 1.7(b) would imply that all \([I | *] \in P\), contradicting our choice of \( I \). Thus \( \{i_1, \ldots, i_{s-1}\} \geq \{r_1, \ldots, r_{s-1}\} \), and similarly \( \{j_1, \ldots, j_{s-1}\} \geq \{c_1, \ldots, c_{s-1}\} \). Since \( I \neq \{r_1, \ldots, r_s\} \), we must also have \( i_s < r_s \).

Note that \( r_{s-1} \leq i_{s-1} < i_s < r_s \), and so \( i_s \notin \{r_1, \ldots, r_{s-1}\} \). Further, \( \{r_1, \ldots, r_{s-1}, i_s\} < \{r_1, \ldots, r_s\} \), and so all \([r_1 \cdots r_{s-1} i_s | *] \in P\) by \((1)\), whence \( J \neq \{r_1, \ldots, r_{s-1}, i_s\} \). Therefore there is some \( b < s \) such that \( j_l = r_l \) for \( l < b \) while \( j_b > r_b \).

Suppose there exists \( d < b \) such that \( j_m = c_m \) for \( m < d \) while \( j_d > c_d \). We will apply Lemma 1.8 with

\[
I_1 = I = \{i_1, \ldots, i_s\} \quad J_1 = J = \{j_1, \ldots, j_s\} \quad I_2 = \{r_1, \ldots, r_d\} \quad J_2 = \{c_1, \ldots, c_d\}.
\]

Note that \(|I_1 \cap I_2| = d-1 = |J_1 \cap J_2|\), and that \(|I_1 \cap I_2| = |J_1 \cap J_2|\) only when \( d = b \). Since \( r_{b-1} < r_b < i_b \), we have \( \{r_1, \ldots, r_b, i_b, \ldots, i_{s-1}\} < I \), and so all \([r_1 \cdots r_b i_b \cdots i_{s-1} | *] \in P\).
by the minimality of $I$. Then Lemma 1.7(b) implies that all $[r_1 \cdots r_b i_b \cdots i_s | s] \in P$. In particular, when $d = b$ we find that $[I_1 \cup I_2 | J_1 \cup J_2] \subseteq P$.

If $J_1 \cap J_2 \subseteq J' \subseteq J_1 \cup J_2$ with $|J'| = s$ and $J' \neq J_1$, then $J' = J \cup \{c_d\} \setminus \{j_p\}$ for some $p \geq d$. In this case, $J' < J$, and so $[I_1 | J'] \subseteq P$ by the minimality of $J$. Hence, case (a) of Lemma 1.8 (if $d > b$) or case (c) (if $d = b$) implies that either $[I_1 | J_1] \subseteq P$ or $[I_2 | J_2] \subseteq P$. But $[I_1 | J_1] \not\subseteq P$ by assumption, and $[I_2 | J_2] \not\subseteq P$ by $(P_d)$, so we have a contradiction. Therefore $j_m = c_m$ for all $m \leq b$.

We will conclude by applying Lemma 1.8 with

$$I_1 = I = \{i_1, \ldots, i_s\} \quad I_2 = \{r_1, \ldots, r_b\}$$

$$J_1 = J = \{j_1, \ldots, j_s\} \quad J_2 = \{c_1, \ldots, c_b\}.$$  

Note that $i_{b-1} = r_{b-1} < r_b < i_b$ implies $r_b \not\subseteq I$, and so $[I_1 \cap I_2] < |J_1 \cap J_2]$. If $I_1 \cap I_2 \subseteq I' \subseteq I_1 \cup I_2$ with $|I'| = s$ and $I' \neq I_1$, then $I' = I \cup \{r_b\} \setminus \{i_p\}$ for some $p \geq b$. In this case, $I' < I$, whence $[I' | J_1] \subseteq P$ by the minimality of $I$. Thus Lemma 1.8(b) implies that either $[I_1 | J_1] \subseteq P$ or $[I_2 | J_2] \subseteq P$, and again we have reached a contradiction.

Therefore $(Q_s)$ must hold, which establishes our induction step. □

1.10. Corollary. Assume that $q \neq \pm1$. Given any completely prime ideal $P \subseteq A$, there is a unique pair $(r, c) \in RC$ such that $K_{r,c} \subseteq P$ and $P \cap D_{r,c} = \emptyset$.

Thus, if $q$ is not a root of unity,

$$\text{spec } A = \bigsqcup_{(r, c) \in RC} \text{spec}_{r,c} A.$$  

Proof. Let $t \leq n$ be maximal such that $P$ does not contain all $t \times t$ quantum minors, let $\{r_1 < \cdots < r_t\}$ and $\{c_1 < \cdots < c_t\}$ be as in Theorem 1.9, and set $r = (r_1, \ldots, r_t)$ and $c = (c_1, \ldots, c_t)$. The theorem implies that $K_{r,c} \subseteq P$ and that $d^r_s c \not\subseteq P$ for $s = 1, \ldots, t$. Since $P$ is completely prime, it follows that $P \cap D_{r,c} = \emptyset$.

Now suppose that we also have $(r', c') \in RC$, for some $t'$ such that $K_{r',c'} \subseteq P$ and $P \cap D_{r',c'} = \emptyset$. Then $P$ contains all $(t' + 1) \times (t' + 1)$ quantum minors but not all $t' \times t'$ quantum minors, whence $t' = t$. Moreover, we have $d^r_c \not\subseteq K_{r',c'}$ and $d^{r',c'} \not\subseteq K_{r,c}$. The first relation implies that $r \geq r'$ and $c \geq c'$, and the second relation yields the reverse inequalities. (Here we have transferred the relation $\leq$ in (1.1) from index sets to sequences in the obvious manner.) Therefore $r' = r$ and $c' = c$. □

2. Structure of $A_{r,c}$

The purpose of this section is to develop a structure theorem for the localizations $A_{r,c}$. We introduce localized factor algebras $R^+_r$ and $R^-_c$ of $A$ (patterned after quantized coordinate rings of unipotent groups of triangular matrices), together with subalgebras $B^+_r \subseteq R^+_r$ and $B^-_c \subseteq R^-_c$ (patterned after quantized coordinate rings of unipotent groups of triangular matrices), and we show that $A_{r,c}$ is isomorphic to an algebra $B_{r,c}$ trapped between $B^+_r \otimes B^-_c$ and $R^+_r \otimes R^-_c$. More precisely, we prove that $B_{r,c}$ is a skew-Laurent extension of $B^+_r \otimes B^-_c$ (this is a key ingredient in establishing that $A_{r,c} \cong B_{r,c}$), and that $R^+_r \otimes R^-_c$ is a skew-Laurent extension of $B_{r,c}$.  


2.1. Fix $t \in \{0, 1, \ldots, n\}$ and $(r, c) \in RC_t$ throughout the section. Set
\[ R^+_{r,0} = A/\langle X_{ij} \mid j > t \text{ or } i < r_j \rangle \quad \text{and} \quad R^-_{c,0} = A/\langle X_{ij} \mid i > t \text{ or } j < c_i \rangle. \]

Write $Y_{ij}$ and $Z_{ij}$ for the images of $X_{ij}$ in $R^+_{r,0}$ and $R^-_{c,0}$, respectively. Note that these algebras are iterated skew polynomial extensions of $k$, hence noetherian domains, the natural indeterminates for these iterated skew polynomial structures being those $Y_{ij}$ and $Z_{ij}$ which are nonzero. These indeterminates, when recorded within an $n \times n$ matrix, display ‘stairstep’ patterns – for example, if $n = 4$ and $r = (1, 2, 4)$, the $Y_{ij}$ may be displayed as follows:
\[
\begin{bmatrix}
Y_{11} & 0 & 0 & 0 \\
Y_{21} & Y_{22} & 0 & 0 \\
Y_{31} & Y_{32} & 0 & 0 \\
Y_{41} & Y_{42} & Y_{43} & 0
\end{bmatrix}.
\]

Since all the information is recorded in the placement of the zero and nonzero positions within this matrix, a convenient abbreviation for this example is to write
\[ R^+_{(1,2,4),0} = k \begin{bmatrix}
+ & 0 & 0 & 0 \\
+ & + & 0 & 0 \\
+ & + & + & 0 \\
+ & + & + & +
\end{bmatrix}. \]

Observe that the $Y_{r,s}$ are regular normal elements in $R^+_{r,0}$, and that the $Z_{sc,s}$ are regular normal elements in $R^-_{c,0}$. More precisely,
\[
Y_{r,s}Y_{ij} = \begin{cases} 
q^{-1}Y_{ij}Y_{r,s} & (i = r, j \neq s) \\
qY_{ij}Y_{r,s} & (i \neq r, j = s) \\
Y_{ij}Y_{r,s} & (i \neq r, j \neq s)
\end{cases}
\]
\[
Z_{sc,s}Z_{lm} = \begin{cases} 
qZ_{lm}Z_{sc,s} & (l = s, m \neq c_s) \\
q^{-1}Z_{lm}Z_{sc,s} & (l \neq s, m = c_s) \\
Z_{lm}Z_{sc,s} & (l \neq s, m \neq c_s).
\end{cases}
\]

(For instance, the first relation above holds when $j > s$ because $Y_{ij} = 0$ in that case. To verify the third relation, observe that $Y_{r,j} = 0$ if $j > s$, while $Y_{is} = 0$ if $i < r_s$.) In particular, the $Y_{r,s}$ commute with each other, and the $Z_{sc,s}$ commute with each other.

Due to the normality of the $Y_{r,s}$ and the $Z_{sc,s}$, we can form Ore localizations
\[ R^+_r = R^+_{r,0}[Y_{r_1,1}, Y_{r_2,2}, \ldots, Y_{r_t,t}] \quad \text{and} \quad R^-_c = R^-_{c,0}[Z_{1c_1}^{-1}, Z_{2c_2}^{-1}, \ldots, Z_{tc_t}^{-1}]. \]

These algebras are noetherian domains, and they may be viewed as quantized coordinate rings of certain locally closed subvarieties of $M_n(k)$. Extending the abbreviated description given for the example above, we display the following abbreviation for $R^+_r$ in that case:
\[ R^+_{(1,2,4)} = k \begin{bmatrix}
+ & 0 & 0 & 0 \\
+ & + & 0 & 0 \\
+ & + & + & 0 \\
+ & + & + & +
\end{bmatrix}. \]

Observe that the standard $\mathbb{Z}^{2n}$-grading on $A$ induces $\mathbb{Z}^{2n}$-gradings on $R^+_r$ and $R^-_c$, which we also refer to as standard.
2.2. Let $\pi^+_{r,0} : A \to R^+_{r,0}$ and $\pi^-_{c,0} : A \to R^-_{c,0}$ be the quotient maps, and define

$$\beta_{r,c} : A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi^+_{r,0} \otimes \pi^-_{c,0}} R^+_{r,0} \otimes R^-_{c,0} \subseteq R^+_r \otimes R^-_c.$$ 

Observe that

$$\beta_{r,c}(X_{ij}) = \sum_{l \leq t, r_i \leq t, c_i \leq j} Y_{il} \otimes Z_{lj}$$

for all $i, j$. In particular, $\beta_{r,c}(X_{ij}) = 0$ when $i < r_1$ or $j < c_1$.

