Totally nonnegative matrices

Dublin, February 2010

- A matrix is **totally positive** if each of its minors is positive.

- A matrix is **totally nonnegative** if each of its minors is nonnegative.

History

- Fekete (1910s)
- Gantmacher and Krein, Schoenberg (1930s): small oscillations, eigenvalues
- Karlin and McGregor (1950s): statistics, birth and death processes
- Lindström (1970s): planar networks
- Gessel and Viennot (1985): binomial determinants, Young tableaux
- Gasca and Peña (1992): optimal checking
- Lusztig (1990s): reductive groups, canonical bases
- Fomin and Zelevinsky (1999/2000): survey articles (eg Math Intelligencer)
- Postnikov (2006): the totally nonnegative grassmannian

- The eigenvalues of a totally positive matrix are positive real and distinct

- Every totally nonnegative matrix is the limit of a sequence of totally positive matrices

- As a result, the eigenvalues of a totally nonnegative matrix are real and nonnegative

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

¿ How much work is involved in checking if a matrix is totally positive?

Eg. $n = 4$:

$$\#\text{minors} \quad = \quad \sum_{k=1}^{n} \binom{n}{k}^2 \quad = \quad \binom{2n}{n} - 1 \quad \approx \quad \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

2×2 **case**

The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has **five** minors: $a, b, c, d, \Delta = ad - bc$.

If $b, c, d, \Delta = ad - bc > 0$ then

$$a = \frac{\Delta + bc}{d} > 0$$

so it is sufficient to check **four** minors.

Theorem (Fekete, 1913)

A matrix is totally positive if each of its **solid minors** is positive.

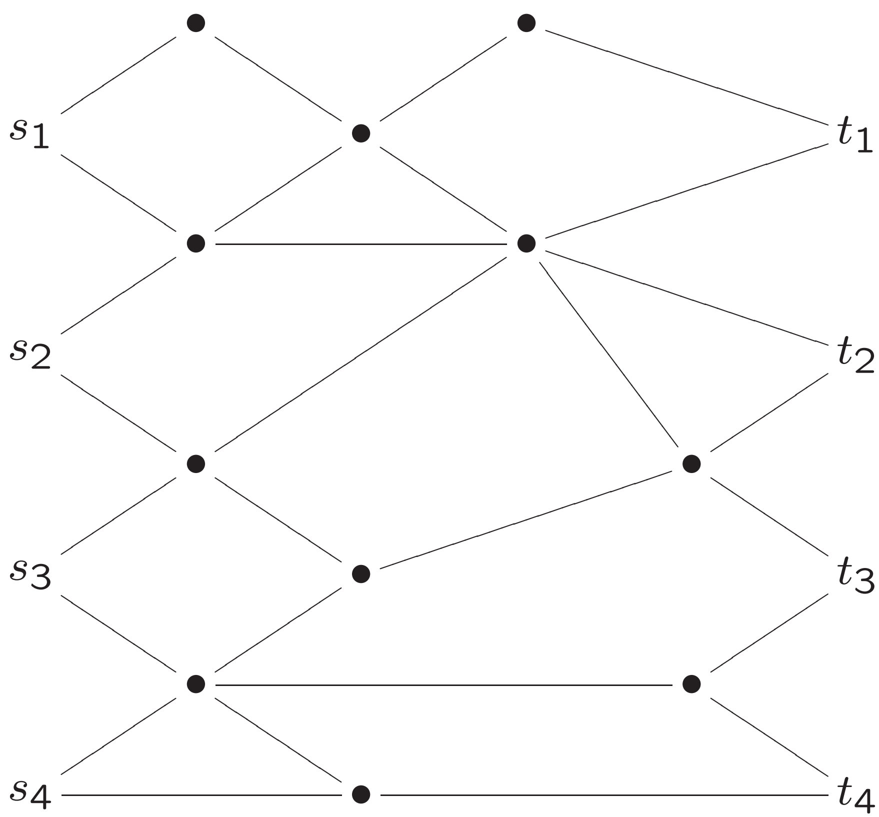
Theorem (Gasca and Peña, 1992)

A matrix is totally positive if each of its **initial minors** is positive.

