ORNSTEIN-UHLENBECK PROCESS ON ABSTRACT WIENER SPACES, HYPERCONTRACTIVITY, AND LOGARITHMIC SOBOLEV INEQUALITY

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Abstract. We briefly go over Ornstein-Uhlenbeck processes on abstract Wiener spaces. Then, we prove the hypercontractivity of the Ornstein-Uhlenbeck semigroup and show its equivalence to the logarithmic Sobolev inequality. This presentation is mainly based on the monograph by Shigekawa [14].

Contents

1. Ornstein-Uhlenbeck process on an abstract Wiener space

Consider the following Langevin equation:

$$dX_t = \alpha dB_t - \beta X_t dt, \qquad X_0 = x_0, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and $\beta > 0$. Its solution X_t given by

$$X_t = e^{-\beta t} x_0 + \alpha \int_0^t e^{-\beta(t-u)} dB_u \tag{1.2}$$

is called an Ornstein-Uhlenbeck process. It is known that the random variable X_t given by (1.2) is Gaussian with mean $x_0 e^{-\beta t}$ and variance $\frac{\alpha^2}{2\beta}(1 - e^{-2\beta t})$. Moreover, the Gaussian measure with mean 0 and variance $\frac{\alpha^2}{2\beta}$ is an invariant measure for the Ornstein-Uhlenbeck process. In the following, we discuss an Ornstein-Uhlenbeck process on an abstract Wiener space.

1.1. **Abstract Wiener space.** We first go over the definition of abstract Wiener spaces introduced by Gross [7]. See [7, 9, 14] for more on this topic. The standard Gaussian measure on $\mathbb{R}^d$ is the unique probability measure $\mu = \mu_d$ such that

$$\int_{\mathbb{R}^d} e^{i\langle x,y\rangle} d\mu(x) = e^{-\frac{1}{2}|y|^2},$$

for all $y \in \mathbb{R}^d$.

Let H be a real separable Hilbert space. An obvious attempt to extend the standard Gaussian measure to H would be to consider a probability measure μ such that

$$\int_{H} e^{i\langle x,h\rangle_H} d\mu(x) = e^{-\frac{1}{2}\|h\|_H^2},$$

for al $h \in H$. However, such a measure does not exists if dim $H = \infty$. Indeed, suppose that such a measure existed. Then, for an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H, we have

$$\int_{H} e^{i\langle x,e_n\rangle_H} d\mu(x) = e^{-\frac{1}{2}}. \tag{1.3}$$

Since $\langle x, e_n\rangle_H \to 0$ as $n \to \infty$ for any $x \in H$, it follows from Lebesgue dominated convergence theorem that the limit of the left-hand side in (1.3) is $e^0 = 1 \neq e^{-\frac{1}{2}}$. This is a contradiction.

We resolve this issue by extending this measure μ on a larger Banach space $B \supset H$. Given a real separable Hilbert space H, let $\mathcal{F}$ denote the set of finite dimensional orthogonal projections $\mathbf{P}$ of H. Then, define a cylinder set E by

$$E = \{x \in H : \mathbf{P}x \in F\}, \tag{1.4}$$

where $\mathbf{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbf{P}H$, and let $\mathcal{R}$ denote the collection of such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. Then, the Gauss measure μ on H is defined by

$$\mu(E) = \mu_{\mathbf{P}}\big(\{x \in \mathbf{P}H : x \in F\}\big)$$

for $E \in \mathcal{R}$ of the form (1.4), where $\mu_{\mathbf{P}}$ is the standard Gaussian measure on $\mathbf{P}H$. It is known that μ is finitely additive but not countably additive in $\mathcal{R}$.

We say that a seminorm $|||\cdot|||$ in H is called measurable if for every $\varepsilon > 0$, there exists $\mathbf{P}_\varepsilon \in \mathcal{F}$ such that

$$\mu_{\mathbf{P}}\big(\{x \in \mathbf{P}H : |||\mathbf{P}x||| > \varepsilon\}\big) < \varepsilon \tag{1.5}$$

for $\mathbf{P} \in \mathcal{F}$ orthogonal to $\mathbf{P}_\varepsilon$. Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite dimensional. Let B be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into B. The triple (H, B, i) is called an *abstract Wiener space*.[1] Now, regarding $y \in B^*$ as an element of $H^* \equiv H$ by restriction, we embed B^* in H. Define the extension of μ onto B (which we still denote by μ) as follows. For a Borel set $F \subset \mathbf{P}H \cong \mathbb{R}^d$, set

$$\mu(\{x \in B : ((x, y_1), \cdots, (x, y_n)) \in F\}) := \mu(\{x \in H : (\langle x, y_1\rangle_H, \cdots, \langle x, y_n\rangle_H) \in F\}),$$

[1] Sometimes, (B, H) or (B, H, μ) is referred to as an abstract Wiener space.

where y_j's are in B^* and $(\cdot,\cdot)$ denote the natural pairing between B and B^*. Let $\mathcal{R}_B$ denote the collection of cylinder sets $\{x \in B : ((x,y_1),\cdots,(x,y_n)) \in F\}$ in B.

Theorem 1.1 (Gross [7])**.** *μ is countably additive in the σ-field generated by $\mathcal{R}_B$.*[2]

Another equivalent definition for an abstract Wiener space is the following,

Definition 1.2. Let B be a real separable Banach space B, H be a real separable Hilbert space that is densely and continuously embedded in B, and μ be a Gaussian measure on B. We say that (B,H,μ) is an *abstract Wiener space* if we have

$$\int_B e^{i\langle x,\varphi\rangle} d\mu(x) = e^{-\frac{1}{2}\|\varphi\|^2_{H^*}} \tag{1.6}$$

for all $\varphi \in B^* \subset H^*$. Here, the bracket $\langle\,\cdot\,,\cdot\,\rangle$ denotes the B-B^* duality pairing. The Hilbert space H is called a reproducing kernel Hilbert space or a Cameron-Martin space.[3]

Example 1 (classical Wiener space)**.** Let $\{X_t\}$ be the Wiener process (starting at 0). Then, noting that the Wiener process is pathwise continuous (almost surely), it induces a probability measure on the path space

$$C_0([0,\infty);\mathbb{R}^d) = \{w : [0,\infty) \to \mathbb{R}^d \,:\, w \text{ is continuous and } w_0 = 0\}$$

The suffix 0 indicates that a path starts at 0. The measure μ defined on $C_0([0,\infty);\mathbb{R}^d)$ in this way is called the Wiener measure and the space $C_0([0,\infty);\mathbb{R}^d)$ coupled with the measure μ is called the Wiener space.

In the following, we restrict our attention to a finite interval $[0,T]$. Let $B = C_0([0,T);\mathbb{R}^d)$ and μ be the Wiener measure defined on B. Given $f = (f_1,\dots,f_d) \in L^2([0,T];\mathbb{R}^d)$, define the Wiener integral $I(f)$ by

$$I(f) = \int_0^T f(t)\cdot dw_t = \sum_{j=1}^d \int_0^T f_j(t)dw_t^j.$$

Then, $I(f)$ is Gaussian with mean 0 and variance $\|f\|_{L^2([0,T])}$.

Riesz representation theorem says that B^* coincides with the set of all signed measures on $(0,T]$ of finite variation. Given $\varphi \in B^*$, define $f_\varphi : [0,T] \to \mathbb{R}$ by

$$f_\varphi(t) = \int_{(0,t]} \nu_\varphi(ds),$$

where ν_φ is the (vector-valued) signed measure on $(0,T]$ corresponding to φ. Note that $f_\varphi \in BV([0,T])$. By Ito formula, we have

$$f_\varphi(T)\cdot w(T) = \int_0^T w_s\cdot\nu_\varphi(ds) + \int_0^T f_\varphi(s)\cdot dw_s.$$

Thus, we have

$$\langle w,\varphi\rangle = \int_0^T \big(f_\varphi(T) - f_\varphi(t)\big)\cdot dw_t. \tag{1.7}$$

[2]Indeed, the σ-field generated by $\mathcal{R}_B$ is the Borel σ-field $\mathcal{B}(B)$ of B. See Theorem 4.2 on p. 74 in [9].

[3]The triple $B^* \subset H^* = H \subset B$ is called a Gel'fand triple. It is, however, more natural to start with a nuclear space $V := B^*$ and view it as $V \subset H \subset V^*$ in this case.

Then, by define h_φ as

$$h_\varphi(t) := \int_0^t \big(f_\varphi(T) - f_\varphi(s)\big) ds, \tag{1.8}$$

we can rewrite (1.7) as

$$\langle w, \varphi \rangle = \int_0^T \dot{h}_\varphi(t) \cdot dw_t. \tag{1.9}$$

Now, define a Hilbert space H by[4]

$$H = \big\{ h \in B : h \text{ is absolutely continuous}, \ h_0 = 0, \text{ and } \dot{h} \in L^2([0,T];\mathbb{R}^d) \big\} \tag{1.10}$$

with the inner product $\langle h, k\rangle_H := \langle \dot{h}, \dot{k} \rangle_{L^2([0,T])}$. Note that (1.8) defines a mapping $\iota^* : B^* \to H$. We use this notation since i^* is the dual operator of the natural inclusion $\iota : H \to B$. Indeed, we have

$$_B\langle \iota h, \varphi \rangle_{B^*} = \langle h, \iota^* \varphi\rangle_H$$

for $h \in H$ and $\varphi \in B^*$. See [14]. Lastly, note that we can use (1.8) and (1.9) and directly verify (1.6), with the understanding that φ on the right-hand side of (1.6) is really given by $\iota^*\varphi = h_\varphi$.