2.3. Lemma. $K_{r,c} \subseteq \ker(\beta_{r,c})$.

Remark. We conjecture that $\ker(\beta_{r,c}) = K_{r,c}$.

Proof. Since $X_{ij} \in \ker(\pi^+_{r,0})$ for $j > t$, all $(t + 1) \times (t + 1)$ and larger quantum minors lie in $\ker(\pi^+_{r,0})$. In view of the rule for comultiplication of quantum minors (see (5.1)(d)), it follows that all $(t + 1) \times (t + 1)$ and larger quantum minors lie in $\ker(\beta_{r,c})$.

Consider an index set $I$ with $|I| = l \leq t$ and $I \not\supseteq \{r_1, \ldots, r_t\}$. Write $I = \{i_1 < \cdots < i_l\}$; then $i_m < r_m$ for some $m \leq l$. Hence, $Y_{i_j, j} = 0$ for all $s \leq m$ and $j \geq m$. This implies that $\pi^+_{r,0}[i_1 \cdots i_m \mid M] = 0$ for all $M$ with $|M| = m$, whence $\pi^+_{r,0}[I \mid K] = 0$ for all $K$ with $|K| = l$ (cf. Lemmas 1.7 or 5.4). Therefore $\beta_{r,c}[I \mid J] = 0$ for all $J$ with $|J| = l$.

Likewise, $\beta_{r,c}[I \mid J] = 0$ whenever $|I| = l \leq t$ and $J \not\supseteq \{c_1, \ldots, c_l\}$. \qed

2.4. Let $B_{r,c}$ denote the $k$-subalgebra of $R^+_r \otimes R^-_c$ generated by the set

$$\{Y_{il} \otimes Z_{lj} \mid l \leq t, \ i \geq r_i, \ j \geq c_j \} \cup \{Y^{-1}_{r_l} \otimes Z^{-1}_{l_c} \mid l \leq t\}.$$ 

We may also express $B_{r,c}$ as the subalgebra of $R^+_r \otimes R^-_c$ generated by

$$\{Y_{il}Y^{-1}_{r_l} \otimes 1 \mid l \leq t, \ i > r_l \} \cup \{1 \otimes Z_{lj}Z^{-1}_{l_c} \mid l \leq t, \ j > c_j \} \cup \{(Y_{r_l} \otimes Z_{l_c})^{-1} \mid l \leq t\}.$$ 

Note that $\beta_{r,c}(X_{ij}) \in B_{r,c}$ for all $i, j$, so that $\beta_{r,c}(A) \subseteq B_{r,c}$.

Let $l \leq t$. Since $Y_{r_i,j} = 0$ for $s \leq l$ and $j > s$, we have

$$\pi^+_{r,0}[r_1 \cdots r_l \mid K] = \begin{cases} Y_{r_1}Y_{r_2} \cdots Y_{r_l} & (K = \{1, \ldots, l\}) \\ 0 & (K \neq \{1, \ldots, l\}). \end{cases}$$

Similarly, $\pi^-_{c,0}[c_1 \cdots c_l] = Z_{1c_1}Z_{2c_2} \cdots Z_{lc_l}$, and therefore

$$\beta_{r,c}(d^r \circ c) = (Y_{r_1} \otimes Z_{1c_1})(Y_{r_2} \otimes Z_{2c_2}) \cdots (Y_{r_l} \otimes Z_{lc_l}).$$

In particular, $\beta_{r,c}(d^r \circ c)$ is invertible in $B_{r,c}$. 
2.5. Lemma. The map $\beta_{r,c}$ induces a surjective $k$-algebra homomorphism

$$\tilde{\beta}_{r,c} : A_{r,c} \rightarrow B_{r,c}.$$

Remark. We shall prove later (Theorem 2.11) that $\tilde{\beta}_{r,c}$ is an isomorphism. Note that the lemma already implies that $A_{r,c}$ is nonzero.

Proof. We have $K_{r,c} \subseteq \ker(\beta_{r,c})$ by Lemma 2.3 and $\beta_{r,c}(A) \subseteq B_{r,c}$ by (2.4), and so $\beta_{r,c}$ induces a homomorphism $\beta'_{r,c} : A/K_{r,c} \rightarrow B_{r,c}$. It also follows from (2.4) that $\beta'_{r,c}$ sends the elements of $\tilde{D}_{r,c}$ to units of $B_{r,c}$, and therefore $\beta'_{r,c}$ does induce a homomorphism $\tilde{\beta}_{r,c} : A_{r,c} \rightarrow B_{r,c}$.

Set $E = \tilde{\beta}_{r,c}(A_{r,c})$; we must show that $E = B_{r,c}$. Note that

$$(Y_{rt} \otimes Z_{lt})^{\pm 1} = \beta_{r,c}(d_{t}^{r,c})^{\pm 1} \beta_{r,c}(d_{t-1}^{r,c})^{\pm 1} \in E$$

for $l \leq t$, where $d_{0}^{r,c} = 1$. It remains to show that $Y_{it} \otimes Z_{ij} \in E$ for all $i, l, j$.

As in (2.4), we see that

$$\beta_{r,c}([r_{1} \cdots r_{l-1} | c_{1} \cdots c_{l}]) = (Y_{r_{1}} \otimes Z_{c_{1}}) \cdots (Y_{r_{l-1}} \otimes Z_{l-1,c_{l-1}})(Y_{it} \otimes Z_{lt})$$

for $l \leq t$ and $i \geq r_{l}$, whence $Y_{it} \otimes Z_{lt} \in E$. Similarly, $Y_{it} \otimes Z_{lj} \in E$ for $j \geq c_{l}$, and therefore $Y_{it} \otimes Z_{lj} = q^{1 - h(j,c_{l})}(Y_{r_{l}}^{-1} \otimes Z_{lt}^{-1})(Y_{it} \otimes Z_{lj})(Y_{it} \otimes Z_{lt}) \in E$, as desired. $\square$

2.6. Lemma. $R_{r}^{+} \otimes R_{c}^{-}$ is a skew-Laurent extension of $B_{r,c}$ of the form

$$R_{r}^{+} \otimes R_{c}^{-} = B_{r,c}[1 \otimes Z_{1c_{1}}^{\pm 1}, \ldots, 1 \otimes Z_{tc_{t}}^{\pm 1}; \tau_{1}, \ldots, \tau_{t}]$$

for some $\tau_{1}, \ldots, \tau_{t} \in \{1\}^{2n} \times H$.

Proof. First observe that there exist $\sigma_{1}, \ldots, \sigma_{t} \in H$ such that $Z_{lc_{l}} r = \sigma_{l}(r) Z_{lc_{l}}$ for $l \leq t$ and $r \in R_{c}^{-0}$. This relation extends to $r \in R_{c}^{-}$, and so

$$(1 \otimes Z_{lc_{l}})w = (1 \times \sigma_{l})(w)(1 \otimes Z_{lc_{l}})$$

for $l \leq t$ and $w \in R_{r}^{+} \otimes R_{c}^{-}$. In particular, if $\tau_{l} = (1, \ldots, 1, \sigma_{l})$ then $1 \otimes Z_{lc_{l}}$ is $\tau_{l}$-normal with respect to $B_{r,c}$.

The standard $\mathbb{Z}^{2n}$-gradings on $R_{r}^{+}$ and $R_{c}^{-}$ (cf. (2.1)) induce a $\mathbb{Z}^{4n}$-grading on $R_{r}^{+} \otimes R_{c}^{-}$. With respect to this grading, $B_{r,c}$ is a homogeneous subalgebra of $R_{r}^{+} \otimes R_{c}^{-}$, and its homogeneous components have degrees of the form $(*, b, b, *)$. On the other hand, $1 \otimes Z_{lc_{l}}$ has degree $(0, 0, \epsilon_{l}, *)$, so the monomials

$$(1 \otimes Z_{1c_{1}})^{m_{1}}(1 \otimes Z_{2c_{2}})^{m_{2}} \cdots (1 \otimes Z_{tc_{t}})^{m_{t}}$$
have degrees \((0, 0, m_1\epsilon_1 + \cdots + m_t\epsilon_t, *)\). It follows that these monomials are left (or right) linearly independent over \(B_{r,c}\). Hence, the subalgebra

\[
C = \sum_{m_1, \ldots, m_t \in \mathbb{Z}} B_{r,c}(1 \otimes Z_{1c l_1})^{m_1} (1 \otimes Z_{2c 2})^{m_2} \cdots (1 \otimes Z_{l c l_t})^{m_t}
\]

of \(R^+_r \otimes R^-_c\) is a skew-Laurent extension of \(B_{r,c}\) of the desired form.

It remains to show that \(C = R^+_r \otimes R^-_c\). First note that \(Y_{ii} \otimes 1 = (Y_{ii} \otimes Z_{t c l})(1 \otimes Z_{t c l}^{-1}) \in C\) for \(l \leq t\) and \(i \geq r_l\), and that

\[
Y_{r_l l}^{-1} \otimes 1 = (Y_{r_l l}^{-1} \otimes Z_{t c l}^{-1})(1 \otimes Z_{t c l}) \in C
\]

for \(l \leq t\). On the other hand,

\[
1 \otimes Z_{t j} = (Y_{r_l l} \otimes Z_{t j})(Y_{r_l l}^{-1} \otimes Z_{t c l}^{-1})(1 \otimes Z_{t c l}) \in C
\]

for \(l \leq t\) and \(j \geq c_l\), and \(1 \otimes Z_{t c l}^{-1} \in C\) for \(l \leq t\). Therefore \(C = R^+_r \otimes R^-_c\). □

2.7. Let \(B^+_r\) and \(B^-_c\) denote the subalgebras of \(R^+_r\) and \(R^-_c\) generated by the respective subsets

\[
\{Y_{ii}Y_{r_l l}^{-1} \mid l \leq t, i > r_l\} \quad \text{and} \quad \{Z_{t j}Z_{t c l}^{-1} \mid l \leq t, j > c_l\}.
\]

The algebra \(B^+_r\), for instance, may be viewed as a quantized coordinate ring of the variety

\[
\{(a_{ij}) \in M_n(k) \mid a_{ij} = 0 \text{ when } j > t \text{ or } i < r_j, \text{ and } a_{r j, j} = 1 \text{ for } j \leq t\}.
\]

(The factor algebra \(R^+_r/(Y_{r_1 1} - 1, \ldots, Y_{r_t 1} - 1)\), which one might expect to appear in the above role, is inappropriate because it collapses to \(k[Y_{r_1 1}^{\pm 1}, \ldots, Y_{r_t 1}^{\pm 1}]\) when \(q \neq 1\).)

As noted in (2.4), \(B_{r,c}\) is generated by its subalgebra \(B^+_r \otimes B^-_c\) together with the elements \((Y_{r_l l} \otimes Z_{t c l})^{\pm 1}\) for \(l \leq t\). In fact:

**Lemma.** \(B_{r,c}\) is a skew-Laurent extension of \(B^+_r \otimes B^-_c\) of the form

\[
B_{r,c} = (B^+_r \otimes B^-_c)[(Y_{r_1 1} \otimes Z_{1 c l_1})^{\pm 1}, \ldots, (Y_{r_t 1} \otimes Z_{t c l_t})^{\pm 1}; \eta_1, \ldots, \eta_t]
\]

for some \(\eta_1, \ldots, \eta_t \in H \times H\).