Theorem (Gasca and Peña, 1992)

A totally nonnegative matrix is totally positive if each of its **corner minors** is positive.

Planar networks Consider an directed graph with no directed cycles, n sources and n sinks.



$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Edges directed left to right.

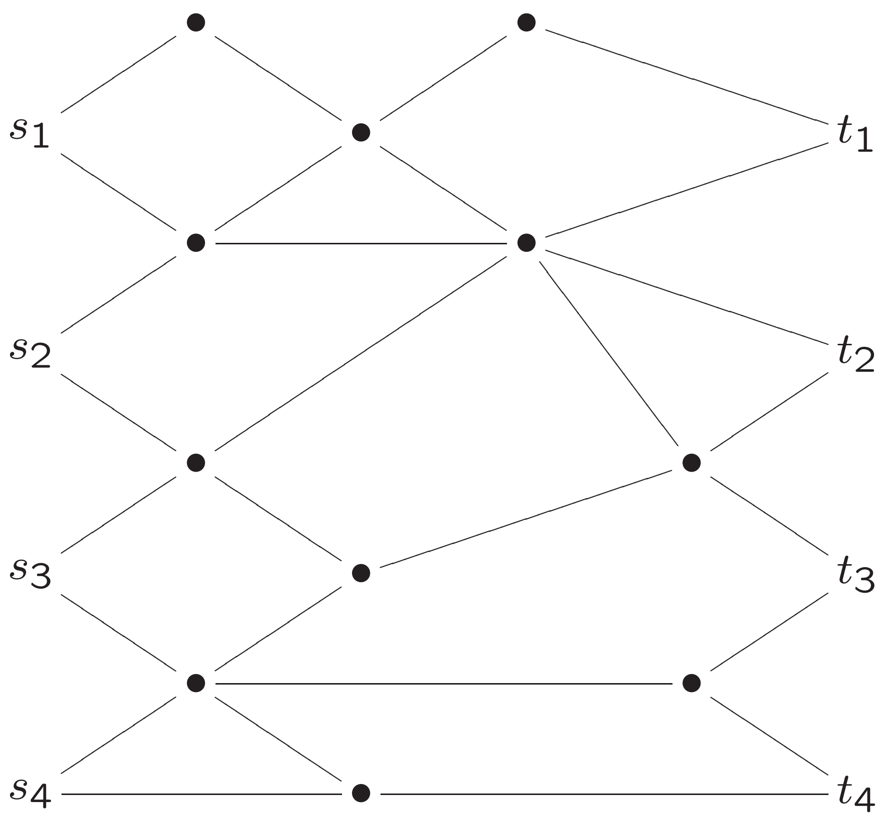
(Skandera: Introductory notes on total positivity)

Notation The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$.

Theorem (Lindström)
The path matrix of any planar network is totally nonnegative. In fact, the minor $[I \mid J]$ is equal to the number of families of nonintersecting paths from sources indexed by I and sinks indexed by J.

If we allow weights on paths then even more is true.

Theorem
Every totally nonnegative matrix is the weighted path matrix of some planar network.

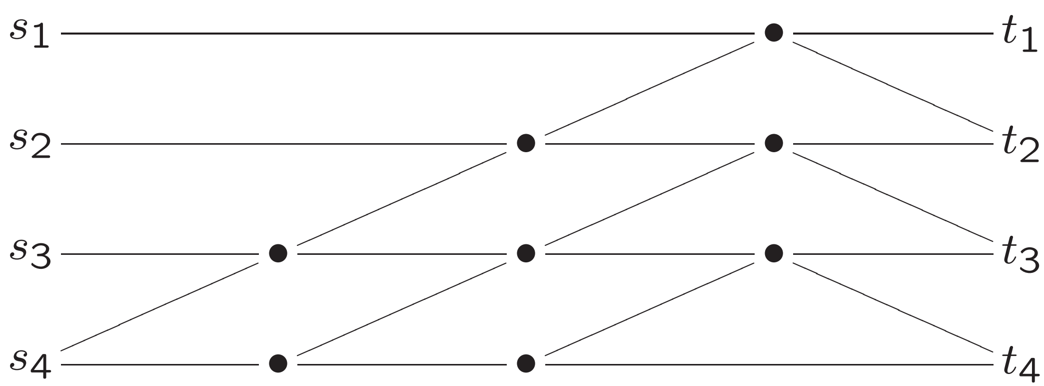


Edges directed left to right.