Example 2. Let $H = \dot{H}^1(\mathbb{T})$, where $\dot{H}^1(\mathbb{T})$ is the homogeneous Sobolev space on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the norm

$$\|u\|_{\dot{H}^1(\mathbb{T})} = \sum_{n \in \mathbb{Z}\setminus\{0\}} |\widehat{u}_n|^2,$$

and consider the Gauss measure μ on H:

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{\dot{H}^1(\mathbb{T})}} du = Z^{-1} e^{-\frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 dx} du.$$

Then, by setting $B = \dot{H}^s(\mathbb{T})$, one can show that (B, H, μ) is an abstract Wiener space if and only if $s < \frac{1}{2}$. In this case, u under μ is the periodic Wiener process[5] on $\mathbb{T}$. See [1] for more discussion on this issue.

Let (B, H, μ) be an abstract Wiener space. Then, an element $\varphi \in B^*$ is a measurable function on B with the Gaussian distribution and thus belongs to $L^2(B, \mu)$. From (1.6), we have

$$\int_B \langle w, \varphi\rangle^2 d\mu(w) = \|\iota^*\varphi\|_H^2. \tag{1.11}$$

With $\iota^* : B^* \to H^*$, we view B^* as a subspace of H^*. Then, (1.11) gives rise to an isometric isomorphism from H^* into $L^2(B, \mu)$. We denote this map by I_1 and call it the *Wiener integral of order 1.* We also denote the image of I_1 by $\mathcal{H}_1$, i.e. $H \cong \mathcal{H}_1$ under I_1. In case of the classical Wiener space, I_1 is given by

$$I_1(h^*) = \int_0^T \dot{h}(s) \cdot dw_s,$$

[4]We can also take $T = \infty$ by replacing $[0,T]$ with $[0, \infty)$.

[5]Compare this with Example 1. In both examples, the Cameron-Martin spaces are given by $\dot{H}^1$ with different boundary conditions. On the one hand, with the Dirichlet boundary condition $h_0 = 0$, we recover the usual Wiener process on $[0,T]$ or $\mathbb{R}$ as in Example 1. On the other hand, the periodic boundary condition gives rise to the periodic Wiener process on $\mathbb{T}$.

where $h^* \in H^*$ denotes the element corresponding to $h \in H$ under the Riesz isomorphism.

We conclude this subsection by stating a few more properties of an abstract Wiener space. We first state the Cameron-Martin Theorem [2].

Theorem 1.3. *Let (B, H, μ) be an abstract Wiener space. Then, for any $h \in H$, the shifted measure $\mu_h := \mu(\,\cdot - h)$ is mutually absolutely continuous with respect to μ. Moreover, the Radon-Nikodým derivative is given by*

$$\frac{d\mu(\,\cdot - h)}{d\mu}(x) = \exp\bigg\{ -\frac{1}{2}\|h\|_H^2 + I_1(h^*)(x)\bigg\},$$

for $x \in B$. Moreover, for $y \in B \setminus H$, the shifted measure $\mu_y = \mu(\,\cdot - y)$ and μ are mutually singular.

Note that Theorem 1.3 shows that the Cameron-Martin space is a natural direction of differentiation in an abstract Wiener space.

The next theorem is due to Fernique [6].

Theorem 1.4. *Let (B, H, μ) be an abstract Wiener space. Give be a continuous seminorm p on B, there exists a constant $\alpha = \alpha(p) > 0$ such that*

$$\int_B e^{\alpha p(x)^2} d\mu(x) < \infty.$$

See [14] for the proofs of Theorems 1.3 and 1.4.

1.2. **Ornstein-Uhlenbeck process.** Let (B, H, μ) be an abstract Wiener space. We will construct an *Ornstein-Uhlenbeck process* on B.

For $t \geq 0$, let μ_t be the induced measure of μ under the map $x \mapsto \sqrt{t}x$. i.e. $\mu_t(A) = \mu(\sqrt{t}x \in A)$. Then, the characteristic function of μ_t is given by

$$\widehat{\mu}_t(\varphi) = \int_B e^{i\langle x, \varphi\rangle} \mu_t(dx) = \exp\bigg\{ -\frac{t}{2}\|\iota^*\varphi\|_{H^*}^2\bigg\}.$$

for $\varphi \in B^*$. We define transition probabilities on B by

$$P_t(x, A) = \int_B \mathbf{1}_A\big(e^{-t}x + \sqrt{1 - e^{-2t}}y\big)\mu(dy) \tag{1.12}$$

for $t \geq 0$, $x \in B$, and $A \in \mathcal{B}(B)$. Then, noting $\mu_t * \mu_s = \mu_{t+s}$, the following Chapman-Kolmogorov equation holds:

$$\int_B P_t(x, dy)P_s(y, A) = P_{t+s}(x, A).$$

Then, Kolmogorov extension theorem guarantees existence of a Markov process associated with $\{P_t(x, A)\}$.

Definition 1.5. The Markov process associated with the transition probabilities $\{P_t(x, dy)\}$ is called an *Ornstein-Uhlenbeck process.*

At this point, the Ornstein-Uhlenbeck process is realized as a measure on $B^[0, \infty)$ for any given starting point $x \in B$. In fact, one can show that given $x \in B$, there exists a constant $C = C(x)$ such that

$$\iint_{B\times B} \|y - z\|^4 P_s(x, dy)P_t(y, dz) \leq C(t^2 + t^4)$$

for any $s, t \geq 0$. Thus, by Kolmogorov's continuity criterion, we can realize the Ornstein-Uhlenbeck process as a measure on $C([0,\infty); B)$.

Remark 1.6. (a) The Markov process associated with the transition probabilities $P_t^B(x, A) = \mu_t(A - x)$ is called a Brownian motion. We can also realize the Brownian motion as a measure on $C([0,\infty); B)$. Denote the laws of the Brownian motion and Ornstein-Uhlenbeck process by P_x^B and P_x. If $\dim B < \infty$, the laws P_x^B and P_x are equivalent to each other in $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ-fields generated by the trajectories up to time t. If $\dim B = \infty$, however, P_x^B and P_x are mutually singular even on $\mathcal{F}_t$.

(b) The space $C([0,\infty); B)$ becomes a separable Fréchet space under the compact-open topology. Both P_x^B and P_x are Gaussian measure on $C([0,\infty); B)$. Denoting the corresponding reproducing kernel Hilbert spaces by $H_{P_x^B}$ and H_{P_x}. Then, arguing as in the classical Wiener space, we see that

$$
\begin{aligned}
H_{P_x^B} &= \big\{ h \in C([0,\infty); B) : h(t) = \int_0^t \dot{h}(s) ds, \ \dot{h} \in L^2([0,\infty); H) \big\}, \\
H_{P_x} &= \big\{ h \in C([0,\infty); B) : h(t) = e^{-t} \int_0^t e^s \dot{h}(s) ds, \ \dot{h} \in L^2([0,\infty); H) \big\}.
\end{aligned}
$$

Recall that the Gaussian measure with mean 0 and variance $\frac{\alpha^2}{2\beta}$ is an invariant measure for the Ornstein-Uhlenbeck process given by (1.1). The next proposition shows that the Gaussian measure μ is the unique invariant measure for the Ornstein-Uhlenbeck process on B.

Proposition 1.7 (invariant measure)**.** *Let (B, H, μ) be an abstract Wiener space. Then, μ is the unique invariant measure for the Ornstein-Uhlenbeck process on B. Namely, we have*

$$
\int_B P_t(x, A) \mu(dx) = \mu(A) \tag{1.13}
$$

for all $A \in \mathcal{B}(B)$.

Proof. From (1.12), we have

$$
\begin{aligned}
\int_B P_t(x, A) \mu(dx) &= \int_B \int_B \mathbf{1}_A\big(e^{-t} x + \sqrt{1 - e^{-2t}} y\big) \mu(dy) \mu(dx) \\
&= \int_B \mathbf{1}_A(x) \mu_{e^{-2t}} \mu_{1 - e^{-2t}}(dx) \\
&= \mu(A).
\end{aligned}
$$

As for the uniqueness part of the statement, simply note from (1.12) that

$$
\lim_{t \to \infty} \int_B F(y) P_t(x, dy) = \int_B F(y) \mu(dy) \tag{1.14}
$$

for all bounded continuous functions F and all $x \in B$. Indeed, if ρ is another invariant measure, satisfying (1.13), then, it follows from (1.14) (and approximating $\mathbf{1}_A$ by bounded continuous functions) that

$$\begin{aligned}\rho(A) &= \int_B P_t(x, A)\rho(dx) = \int_B \int_B \mathbf{1}_A(y) P_t(x, dy)\rho(dx) \\ &\to \int_B \int_B \mathbf{1}_A(y)\mu(dy)\rho(dx) = \mu(A)\end{aligned}$$

for any measurable set A, as $t \to \infty$. $\square$

Remark 1.8. The uniqueness of the invariant measure μ implies that μ is ergodic. The following argument is taken from Theorem 5.16 in [4].

Suppose that μ is not ergodic. Then, there exists a non-trivial invariant set[6] $\Gamma \in \mathcal{B}(B)$. Define a measure μ_Γ by

$$\mu_\Gamma(A) = \frac{1}{\mu(\Gamma)}\mu(A \cap \Gamma). \tag{1.15}$$

for $A \in \mathcal{B}(B)$. In the following, we will show that μ_Γ is also an invariant measure, which shows a contradiction.