**Proof.** This is proved in the same manner as Lemma 2.6. □

2.8. We use the above structure of \(B_{r,c}\) in constructing a homomorphism \(B_{r,c} \rightarrow A_{r,c}\) which will be the inverse of \(\tilde{\alpha}_{r,c}\). To begin the construction, we will define suitable homomorphisms from \(B^+_r\) and \(B^-_c\) to \(A_{r,c}\) whose images centralize each other. For that purpose, we need to know the defining relations for \(B^+_r\) and \(B^-_c\), as well as certain commutation relations in \(A_{r,c}\).
Set $y_{ij} = Y_{ij}Y_{rj}^{-1}$ for $j \leq t$ and $i > r_j$. In view of the basic commutation relations satisfied by the $Y_{lm}$, it is easily checked that the $y_{ij}$ satisfy the following relations:

$$
\begin{align*}
  y_{ij}y_{im} &= qy_{im}y_{ij} & (j < m) \\
  y_{ij}y_{lj} &= qy_{lj}y_{ij} & (i < l) \\
  y_{ij}y_{lm} &= y_{lm}y_{ij} & (i < l, j > m) \\
  y_{ij}y_{lm} &= \begin{cases} 
  y_{lm}y_{ij} & (i < r_m) \\
  q^{-1}y_{lm}y_{ij} + \tilde{q}y_{lj} & (i = r_m) \\
  y_{lm}y_{ij} + \tilde{q}y_{im}y_{lj} & (i > r_m)
  \end{cases} & (i < l, j < m).
\end{align*}
$$

(1)

**Lemma.** The relations (1) are defining relations for the elements $y_{ij}$ generating the algebra $B_r^+$. 

**Proof.** Let $S$ be the $k$-algebra presented by generators $s_{ij}$ for $j \leq t$ and $i > r_j$ satisfying the analogs of (1). Then there is a $k$-algebra homomorphism $\phi : S \to B_r^+$ such that $\phi(s_{ij}) = y_{ij}$ for all $i, j$.

In $R_r^+$, list the $Y_{ij}$ lexicographically, and observe that the ordered monomials in the $Y_{ij}$ are linearly independent. This remains true for ordered monomials in which we allow the $Y_{rj}$ to have negative exponents. Since the $Y_{rj}$ commute up to scalars with the $Y_{lm}$ (recall (2.1)), it follows that the ordered monomials in the $y_{ij}$ are linearly independent, and so these monomials form a basis for $B_r^+$. On the other hand, there are sufficient commutation relations for the $s_{ij}$ to show that the ordered monomials in the $s_{ij}$ span $S$. Hence, $\phi$ maps a spanning subset of $S$ to a basis for $B_r^+$. Therefore $\phi$ is an isomorphism, and the lemma is proved. □

2.9. **Lemma.** Let $M = [I \cup \{a\} \mid J]$ and $N = [I' \mid J']$ be quantum minors in $A$ with $I \subset I'$ and $J \subset J'$. Assume that $a \notin I'$ and that $b = \max(I') \notin I$.

(a) If $a > b$, then $MN = q^{-1}NM$.

(b) If $a < b$, then

$$
MN - q^{-1}NM \equiv \widehat{q}(-q)^{(I' \cap A) \cap (a, b)}[I \cup \{b\} \mid J][I' \cup \{a\} \setminus \{b\} \mid J']
$$

modulo the ideal $L := \langle [I' \cup \{a\} \setminus \{i'\} \mid J'] \mid i' \in I' \cap (a, b) \rangle$.

**Proof.** (a) Expand $M$ using the $q$-Laplace relation of Corollary 5.5(b2) with $r = a$. Since $(a, n) \cap I = \emptyset$, we get

$$
M = \sum_{j \in J} (-q)^{(j, n) \cap J}[I \mid J \setminus \{j\}]X_{aj}. \tag{1}
$$

Note that all the $[I \mid J \setminus \{j\}]$ commute with $N$. Lemma 5.2(c1) implies that $X_{aj}N = q^{-1}NX_{aj}$ for all $j \in J$, and thus it follows from (1) that $MN = q^{-1}NM$. 

(b) Set \( \alpha = |(a, n) \cap I| \), and again expand \( M \) using Corollary 5.5(b2) with \( r = a \). This time, we get
\[
(2) \quad (-q)^\alpha M = \sum_{j \in J} (-q)^{|(j, n) \cap J|} [I | J \setminus \{j\}] \mathcal{X}_{aj}.
\]
For all \( j \in J \), Lemma 5.2(c2) implies that
\[
NX_{aj} - qX_{aj}N = \hat{q} \sum_{i' \in I'} (-q)^{|I' \cap [a, i']|} X_{i'j}[I' \cup \{a\} \setminus \{i'\} | J'],
\]
whence
\[
(3) \quad X_{aj}N - q^{-1}NX_{aj} \equiv \hat{q}(-q)^\beta X_{bj}[I' \cup \{a\} \setminus \{b\} | J'] \quad (\text{mod } L),
\]
where \( \beta = |I' \cap [a, b]| - 1 = |I' \cap (a, b)| \). Note that \( \beta - \alpha = |(I' \setminus I) \cap (a, b)| \).

Combining (2) and (3), we obtain
\[
(4) \quad MN \equiv \sum_{j \in J} (-q)^{|(j, n) \cap J| - \alpha} [I | J \setminus \{j\}] \left( q^{-1}NX_{aj} + \hat{q}(-q)^\beta X_{bj}[I' \cup \{a\} \setminus \{b\} | J'] \right) \quad (\text{mod } L).
\]
modulo \( L \). Since all the \([I | J \setminus \{j\}]\) commute with \( N \), it follows from (2) that
\[
(5) \quad \sum_{j \in J} (-q)^{|(j, n) \cap J| - \alpha} [I | J \setminus \{j\}] q^{-1}NX_{aj} = q^{-1}NM.
\]
Further, an application of Corollary 5.5(b2) with \( r = b \) yields
\[
(6) \quad \sum_{j \in J} (-q)^{|(j, n) \cap J|} [I | J \setminus \{j\}] X_{bj} = [I \cup \{b\} | J].
\]
Combining (4), (5), and (6), we complete the proof. \( \square \)

2.10. Corollary. Let \( M = [I | J \cup \{a\}] \) and \( N = [I' | J'] \) be quantum minors in \( A \) with \( I \subseteq I' \) and \( J \subset J' \). Assume that \( a \notin J' \) and that \( b = \max(J') \notin J \).

(a) If \( a > b \), then \( MN = q^{-1}NM \).

(b) If \( a < b \), then
\[
MN - q^{-1}NM \equiv \hat{q}(-q)^{|J \setminus \{a\} \cap (a, b)|} [I | J \cup \{b\}] [I' | J' \cup \{a\} \setminus \{b\}] \quad (\text{mod } L)
\]
modulo the ideal \( L := \left\langle [I' \cup \{a\} \setminus \{j'\}] \mid j' \in J' \cap (a, b) \right\rangle \).

Proof. This follows from Lemma 2.9 by symmetry. \( \square \)
2.11. Theorem. The map $\tilde{\beta}_{r,c} : A_{r,c} \rightarrow B_{r,c}$ is an isomorphism.

Proof. Recall the notation $d_i^{r,c} = d_i^{r,c} + K_{r,c}$ from (1.2). Similarly, we shall use tildes to denote other cosets in $A/K_{r,c}$. To abbreviate the relation of congruence modulo $K_{r,c}$, we adopt the notation $\equiv_{r,c}$. We shall use the same symbols for elements of $A/K_{r,c}$ and their images in the localization $A_{r,c}$, which does not cause problems as long as we only transfer equations from $A/K_{r,c}$ to $A_{r,c}$ and not in the reverse direction.

We proceed to construct, in several steps, a $k$-algebra homomorphism $\phi : B_{r,c} \rightarrow A_{r,c}$ that will be an inverse for $\beta_{r,c}$. The construction of $\phi$ is based on the skew-Laurent structure of $B_{r,c}$ given in Lemma 2.7. The first ingredient will be a homomorphism from $B_{r}^+$ to $A_{r,c}$. To describe it, set

$$u_{ij} = [r_1 \cdots r_{j-1} i | c_1 \cdots c_j] \in A \quad (j \leq t, \ i \geq r_j)$$

$$v_{ij} = \tilde{u}_{ij}(d_j^{r,c})^{-1} \in A_{r,c} \quad (j \leq t, \ i > r_j)$$

$$y_{ij} = Y_{ij}Y_j^{-1} \in R_{r}^+ \quad \text{(as in (2.8))} \quad (j \leq t, \ i > r_j).$$

(While the given expressions for $v_{ij}$ and $y_{ij}$ also make sense for $j \leq t$ and $i = r_j$, they yield $v_{r_j j} = 1 \in A_{r,c}$ and $y_{r_j j} = 1 \in R_{r}^+$. It is more convenient for the proof to exclude these possibilities.) Recall that the $y_{ij}$ generate $B_{r}^+$.

Claim 1: There exists a homomorphism $\phi^+ : B_{r}^+ \rightarrow A_{r,c}$ such that $\phi^+(y_{ij}) = v_{ij}$ for $j \leq t$ and $i > r_j$.

To prove this, we must show that the $v_{ij}$ satisfy the analogs of the relations (2.8)(1), i.e., the corresponding equations with all $y$’s replaced by $v$’s. We first check that the $u_{ij}$ satisfy the relations (1) below. The first relation follows from Lemma 2.9(a); for the second, observe that $\{r_1, \ldots, r_{j-1}, i\} \subset \{r_1, \ldots, r_{m-1}, l\}$ and $\{c_1, \ldots, c_j\} \subset \{c_1, \ldots, c_m\}$ in that case.

$$u_{lm}u_{ij} = \begin{cases} q^{-1}u_{ij}u_{lm} & (i < l, \ j \geq m) \\ u_{ij}u_{lm} & (i \leq l, \ j < m, \ i \in \{r_j, \ldots, r_{m-1}, l\}) \end{cases}. \tag{1}$$

Furthermore, when $i < l$ and $j < m$ but $i \notin \{r_j, \ldots, r_{m-1}\}$, Lemma 2.9(b) implies that

$$u_{ij}u_{lm} - q^{-1}u_{lm}u_{ij} \equiv_{r,c} \begin{cases} \hat{q}u_{ij}u_{lm} & (i \geq r_m) \\ 0 & (i < r_m). \end{cases} \tag{2}$$

Since $u_{rl} = d_l^{r,c}$ for $l \leq t$, the relations (1) and (2) yield commutation relations for the $d_l^{r,c}$ and $u_{ij}$, which we combine in the following form:

$$d_l^{r,c}u_{ij} = u_{ij}d_l^{r,c} \quad \begin{cases} (l < j), \ or \\ (l \geq j \text{ and } i \notin \{r_j, \ldots, r_l\}) \end{cases} \tag{3}$$

$$d_l^{r,c}u_{ij} \equiv_{r,c} qu_{ij}d_l^{r,c} \quad (l \geq j \text{ and } i \notin \{r_j, \ldots, r_l\}).$$

(Note that when $l < j$, we have $i \geq r_j > r_l$.)
It follows from (1), (2), and (3) that the elements $v_{ij} \in A_{r,c}$ indeed satisfy the analogs of the relations (2.8)(1), as desired. This establishes Claim 1.