$M = \big(m_{ij}\big)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Binomial example



Edges directed left to right.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix}$$

Let $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ be the set of totally nonnegative $m \times p$ real matrices.

Let Z be a subset of minors. The **cell** S_Z^o is the set of matrices in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero).

Some cells may be empty. The space $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ is partitioned by the nonempty cells.

Example In $\mathcal{M}_2^{\text{tnn}}$ the cell $S^{\circ}_{\{[2,2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

Exercise There are 14 nonempty cells in $\mathcal{M}_2^{\text{tnn}}$.

Consider the 2×2 minors of the matrix

$$P := \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix}$$

$$[12] = 1, \quad [13] = a \quad [14] = b$$

$$[23] = c, \quad [24] = d \quad [34] = ad - bc$$

So, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is totally nonnegative if and only if the **Plücker coordinates** of the point P in the **grassmannian** $\mathcal{G}(2,4)$ are all nonnegative.

- $\mathcal{G}(2,4)$ is the grassmannian of 2-spaces in 4-space

- Specify P by two linearly independent vectors $\mathbf{v_1}, \mathbf{v_2}$

- Display in a 2×4 matrix

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{pmatrix}$$

- Many such matrices give the same P (change of basis, left multiplication by $GL(2)$)

- Let $[ij]$ be the 2×2 minor using columns i and j

- The ratios [12] : [13] : [14] : [23] : [24] : [34] specify P uniquely

- There is a **Plücker relation** $[12][34] - [13][24] + [14][23] = 0$

We can choose a **normal form** for $P \in \mathcal{G}(2,4)$ by reducing to echelon form: the generic case is

$$\begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \end{pmatrix} \quad \approx \quad M_2(k) \quad \approx \quad k^4$$

The **big cell** of $\mathcal{G}(2,4)$ is the space of 2×2 matrices

The next most likely case is

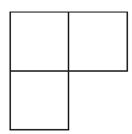
$$\begin{pmatrix} 1 & \star & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix} \quad \approx \quad k^3$$

Continuing, we obtain

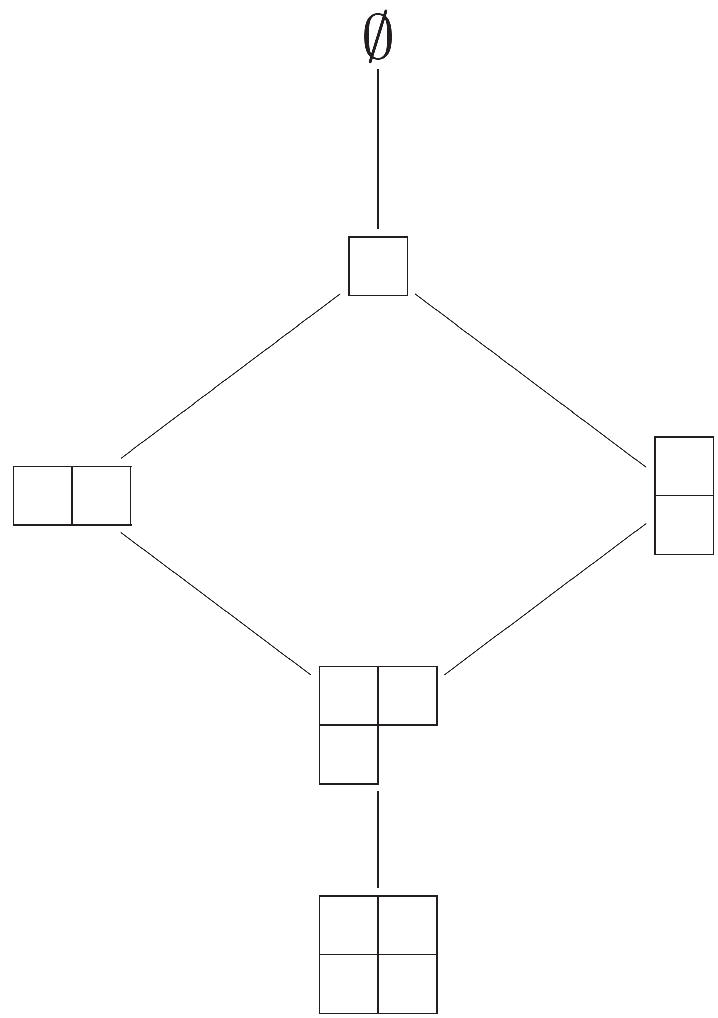
$$\mathcal{G}(2,4) \quad \approx \quad k^4 \sqcup k^3 \sqcup k^2 \sqcup k^2 \sqcup k^1 \sqcup k^0$$