In view of (1.13), we need to show

$$\mu_\Gamma(A) = \int_B P_t(x, A)\mu_\Gamma(dx) \tag{1.16}$$

for all $A \in \mathcal{B}(B)$. In view of (1.15), we see that (1.16) is equivalent to

$$\mu(A \cap \Gamma) = \int_\Gamma P_t(x, A)\mu(dx). \tag{1.17}$$

Since Γ is invariant, we have

$$T_t \mathbf{1}_\Gamma = \mathbf{1}_\Gamma \qquad \text{and} \qquad T_t \mathbf{1}_{\Gamma^c} = \mathbf{1}_{\Gamma^c}.$$

for $t \geq 0$. Noting that $P_t(x, A) = T_t \mathbf{1}_A(x)$,

$$P_t(x, \Gamma) = \mathbf{1}_\Gamma(x) \qquad \text{and} \qquad P_t(x, \Gamma^c) = \mathbf{1}_{\Gamma^c}(x). \tag{1.18}$$

From (1.18) and (1.13), we have

$$\begin{aligned}\int_\Gamma P_t(x, A)\mu(dx) &= \int_\Gamma P_t(x, A \cap \Gamma)\mu(dx) + \int_\Gamma \underbrace{P_t(x, A \cap \Gamma^c)}_{=0\ \mu\text{-a.e.}}\mu(dx) \\ &= \int_\Gamma P_t(x, A \cap \Gamma)\mu(dx) = \int_B P_t(x, A \cap \Gamma)\mu(dx) = \mu(A \cap \Gamma).\end{aligned}$$

Therefore, (1.17) holds, yielding a contradiction.

2. Ornstein-Uhlenbeck semigroup: hypercontractivity

2.1. **Ornstein-Uhlenbeck semigroup and operator.** Let $\{T_t\}_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup defined by

$$T_t F(x) = \int_B F(y) P_t(x, dy) = \int_B F\big(e^{-t}x + \sqrt{1 - e^{-2t}}y\big)\mu(dy) \tag{2.1}$$

for a non-negative Borel measurable function F. For a general Borel measurable function F, by writing F as $F = F_+ - F_-$, we define

$$T_t F(x) = T_t F_+(x) - T_t F_-(x).$$

[6] See Theorem 5.15 in [4].

Here, $F_+ = F \vee 0$ and $F_- = (-F) \vee 0$. If $T_t F_+(x) = T_t F_-(x) = \infty$, then we set $T_t F(x) = \infty$ by convention.

Next, we introduce several classes of functions.

Definition 2.1. (i) We define $\mathcal{S}$ to be the collection of functions $F : B \to \mathbb{R}$ such that there exists $d \in \mathbb{N}$, $\varphi_1, \dots, \varphi_d \in B^*$, and $f \in C^\infty(\mathbb{R}^d)$ such that

$$F(x) = f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle) \tag{2.2}$$

for $x \in B$. Here, we assume that f and its derivatives has at most polynomial growth.

(ii) We define $\mathcal{S}_0 \subset \mathcal{S}$ such that f as above as a compact support.

(iii) We say $F \in \mathcal{S}$ is a polynomial if f as above is a polynomial. We denote the collection of polynomials on B by $\mathcal{P}$.

Note that, for any $p \geq 1$, we have $\mathcal{P}, \mathcal{S}_0, \mathcal{S} \subset L^p(B, \mu)$ and they are all dense in $L^p(B, \mu)$.

Proposition 2.2. *Let $p \geq 1$. Then, the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is a strongly continuous contraction semigroup in $L^p(B, \mu)$. Namely, we have*

$$\|T_t F\|_p \leq \|F\|_p, \tag{2.3}$$

$$\lim_{t\downarrow 0} \|T_t F - F\|_p = 0. \tag{2.4}$$

Proof. By Jensen's inequality, we have

$$|T_t F(x)|^p = \left| \int_B F(y) P_t(x, dy) \right|^p \leq \int_B |F(y)|^p P_t(x, dy).$$

Hence, by Proposition 1.7, we have

$$\begin{aligned} \|T_t F(x)\|_p^p = \int_B |T_t F(x)|^p \mu(dx) &\leq \int_B \int_B |F(y)|^p P_t(x, dy) \mu(dx) \\ &\leq \int_B |F(y)|^p \mu(dy) = \|F\|_p^p. \end{aligned}$$

This proves (2.3). As for (2.4), we use the density argument. For $F \in \mathcal{S}_0$, we have $\lim_{t\downarrow 0} T_f(x) = F(x)$. Since $\{T_t\}_{t\geq 0}$ is uniformly bounded, it follows from Lebesgue dominated convergence theorem that $\lim_{t\downarrow 0} \|T_t F - F\|_p = 0$. Then, (2.3) follows from the density of $\mathcal{S}_0$ in $L^p(B, \mu)$. □

Let L denote the generator (called the Ornstein-Uhlenbeck operator[7]) of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. Since the semigroup is define on $L^p(B, \mu)$, we should specify p and denote the generator by L_p with its domain $\mathrm{Dom}(L_p)$. If there is no confusion, however, we simply use L. In the next subsection, we will obtain a concrete expression for the Ornstein-Uhlenbeck operator L.

[7]It is also called the Hartree-Fock operator.

2.2. **Differentiation.** Recall the definition of a Gâteaux derivative. We say that a function $F : B \to \mathbb{R}$ is Gâteaux differentiable at $x \in B$ if there exists $\varphi \in B^*$ such that

$$\frac{d}{dt} F(x + ty)\bigg|_{t=0} = \langle y, \varphi \rangle$$

for all $y \in B$. Here, $\langle \cdot, \cdot \rangle$ denotes the B-B^* duality pairing. In this case, we say that φ is a Gâteaux derivative of F at x, denoted by $F'(x)$. Note that the a Gâteaux derivative corresponds to a directional derivative, while a Fréchet derivative corresponds to a total derivative.

In the following, we use Cameron-Martin theorem (Theorem 1.3) and extend the notion of differentiation called H-differentiation. Given an abstract Wiener space (B, H, μ), Cameron-Martin theorem states that the Gaussian measure μ is absolutely continuous under a shift in the direction of H.

Definition 2.3. A function $F : B \to \mathbb{R}$ is said to be *H-differentiable* at $x \in B$ if there exists $DF(x) \in H^*$ such that

$$\frac{d}{dt} F(x + th)\bigg|_{t=0} = \langle h, DF(x) \rangle$$

for all $h \in H$. We say that $DF(x)$ is the H-derivative of F at x.

We can also introduce higher order H-derivatives. Given $k \in \mathbb{N}$, we use $\mathcal{L}^k(H; \mathbb{R})$ to denote the collection of k-linear map $\Phi : \underbrace{H \times H}_{k \text{ times}} \to \mathbb{R}$. Then, F is said to be k-times H-differentiable if there exists $\Phi \in \mathcal{L}^k(H; \mathbb{R})$ such that

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(x + t_1 h_1 + \cdots t_k h_k)\bigg|_{t_1 = \cdots = t_k = 0} = \Phi(h_1, \dots, h_k)$$

for all $h_1, \dots, h_k \in H$. Such Φ is called the kth H-derivative of F at x, denoted by $D^k F(x)$.

We say that $\Phi \in \mathcal{L}^2(H; \mathbb{R})$ is of trace class if

$$\sup \sum_{n=1}^{\infty} |\Phi(h_n, k_n)| < \infty,$$

where $\{h_n\}$ and $\{k_n\}$ range over all complete orthonormal system in of H. The set of all trace class operators is denoted by $\mathcal{L}_{(1)}(H)$. Given $\Phi \in \mathcal{L}_{(1)}(H)$, the trace of Φ is defined by

$$\operatorname{tr} \Phi = \sum_{n=1}^{\infty} \Phi(h_n, h_n)$$

where $\{h_n\}$ is a complete orthonormal systems in H. Note that $\operatorname{tr} \Phi$ does not depend on the choice of $\{h_n\}$.

Given $F \in \mathcal{S}$ of the form (2.2), its Gâteaux derivative of F at $x \in B$ is given by

$$F'(x) = \sum_{j=1}^{d} \partial_j f(\langle x, \varphi_1 \rangle, \dots, \langle x, \varphi_d \rangle) \varphi_j. \tag{2.5}$$

Recall that $\varphi_j \in B^*$. The H-derivative of F at $x \in B$ is given by

$$DF(x) = \sum_{j=1}^{d} \partial_j f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle)\varphi_j, \tag{2.6}$$

where φ_j is regarded as an element in H^*. Strictly speaking, we should use $\iota^*\varphi_j$ instead of φ_j.

Similarly, the kth H-derivative of F at $x \in B$ is given by

$$D^k F(x) = \sum_{j_1, \dots, j_k = 1}^{d} \partial_{j_1} \cdots \partial_{j_k} f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle)\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}.$$

Without loss of generality, assume that $\{\varphi_j\}$ is an orthonormal system in H^*. Then, we have

$$\begin{aligned} \operatorname{tr} D^2 F(x) &= \sum_{j=1}^{d} \partial_j^2 f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle) \\ &= \Delta_d f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle), \end{aligned}$$

where Δ_d denotes the usual Laplacian on $\mathbb{R}^d$.

Now, we are ready to compute a concrete expression for the Ornstein-Uhlenbeck operator L.