Next, set

$$w_{lm} = [r_1 \ldots r_l \mid c_1 \ldots c_{l-1} m] \in A$$

$$t_{lm} = \tilde{w}_{lm}(d_i^{r,c})^{-1} \in A_{r,c}$$

$$z_{lm} = Z_{lm}Z_{tc_{l-1}}^{-1} \in R_c^-$$

and recall that the $z_{lm}$ generate $B_c^-$.

**Claim 2.** There exists a homomorphism $\phi^- : B_c^- \rightarrow A_{r,c}$ such that $\phi^-(z_{lm}) = t_{lm}$ for $l \leq t$ and $m > c_l$.

This follows from Claim 1 by symmetry.

**Claim 3.** Each $v_{ij}$ commutes with each $t_{lm}$.

We first collect the following commutation relations between the $u_{ij}$ and the $w_{lm}$:

$$u_{ij}w_{lm} = w_{lm}u_{ij} \quad \left\{ \begin{array}{ll}
(j = l), & \\
(j < l \text{ and } i \in \{r_{j+1}, \ldots, r_l\}), & \\
(j > l \text{ and } m \in \{c_{l+1}, \ldots, c_j\}), & \\
(q^{-1}w_{lm}u_{ij}), & \\
(qw_{lm}u_{ij}) & \left( j < l \text{ and } i \notin \{r_{j+1}, \ldots, r_l\} \right) \\
&qw_{lm}u_{ij} & \left( j > l \text{ and } m \notin \{c_{l+1}, \ldots, c_j\} \right).
\end{array} \right.$$

The first equation in (4) follows from Lemma 5.3(c). The next two equations hold because

$$\{r_1, \ldots, r_{j-1}, i\} \subset \{r_1, \ldots, r_l\} \text{ and } \{c_1, \ldots, c_j\} \subset \{c_1, \ldots, c_{l-1}, m\}$$

in the first case, while those inclusions are reversed in the second case. Finally, the last two relations follow from Lemma 2.9 and Corollary 2.10.

Commutation relations between the $d_i^{r,c}$ and the $u_{ij}$ are given in (3), and, by symmetry, we also have

$$d_j^{r,c}w_{lm} = w_{lm}d_j^{r,c} \quad \left\{ \begin{array}{ll}
(j < l), & \\
(j \geq l \text{ and } m \in \{c_l, \ldots, c_j\}), & \\
(q^{-1}w_{lm}d_j^{r,c}), & \\
(qw_{lm}d_j^{r,c}) & \left( j \geq l \text{ and } m \notin \{c_l, \ldots, c_j\} \right).
\end{array} \right.$$
In view of the relations given in (2.1), we see that

\[
Y_{rs,s} y_{ij} Y_{rs,s}^{-1} = \left\{ \begin{array}{ll}
q^{-1} y_{ij} & (i = r_s) \\
q y_{ij} & (j = s) \\
y_{ij} & (i \neq r_s, j \neq s)
\end{array} \right.
\]

(6)

\[
Z_{sc_s} z_{lm} Z_{sc_s}^{-1} = \left\{ \begin{array}{ll}
q z_{lm} & (l = s) \\
q^{-1} z_{lm} & (m = c_s) \\
z_{lm} & (l \neq s, m \neq c_s).
\end{array} \right.
\]

On the other hand, it follows from (3) and (5) that the units \( \tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1} \) in \( A_{r,c} \) normalize the elements \( v_{ij} \) and \( t_{lm} \) in exactly the same way, that is,

\[
\tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1} v_{ij} \tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1} = \left\{ \begin{array}{ll}
qu^{-1} v_{ij} & (i = r_s) \\
q v_{ij} & (j = s) \\
v_{ij} & (i \neq r_s, j \neq s)
\end{array} \right.
\]

(7)

\[
\tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1} t_{lm} \tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1} = \left\{ \begin{array}{ll}
q t_{lm} & (l = s) \\
q^{-1} t_{lm} & (m = c_s) \\
t_{lm} & (l \neq s, m \neq c_s).
\end{array} \right.
\]

Claim 4 follows from (6), (7), and Lemma 2.7.

Claim 5. \( \phi = \tilde{\beta}_{r,c}^{-1} \).

Since \( \tilde{\beta}_{r,c} \) is surjective (Lemma 2.5), it is enough to show that \( \phi \tilde{\beta}_{r,c} \) is the identity on \( A_{r,c} \). First note, using (2.4), that

\[
\phi \tilde{\beta}_{r,c} (\tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1}) = \phi (Y_{rs,s} \otimes Z_{sc_s}) = \tilde{d}^{r,c}_s (\tilde{d}^{r,c}_{s-1})^{-1}
\]

for all \( s \), whence \( \phi \tilde{\beta}_{r,c} (\tilde{d}^{r,c}_s) = \tilde{d}^{r,c}_s \) for all \( s \). Next, for \( j \leq t \) and \( i > r_j \) we have

\[
\phi \tilde{\beta}_{r,c} (\tilde{u}_{ij}) = \phi ((Y_{r_1} \otimes Z_{1c_1}) \cdots (Y_{r_{j-1},j-1} \otimes Z_{j-1,c_{j-1}})(Y_{ij} \otimes Z_{jc_j}))
\]

\[
= \phi ((y_{ij} \otimes 1)(Y_{r_1} \otimes Z_{1c_1}) \cdots (Y_{r_{j-1},j-1} \otimes Z_{j-1,c_{j-1}})(Y_{ij} \otimes Z_{jc_j}))
\]

\[
= \phi ((y_{ij} \otimes 1) \beta_{r,c} (\tilde{d}^{r,c}_j)) = \phi v_{ij} \tilde{d}^{r,c}_j = \tilde{u}_{ij}.
\]

By symmetry, \( \phi \tilde{\beta}_{r,c} (\tilde{w}_{lm}) = \tilde{w}_{lm} \) for \( l \leq t \) and \( m > c_l \). Therefore \( \phi \tilde{\beta}_{r,c} \) at least equals the identity on the subalgebra \( C \) of \( A_{r,c} \) generated by (the image of) the set

\[
\{(\tilde{d}^{r,c}_s)^{\pm 1} \mid s \leq t\} \cup \{\tilde{u}_{ij} \mid j \leq t, i > r_j\} \cup \{\tilde{w}_{lm} \mid l \leq t, m > c_l\}.
\]

To finish the proof, we just need to show that \( C = A_{r,c} \); that is, that \( \tilde{X}_{ij} \in C \) for all \( i,j \). This is clear in case \( i < r_1 \) or \( j < c_1 \), since in those cases \( \tilde{X}_{ij} = 0 \). We also have
\( \tilde{X}_{ic} = \check{u}_{ij} \in C \) for \( i \geq r_1 \) and \( \tilde{X}_{rj} = \check{v}_{1j} \in C \) for \( j \geq c_1 \). Hence, \( \tilde{X}_{ij} \in C \) whenever \( i \leq r_1 \) or \( j \leq c_1 \).

Now let \( 1 < l \leq t \) and assume that \( \tilde{X}_{ij} \in C \) whenever \( i \leq r_{l-1} \) or \( j \leq c_{l-1} \). For \( r_{l-1} < i \leq t \) and \( c_{l-1} < j \leq c_l \), it follows from Corollary 5.5(b1) that

\[
(-q)^{l-1}[r_1 \cdots r_{l-1} i \mid c_1 \cdots c_{l-1} j] = (-q)^{l-1}X_{ij}[r_1 \cdots r_{l-1} \mid c_1 \cdots c_{l-1}]
+ \sum_{s=1}^{l-1}(-q)^{s-1}X_{ic}[r_1 \cdots r_{l-1} \mid c_1 \cdots c_s \cdots c_{l-1} j].
\]

Most of the cosets of the factors in (8) can be seen to lie in \( C \) right away. For instance, the coset of the left hand side is zero if either \( i < r_l \) or \( j < c_l \), and it equals \( \check{u}_{il} \) if \( i \geq r_l \) and \( j = c_l \). For \( s < t \), we have \( \tilde{X}_{ic} \in C \) by the inductive hypothesis, and similarly the coset \([r_1 \cdots r_{l-1} \mid c_1 \cdots c_s \cdots c_{l-1} j] + K_{r,c} \) is in \( C \) since it is a linear combination of products of elements \( \tilde{X}_{a_b} \) with \( a \leq r_{l-1} \). Consequently, we obtain from (8) that

\[
X_{ij}[r_1 \cdots r_{l-1} \mid c_1 \cdots c_{l-1}] + K_{r,c} \in C.
\]

In other words, \( \tilde{X}_{ij}d_{l-1}^{r,c} \in C \), whence \( \tilde{X}_{ij} \in C \). Thus, \( \tilde{X}_{ij} \in C \) whenever \( j \leq c_l \). By symmetry, \( \tilde{X}_{ij} \in C \) whenever \( i \leq r_l \).

The above induction proves that \( \tilde{X}_{ij} \in C \) whenever \( i \leq r_t \) or \( j \leq c_t \). If there exist indices \( i > r_t \) and \( j > c_t \), we have

\[
(-q)^{t}[r_1 \cdots r_t i \mid c_1 \cdots c_t j] = (-q)^{t}X_{ij}[r_1 \cdots r_t \mid c_1 \cdots c_t]
+ \sum_{s=1}^{t}(-q)^{s-1}X_{ic}[r_1 \cdots r_t \mid c_1 \cdots c_s \cdots c_t j]
\]

by Corollary 5.5(b1), from which we see as above that \( \tilde{X}_{ij} \in C \). (Note that the left hand side of (9) necessarily lies in \( K_{r,c} \) because it involves a \( (t+1) \times (t+1) \) quantum minor.)

Therefore all \( \tilde{X}_{ij} \in C \), and the proof is complete. \( \square \)

3. Tensor product decompositions of \( H \)-primes

Throughout this section, we assume that \( q \) is not a root of unity; we shall place reminders of this hypothesis in the relevant results. Thus, by [8, Theorem 3.2], all primes of \( A \) are completely prime. Since this property survives in factors and localizations, all primes in the algebras \( A_{r,c} \), \( R^+_r \), and \( R^-_c \) are completely prime, and also in \( B_{r,c} \) because of Theorem 2.11. We have already observed that the algebras \( R^+_{r,0} \) and \( R^-_{c,0} \) are iterated skew polynomial algebras over \( k \), and so is their tensor product. The iterated skew polynomial structure of \( R^+_{r,0} \otimes R^-_{c,0} \) is easily seen to satisfy the hypotheses of [8, Theorem 2.3], and thus all its primes are completely prime. Consequently, all primes in the localizations \( R^+_r \otimes R^-_c \) are completely prime.
3.1. In order to deal with $H$-primes in tensor products, we need the following rationality property. Suppose that $S$ is a noetherian $k$-algebra and that $G$ is a group acting on $S$ by $k$-algebra automorphisms. We say that a $G$-prime $P$ of $S$ is strongly $G$-rational provided the algebra $Z(\text{Fract} S/P)^G$, the fixed ring of the center of the Goldie quotient ring of $S/P$ under the induced $G$-action, equals $k$.