To each possible case, we associate a Young diagram (or partition). For example,

$$\begin{pmatrix} 1 & \star & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix} \approx$$



We get all possible Young diagrams that fit into a 2×2 box.



The **totally nonnegative grassmannian** $\mathcal{G}_{kn}^{\text{tnn}}$ consists of the k-dimensional subspace of $\mathbb{R}^n$ which can be represented by a $k \times n$ matrix A all of whose $k \times k$ minors are ≥ 0

Cell decomposition: Cells are specified by stating which $k \times k$ minors are zero (and the rest are nonzero)

Again, if Z is a subset of $k \times k$ minors then S_Z° is the cell where the minors in Z are zero and the rest are > 0

Postnikov (arXiv:math/0609764) defines **Le-diagrams**: a Young diagram that fits into a $k \times n$ array that is filled with entries either 0 or 1 is said to be a **Le-diagram** if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0.

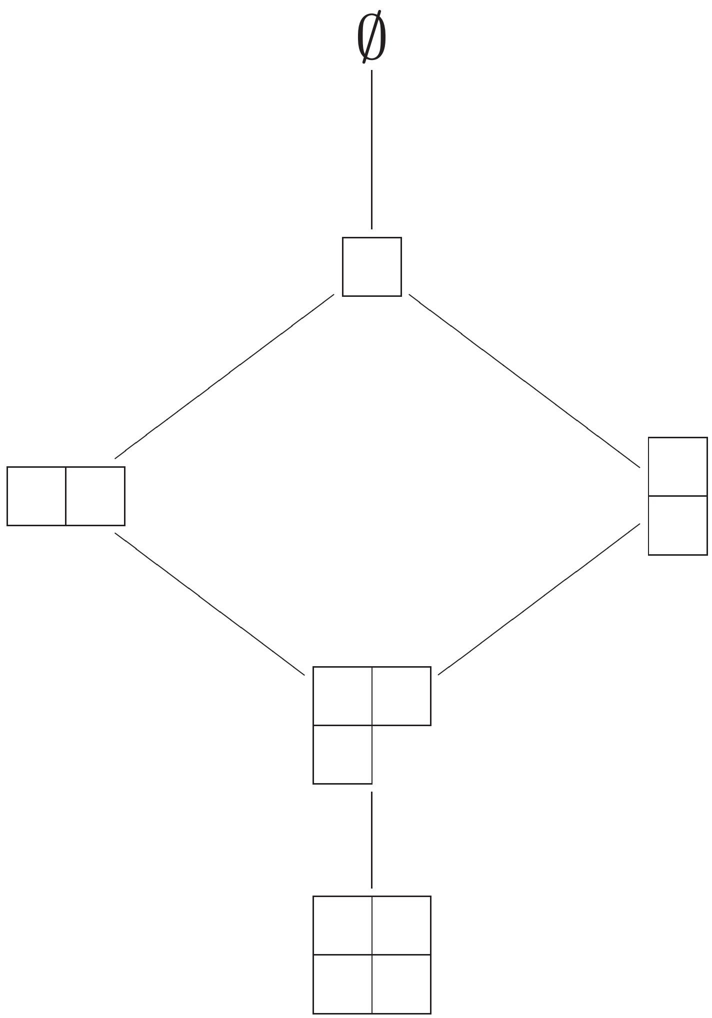
An example and a nonexample of a Le-diagram inside a 5×7 array