Proposition 2.4. *Let $F \in \mathcal{S}$. Then, we have*

$$LF(x) = \operatorname{tr} D^2 F(x) - {}_B\langle x, F'(x)\rangle_{B^*}. \tag{2.7}$$

Proof. Let $F \in \mathcal{S}$ be given by (2.2). As before, we assume that $\{\varphi_j\}$ is an orthonormal system in H^*. Let $\xi = (\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle)$. Then, from (2.1), we have

$$T_t F(x) = \int_B F(y) P_t(x, dy) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) e^{-\frac{|\eta|^2}{2}} d\eta. \tag{2.8}$$

Then, for $t > 0$, we have

$$\begin{aligned} \frac{d}{dt} T_t F(x) &= \frac{d}{dt} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) e^{-\frac{|\eta|^2}{2}} d\eta \\ &= -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sum_{j=1}^{d} e^{-t}\xi^j \partial_j f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) e^{-\frac{|\eta|^2}{2}} d\eta \\ &\quad + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sum_{j=1}^{d} \partial_j f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) \frac{\eta^j e^{-2t}}{\sqrt{1 - e^{-2t}}} e^{-\frac{|\eta|^2}{2}} d\eta \\ &= -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sum_{j=1}^{d} e^{-t}\xi^j \partial_j f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) e^{-\frac{|\eta|^2}{2}} d\eta \\ &\quad - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sum_{j=1}^{d} \partial_j f\big(e^{-t}\xi + \sqrt{1 - e^{-2t}}\eta\big) \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \partial_j e^{-\frac{|\eta|^2}{2}} d\eta \end{aligned}$$

By integration by parts,

$$= -\frac{1}{(2\pi)^{\frac{d}{2}}} e^{-t} \sum_{j=1}^{d} \xi^j \int_{\mathbb{R}^d} \partial_j f\big(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta\big) e^{-\frac{|\eta|^2}{2}} d\eta$$
$$+ \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \Delta_d f\big(e^{-t}\xi + \sqrt{1-e^{-2t}}\eta\big) \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \partial_j e^{-\frac{|\eta|^2}{2}} d\eta.$$

This computation makes sense pointwise (for each $x \in B$). By Lebesgue dominated convergence theorem, we see that the above computation holds in $L^p(B,\mu)$, providing LT_tF for $t > 0$. By letting $t \to 0$, we obtain

$$LF(x) = \Delta_d f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle) - \sum_{j=1}^{d} \langle x, \varphi_j\rangle \partial_j f(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_d\rangle). \tag{2.9}$$

This is exactly (2.7) for $F \in \mathcal{S}$ of the form (2.2). $\square$

Remark 2.5. As we saw in the proof of Proposition 2.4, a question on (infinite dimensional) B can be reduced to that on $\mathbb{R}^d$ by choosing $\varphi_1, \dots, \varphi_d$ to be an orthonormal system in $\in H^*$. We will use this reduction in the following as well. In doing so, we must be careful to make sure that constants are independent of dimensions.

2.3. **Hypercontractivity.** Previously, we proved the contractivity of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. The following proposition due to Nelson [12] shows that much more is true for the Ornstein-Uhlenbeck semigroup. Namely, the Ornstein-Uhlenbeck semigroup is hypercontractive.[8]

Proposition 2.6 (Hypercontractivity)**.** *Given $p > 1$ and $t \geq 0$, set $q(t) = e^{2t}(p-1)+1$. Then, we have*

$$\|T_tF\|_{q(t)} \leq \|F\|_p \tag{2.10}$$

for any $F \in L^p(B,\mu)$.

Remark 2.7. Another way to state Proposition 2.6 is as follows: Given $q > p > 1$, let $t \geq \frac{1}{2}\log\big(\frac{q-1}{p-1}\big)$. Then, we have

$$\|T_tF\|_q \leq \|F\|_p$$

for any $F \in L^p(B,\mu)$.

The following proof is due to Neveu [13].

Proof. By density of $\mathcal{S}$ in $L^p(B,\mu)$, it suffices to prove (2.10) when $B = \mathbb{R}^d$. See Remark 2.5.

[8] Let (X,μ) be a measure space and let H be a non-negative self-adjoint operator on $L^2(X,\mu)$. The semigroup e^{-tH} is called hypercontractive if (i) e^{-tH} is a contraction: $L^p \cap L^2 \to L^p$, $1 \leq p \leq \infty$, and (ii) there exists $t > 0$ such that e^{-tH} is bounded from L^2 to L^4. It is known that if e^{-tH} is hypercontractive, then given $1 < p \leq q < \infty$, there exists $T(p,q) < \infty$ such that e^{-tH} is a contraction from L^p to L^q for $t \geq T(p,q)$. See [5].

Let B_t and $\widetilde{B}_t$, $0 \leq t \leq 1$ be independent d-dimensional Brownian motion with $B_0 = \widetilde{B}_0 = 0$. For $\lambda = \lambda(t) \in (0,1)$ to be chosen later, let

$$q = \frac{p-1}{\lambda^2} = 1. \tag{2.11}$$

Furthermore, let f and g be Borel measurable functions on $\mathbb{R}^d$ and assume that there exist $b > a > 0$ such that $a \leq f(x), g(x) \leq b$ for all $x \in \mathbb{R}^d$. Now, define a new Brownian motion $\widehat{B}_t$ by

$$\widehat{B}_t = \lambda B_t + \sqrt{1-\lambda^2}\widetilde{B}_t. \tag{2.12}$$

Setting $\mathcal{F}_t^B = \sigma(B_s : 0 \leq s \leq t)$ and $\mathcal{F}_t^{\widehat{B}} = \sigma(\widehat{B}_s : 0 \leq s \leq t)$, define continuous martingale M_t and N_t by

$$M_t = \mathbb{E}[f^p(\widehat{B}_1)|\mathcal{F}_t^{\widehat{B}}] \qquad \text{and} \qquad N_t = \mathbb{E}[g^{q'}(B_1)|\mathcal{F}_t^B]$$

for $0 \leq t \leq 1$. Then, by the martingale representation theorem[9], we have

$$\begin{aligned} M_t &= M_0 + \int_0^t \varphi_s d\widehat{B}_s, \\ N_t &= N_0 + \int_0^t \psi_s dB_s. \end{aligned}$$

Note that[10]

$$\langle M\rangle_t = \int_0^t \varphi_s^2 ds, \quad \langle N\rangle_t = \int_0^t \psi_s^2 ds, \quad \text{and} \quad \langle M, N\rangle_t = \lambda \int_0^t \varphi_s \psi_s ds.$$

Then, by Ito's formula, we have

$$\begin{aligned} d(M_t^{\frac{1}{p}} N_t^{\frac{1}{q'}}) = \frac{1}{p} M_t^{\frac{1}{p}-1} N_t^{\frac{1}{q'}} dM_t + \frac{1}{q'} M_t^{\frac{1}{p}} N_t^{\frac{1}{q'}-1} dN_t \\ + \frac{1}{2}\frac{1}{p}\bigg(\frac{1}{p}-1\bigg) M_t^{\frac{1}{p}-2} N_t^{\frac{1}{q'}} d\langle M\rangle_t + \frac{1}{p}\frac{1}{q'} M_t^{\frac{1}{p}-1} N_t^{\frac{1}{q'}-1} d\langle M, N\rangle_t \\ + \frac{1}{2}\frac{1}{q'}\bigg(\frac{1}{q'}-1\bigg) M_t^{\frac{1}{p}} N_t^{\frac{1}{q'}-2} d\langle N\rangle_t. \end{aligned}$$

[9]Let M_t, $a \leq t \leq b$, be a square integrable martingale with respect to $\{\mathcal{F}_t^B : a \leq t \leq b\}$ and $M_a = 0$. Then, M_t has a continuous version $\widetilde{M}_t$ given by $\widetilde{M}_t = \int_0^t \theta_s dB_s$ for some $\theta_t \in L^2_{\text{ad}}([a,b]\times\Omega)$ adapted to the filtration.

[10]Given two local martingales M_t and N_t, i.e. locally square integrable continuous martingales, the quadratic variation $\langle M, N\rangle_t$ is defined such that $M_tN_t - \langle M, N\rangle_t$ is a martingale. If one of them is a process of bounded variation, we set $\langle M, N\rangle_t = 0$ by convention.

Hence,

$$\begin{aligned}
&\mathbb{E}\big[M_t^{\frac{1}{p}} N_t^{\frac{1}{q'}}\big] - \mathbb{E}\big[M_0^{\frac{1}{p}} N_0^{\frac{1}{q'}}\big] \\
&\quad = -\frac{1}{2}\mathbb{E}\Big[\int_0^t M_s^{\frac{1}{p}-2} N_s^{\frac{1}{q'}-2}\Big\{\frac{1}{p}\Big(1-\frac{1}{p}\Big)N_s^2\varphi_s^2 \\
&\qquad\qquad - 2\frac{1}{p}\frac{1}{q'}M_sN_s\lambda\varphi_s\psi_s + \frac{1}{q'}\Big(1-\frac{1}{q'}\Big)M_s^2\psi_s^2\Big\}ds\Big] \\
&\quad = -\frac{1}{2}\mathbb{E}\Big[\int_0^t M_s^{\frac{1}{p}-2} N_s^{\frac{1}{q'}-2}\Big\{\Big(\frac{\sqrt{p-1}}{p}N_s\varphi_s - \frac{\sqrt{q'-1}}{q'}M_s\psi_s\Big)^2 \\
&\qquad\qquad + 2\frac{\sqrt{p-1}}{p}\frac{\sqrt{q'-1}}{q'}N_s\varphi_sM_s\psi_s - 2\frac{1}{p}\frac{1}{q'}\lambda N_s\varphi_sM_s\psi_s\Big\}ds\Big] \qquad (2.13)
\end{aligned}$$

From (2.11), we have

$$\sqrt{(p-1)(q'-1)} = \sqrt{(p-1)\Big(\frac{q}{q-1}-1\Big)} = \sqrt{\frac{p-1}{q-1}} = \sqrt{\frac{p-1}{\lambda^{-2}(p-1)}} = \lambda. \qquad (2.14)$$

Therefore, from (2.13) and (2.14), we have Hence,

$$\begin{aligned}
&\mathbb{E}\big[M_t^{\frac{1}{p}} N_t^{\frac{1}{q'}}\big] - \mathbb{E}\big[M_0^{\frac{1}{p}} N_0^{\frac{1}{q'}}\big] \\
&\quad = -\frac{1}{2}\mathbb{E}\Big[\int_0^t M_s^{\frac{1}{p}-2} N_s^{\frac{1}{q'}-2}\Big(\frac{\sqrt{p-1}}{p}N_s\varphi_s - \frac{\sqrt{q'-1}}{q'}M_s\psi_s\Big)^2 ds\Big] \le 0. \qquad (2.15)
\end{aligned}$$

Noting that $M_1 = f^p(\widehat{B}_1)$, $M_0 = \mathbb{E}[f^p(\widehat{B}_1)]$ $N_1 = g^{q'}(B_1)$, and $N_0 = \mathbb{E}[g^{q'}(B_1)]$, it follows from (2.15) that

$$\mathbb{E}[f(\widehat{B}_1)g(B_1)] \le \mathbb{E}[f^p(\widehat{B}_1)]^{\frac{1}{p}}\mathbb{E}[g^{q'}(B_1)]^{\frac{1}{q'}}, \qquad (2.16)$$