By [9, (5.7)(i)] (cf. [1, Theorem II.5.14]) and [1, Corollary II.6.5], $A$ has only finitely many $H$-primes, and they are all completely prime and strongly $H$-rational. These properties carry over to $A_{r,c}$, $R^+_r$, and $R^-_c$. Analogous results [1, Theorems II.5.12 and II.6.4] imply that $R^+_{r,0} \otimes R^-_{c,0}$ has only finitely many $(H \times H)$-primes, and they are all completely prime and strongly $(H \times H)$-rational. These properties now carry over to $R^+_r \otimes R^-_c$.

Identify $H$ with the subgroup

$$\tilde{H} = ((k^X \times \{1\}^n) \times \{1\}^n \times (k^X)^n) \triangleleft H \times H$$

in the obvious way. With this identification, $\beta_{r,c}$ and $\tilde{\beta}_{r,c}$ are $H$-equivariant. In particular, it follows that $B_{r,c}$ has only finitely many $\tilde{H}$-primes, and they are all completely prime and strongly $\tilde{H}$-rational.

3.2. Lemma. For $i = 1, 2$, let $A_i$ be a $k$-algebra, $H_i$ a group acting on $A_i$ by $k$-algebra automorphisms, and $P_i$ an $H_i$-prime ideal of $A_i$. Set $P = (P_1 \otimes A_2) + (A_1 \otimes P_2)$, and let $H_1 \times H_2$ act on $A_1 \otimes A_2$ in the natural manner.

(a) If each $A_i/P_i$ is $H_i$-simple and $Z(A_1/P_1)^{H_1} = k$, then $(A_1 \otimes A_2)/P$ is $(H_1 \times H_2)$-simple.

(b) If each $A_i$ is noetherian and $P_i$ is strongly $H_i$-rational, then $P$ is an $(H_1 \times H_2)$-prime ideal of $A_1 \otimes A_2$. Moreover, $P$ is the only $(H_1 \times H_2)$-prime ideal of $A_1 \otimes A_2$ that contracts to $P_1 \otimes 1$ in $A_1 \otimes 1$ and to $1 \otimes P_2$ in $1 \otimes A_2$.

(c) If each $A_i$ is noetherian and each $P_i$ is strongly $H_i$-rational, then $P$ is strongly $(H_1 \times H_2)$-rational.

Proof. Since $(A_1 \otimes A_2)/P \cong (A_1/P_1) \otimes (A_2/P_2)$, there is no loss of generality in assuming that each $P_i = 0$.

(a) This is a standard shortest length argument. Let $I$ be a nonzero $(H_1 \times H_2)$-ideal of $A_1 \otimes A_2$, and let $m$ be the shortest length for nonzero elements of $I$ (as sums of pure tensors). Choose a nonzero element

$$x = b_1 \otimes c_1 + \cdots + b_m \otimes c_m \in I$$

of length $m$, where the $b_j \in A_1$ and $c_j \in A_2$, and note that the $c_j$ are linearly independent over $k$. Now the set

$$\{b \in A_1 \mid (b \otimes c_1 + A_1 \otimes c_2 + \cdots + A_1 \otimes c_m) \cap I \neq \emptyset\}$$

is a nonzero $H_1$-ideal of $A_1$, and so it equals $A_1$. Thus, without loss of generality, $b_1 = 1$. For any $a \in A_1$, we now have $x(a \otimes 1) - (a \otimes 1)x$ in $I$ with length less than $m$, whence
Consequently, the set $A$ is an $(A, C)$-ideal of $A$, and therefore $h$ is a semiprime ideal of $A$, disjoint from both $A_1$ and $1 \otimes A_2$. It follows that each nonzero $(H_1 \times H_2)$-ideal of $A$ meets $C$; in particular, $A_1 \otimes A_2$ is an $(H_1 \times H_2)$-ideal of $A$. Now let $Q$ be any $(H_1 \times H_2)$-prime ideal of $A$ that contracts to zero in both $A_1 \otimes 1$ and $1 \otimes A_2$. Then $Q$ is a semiprime ideal, disjoint from both $C_1 \otimes 1$ and $1 \otimes C_2$. If some prime ideal $Q_0$ minimal over $Q$ meets $C_1 \otimes 1$, then $h(Q_0)$ meets $C_1 \otimes 1$ for all $h \in H_1 \times H_2$. But since $Q$ is a finite intersection of some of the $h(Q_0)$, it would follow that $Q$ meets $C_1 \otimes 1$, a contradiction. Therefore $C_1 \otimes 1$ is disjoint from all primes minimal over $Q$, whence $C_1 \otimes 1$ is regular modulo $Q$. Likewise, $1 \otimes C_2$ is regular modulo $Q$. It follows that $C$ is disjoint from $Q$, and therefore $Q = 0$.

(c) After localization, we can assume that each $A_i$ is $H_i$-simple artinian. By part (a), $A_1 \otimes A_2$ is now $(H_1 \times H_2)$-localization. Consider an element $u$ in $Z(\text{Fract}(A_1 \otimes A_2))^{H_1 \times H_2}$. The set $\{ a \in A_1 \otimes A_2 \mid au \in A_1 \otimes A_2 \}$ is a nonzero $(H_1 \times H_2)$-ideal of $A_1 \otimes A_2$, and so it equals $A_1 \otimes A_2$. Therefore $u \in A_1 \otimes A_2$. Now write $u = v_1 \otimes w_1 + \cdots + v_t \otimes w_t$ for some $v_i \in A_1$ and some linearly independent $w_j \in A_2$. Since $u$ is fixed by $H_1 \times 1$ and commutes with $A_1 \otimes 1$, we see that all $v_j \in Z(A_1)^{H_1} = k$. Hence, $u = 1 \otimes w$ for some $w \in A_2$. But then $w \in Z(A_2)^{H_2} = k$, and therefore $u \in k$.

3.3. Proposition. For $i = 1, 2$, let $A_i$ be a noetherian $k$-algebra and $H_i$ a group acting on $A_i$ by $k$-algebra automorphisms. Assume that all $H_i$-primes of $A_i$ are strongly $H_i$-rational. Then the rule $(P_1, P_2) \mapsto (P_1 \otimes A_2) + (A_1 \otimes P_2)$ provides a bijection

$$(H_1 \text{-spec } A_1) \times (H_2 \text{-spec } A_2) \longrightarrow (H_1 \times H_2) \text{-spec } (A_1 \otimes A_2).$$

Proof. Lemma 3.2(b) shows that the given rule maps $(H_1 \text{-spec } A_1) \times (H_2 \text{-spec } A_2)$ to $(H_1 \times H_2) \text{-spec } (A_1 \otimes A_2)$.

Now consider an $(H_1 \times H_2)$-prime $P$ of $A_1 \otimes A_2$. Let $P_1$ and $P_2$ be the inverse images of $P$ under the natural maps $A_i \twoheadrightarrow A_1 \otimes A_2$. Then each $P_i$ is an $H_i$-ideal of $A_i$, and $(P_1 \otimes A_2) + (A_1 \otimes P_2) \subseteq P$. There are $H_1$-primes $Q_1, \ldots, Q_t$ in $A_1$, containing $P_1$, such that $Q_1 Q_2 \cdots Q_t \subseteq P_1$. Then the $Q_i \otimes A_2$ are $(H_1 \times H_2)$-ideals of $A_1 \otimes A_2$ such that

$$(Q_1 \otimes A_2) (Q_2 \otimes A_2) \cdots (Q_t \otimes A_2) \subseteq P_1 \otimes A_2 \subseteq P.$$
Consequently, some \( Q_j \otimes A_2 \subseteq P \), whence \( Q_j \subseteq P_1 \), and so \( Q_j = P_1 \). This shows that \( P_1 \) is \( H_1 \)-prime. Similarly, \( P_2 \) is \( H_2 \)-prime.

By Lemma 3.2(b), \((P_1 \otimes A_2) + (A_1 \otimes P_2)\) is an \((H_1 \times H_2)\)-prime of \( A_1 \otimes A_2 \), and it is the only \((H_1 \times H_2)\)-prime of \( A_1 \otimes A_2 \) that contracts to \( P_1 \otimes 1 \) in \( A_1 \otimes 1 \) and to \( 1 \otimes P_2 \) in \( 1 \otimes A_2 \). Therefore \( P = (P_1 \otimes A_2) + (A_1 \otimes P_2) \). It is clear that \( P_1 \) and \( P_2 \) are unique, since \( P_i \) equals the inverse image of \((P_1 \otimes A_2) + (A_1 \otimes P_2)\) under the natural map \( A_i \to A_1 \otimes A_2 \).

**3.4. Lemma.** [\( q \) not a root of unity] Let \((r, c) \in \mathbf{RC}\). If \( \tilde{P} \) is an \( \tilde{H} \)-prime of \( B_{r, c} \), then there exists a unique \((H \times H)\)-prime \( Q \) of \( R_r^+ \otimes R_c^- \) such that \( Q \cap B_{r, c} = \tilde{P} \).

*Proof.* Since \( B_{r, c} \) has only finitely many \( \tilde{H} \)-primes and \( H \times H \) just permutes them, the \((H \times H)\)-orbit of \( \tilde{P} \) in \( \text{spec} B_{r, c} \) is finite. Since \( \tilde{P} \) is prime, it now follows from [1, Proposition II.2.9] that \( \tilde{P} \) must be invariant under \( H \times H \). In view of Lemma 2.6, \( Q = \tilde{P}(R_r^+ \otimes R_c^-) \) is an \((H \times H)\)-invariant prime of \( R_r^+ \otimes R_c^- \) such that \( Q \cap B_{r, c} = \tilde{P} \). It remains to show that if \( Q' \) is any \((H \times H)\)-prime of \( R_r^+ \otimes R_c^- \) that contracts to \( \tilde{P} \), then \( Q' = Q \). Note that \( Q' \supseteq Q \) by definition of \( Q \).