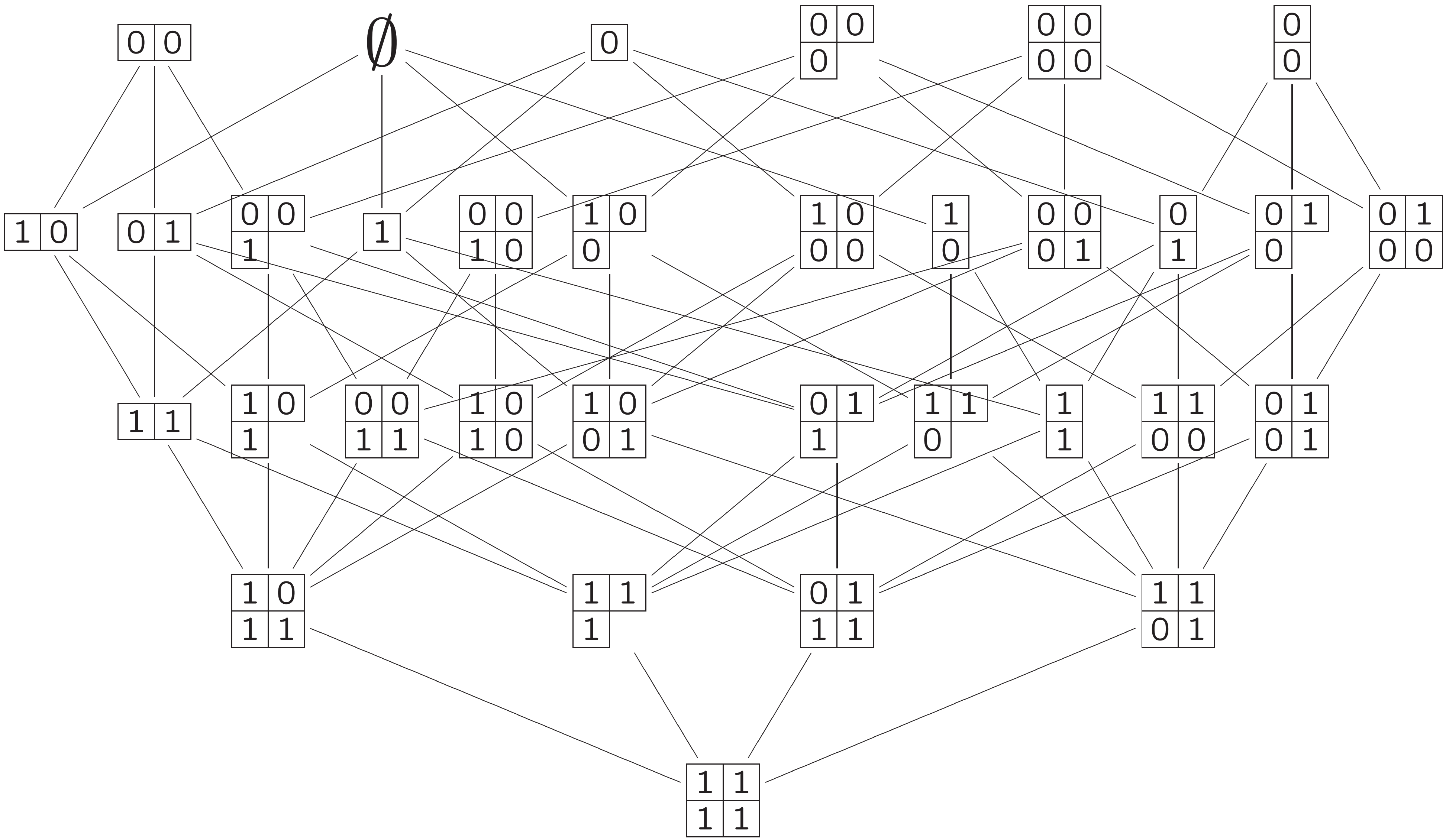
1	1	0	1	0	1	1
0	0	0	1	0	1	1
1	1	1	1	0		
0	0	0	1	0		
1	1	1				

1	1	0	1	0	1	1
0	0	0	1	0	1	0
1	1	1	1	0		
0	0	0	1	0		
1	1	1				

Theorem There is a bijection between Le-diagrams on Young diagrams that fit into a $k \times n$ array and nonempty cells S_Z° in $\mathcal{G}_{kn}^{\text{tnn}}$.