In view of (2.12), we can write (2.16) as

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\big(\lambda\xi + \sqrt{1-\lambda^2}\eta\big)g(\xi)\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|\xi|^2}{2}}\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|\eta|^2}{2}}d\xi d\eta \\
&\qquad \le \Big\{\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f^p(\xi)e^{-\frac{|\xi|^2}{2}}d\xi\Big\}^{\frac{1}{p}}\Big\{\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} g^{q'}(\xi)e^{-\frac{|\eta|^2}{2}}d\eta\Big\}^{\frac{1}{q'}} \\
&\qquad = \|f\|_p\|g\|_{q'}. \qquad (2.17)
\end{aligned}$$

In view of the definition (2.1) of the Ornstein-Uhlenbeck semigroup, we set $\lambda = e^{-t}$ and thus $q = q(t)$. Then, (2.17) yields

$$\int_{\mathbb{R}^d} T_tf(\xi)g(\xi)\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|\xi|^2}{2}}d\xi \le \|f\|_p\|g\|_{q(t)'}. \qquad (2.18)$$

By taking the limits, we see that (2.18) holds for all non-negative functions $f, g \ge 0$. For general Borel measurable functions, noting that $|T_tf(\xi)| \le T_t|f|(\xi)$, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d} T_tf(\xi)g(\xi)\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|\xi|^2}{2}}d\xi\Big| &\le \int_{\mathbb{R}^d} T_t|f|(\xi)|g(\xi)|\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|\xi|^2}{2}}d\xi \\
&\le \|f\|_p\|g\|_{q(t)'}.
\end{aligned}$$

By duality this proves (2.10). $\square$

Remark 2.8. There is also a notion of ultracontractivity. Given a measure space (X, μ), let H be a non-negative self-adjoint operator on $L^2(X, \mu)$. Then, the semigroup e^{-tH} is called ultracontractive if (i) e^{-tH} is a contraction: $L^p \to L^p$, $1 \le p \le \infty$, and (ii) e^{-tH} is bounded from L^2 to L^∞ for all $t > 0$. It is known that the Ornstein-Uhlenbeck semigroup is not ultracontractive. See Theorem 4.3.2 in [5].[11]

3. Logarithmic Sobolev inequality

In the following, we prove that the hypercontractivity (Proposition 2.6) implies the following logarithmic Sobolev inequality.

Proposition 3.1. *Let $p > 1$. For $F \in \mathrm{Dom}\,(L_p)$, i.e. F belongs to the domain of L in $L^p(B, \mu)$, we have*

$$\int_B |F(x)|^p \log |F(x)| \mu(dx) \le -\frac{p}{2(p-1)} \int_B F_p(x) LF(x) \mu(dx) + \|F\|_p^p \log \|F\|_p^p, \tag{3.1}$$

where

$$F_p(x) = |F(x)|^{p-1} \operatorname{sgn}(F(x)) = \begin{cases} |F(x)|^{p-1}, & \text{if } F(x) > 0, \\ 0, & \text{if } F(x) = 0, \\ -|F(x)|^{p-1}, & \text{if } F(x) < 0. \end{cases}$$

Gross [8] gave a direct proof of the logarithmic Sobolev inequality (3.1). Moreover, he showed that the logarithmic Sobolev inequality is equivalent to the hypercontractivity.

For $p \ge 2$, we can rewrite (3.1) as

$$\int_B |F(x)|^p \log |F(x)| \mu(dx) \le \frac{p}{2} \int_B |F(x)|^{p-2} \|DF(x)\|_{H^*}^2 \mu(dx) + \|F\|_p^p \log \|F\|_p^p. \tag{3.2}$$

The usual Sobolev inequality states that

$$\|u\|_{L^q(\mathbb{R}^d; m)} \le C(d, p, q) \big\| |\nabla|^s u \big\|_{L^p(\mathbb{R}^d; m)}$$

if $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$, $1 < p < q < \infty$. Here, m denotes the Lebesgue measure on $\mathbb{R}^d$. For fixed s, say $s = 1$, the gain in integrability tends to 0 as $d \to \infty$. Indeed, the logarithmic gain in integrability in (3.1) and (3.3) is the best we can have. See Remark 3.2 below.

Proof of Proposition 3.1. It suffices to prove (3.1) for $F \in \mathcal{P}$. Proposition 2.6 states that

$$\frac{1}{t}\big(\|T_t F\|_{q(t)} - \|F\|_p\big) \le 0,$$

where $q(t)$ is given by

$$q(t) = e^{2t}(p-1) + 1. \tag{3.3}$$

Thus, we have

$$\frac{d}{dt} \|T_t F\|_{q(t)} \bigg|_{t=0} \le 0. \tag{3.4}$$

[11] We can assume that $F \in \mathcal{S}$ of the form (1.13) with $d = 1$. Then, we can apply the one-dimensional result (Theorem 4.3.2 in [5]). Note that a direct computation shows $\widetilde{H} = -\partial_x^2 + x\partial_x$ in Theorem 4.3.2 of [5].

In the following, we compute this derivative explicitly. Define ϕ by

$$\phi(t) = \|T_tF\|_{q(t)}^{q(t)} = \int_B |T_tF(x)|^{q(t)}\mu(dx). \tag{3.5}$$

Then, we have

$$\begin{aligned}\phi'(t) = \int_B \Big\{ &|T_tF(x)|^{q(t)} \log|T_tF(x)| q'(t) \\ &+ q(t)|T_tF(x)|^{q(t)-1} \operatorname{sgn}(T_tF(x)) \frac{d}{dt} T_tF(x) \Big\} \mu(dx).\end{aligned} \tag{3.6}$$

Noting that $q'(t) = 2e^{2t}(p-1)$, we have, for $t = 0$,

$$\phi'(0) = \int_B \Big\{ 2(p-1)|F(x)|^p \log|F(x)| + p \underbrace{|F(x)|^{p-1} \operatorname{sgn}(F(x))}_{=F_p(x)} LF(x) \Big\} \mu(dx). \tag{3.7}$$

On the other hand, we have

$$\frac{d}{dt}\|T_tF\|_{q(t)} = \frac{d}{dt}\big(\phi(t)^{\frac{1}{q(t)}}\big) = \frac{1}{q(t)}\phi(t)^{\frac{1}{q(t)}-1}\phi'(t) + \phi(t)^{\frac{1}{q(t)}} \log\phi(t) \frac{-q'(t)}{q(t)^2}. \tag{3.8}$$

Putting (3.4), (3.7), and (3.8) together, we obtain

$$\begin{aligned}&\frac{d}{dt}\|T_tF\|_{q(t)}\bigg|_{t=0} \\ &\quad= \frac{1}{p}\|F\|_p^{p(\frac{1}{p}-1}\bigg\{ \int_B \Big\{ 2(p-1)|F(x)|^p \log|F(x)| + pF_p(x)LF(x)\Big\}\mu(dx)\bigg\} \\ &\quad\quad - \frac{2(p-1)}{p^2}\|F\|_p \log\|F\|_p^p \\ &\quad\le 0.\end{aligned} \tag{3.9}$$

Then, (3.1) follows from multiplying (3.9) by $\frac{p}{2(p-1)}\|F\|_p^{p-1}$. ☐

The converse statement that the logarithmic Sobolev inequality (3.1) implies the hypercontractivity (2.10) basically follows from (a slight modification of) reversing the proof of Proposition 3.1. See Theorem 1 in [8] for details.

Given $f \in L^\infty(B,\mu)$ and $g \in C_c^\infty(0,\infty)$, let $F = \int_0^\infty g(t)T_tf\,dt$. Then, from Proposition 2.2, we see that $F \in L^q(B,\mu)$ for each $q > 1$. Let $\mathcal{D}$ denote the linear span of all such F. Note that $\mathcal{D}$ is invariant under T_t and that T_tF, $F \in \mathcal{D}$ is differentiable in $L^q(B,\mu)$ for all $q > 1$. In particular, in view of Hille-Yosida theorem, we have $\mathcal{D} \subset \operatorname{Dom}(L_q)$ for any $q > 1$.

Given $F \in \mathcal{D}$, it follows from the logarithmic Sobolev inequality (3.1) with (3.5), (3.6), and (3.8) that

$$\begin{aligned}\frac{d}{dt}\|T_tF\|_{q(t)} = \frac{2(q(t)-1)}{q(t)\|T_tF\|_{q(t)}^{q(t)-1}} \bigg\{ &\int_B |T_tF(x)|^{q(t)} \log|T_tF(x)|\mu(dx) \\ &+ \frac{q(t)}{2(q(t)-1)}\int_B (T_tF)_{q(t)}(x)LT_tF(x)\mu(dx) - \|T_tF\|_{q(t)}^{q(t)}\log\|T_tF\|_{q(t)}\bigg\} \\ \le 0 \hspace{8em}&\end{aligned}$$

for $t \geq 0$, since $T_t F \in \mathrm{Dom}\,(L_{q(t)})$. This shows (2.10) for $F \in \mathcal{D}$. Then, by density of $\mathcal{D}$, (2.10) extends to all $F \in L^p(B, \mu)$. Indeed, if $F \in L^p(B, \mu)$, given $\varepsilon > 0$ and $t > 0$, let $F_\varepsilon \in \mathcal{D}$ such that

$$\|F - F_\varepsilon\|_p + \|F - F_\varepsilon\|_{q(t)} < \varepsilon.$$

Then, from (2.10) for F_ε and Proposition 2.2, we have

$$\|T_t F\|_{q(t)} \leq \|T_t F_\varepsilon\|_{q(t)} + \|T_t F - T_t F_\varepsilon\|_{q(t)} < \|F_\varepsilon\|_p + \varepsilon < \|F\|_p + 2\varepsilon,$$

for any $\varepsilon > 0$. This proves (2.10).