Set \( G = (k^x)^n \), let \( \phi : G \to \{(1)^n \times (k^x)^n\} \times ((k^x)^n \times \{1\}^n) \subseteq H \times H \) be the homomorphism given by the rule

\[
\phi(\alpha_1, \ldots, \alpha_n) = (1, \ldots, 1, \alpha_1^{-1}, \ldots, \alpha_n^{-1}, \alpha_1, \ldots, \alpha_n, 1, \ldots, 1),
\]

and use \( \phi \) to pull back the action of \( H \times H \) on \( R_r^+ \otimes R_c^- \) to an action of \( G \). With respect to this \( G \)-action, \( B_{r, c} \) is generated by fixed elements, and each of the elements \( 1 \otimes Z_{sc} \) is a \( G \)-eigenvector with eigenvalue equal to the projection \((\alpha_1, \ldots, \alpha_n) \mapsto \alpha_s \). In view of Lemma 2.6 and the fact that \( k \) is infinite, it follows that the \( G \)-eigenspaces of \( R_r^+ \otimes R_c^- \) are the subspaces \( B_{r, c}(1 \otimes (Z_{1e_1}^m Z_{2e_2}^{m_2} \cdots Z_{le_l}^{m_l})) \) for \((m_1, \ldots, m_l) \in \mathbf{Z}^l \). Consequently, any \( G \)-eigenvector in \( R_r^+ \otimes R_c^- \) has the form \( du \) where \( d \in B_{r, c} \) and \( u \) is a unit. If \( du \in Q' \), then \( d \in Q' \cap B_{r, c} = \tilde{P} \), whence \( du \in Q \). Since \( Q' \) is \( G \)-invariant, we conclude that \( Q' = Q \), as desired. \( \Box \)

**3.5.** Set \( H\text{-spec}_{r, c} A = (H\text{-spec} A) \cap (\text{spec}_{r, c} A) \) for \((r, c) \in \mathbf{RC} \). These sets partition \( H\text{-spec} A \) because of Corollary 1.10.

**Theorem.** [\( q \) not a root of unity] For each \((r, c) \in \mathbf{RC} \), there is a bijection

\[
(H\text{-spec} R_r^+) \times (H\text{-spec} R_c^-) \longrightarrow H\text{-spec}_{r, c} A
\]

given by the rule \((Q^+, Q^-) \mapsto \beta_{r, c}^{-1}((Q^+ \otimes R_c^-) + (R^+ \otimes Q^-)) \).

*Proof.* If \( Q^+ \in H\text{-spec} R_r^+ \) and \( Q^- \in H\text{-spec} R_c^- \), then Proposition 3.3 shows that the ideal \( Q = (Q^+ \otimes R_c^-) + (R^+_r \otimes Q^-) \) is an \((H \times H)\)-prime of \( R_r^+ \otimes R_c^- \). In particular, \( Q \) is completely prime, and so \( Q \cap B_{r, c} \) is an \( \tilde{H} \)-prime of \( B_{r, c} \), whence \( \beta_{r, c}^{-1}(Q) = \beta_{r, c}^{-1}(Q \cap B_{r, c}) \) is an \( H \)-prime of \( A \) lying in \( \text{spec}_{r, c} A \). This shows that the given rule defines a map from \((H\text{-spec} R_r^+) \times (H\text{-spec} R_c^-) \) to \( H\text{-spec}_{r, c} A \).

Now consider an arbitrary \( H \)-prime \( P \) in \( H\text{-spec}_{r, c} A \). Then \( P \) induces an \( H \)-prime in \( A_{r, c} \) that contracts to \( P \) under the localization map. In view of Theorem 2.11, it follows
that $\beta_{r,c}(P)$ induces an $\tilde{H}$-prime $\tilde{P}$ of $B_{r,c}$ such that $\beta_{r,c}^{-1}(\tilde{P}) = P$. By Lemma 3.4, there is a unique $(H \times H)$-prime $Q$ of $R^+_r \otimes R^-_c$ such that $Q \cap B_{r,c} = \tilde{P}$. Then Proposition 3.3 implies that $Q = (Q^+ \otimes R^-_c) + (R^+_r \otimes Q^-)$ for some $H$-primes $Q^+$ in $R^+_r$ and $Q^-$ in $R^-_c$. Thus,

$$\beta_{r,c}^{-1}((Q^+ \otimes R^-_c) + (R^+_r \otimes Q^-)) = \beta_{r,c}^{-1}(Q) = \beta_{r,c}^{-1}(Q \cap B_{r,c}) = \beta_{r,c}^{-1}(\tilde{P}) = P.$$

It remains to show that $Q^+$ and $Q^-$ are unique. Consider any $T^+ \in H$-spec $R^+_r$ and $T^- \in H$-spec $R^-_c$ such that $P = \beta_{r,c}^{-1}(T)$ where $T = (T^+ \otimes R^-_c) + (R^+_r \otimes T^-)$. As in the first paragraph of the proof, $T$ is an $(H \times H)$-prime of $R^+_r \otimes R^-_c$ and $T \cap B_{r,c}$ is an $\tilde{H}$-prime of $B_{r,c}$. Since $\beta_{r,c}^{-1}(T \cap B_{r,c}) = P$, we must have $T \cap B_{r,c} = \tilde{P}$, whence $T = Q$ by the uniqueness of $Q$. Therefore Proposition 3.3 shows that $T^+ = Q^+$ and $T^- = Q^-$, as desired. \hfill \Box

**3.6.** Fix $t \in \{0, 1, \ldots, n\}$, and let $H$-spec$^t A$ be the set of those $H$-primes of $A$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. By Corollary 1.10,

$$H\text{-spec}^t A = \bigcup_{(r,c) \in RC_t} H\text{-spec}_{r,c} A,$$

and consequently Theorem 3.5 implies that

$$|H\text{-spec}^t A| = \sum_{(r,c) \in RC_t} |H\text{-spec} R^+_r| \cdot |H\text{-spec} R^-_c|.$$

If $R_t$ denotes the set of sequences $(r_1, \ldots, r_t) \in \mathbb{N}^t$ with $1 \leq r_1 < \cdots < r_t \leq n$, then $RC_t = R_t \times R_t$. For each $r \in R_t$, the automorphism $\tau$ of $A$ discussed in (5.1) induces an isomorphism $\overline{\tau} : R^+_r \to R^-_r$. While $\overline{\tau}$ is not $H$-equivariant, there is an automorphism $\gamma$ of $H$, given by $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \mapsto (\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n)$, such that the following diagram commutes:

$$\begin{array}{ccc}
H & \xrightarrow{\gamma} & H \\
\downarrow & & \downarrow \\
\text{Aut } R^+_r & \xrightarrow{\overline{\tau}} & \text{Aut } R^-_r \\
\end{array}$$

(Here the vertical arrows denote the standard actions of $H$ on $R^+_r$ and $R^-_r$.) Hence, $\overline{\tau}$ provides a bijection of $H$-spec $R^+_r$ onto $H$-spec $R^-_r$. Therefore

$$|H\text{-spec}^t A| = \sum_{r,c \in R_t} |H\text{-spec} R^+_r| \cdot |H\text{-spec} R^-_c| = \left( \sum_{r \in R_t} |H\text{-spec} R^+_r| \right)^2,$$

a perfect square. These numbers are known in three cases:

$$|H\text{-spec}^t A| = \begin{cases} 
1 & (t = 0) \\
(2^n - 1)^2 & (t = 1) \\
(n!)^2 & (t = n).
\end{cases}$$
The case when \( t = 0 \) is trivial, and the case when \( t = 1 \) is given by [5, Corollary 3.5]. For the remaining case, note first that \( \text{H-spec}^{[n]} A \approx \text{H-spec} \mathcal{O}_q(GL_n(k)) \). It can be checked that there is a bijection between \( \text{H-spec} \mathcal{O}_q(GL_n(k)) \) and the set of winding-invariant primes of \( \mathcal{O}_q(SL_n(k)) \) (e.g., see [1, Lemma II.5.16]), and it follows from the work of Hodges and Levasseur [11] that the latter set is in bijection with the double Weyl group \( S_n \times S_n \) (cf. [1, Corollary II.4.12]).

3.7. As a corollary of Theorem 3.5, we obtain the following less specific but more digestible result.

**Corollary.** \([q \text{ not a root of unity}]\) Set

\[
R^+ = A/\langle X_{ij} \mid i < j \rangle \quad \text{and} \quad R^- = A/\langle X_{ij} \mid i > j \rangle,
\]

let \( \pi^\pm : A \to R^\pm \) denote the quotient maps, and let \( \beta \) denote the composition

\[
A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi^+ \otimes \pi^-} R^+ \otimes R^-.
\]

Given any \( \text{H-prime} P \) in \( A \), there exist \( \text{H-primes} P^\pm \) in \( R^\pm \) such that

\[
P = \beta^{-1}\left((P^+ \otimes R^-) + (R^+ \otimes P^-)\right).
\]

**Proof.** By Corollary 1.10 and Theorem 3.5, \( P = \beta_{r,c}^{-1}\left(((Q^+ \otimes R^-_c) + (R^+_r \otimes Q^-))\right) \) for some \((r,c) \in RC\) and some \( \text{H-primes} Q^+ \) in \( R^+_r \) and \( Q^- \) in \( R^-_c \). Observe that \( X_{ij} \in \ker \pi^+_{r,0} \) when \( i < j \), and that \( X_{ij} \in \ker \pi^-_{c,0} \) when \( i > j \). Hence, there are surjective \( k \)-algebra homomorphisms \( \tau^+ : R^+ \to R^+_r,0 \) and \( \tau^- : R^- \to R^-_c,0 \) such that \( \tau^+ \pi^+ = \pi^+_{r,0} \) and \( \tau^- \pi^- = \pi^-_{c,0} \). Consequently, \((\tau^+ \otimes \tau^-)\beta = \beta_{r,c}\) (with the obvious adjustment of codomains).

Next, observe that \( Q^+_0 = Q^+ \cap R^+_r,0 \) and \( Q^-_0 = Q^- \cap R^-_c,0 \) are \( \text{H-primes} \) of \( R^+_r,0 \) and \( R^-_c,0 \), respectively, whence the ideals \( P^\pm = (\tau^\pm)^{-1}(Q_0^\pm) \) are \( \text{H-primes} \) in \( R^\pm \). Finally,

\[
P = (\tau^+ \otimes \tau^-)^{-1}\left((Q^+_0 \otimes R^-_c,0) + (R^+_r,0 \otimes Q^-_0)\right) = \beta^{-1}\left((P^+ \otimes R^-) + (R^+ \otimes P^-)\right),
\]

as desired. \( \square \)

4. Illustration: \( \mathcal{O}_q(M_2(k)) \)

Theorem 3.5 opens a potential route to computing the \( \text{H-primes} \) of \( A \) in the generic case: If we can find all the \( \text{H-primes} \) in each \( R^+_r \) and \( R^-_c \), we immediately obtain descriptions of all the \( \text{H-primes} \) in \( A \). Since these descriptions would be in terms of pullbacks of \( \text{H-primes} \) from the algebras \( R^+_r \otimes R^-_c \), it would still remain to find generating sets for these ideals.