Example There are 33 nonempty cells in $\mathcal{G}_{kn}^{\text{tnn}}$.





Postnikov's algorithm starts with a Le-Diagram and produces a planar network from which one generates a point in the non-negative grassmannian, and hence a nonempty cell.

Example in the big cell, or matrix, case

	0	
0	0	

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Lauren Williams (Advances (2005)) has counted the number of nonempty cells in $\mathcal{G}_{kn}^{\mathrm{tnn}}$ by counting the number of Le-diagrams on Young diagrams fitting in a $k \times (n-k)$ array, by using a recursion formula on the size of the Young diagrams.

• Let $A_{k,n}(q)$ be the polynomial in q whose q^r coefficient is the number of nonempty totally nonnegative cells in $\mathcal{G}_{kn}^{\mathrm{tnn}}$ that have dimension r. Then

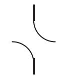
$$A_{k,n}(q) = \sum_{i=0}^{n} \binom{n}{i} q^{-(k-i)^2} \left\{ [i-k]_q^i [k-i+1]_q^{n-i} - [i-k-1]_q^i [k-i]_q^{n-i} \right\}$$

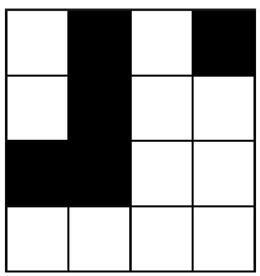
where $[m]_q = 1 + q + \cdots + q^{m-1}$

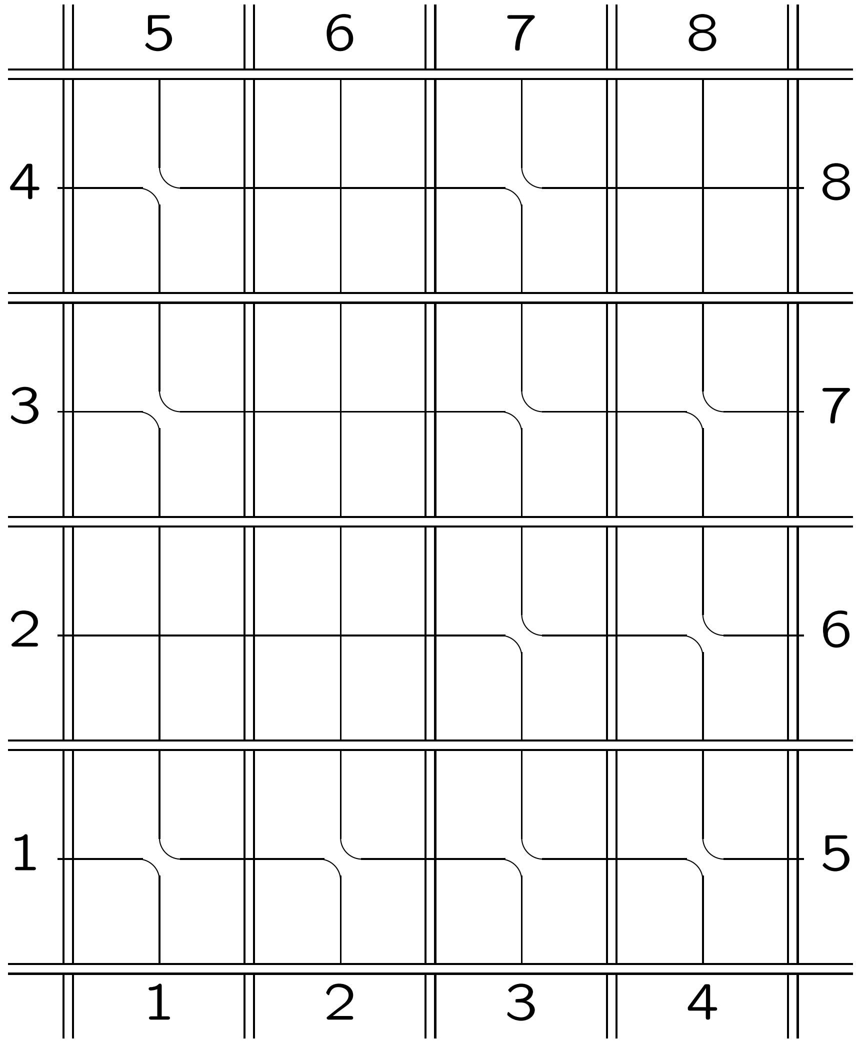
Setting $q = 1$ in the formula gives the total number of nonempty totally nonnegative cells in $\mathcal{G}_{kn}^{\mathrm{tnn}}$