Remark 3.2. Let $q > p$. Then, the following inequality

$$\|F\|_p \leq C\big(\|D^k F\|_p + \|F\|_p\big) \tag{3.10}$$

can not hold, no matter how large we take the constant C or $k \in \mathbb{N}$.

We consider the case $B = \mathbb{R}$. Let $F(x) = e^{\alpha x}$, $\alpha > 0$. Then, we have

$$\|F\|_q^q = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{q\alpha x} e^{-\frac{x^2}{2}}\, dx = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}q^2\alpha^2} \int_{\mathbb{R}} e^{-\frac{(x-q\alpha)^2}{2}}\, dx = e^{\frac{1}{2}q^2\alpha^2}.$$

On the other hand, we have

$$\|D^k F\|_p^p = \alpha^{kp}\|F\|_p^p = \alpha^{kp} e^{\frac{1}{2}p^2\alpha^2}$$

Noting that $q > p$, we have

$$\frac{\|F\|_q}{\|D^k F\|_p + \|F\|_p} = \frac{e^{\frac{1}{2}q\alpha^2}}{(\alpha^k + 1)e^{\frac{1}{2}p\alpha^2}} \to \infty,$$

as $\alpha \to \infty$. This shows that (3.10) can not hold.

Appendix A. Eigenfunction of the Ornstein-Uhlenbeck operator: multiple Wiener integrals

A.1. **Hermite polynomials and multiple Wiener integrals.** The Hermite polynomial $H_n(\xi)$, $n \in \mathbb{Z}_{\geq 0}$, $\xi \in \mathbb{R}$, is defined by[12]

$$H_n(\xi) = \frac{(-1)^n}{n!} e^{\frac{\xi^2}{2}} \frac{d^n}{d\xi^n} e^{-\frac{\xi^2}{2}}.$$

The first few polynomials are

$$H_0(\xi) = 1, \quad H_1(\xi) = \xi, \quad H_2(\xi) = \tfrac{1}{2}(\xi^2 - 1), \quad H_3(\xi) = \tfrac{1}{6}(\xi^3 - \xi).$$

We list some of the fundamental properties of the Hermite polynomials:

[12]There are slightly different definitions of the Hermite polynomials with different multiplicative constants.

(1) Generating function: $$e^{t\xi-\frac{t^2}{2}}=\sum_{n=0}^{\infty}t^nH_n(\xi), \tag{A.1}$$

(2) Derivative: $$\frac{d}{d\xi}H_n(\xi)=H_{n-1}(\xi), \tag{A.2}$$

(3) Recursive formula: $$(n+1)H_{n+1}(\xi)-\xi H_n(\xi)+H_{n-1}(\xi)=0, \tag{A.3}$$

(4) Orthogonality: $$\int_{\mathbb{R}}H_n(\xi)H_m(\xi)\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi=\frac{1}{n!}\delta_{nm}, \tag{A.4}$$

(5) Characteristic function: $$\int_{\mathbb{R}}e^{i\eta\xi}H_n(\xi)\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi=e^{-\frac{\eta^2}{2}}\frac{i^n}{n!}\eta^n. \tag{A.5}$$

See Chapter 9 in [10] for more on the Hermite polynomials (and the multiple Wiener-Ito integrals).

Let $\Lambda=\{a=(a_1,a_2,\dots):\ a_j\in\mathbb{Z}_{\geq 0},\ a_j=0$ except for finitely many $j\}$. For $a\in\Lambda$, we define $a!:=\prod_{j=1\infty}(a_j!)$ and $|a|:=\sum_{j=1}^{\infty}a_j$. We fix a complete orthonormal system $\{\varphi_j\}\subset B^*$ in H^* in the following. Given $a\in\Lambda$, We define a *Fourier-Hermite polynomial* $\mathbf{H}_a$ by

$$\mathbf{H}_a(x)=\prod_{j=1}^{\infty}H_{a_j}(\langle x,\varphi_j\rangle). \tag{A.6}$$

Note that the infinite product above is actually a finite product since $H_0(\xi)=1$.

Definition A.1. Give $n\in\mathbb{Z}_{\geq 0}$, the closed subspace spanned by $\{\mathbf{H}_a(x):\ |a|=n\}$ in $L^2(B,\mu)$ is called the space of *multiple Wiener integrals*[13] of degree n and is denoted by $\mathcal{H}_n$. We denote by J_n the orthogonal projection to $\mathcal{H}_n$.

Note that $\mathcal{H}_1$ defined in Subsection 1.1 coincides with the $\mathcal{H}_1$ in Definition A.1. See Subsection B.1 for more properties of $\mathcal{H}_n$ and J_n.

Proposition A.2 (Wiener-Ito theorem)**.** (i) *The collection* $\{\sqrt{a!}\mathbf{H}_a:\ a\in\Lambda\}$ *is a complete orthonormal system in* $L^2(B,\mu)$.
(ii) *The collection* $\{\sqrt{a!}\mathbf{H}_a:\ a\in\Lambda,\ |a|=n\}$ *is a complete orthonormal system in* $\mathcal{H}_n$.
(iii) *The space* $L^2(B,\mu)$ *has the following direct sum decomposition:*

$$L^2(B,\mu)=\bigoplus_{n=1}^{\infty}\mathcal{H}_n.$$

This decomposition is called the Wiener-Ito expansion.

A.2. **Kernel expression of multiple Wiener integrals.** Let $\mathcal{L}^n_{(2)}(H;\mathbb{R})$ be the collection of all n-linear Hilbert-Schmidt operators on $H^{\otimes n}$. It is a Hilbert space with the following Hilbert-Schmidt inner product; Then,

$$(S,T)_{\mathrm{HS}}=(S,T)_{\mathcal{L}^n_{(2)}(H;\mathbb{R})}:=\sum_{j_1,\dots,j_n=1}^{\infty}S(e_{j_1},\dots,e_{j_n})T(e_{j_1},\dots,e_{j_n}),$$

[13]They are also called multiple Wiener-Ito integrals of degree n or homogeneous Wiener chaoses of order n.

where $\{e_j\} \subset H$ is a complete orthonormal system in H. We denote the induced norm by $\|\cdot\|_{\text{HS}}$. Given $T \in \mathcal{L}^n_{(2)}(H;\mathbb{R})$, we define its symmetrization $\mathcal{S}T$ by

$$\mathcal{S}T(h_1,\dots,h_n) = \frac{1}{n!}\sum_{\sigma\in S_n} T(h_{\sigma(1)},\dots,h_{\sigma(n)}), \qquad h_1,\dots,h_n \in H,$$

where S_n is the symmetric group of order n. We say that $T \in \mathcal{L}^n_{(2)}(H;\mathbb{R})$ is symmetric if $\mathcal{S}T = T$. We denote by $\mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$ the collection of all symmetric Hilbert-Schmidt operators in $\mathcal{L}^n_{(2)}(H;\mathbb{R})$. This is a closed subspace of $\mathcal{L}^n_{(2)}(H;\mathbb{R})$.

Given a fixed complete orthonormal system $\{\varphi_j\} \subset B^*$ in H^*, define $\otimes^n_{k=1}\varphi_{j_k} \in \mathcal{L}^n_{(2)}(H;\mathbb{R})$ by

$$\otimes^n_{k=1}\varphi_{j_k}(h_,\dots,h_n) = \langle h_1,\varphi_{j_1}\rangle\cdots\langle h_n,\varphi_{j_n}\rangle,$$

for $h_1,\dots,h_n \in H$. Then,

$$\big\{\otimes^n_{k=1}\varphi_{j_k} : j_1,\dots,j_n = 1,2,\dots\big\}$$

forms a complete orthonormal system of $\mathcal{L}^n_{(2)}(H;\mathbb{R})$.

We can also construct a complete orthonormal system of $\mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$ in a similar manner. Given $a \in \Lambda$ with $|a| = n$, define $\varphi^a \in \mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$ by

$$\varphi^a = \mathcal{S}(\varphi_1^{\otimes a_1}\otimes\varphi_2^{\otimes a_2}\otimes\cdots). \tag{A.7}$$

Note that $\|\varphi^a\|_{\text{HS}} = \sqrt{a!/n!}$. Then, the collection

$$\big\{\sqrt{a!/n!}\varphi^a : a \in \Lambda,\, |a| = n\big\} \tag{A.8}$$

forms a complete orthonormal system of $\mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$.

We now consider the kernel representation of multiple Wiener integrals. Given $F \in L^2(B,\mu)$, we define a functional τF on H^* by

$$\tau F(\ell) = e^{\frac{|\ell|^2}{2}}\int_B e^{iI_1(\ell)(x)}F(x)\mu(dx),$$

for $\ell \in H^*$. Then, given $a \in \Lambda$ with $|a| = n$, it follows from (A.5) and (A.7) that

$$\tau\mathbf{H}_a(\ell) = i^n\frac{1}{a!}(\varphi^a,\ell^{\otimes n})_{\text{HS}}.$$

Thus, we have

$$\tau\sqrt{a!}\mathbf{H}_a(\ell) = i^n\bigg(\frac{\varphi^a}{\sqrt{a!}},\ell^{\otimes n}\bigg)_{\text{HS}}. \tag{A.9}$$

In view of Proposition A.2 (ii) and (A.8), we see that τ is an isomorphism between $\mathcal{H}_n$ and $\mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$. Namely, given $F \in \mathcal{H}_n$, there exists $T \in \mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$ such that

$$\tau\sqrt{a!}F(\ell) = i^n(T,\ell^{\otimes n})_{\text{HS}}. \tag{A.10}$$

Definition A.3. Using the correspondence (A.10), we define $I_n : \mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R}) \to \mathcal{H}_n$ by $I_n(T) = F$, called a *multiple Wiener integral* (of T) of degree n An operator $T \in \mathcal{S}\mathcal{L}^n_{(2)}(H;\mathbb{R})$ is called the *kernel* of F. If $T \in \mathcal{L}^n_{(2)}(H;\mathbb{R})$ is not symmetric, we define $I_n(T) := I_n(\mathcal{S}T)$.