To illustrate the procedure, we sketch the case where \( n = 2 \), for which \( \text{H-spectrum} \) \( A \) is already known. In [7], we use the above process to compute \( \text{H-spectrum} \) \( A \) when \( n = 3 \).
4.1. Assume that $q$ is not a root of unity, and fix $n = 2$. There are only four choices for $r$ and $c$, namely $\varnothing$, $(1)$, $(2)$, and $(1, 2)$. The corresponding algebras $R^+_r$ and $R^-_c$ are

$$
R^+_\varnothing = A/\langle X_{11}, X_{12}, X_{21}, X_{22} \rangle = k \\
R^+_{(1)} = (A/\langle X_{12}, X_{22} \rangle)[X_{11}^{-1}] = k[Y_{11}^{\pm 1}, Y_{21}] \\
R^+_{(2)} = (A/\langle X_{11}, X_{12}, X_{22} \rangle)[X_{21}^{-1}] = k[Y_{21}^{\pm 1}] \\
R^+_{(1,2)} = (A/\langle X_{12} \rangle)[X_{11}^{-1}, X_{22}^{-1}] = k[Y_{11}^{\pm 1}, Y_{21}, Y_{22}^{\pm 1}] \\
R^-_{\varnothing} = k \\
R^-_{(1)} = k[Z_{11}^{\pm 1}, Z_{12}] \\
R^-_{(2)} = k[Z_{12}^{\pm 1}] \\
R^-_{(1,2)} = k[Z_{11}^{\pm 1}, Z_{12}, Z_{22}^{\pm 1}].
$$

The $H$-primes in these algebras are easily computed:

$$
H\text{-spec } R^+_{\varnothing} = \{ \langle 0 \rangle \} \\
H\text{-spec } R^+_{(1)} = \{ \langle 0 \rangle, \langle Y_{21} \rangle \} \\
H\text{-spec } R^+_{(2)} = \{ \langle 0 \rangle \} \\
H\text{-spec } R^+_{(1,2)} = \{ \langle 0 \rangle, \langle Y_{21} \rangle \} \\
H\text{-spec } R^-_{\varnothing} = \{ \langle 0 \rangle \} \\
H\text{-spec } R^-_{(1)} = \{ \langle 0 \rangle, \langle Z_{12} \rangle \} \\
H\text{-spec } R^-_{(2)} = \{ \langle 0 \rangle \} \\
H\text{-spec } R^-_{(1,2)} = \{ \langle 0 \rangle, \langle Z_{12} \rangle \}.
$$

The only choice for $(r, c) \in RC_0$ is $r = c = \varnothing$. In this case, the only $H$-primes in $R^+_r$ and $R^-_c$ are the zero ideals, and $\beta^{-1}_{\varnothing, \varnothing}(\langle 0 \rangle) = \langle X_{11}, X_{12}, X_{21}, X_{22} \rangle$, the augmentation ideal of $A$. We record this $H$-prime using the symbol

$$
\bullet \bullet
$$

to denote the generating set $\{X_{11}, X_{12}, X_{21}, X_{22}\}$, the bullet in position $(i, j)$ being a marker for the element $X_{ij}$.

Corresponding to the four pairs $(r, c) \in RC_1$, there are nine $H$-primes in $A$ of the form

$$
\beta^{-1}_{r, c}(Q^+ \otimes R^-_c + (R^+_r \otimes Q^-))
$$

where $Q^+$ is an $H$-prime in $R^+_r$ and $Q^-$ is an $H$-prime in $R^-_c$. We can record generating sets for these ideals as follows, continuing the notation introduced in the previous paragraph; here $\square$ is a placeholder and $\varnothing$ denotes the 2 × 2 quantum determinant.

$$
\begin{array}{c|c|c}
R^-_{(1)} & R^-_{(2)} \\
\langle 0 \rangle & \langle Z_{12} \rangle & \langle 0 \rangle \\
R^+_{(1)} & \langle 0 \rangle & \varnothing \bullet \bullet & \bullet \\
& \langle Y_{21} \rangle & \bullet \bullet \bullet & \bullet \\
R^+_{(2)} & \langle 0 \rangle & \bullet \bullet \bullet & \bullet \\
\end{array}
$$
(See, e.g., [5, Theorem 1.1] for a proof that the quantum determinant generates the kernel of $\beta_{1,1}$. We leave it to the reader to check that the other $H$-primes are generated as indicated.)

Finally, the only choice for $(r, c) \in RC_2$ is $r = c = (1, 2)$, and there are four $H$-primes in $A$ of the form

$$\beta_{1,2}^{-1}((Q^+ \otimes R_{1,2}^-) + (R_{1,2}^+ \otimes Q^-))$$

with $Q^\pm \in H\text{-spec } R_{1,2}^\pm$. We record generating sets for these ideals as follows:

<table>
<thead>
<tr>
<th>$\langle 0 \rangle$</th>
<th>$\langle Z_{12} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 0 \rangle$</td>
<td>$\circ \circ \bullet \bullet$</td>
</tr>
<tr>
<td>$\langle Y_{21} \rangle$</td>
<td>$\circ \circ \circ \bullet$</td>
</tr>
</tbody>
</table>

We now conclude from Theorem 3.5 that we have found all the $H$-primes of $A = O_q(M_2(k))$. There are 14 in total, which we can display as follows:

For a display showing the inclusions among these ideals, see [5, (3.6)].

5. Appendix. Relations in $O_q(M_n(k))$

The proofs in this paper rely on a number of relations among the generators and quantum minors in quantum matrix algebras. We record and/or derive those relations in this appendix. Throughout, let $A = O_q(M_n(k))$ with $k$ an arbitrary field and $q \in k^\times$ an arbitrary nonzero scalar.

5.1. (a) We present the algebra $A$ with generators $X_{ij}$ for $i, j = 1, \ldots, n$ and relations

$$X_{ij}X_{lj} = qX_{lj}X_{ij} \quad (i < l)$$
$$X_{ij}X_{im} = qX_{im}X_{ij} \quad (j < m)$$
$$X_{ij}X_{lm} = X_{lm}X_{ij} \quad (i < l, j > m)$$
$$X_{ij}X_{lm} - X_{lm}X_{ij} = (q - q^{-1})X_{im}X_{ij} \quad (i < l, j < m).$$

As is well known, $A$ is in fact a bialgebra, with comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to k$ such that

$$\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj} \quad \text{and} \quad \epsilon(X_{ij}) = \delta_{ij}.$$
(b) The algebra $A$ possesses various symmetries. In particular, it supports a $k$-algebra automorphism $\tau$ such that $\tau(X_{ij}) = X_{ji}$ for all $i, j$ \cite[Proposition 3.7.1]{16}. Pairs of results which imply each other just through applications of $\tau$ will be simply referred to as “symmetric”.

(c) Given $U, V \subseteq \{1, \ldots, n\}$ with $|U| = |V| = t$, let $\mathcal{O}_q(M_{U,V}(k))$ denote the $k$-subalgebra of $A$ generated by those $X_{ij}$ with $i \in U$ and $j \in V$. There is a natural isomorphism $\mathcal{O}_q(M_{t}(k)) \to \mathcal{O}_q(M_{U,V}(k))$, which sends the quantum determinant of $\mathcal{O}_q(M_{t}(k))$ to the quantum minor in $A$ involving rows from $U$ and columns from $V$. As in \cite{6}, we denote this quantum minor by $[U \mid V]$, or in the form $[u_1 \cdots u_t \mid v_1 \cdots v_t]$ if we wish to list the elements of $U$ and $V$.

We shall also use the isomorphism $\mathcal{O}_q(M_{t}(k)) \to \mathcal{O}_q(M_{U,V}(k))$ to simplify various proofs, since it will allow us to work in smaller quantum matrix algebras than $A$ on occasion. For example, since the quantum determinant in $\mathcal{O}_q(M_{t}(k))$ is central (e.g., \cite[Theorem 4.6.1]{16}), the quantum minor $[U \mid V]$ commutes with $X_{ij}$ for all $i \in U$ and $j \in V$. Consequently,

$$[U \mid V][I \mid J] = [I \mid J][U \mid V] \quad (I \subseteq U, J \subseteq V).$$

(d) Recall from \cite[Equation 1.9]{15} the comultiplication rule for quantum minors:

$$\Delta([I \mid J]) = \sum_{|K| = |I|} [I \mid K] \otimes [K \mid J].$$

5.2. We next restate some identities from \cite{16}, given there for generators and maximal minors, in a form that applies to minors of arbitrary size. Note the difference between our choice of relations for $A$ (see (5.1)(a)) and that in \cite[p. 37]{16}. Because of this, we must interchange $q$ and $q^{-1}$ whenever carrying over a formula from \cite{16}.

Lemma. Let $r, c \in \{1, \ldots, n\}$ and $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J| \geq 1$.

(a) If $r \in I$ and $c \in J$, then $X_{rc}[I \mid J] = [I \mid J]X_{rc}$.

(b) If $r \in I$ and $c \notin J$, set $J^+ = J \cup \{c\}$. Then

$$X_{rc}[I \mid J] - q^{-1}[I \mid J]X_{rc} = (q^{-1} - q) \sum_{j \in J \setminus \{c\}} (-q)^{-|J \cap \{c\}|} [I \mid J^+ \setminus \{j\}]X_{rj}$$

(2) $[I \mid J]X_{rc} - qX_{rc}[I \mid J] = (q - q^{-1}) \sum_{j \in J \setminus \{c\}} (-q)^{|J \cap \{c\}|} X_{rj}[I \mid J^+ \setminus \{j\}]$

(c) If $r \notin I$ and $c \in J$, set $I^+ = I \cup \{r\}$. Then

$$X_{rc}[I \mid J] - q^{-1}[I \mid J]X_{rc} = (q^{-1} - q) \sum_{i \in I \setminus \{r\}} (-q)^{-|I \cap \{r\}|} [I^+ \setminus \{i\} \mid J]X_{ic}$$
3. Lemma. Part (c) is symmetric to (b).

5.3. Lemma. Let $U, V \subseteq \{1, \ldots, n\}$ with $|U| = |V|$, and let $u_1, u_2 \in U$ and $v_1, v_2 \in V$.

Set $U_s = U \setminus \{u_s\}$ and $V_s = V \setminus \{v_s\}$ for $s = 1, 2$.

(a) If $u_1 < u_2$, then $[U_1 | V_1][U_2 | V_1] = q^{-1}[U_2 | V_1][U_1 | V_1]$.

(b) If $v_1 < v_2$, then $[U_1 | V_1][U_2 | V_2] = q^{-1}[U_1 | V_2][U_1 | V_1]$.

(c) If $u_1 < u_2$ and $v_1 > v_2$, then $[U_1 | V_1][U_2 | V_2] = [U_2 | V_2][U_1 | V_1]$.

(d) If $u_1 < u_2$ and $v_1 < v_2$, then

$$[U_1 | V_1][U_2 | V_2] - [U_2 | V_2][U_1 | V_1] = (q^{-1} - q)[U_2 | V_1][U_1 | V_2].$$

Proof. Since we may work in $O_q(M_{1, V}(k))$, there is no loss of generality in assuming that $U = V = \{1, \ldots, n\}$. The result then follows from [16, Theorem 5.2.1]. □

5.4. We also require the form of the $q$-Laplace relations given in [15]. For index sets $I$ and $J$, set

$$\ell(I; J) = |\{(i, j) \in I \times J | i > j\}|.$$

Lemma. ($q$-Laplace relations) Let $I, J \subseteq \{1, \ldots, n\}$.