For example, there are 883 cells in the 3×6 nonnegative real grassmannian

Restricted permutations versus Le-diagrams

Replace ■ by $+$ and □ by 

$C =$ 

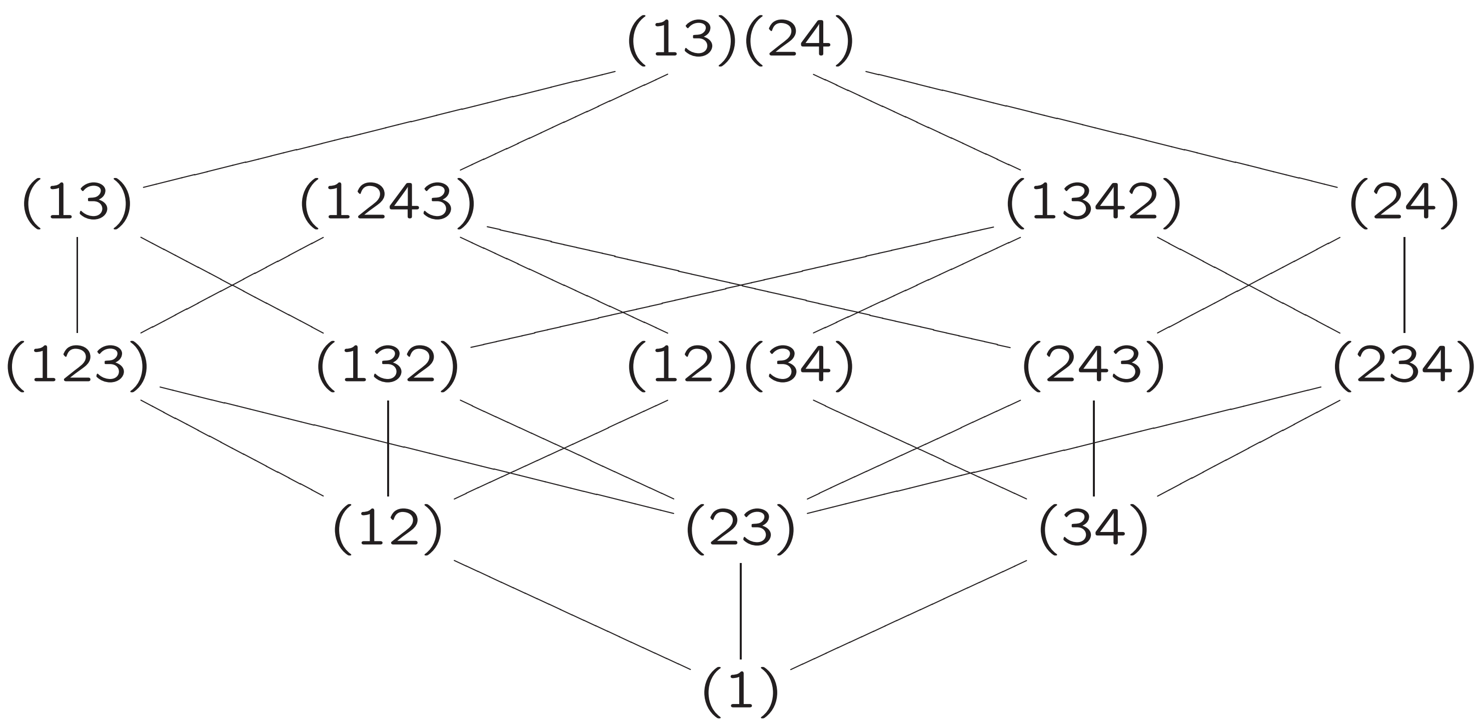


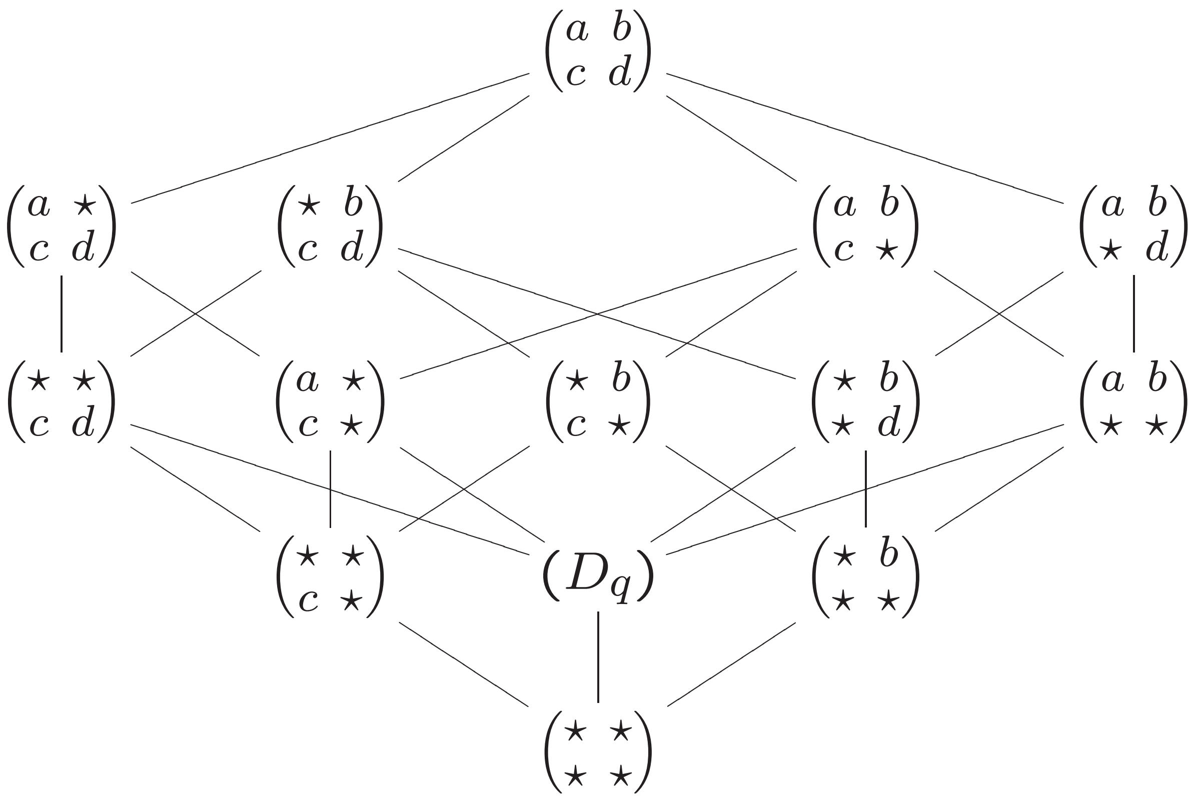
Restricted permutations

$w \in S_4$ with

$$-2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4.$$

There are 14 such permutations:





To interpret this picture, note that, for example, $\begin{pmatrix} a & b \\ c & \star \end{pmatrix}$ denotes the cell in which a, b and c are all zero, and all consequences of these zeros (eg the determinant must also be zero).

Related articles

- A Knutson, T Lam, and D E Speyer: *Positroid varieties I: juggling and geometry*, htpp://arxiv.org/abs/0903.3694.

- S Launois and T H Lenagan, *From totally nonnegative matrices to quantum matrices and back, via Poisson geometry*, http://arxiv.org/abs/0911.2990

- S Oh, *Positroids and Schubert matroids*, http://arxiv.org/abs/0803.1018

- K Talaska, *Combinatorial formulas for Le-coordinates in a totally nonnegative Grassmannian*, http://arxiv.org/abs/0812.0640