Note that we have

$$\|I_n(T)\|_{L^2(B,\mu)} = \sqrt{n!}\|T\|_{\mathrm{HS}}.$$

Furthermore, from (A.9), we have $I_n(\varphi^a) = a!\mathbf{H}_a$ for $a \in \Lambda$ with $|a| = n$.

The operator I_n corresponds to the so-called Wick product[14] As the Wick product is usually denoted by $: \cdot :$, we may write $: T(x) :$ in place of $I_n(T)$. For instance, given $\varphi_1, \dots, \varphi_n \in H^*$, we may write

$$I_n(\varphi_1 \otimes \cdots \otimes \varphi_n)(x) \qquad \text{or} \qquad :\langle x, \varphi_1\rangle \cdots \langle x, \varphi_n\rangle : .$$

Remark A.4. Note that we have slightly abused notations. Given $\varphi \in H^*$, the expression $\langle x, \varphi\rangle$ does not necessarily make sense for $x \in B \supset H$. φ can be approximated by a sequence $\{\psi_n\} \subset B^*$ and $\langle x, \psi_n\rangle$ converges to $I_1(\varphi)$ in $L^2(B,\mu)$. Then, we define $\langle x, \varphi\rangle := I_1(\varphi)$. This is called a Wiener integral and is defined μ-almost everywhere.

Using the notation above, we have

$$:\langle x, \xi_1\rangle^{a_1} \cdots \langle x, \xi_n\rangle^{a_n}: = \prod_{j=1}^{\infty} a_j! H_{a_j}(\langle x, \xi_j\rangle). \tag{A.11}$$

where $\eta_1, \dots, \eta_n$ are orthonormal in H^*.

Example 3. Let $\xi, \eta \in H^*$. Then, we have

$$: \langle x, \xi\rangle\langle x, \eta\rangle : = \langle x, \xi\rangle\langle x, \eta\rangle - (\xi, \eta)_{H^*}. \tag{A.12}$$

If $\xi \perp \eta$, then (A.12) is obvious. Otherwise, write η as

$$\eta = \frac{(\eta, \xi)}{|\xi|^2}\xi + \underbrace{\eta - \frac{(\eta, \xi)}{|\xi|^2}\xi}_{=:\zeta}.$$

Noting that $\xi \perp \zeta$, it follows from (A.11) that

$$\begin{aligned} : \langle x, \xi\rangle\langle x, \eta\rangle : &= (\eta, \xi) :\big\langle x, \tfrac{\xi}{|\xi|}\big\rangle^2 : + \langle x, \xi\rangle\langle x, \zeta\rangle = \langle x, \xi\rangle\big\langle x, (\eta, \xi)\tfrac{\xi}{|\xi|^2} + \zeta\big\rangle - (\eta, \xi) \\ &= \langle x, \xi\rangle + \langle x, \eta\rangle - (\eta, \xi). \end{aligned}$$

This proves (A.12).

Example 4 (classical Wiener space)**.** In the case of the classical Wiener space $B = C_0([0,\infty);\mathbb{R}^d)$, we can obtain concrete expressions for multiple Wiener integrals.

Given n-linear map T on $H^{\otimes n}$, there exists a unique integral kernel $\Phi = (\Phi_{j_1,\dots,j_n}) \in L^2([0,\infty);(\mathbb{R}^d)^{\otimes n})$ such that[15]

$$T(h_1, \dots, h_n) = \sum_{j_1,\dots,j_n}^{d} \int_0^\infty \cdots \int_0^\infty \Phi_{j_1,\dots,j_n}(t_1, \dots, t_n)\dot{h}_1^{j_1}(t_1) \cdots \dot{h}_n^{j_n}(t_n)dt_1 \cdots dt_n. \tag{A.13}$$

for $h_1, \dots, h_n \in H$.

[14] It is also called Wick ordering or Wick renormalization.

[15] Recall the definition of the Cameron-Martin space in (1.10). The superscript j on $\dot{h}^j(t)$ denotes the jth coordinate of $\dot{h}(t)$.

If T is symmetric, then for any permutation $\sigma \in S_n$, we have

$$
\begin{aligned}
& T(h_{\sigma(1)}, \dots, h_{\sigma(n)}) \\
& \quad = \sum_{j_1, \dots, j_n}^{d} \int_0^\infty \cdots \int_0^\infty \Phi_{j_1, \dots, j_n}(t_1, \dots, t_n) \dot{h}^{j_1}_{\sigma(1)}(t_1) \cdots \dot{h}^{j_n}_{\sigma(n)}(t_n) dt_1 \cdots dt_n \\
& \quad = \sum_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}^{d} \int_0^\infty \cdots \int_0^\infty \Phi_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}(t_1, \dots, t_n) \dot{h}^{j_{\sigma(1)}}_{\sigma(1)}(t_1) \cdots \dot{h}^{j_{\sigma(n)}}_{\sigma(n)}(t_n) dt_1 \cdots dt_n
\end{aligned}
$$

By rearranging the order of $\dot{h}^{j_k}_k$'s,

$$
\begin{aligned}
& = \sum_{j_1, \dots, j_n}^{d} \int_0^\infty \cdots \int_0^\infty \Phi_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}(t_1, \dots, t_n) \dot{h}^{j_1}_1(t_{\sigma^{-1}(1)}) \cdots \dot{h}^{j_n}_n(t_{\sigma^{-1}(n)}) dt_1 \cdots dt_n \\
& = \sum_{j_1, \dots, j_n}^{d} \int_0^\infty \cdots \int_0^\infty \Phi_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}(t_{\sigma(1)}, \dots, t_{\sigma(n)}) \dot{h}^{j_1}_1(t_n) \cdots \dot{h}^{j_n}_n(t_n) dt_1 \cdots dt_n.
\end{aligned} \tag{A.14}
$$

Hence, from (A.13) and (A.14), we have

$$\Phi_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}(t_{\sigma(1)}, \dots, t_{\sigma(n)}) = \Phi_{j_1, \dots, j_n}(t_1, \dots, t_n).$$

We denote the collection of all such functions by $\hat{L}^2([0, \infty); (\mathbb{R}^d)^{\otimes n})$.

Using this relation, we have the following expression for multiple Wiener integrals.

Proposition A.5. *Let $\Phi \in \hat{L}^2([0, \infty); (\mathbb{R}^d)^{\otimes n})$. Then, we have*

$$I_n(\Phi) = \sum_{j_1, \dots, j_n}^{d} \int_0^\infty dw^{j_n}(t_n) \cdots \int_0^{t_3} dw^{j_2}(t_2) \int_0^{t_2} \Phi_{j_{\sigma(1)}, \dots, j_{\sigma(n)}}(t_{\sigma(1)}, \dots, t_{\sigma(n)}) dw^{j_1}(t_1).$$

See [14] for the proof.

A.3. **Eigenfunctions of the Ornstein-Uhlenbeck operator.** The eigenvalues and eigenfunctions of the Ornstein-Uhlenbeck operator L are completely known. In fact, they are given in terms of the Fourier-Hermite polynomials (A.6).

Lemma A.6. *Let $\mathbf{H}_a(x)$, $a \in \Lambda$, be a Fourier-Hermite polynomial defined by* (A.6). *Then, we have*

$$L\mathbf{H}_a(x) = -|a|\mathbf{H}_a(x), \tag{A.15}$$

and consequently,

$$T_t\mathbf{H}_a(x) = e^{-|a|t}\mathbf{H}_a(x) \tag{A.16}$$

for $T \geq 0$.

Proof. The identity (A.15) is immediate from (2.9) with (A.2) and (A.3), while (A.16) follows from (A.15) and the definition of $T_t = e^{tL}$. ∎

In the following, we use Lemma A.6 and show the symmetry of the semigroup $\{T_t\}$. Given $F, G \in L^2(B, \mu)$, we define the inner product by

$$(F, G) = \int_B F(x), G(x)\mu(dx).$$

With a light abuse of notation, we also use (F, G) to denote the natural pairing of $F \in L^p(B, \mu)$ and $G \in L^{p'}(B, \mu)$.

Proposition A.7. *Let $p > 1$. Then, we have*

$$(T_t F, G) = F, T_t G), \tag{A.17}$$

for $F \in L^p(B, \mu)$ and $G \in L^{p'}(B, \mu)$. Moreover, if $F \in L^p(B, \mu)$ belongs to Dom(L_p) of the Ornstein-Uhlenbeck operator L and $G \in L^{p'}(B, \mu)$ belongs to Dom$(L_{p'})$, then we have

$$(LF, G) = (F, LG). \tag{A.18}$$

Proof. Let $\mathbf{H}_a(x)$ and $\mathbf{H}_b(x)$ be Fourier-Hermite polynomials. Then, from Proposition A.2, we have

$$(T_t\mathbf{H}_a, \mathbf{H}_b) = e^{-t|a|}(\mathbf{H}_a, \mathbf{H}_b) = e^{-t|a|}\frac{1}{a!}\delta_{ab} = e^{-t|b|}\frac{1}{b!}\delta_{ab} = (\mathbf{H}_a, T_t\mathbf{H}_b).$$

Then, (A.17) follows from the density of Fourier-Hermite polynomials. The second identity (A.18) follows from differentiating (A.17) at time 0. □

Proposition A.8. *Let $F, g \in \mathcal{S}$. Then, we have*

$$L(FG)(x) = LF(x)G(x) + F(x)LG(x) + 2(DF(x), DG(x))_{H^*}, \tag{A.19}$$

$$(LF, G) = -(DF, DG). \tag{A.20}$$

Here, $(DF, DG) = \int_B (DF(x), DG(x))_{H^*}\mu(dx)$.