(a) If $J_1, J_2 \subseteq \{1, \ldots, n\}$ with $|J_1| + |J_2| = |I|$, then

$$\sum_{I_1 \cup I_2 = I} \binom{(-q)^{\ell(I_1; I_2)}[I_1 | J_1][I_2 | J_2]}{I_1 \cup I_2 = I} = \begin{cases} (-q)^{\ell(J_1; J_2)}[I | J_1 \cup J_2] & (J_1 \cap J_2 = \emptyset) \\ 0 & (J_1 \cap J_2 \neq \emptyset). \end{cases}$$

(b) If $I_1, I_2 \subseteq \{1, \ldots, n\}$ with $|I_1| + |I_2| = |J|$, then

$$\sum_{I_1 \cup I_2 = J} \binom{(-q)^{\ell(I_1; I_2)}[I_1 | I_1 \cup I_2 | J]}{I_1 \cup I_2 = J} = \begin{cases} (-q)^{\ell(I_1; I_2)}[I_1 \cup I_2 | J] & (I_1 \cap I_2 = \emptyset) \\ 0 & (I_1 \cap I_2 \neq \emptyset). \end{cases}$$

Proof. [15, Proposition 1.1]. □

5.5. The $q$-Laplace relations simplify somewhat when one of the index sets is a singleton, as follows.
Corollary. Let \( r, c \in \{1, \ldots, n\} \) and \( I, J \subseteq \{1, \ldots, n\} \). 

(a) If \(|I| = |J| + 1\), then

\[
\sum_{i \in I} (-q)^{|[1,i] \cap I|} X_{ic}(I \setminus \{i\} | J) = \begin{cases} (-q)^{|[1,c] \cap J|} [I | J \cup \{c\}] & (c \notin J) \\
0 & (c \in J) \end{cases} \quad \text{and} \]

\[
\sum_{i \in I} (-q)^{|[i,n] \cap I|} X_{ic}(I \setminus \{i\} | J) = \begin{cases} (-q)^{|[c,n] \cap J|} [I | J \cup \{c\}] & (c \notin J) \\
0 & (c \in J) \end{cases}
\]

(b) If \(|J| = |I| + 1\), then

\[
\sum_{j \in J} (-q)^{|[1,j] \cap J|} X_{jr}(I \setminus \{j\} | J) = \begin{cases} (-q)^{|[1,r] \cap I|} [I \cup \{r\} | J] & (r \notin I) \\
0 & (r \in I) \end{cases} \quad \text{and} \]

\[
\sum_{j \in J} (-q)^{|[j,n] \cap J|} X_{jr}(I \setminus \{j\} | J) = \begin{cases} (-q)^{|[r,n] \cap I|} [I \cup \{r\} | J] & (r \notin I) \\
0 & (r \in I) \end{cases}
\]

Proof. (a) For the first case, fix \( J_1 = \{c\} \) and \( J_2 = J \). We will use Lemma 5.4(a), which involves a sum over \( I_1 \cup I_2 = I \) with \(|I_1| = 1\); thus \( I_1 = \{i\} \) and \( I_2 = I \setminus \{i\} \) for some \( i \in I \). In that case, \( \ell(I_1; I_2) = |[1, i] \cap I| \) and \( \ell(J_1; J_2) = |[1, c] \cap J| \). Thus, formula (1) follows directly from Lemma 5.4(a). Formula (2) follows similarly, where this time we fix \( J_1 = J \) and \( J_2 = \{c\} \).

(b) These follow from (a) by symmetry. \( \square \)

5.6. Lemma. Let \( U, V \subseteq \{1, \ldots, n\} \) with \(|U| = |V|\).

(a) Let \( U = I \cup K \), and let \( J_1, J_2 \subseteq V \) such that \(|J_1| + |J_2| = 2|I| + |K|\). Then

\[
\sum_{K = K' \cup K'' \mid |K'| = |J_1| - |I|} (-q)^{\ell(I; K') + \ell(K' \cup I | J_1)} [I \cup K' \mid J_1][K'' \cup I \mid J_2] = \begin{cases} (-q)^{\ell(J_1 \setminus J_2; J_1 \setminus J_2)} [I \cup J_1 \setminus J_2][U \mid V] & (|J_1 \cap J_2| = |I|) \\
0 & (|J_1 \cap J_2| > |I|). \end{cases}
\]

(b) Let \( V = J \cup L \), and let \( I_1, I_2 \subseteq U \) such that \(|I_1| + |I_2| = 2|J| + |L|\). Then

\[
\sum_{L = L' \cup L'' \mid |L'| = |I_1| - |J|} (-q)^{\ell(I; L')} [I_1 \cup L' \mid J \cup L''][I_2 \mid J \cup L''] = \begin{cases} (-q)^{\ell(I_1 \setminus I_2; I_1 \setminus I_2)} [I_1 \cap I_2 \mid J][U \mid V] & (|I_1 \cap I_2| = |J|) \\
0 & (|I_1 \cap I_2| > |J|). \end{cases}
\]

Proof. By symmetry, we need only prove (a). Note that

\[|J_1| + |J_2| = 2|I| + |K| = |I| + |U| \geq |I| + |J_1 \cup J_2| = |I| + |J_1| + |J_2| - |J_1 \cap J_2|,\]
whence \( |J_1 \cap J_2| \geq |I| \). Let (1) denote the left hand side of the formula to be established, and (2) the first choice on the right hand side.

Expand each term \((-q)^{\ell(I;K')} [I \cup K' | J_1]\) using Lemma 5.4(b) and insert into (1). Thus, (1) equals

\[
\sum_{K = K' \cup K'' \atop J_1 = J'_1 \cup J''_1} (-q)^{\ell(J'_1;J''_1)} [I | J'_1][K' | J'_1][K'' \cup I | J_2].
\]

We next claim that the sum

\[
\sum_{U = I_1 \cup I_2 \atop I_1 \subseteq K} (-q)^{\ell(I_1;I_2)} [I_1 | J'_1][I_2 | J_2]
\]
equals (3). For \( I_1 \) as in (4), we have \(|I_1| = |J'_1| + |J''_1| = |J_1|\), and so Lemma 5.4(b) gives

\[
\sum_{J_1 = J'_1 \cup J''_1} (-q)^{\ell(J'_1;J''_1)} [I | J'_1][I_1 | J''_1] = 0 \quad (I_1 \not\subseteq K),
\]
because \( I \cap I_1 \neq \emptyset \) in this case. On the other hand, for fixed \( J'_1, J''_1 \) as in (4), we have

\[
\sum_{U = I_1 \cup I_2 \atop I_1 \subseteq K} (-q)^{\ell(I_1;I_2)} [I_1 | J''_1][I_2 | J_2]
\]
equals \( \sum_{K = K' \cup K''} (-q)^{\ell(K';K'' \cup I)} [K' | J''_1][K'' \cup I | J_2] \).

It follows from (5) and (6) that (4) = (3) as claimed, and thus (1) = (4).

For \( J''_1 \) as in (4), we have \(|J''_1| = |K'| = |K| - |K''| = |U| - |K'' \cup I| = |V| - |J_2|\). Hence, Lemma 5.4(a) says that

\[
\sum_{U = I_1 \cup I_2} (-q)^{\ell(I_1;I_2)} [I_1 | J''_1][I_2 | J_2] = \begin{cases} (-q)^{\ell(J''_1;J_2)} [U | V] & (J''_1 \cap J_2 = \emptyset) \\ 0 & (J''_1 \cap J_2 \neq \emptyset). \end{cases}
\]

Substituting (7) into (4), it follows that (1) is equal to the sum

\[
\sum_{J_1 = J'_1 \cup J''_1} (-q)^{\ell(J'_1;J''_1) + \ell(J''_1;J_2)} [I | J'_1] d(J''_1),
\]
where \( d(J''_1) = [U | V] \) if \( J''_1 \) and \( J_2 \) are disjoint, but \( d(J''_1) = 0 \) otherwise.

If \(|J_1 \cap J_2| > |I|\), then since any \(|J'_1| = |I|\), we see that \( J_1 \cap J_2 \not\subseteq J'_1 \) and so \( J''_1 \cap J_2 \neq \emptyset \). Thus in this case all \( d(J''_1) = 0 \), and so (1) = (8) = (2) in this case. \( \Box \)
5.7. Lemma. Let \( r, c \in \{1, \ldots, n\} \) and \( I, J \subseteq \{1, \ldots, n\} \) with \(|I| = |J| \geq 1\). If \( r > \max(I) \) and \( c > \max(J) \), then
\[
[I \mid J]X_{rc} - q^2X_{rc}[I \mid J] = (1 - q^2)[I \sqcup \{r\} \mid J \sqcup \{c\}].
\]

Proof. Since we may work in \( \mathcal{O}_q(M_{I \cup \{r\}, J \cup \{c\}}(k)) \), it suffices to consider the case that \( r = c = n \) and \( I = J = \{1, \ldots, n - 1\} \). Now \([I \mid J] = A(nn)\) in the notation of [16, (4.3)].

Set \( D_q = [I \sqcup \{r\} \mid J \sqcup \{c\}] = [1 \cdots n | 1 \cdots n] \).

The first two \( q \)-Laplace relations in [16, Corollary 4.4.4] yield
\[
\sum_{j=1}^{n} (-q)^{j-n} X_{nj}A(nj) = \sum_{j=1}^{n} (-q)^{n-j} A(nj)X_{nj} = D_q.
\]

Solving for \( X_{nn}A(nn) \) and \( A(nn)X_{nn} \), we obtain
\[
X_{nn}A(nn) = D_q - \sum_{j<n} (-q)^{j-n} X_{nj}A(nj)
\]
\[
A(nn)X_{nn} = D_q - \sum_{j<n} (-q)^{n-j} A(nj)X_{nj}.
\]

For any \( j \), the third relation of [16, Lemma 5.1.2] implies that
\[
X_{nj}A(nj) = A(nj)X_{nj} + (1 - q^{-2}) \sum_{l<j} (-q)^{j-l} A(nl)X_{nl}.
\]

Substituting (4) into (2) for all \( j < n \), we obtain
\[
X_{nn}A(nn) = D_q - \sum_{j<n} (-q)^{j-n} A(nj)X_{nj}
\]
\[- (1 - q^{-2}) \sum_{j<n} \sum_{l<j} (-q)^{2j-l-n} A(nl)X_{nl}
\]
\[
= D_q - \sum_{l<n} \left[ (-q)^{l-n} + (1 - q^{-2}) \sum_{l<j<n} (-q)^{2j-l-n} \right] A(nl)X_{nl}.
\]

The expression in square brackets can be simplified as follows:
\[
(-q)^{l-n} + (1 - q^{-2}) \sum_{l<j<n} (-q)^{2j-l-n} = (-q)^{l-n} \left[ 1 + (1 - q^{-2}) \sum_{0<m<n-l} (-q)^{2m} \right]
\]
\[
= (-q)^{n-l-2}.
\]

Substituting (6) into (5) and replacing \( l \) by \( j \), we obtain
\[
X_{nn}A(nn) = D_q - \sum_{j<n} (-q)^{n-j-2} A(nj)X_{nj}.
\]

Finally, combining (3) with (7), we conclude that
\[
A(nn)X_{nn} - q^2X_{nn}A(nn) = (1 - q^2)D_q,
\]
as desired. \( \square \)
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