Proof. We only consider when $B = \mathbb{R}^d$. The first identity (A.19) follows from (2.9), and the second identity follows from integrating (A.20) over B and use (A.18). □

Appendix B. Applications

In this appendix, we discuss several applications of the hypercontractivity of the Ornstein-Uhlenbeck semigroup (Proposition 2.6).

B.1. **Multiple Wiener integrals.** Recall from Definition A.1 that $\mathcal{H}_n$ is the closed subspace of $L^2(B, \mu)$ spanned by $\{\mathbf{H}_a : |a| = n\}$ and that J_n denotes the orthogonal projection onto $\mathcal{H}_n$ (in $L^2(B, \mu)$).

Proposition B.1. *The space $\mathcal{H}_n$ of multiple Wiener integrals of order n is a closed subspace in $L^p(B, \mu)$, $p > 1$, and the norms $\|\cdot\|_p$ of $L^p(B, \mu)$, $p > 1$, are equivalent to each other.*

Proof. Let F be a finite linear combination of $\{\mathbf{H}_a : |a| = n\}$. By Lemma A.6, we have $T_t F = e^{-nt}F$. Given $1 < p < q$, let t such that $q = e^{2t}(p-1) + 1$. Then, by Proposition 2.6, we have

$$\|F\|_p \geq \|T_t F\|_q = e^{-nt}\|F\|_q,$$

i.e. $\|F\|_p \leq \|F\|_q \leq e^{nt}\|F\|_p$. This shows that the closed subspace of $L^p(B, \mu)$ spanned by $\{\mathbf{H}_a : |a| = n\}$ is independent of p. Since $\mathcal{H}_n$ is the closed subspace of $L^2(B, \mu)$ spanned by

$\{\mathbf{H}_a : |a| = n\}$, it is also a closed subspace of $L^p(B,\mu)$. The equivalence of norms follows from the above computation as well. $\square$

Proposition B.2. *The orthogonal projection J_n is bounded on $L^p(B,\mu)$, $p > 1$. Moreover, we have*

$$J_n J_m = J_m J_n = \delta_{nm} J_n, \tag{B.1}$$

$$T_t J_n = J_n T_t = e^{-nt} J_n. \tag{B.2}$$

Proof. We first clarify the meaning of the statement. On the one hand, for $1 < p < 2$, this means that J_n is to be extended to a bounded operator in $L^p(B,\mu)$, since $L^p(B,\mu) \supset L^2(B,\mu)$. On the other hand, for $p > 2$, the restriction of J_n to $L^p(B,\mu)$ is to be bounded.

Let $1 < p < 2$. Choose t such that $2 = e^{2t}(p-1)+1$. Then, given $F \in L^2(B,\mu)$, noting that (B.2) holds for $p = 2$ and that $T_t J_n F = e^{-nt} J_n F$, it follows from Proposition 2.6 that

$$\|e^{-nt} J_n F\|_p = \|T_t J_n F\|_p \le \|T_t J_n F\|_2 = \|J_n T_t F\|_2 \le \|T_t F\|_2 \le \|F\|_p.$$

Hence, we have

$$\|J_n F\|_p \le e^{nt} \|F\|_p.$$

This shows that J_n can be extended to a bounded operator in $L^p(B,\mu)$.

Next, let $p > 2$. Choose t such that $p = e^{2t}+1$. Then, given $F \in L^p(B,\mu)$, Proposition 2.6 yields

$$\|e^{-nt} J_n F\|_p = \|T_t J_n F\|_p \le \|J_n F\|_2 \le \|F\|_2 \le \|F\|_p.$$

Hence, we have

$$\|J_n F\|_p \le e^{nt} \|F\|_p.$$

Namely, the restriction of J_n to $L^p(B,\mu)$ is a bounded operator.

The identities (B.1) and (B.2) can be easily seen to hold in $L^p(B,\mu)$ for (i) $1 < p < 2$ by the density of $L^2(B,\mu)$ and (ii) $p > 2$ since $L^p(B,\mu) \subset L^2(B,\mu)$. $\square$

B.2. **Applications in nonlinear dispersive PDEs.** The hypercontractivity of the Ornstein-Uhlenbeck semigroup has recently played an important role in the study of nonlinear dispersive partial differential equations (PDEs) such as the nonlinear Schrödinger equations (NLS). In particular, it appears in (i) the construction of the Gibbs measures [16, 15, 11]:

$$\text{“} d\rho = Z^{-1} e^{-H(u)} du \text{''},$$

where H is the Hamiltonian for a given nonlinear dispersive PDE, and (ii) probabilistic construction of solutions in a low regularity setting [3]. In this subsection, we briefly discuss the main tools that appear as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup. See [16, 3, 15, 11] for more details.

Proposition 2.6 along with Lemma A.6 yields that

$$\|F\|_p \le (p-1)^{\frac{k}{2}} \|F\|_2 \tag{B.3}$$

for any $F \in \mathcal{H}_n := \overline{\text{span}\,\{\mathbf{H}_a : |a| = n\}}$. Note that this closure does not depend on $L^p(B,\mu)$ thanks to Proposition B.1.

The following proposition is an extension of the estimate (B.3).

Proposition B.3. *Fix $k \in \mathbb{N}$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Given $N \in \mathbb{N}$, let $\{g_n\}_{n=1}^N$ be a sequence of independent standard complex-valued Gaussian random variables. Define $S_k(\omega)$ by*

$$S_k(\omega) = \sum_{A(k,N)} c(n_1, \dots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega), \tag{B.4}$$

where $A(k, N)$ is defined by

$$A(k,N) = \big\{(n_1, \dots, n_k) \in \{1, \dots, N\}^k : n_1 \le \cdots \le n_k\big\}.$$

Then, for all $N \ge 1$ and $p \ge 2$, we have

$$\|S_k\|_{L^p(\Omega)} \le \sqrt{k+1}(p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}. \tag{B.5}$$

The main point is that (B.5) is independent of N. The proof is based on viewing the real part x_n and imaginary part y_n of $\{g_n\}_{n=1}^N$ as elements in an abstract Wiener space B. Then, by Gram-Schmidt orthogonalization process, we can express $g_{n_1} \cdots g_{n_k}$ as a linear combination of Fourier-Hermite polynomial $\mathbf{H}_a$ for $|a| \le k$ and apply (B.3). See Proposition 2.4 in [15] for details.

The following simple lemma plays an important role in converting (B.5) into a probabilistic statement.

Lemma B.4. *Let (B, H, μ) be an abstract Wiener space. Given a measurable function F, suppose that there exist $\alpha, N > 0$, $k \in \mathbb{N}$, and $C > 0$ such that*

$$\|F\|_p \le C N^{-\alpha} p^{\frac{k}{2}}, \tag{B.6}$$

for all $p \ge 2$. Then, there exists $\delta > 0$, C_1, independent of N and α such that

$$\int_B e^{\delta N^{2\alpha} k |F(x)|^{\frac{2}{k}}} \mu(dx) \le C_1. \tag{B.7}$$

As a consequence, we have

$$\mu\big(\{x \in B : |F(x)| > \lambda\}\big) \le C_1 e^{-\delta N^{\frac{2\alpha}{k}} \lambda^{\frac{2}{k}}} \tag{B.8}$$

for all $\lambda > 0$.

The proof of (B.7) follows from expanding the exponential in the Taylor series and applying (B.6). See Lemma 4.5 in [16] for details.

Combining Proposition B.3 and Lemma B.4, we obtain the following tail estimate on the L^∞_ω-norm of S_k; there exist $c, C > 0$ such that

$$P\big(|S_k| > \lambda\big) \le C \exp\Big(- c \|S_k\|_{L^2(\Omega)}^{-\frac{2}{k}} \lambda^{\frac{2}{k}}\Big). \tag{B.9}$$

This estimate some played an important role in constructing the Gibbs measures for some nonlinear dispersive PDEs, in particular, in showing the integrability of a weight with respect to the periodic (fractional) Brownian motion. This topic is related to Euclidean quantum field theory.

Another consequence of (B.9) is the following probabilistic improvement of Young's inequality. Such a probabilistic improvement was essential in the probabilistic construction of solutions to the (Wick ordered) cubic NLS in a low regularity setting [3]. For simplicity, we consider a trilinear case.

Proposition B.5. *Let $a_n, b_n, c_n \in \ell^2(\mathbb{Z};\mathbb{C})$. Given a sequence $\{g_n\}_{n\in\mathbb{Z}}$ of independent standard complex-valued Gaussian random variables, define $a_n^\omega = g_n a_n$, $b_n^\omega = g_n b_n$, and $c_n^\omega = g_n c_n$, $n \in \mathbb{Z}$. Then, given $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \subset \Omega$ with $P(\Omega_\varepsilon^c) < \varepsilon$ and $C_\varepsilon > 0$ such that*[16]

$$\big\|a_n^\omega * b_n^\omega * c_n^\omega\big\|_{\ell^2} \le C_\varepsilon \|a_n\|_{\ell^2}\|b_n\|_{\ell^2}\|c_n\|_{\ell^2} \tag{B.10}$$

for all $\omega \in \Omega_\varepsilon$.

The proof is immediate from (B.9). Without randomization, Young's inequality only yields

$$\big\|a_n * b_n * c_n\big\|_{\ell^2} \le \|a_n\|_{\ell^2}\|b_n\|_{\ell^1}\|c_n\|_{\ell^1}.$$

Thus, we see that there is a significant improvement[17] in (B.10) under randomization of the sequences, which was a key in establishing a crucial nonlinear estimate in a probabilistic manner in [3].

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[16] One can choose $C_\varepsilon = \big(\frac{1}{c}\log\frac{C}{\varepsilon}\big)^{\frac{3}{2}}$.

[17] Recall that $\ell^1 \subset \ell^2$.

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